Computational Finance

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1 Question 1: The Binomial Method

Problem statement

Question 1.1. *What is the proce of the zero-coupon bond $P(t_i, t_m)$?*

Solution 1.1.

$$P(t_i, t_m) = Fe^{-r(m-i)\Delta t} \tag{1}$$

Question 1.2. *For what values of u, d and p do we obtain the risk-neutral dynamics?*

Solution 1.2. *Parameteres must satisfy following equation*

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2}$$

Question 1.3. *Price a European call at $t_0 = 0$ with maturity $t_M = 1$, strike $K = 70$, interest rate $r = 0.01$, upward move $u = 1.05$ and current spot $S_0 = 50$. Derive numerically the convergence rate as $M \to \infty$*

Solution 1.3.

$$C(M) = e^{-rT}\mathbb{E}\left[(S_{t_M} - K)^+\right]$$

$$\mathbb{P}\left(S_{t_M} = S_0 u^k d^{M-k}\right) = \binom{n}{k} p^k (1-p)^{M-k}$$

$$C(M) = e^{-rT}\sum_{k}^{M}\binom{n}{k} p^k (1-p)^{M-k}\left(S_{t_M} - K\right)^+ = e^{-rT}\sum_{a}^{M}\binom{n}{k} p^k (1-p)^{M-k}\left(S_{t_M} - K\right)$$

$$\text{where } a = \min\left(j \in \overline{1,n} | j \geq b\right) \; b = \frac{\ln\left(K/S_0\right) - n\ln d}{\ln\left(u/d\right)}$$

2 Question 2: PDEs Method

Assume a constant continuously compounded interest rate $r > 0$ and GBM mode for the stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3}$$

Question 2.1. *Derive the Black-Scholes PDE for the present value $V(t, S_t)$ of a European call or put option:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV \tag{4}$$

Solution 2.1. *The solution can be derived from very useful statement of Feynman-Kac theorem which connects stochastic differential equations with PDE. The BS PDE can be directly obtained by this theorem, but it is useful to repeat that calculation once more.*

Theorem 2.1 (Feynman-Kac).

$$dX_t = \beta(u, X_t)du + \gamma(u, X(u))dW_u \tag{5}$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$ and let $t \in [0, T]$.

$$V(t, X(t)) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h(X(T))\right] \tag{6}$$

Then $V(t, X(t))$ satisfies the partial differential equation

$$V_t(t,x) + \beta V_x(t,x) + \frac{1}{2}\gamma^2 V_{xx}(t, x = r)V(t,x) \tag{7}$$

and the terminal condition $f(T,x) = h(x) \quad \forall x$

$$V(t, X(t)) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h((T))|\mathcal{F}(t)\right] \tag{8}$$

$$V(t, X(t)) \textit{ is not a martingale} \tag{9}$$

$$\mathbb{E}^{t,x}\left[e^{-r(T-t)}h((T))|\mathcal{F}(s)\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-r(T-t)}h((T))|\mathcal{F}(t)\right]|\mathcal{F}(s) \neq \mathbb{E}\left[e^{-r(T-s)}h((T))|\mathcal{F}(t)\right]|\mathcal{F}(s)\right] \tag{10}$$

To get rid of the dependency from the conditioning we multiply both side of the equation by e^{-rt}

$$d\left(e^{-rt}V(t, X(t))\right) = e^{-rt}\left[-rVdt + f_t dt + f_x dX + \frac{1}{2}V_{xx}dXdX\right] \tag{11}$$

$$= e^{-rt}\left[-rV + V_t + V_x + \frac{1}{2}\gamma^2 V_{xx}\right]dt + e^{-rt}\gamma V_x dW \tag{12}$$

Setting the dt term to zero we obtain the equation $V_t + V_x + \frac{1}{2}\gamma^2 V_{xx} = -rV$

$$X(t) = S(t)$$
$$\gamma = \sigma S(t)$$
$$\beta = \mu$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

$$\tau = T - \frac{2t}{\sigma^2}$$
$$S = Ee^x V(t,S) = Ev(\tau, x)$$

$$\frac{\partial x}{\partial S} = \frac{1}{S/E}\frac{1}{E} = \frac{1}{S}, \qquad \frac{\partial x}{\partial t} = 0$$

$$\frac{\partial \tau}{\partial t} = -\tfrac{1}{2}\sigma^2 \qquad \frac{\partial \tau}{\partial S} = 0$$

$$\frac{\partial V}{\partial t} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}\right) = -\tfrac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau},$$

$$\frac{\partial V}{\partial S} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial S}\right) = \frac{E}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = E\frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right)$$

$$E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} + \frac{1}{S}\frac{\partial^2 v}{\partial \tau \partial x}\frac{\partial \tau}{\partial S}\right) = E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right)$$

Black Scholes PDE

$$-\frac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau}+\frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial v}{\partial x}+\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right)+rS\frac{E}{S}\frac{\partial v}{\partial x}-rEv=0$$

$$k=\frac{2r}{\sigma^2}$$

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv$$

$$v(\tau,x)=e^{\alpha x+\beta\tau}y(\tau,x)$$

$$\frac{\partial v}{\partial \tau}=e^{\alpha x+\beta\tau}\left(\beta y+\frac{\partial y}{\partial \tau}\right)$$

$$\frac{\partial v}{\partial x}=e^{\alpha x+\beta\tau}\left(\beta y+\frac{\partial y}{\partial \tau}\right)$$

$$\frac{\partial^2 v}{\partial x^2}=e^{\alpha x+\beta\tau}\left(\alpha^2 y+2\alpha\frac{\partial y}{\partial x}+\frac{\partial^2 y}{\partial x^2}\right)$$

$$\frac{\partial y}{\partial \tau}=\frac{\partial^2 y}{\partial x^2}+(k-1+2\alpha)\frac{\partial y}{\partial x}+\left(\alpha^2-\alpha k-\alpha-k-\beta\right)y$$

$$\alpha=\frac{1-k}{2}\beta=\alpha^2-\alpha k-\alpha-k=\left(\frac{1-k}{2}\right)\left(\frac{1+k}{2}\right)-\left(\frac{1+k}{2}\right)=-\frac{1}{4}(1+k)^2$$

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}$$

2.1 Boundary Conditions

Call

$$\begin{cases}V(T,S)=\max(S-E,0)\\ V(t,0)=0\\ V(t,S)\sim S, S\to\infty\end{cases}\qquad\begin{cases}y(0,x)=e^{-\alpha x}\max(e^x-1,0)=\max\left(e^{(k+1)x/2}-e^{(k-1)x/2},0\right)\\ y(\tau,x)\to 0, x\to-\infty\\ y(\tau,S)\sim e^{(k+1)x/2}, x\to\infty\end{cases}\tag{13}$$

Put

$$\begin{cases}V(T,S)=\max(E-S,0)\\ V(t,0)=Ee^{-r(T-t)}\\ V(t,S)\to 0, S\to\infty\end{cases}\qquad\begin{cases}y(0,x)=e^{-\alpha x}\max(e^x-1,0)=\max\left(e^{(k-1)x/2}-e^{(k+1)x/2},0\right)\\ y(\tau,x)\to 0, x\to\infty\\ y(\tau,S)\sim e^{(k-1)^2/4}, x\to-\infty\end{cases}\tag{14}$$