

# Computational Finance

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## 1 Question 1: The Binomial Method

Problem statement

**Question 1.1.** What is the price of the zero-coupon bond  $P(t_i, t_m)$ ?

**Solution 1.1.**

$$P(t_i, t_m) = Fe^{-r(m-i)\Delta t} \quad (1)$$

**Question 1.2.** For what values of  $u, d$  and  $p$  do we obtain the risk-neutral dynamics?

**Solution 1.2.** Parameters must satisfy following equation

$$p = \frac{e^{r\Delta t} - d}{u - d} \quad (2)$$

**Question 1.3.** Price a European call at  $t_0 = 0$  with maturity  $t_M = 1$ , strike  $K = 70$ , interest rate  $r = 0.01$ , upward move  $u = 1.05$  and current spot  $S_0 = 50$ . Derive numerically the convergence rate as  $M \rightarrow \infty$

**Solution 1.3.**

$$\begin{aligned} C(M) &= e^{-rT} \mathbb{E} \left[ (S_{t_M} - K)^+ \right] \\ \mathbb{P} (S_{t_M} = S_0 u^k d^{M-k}) &= \binom{n}{k} p^k (1-p)^{M-k} \\ C(M) &= e^{-rT} \sum_k \binom{n}{k} p^k (1-p)^{M-k} (S_{t_M} - K)^+ = e^{-rT} \sum_a^M \binom{n}{k} p^k (1-p)^{M-k} (S_{t_M} - K) \\ &\text{where } a = \min (j \in \overline{1, n} | j \geq b) \quad b = \frac{\ln (K/S_0) - n \ln d}{\ln (u/d)} \end{aligned}$$

## 2 Question 2: PDEs Method

Assume a constant continuously compounded interest rate  $r > 0$  and GBM model for the stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (3)$$

**Question 2.1.** Derive the Black-Scholes PDE for the present value  $V(t, S_t)$  of a European call or put option:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} = rV \quad (4)$$

**Solution 2.1.** The solution can be derived from very useful statement of Feynman-Kac theorem which connects stochastic differential equations with PDE. The BS PDE can be directly obtained by this theorem, but it is useful to repeat that calculation once more.

**Theorem 2.1** (Feynman-Kac).

$$dX_t = \beta(u, X_t)du + \gamma(u, X(u))dW_u \quad (5)$$

Let  $h(y)$  be a Borel-measurable function. Fix  $T > 0$  and let  $t \in [0, T]$ .

$$V(t, X(t)) = \mathbb{E}^{t,x} \left[ e^{-r(T-t)} h(X(T)) \right] \quad (6)$$

Then  $V(t, X(t))$  satisfies the partial differential equation

$$V_t(t, x) + \beta V_x(t, x) + \frac{1}{2} \gamma^2 V_{xx}(t, x) = rV(t, x) \quad (7)$$

and the terminal condition  $f(T, x) = h(x) \quad \forall x$

$$V(t, X(t)) = \mathbb{E}^{t,x} \left[ e^{-r(T-t)} h((T)) | \mathcal{F}(t) \right] \quad (8)$$

$$V(t, X(t)) \text{ is not a martingale} \quad (9)$$

$$\mathbb{E}^{t,x} \left[ e^{-r(T-t)} h((T)) | \mathcal{F}(s) \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{-r(T-t)} h((T)) | \mathcal{F}(t) \right] | \mathcal{F}(s) \right] \neq \mathbb{E} \left[ e^{-r(T-s)} h((T)) | \mathcal{F}(t) \right] | \mathcal{F}(s) \quad (10)$$

To get rid of the dependency from the conditioning we multiply both side of the equation by  $e^{-rt}$

$$d(e^{-rt} V(t, X(t))) = e^{-rt} \left[ -rV dt + f_t dt + f_x dX + \frac{1}{2} V_{xx} dX dX \right] \quad (11)$$

$$= e^{-rt} \left[ -rV + V_t + V_x + \frac{1}{2} \gamma^2 V_{xx} \right] dt + e^{-rt} \gamma V_x dW \quad (12)$$

Setting the  $dt$  term to zero we obtain the equation  $V_t + V_x + \frac{1}{2} \gamma^2 V_{xx} = -rV$

$$X(t) = S(t)$$

$$\gamma = \sigma S(t)$$

$$\beta = \mu$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} = rV$$

**Question 2.2.** Transform to heat equation

**Solution 2.2.**

$$\tau = T - \frac{2t}{\sigma^2}$$

$$S = Ee^x$$

$$V(t, S) = Ev(\tau, x)$$

$$\begin{aligned}
\frac{\partial x}{\partial S} &= \frac{1}{S/E} \frac{1}{E} = \frac{1}{S}, & \frac{\partial x}{\partial t} &= 0 \\
\frac{\partial \tau}{\partial t} &= -\frac{1}{2}\sigma^2 & \frac{\partial \tau}{\partial S} &= 0 \\
\frac{\partial V}{\partial t} &= E \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial t} \right) = -\frac{1}{2}\sigma^2 E \frac{\partial v}{\partial \tau}, \\
\frac{\partial V}{\partial S} &= E \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial S} + \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial S} \right) = \frac{E}{S} \frac{\partial v}{\partial x} \\
\frac{\partial^2 V}{\partial S^2} &= E \frac{\partial}{\partial S} \left( \frac{1}{S} \frac{\partial v}{\partial x} \right) \\
E \left( -\frac{1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S} \frac{\partial^2 v}{\partial x^2} \frac{\partial x}{\partial S} + \frac{1}{S} \frac{\partial^2 v}{\partial \tau \partial x} \frac{\partial \tau}{\partial S} \right) &= E \left( -\frac{1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S^2} \frac{\partial^2 v}{\partial x^2} \right)
\end{aligned}$$

Black Scholes PDE

$$\begin{aligned}
-\frac{1}{2}\sigma^2 E \frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \left( -\frac{1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S^2} \frac{\partial^2 v}{\partial x^2} \right) + rS \frac{E}{S} \frac{\partial v}{\partial x} - rEv &= 0 \\
k &= \frac{2r}{\sigma^2} \\
\frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{\partial v}{\partial x} - kv
\end{aligned}$$

$$\begin{aligned}
v(\tau, x) &= e^{\alpha x + \beta \tau} y(\tau, x) \\
\frac{\partial v}{\partial \tau} &= e^{\alpha x + \beta \tau} \left( \beta y + \frac{\partial y}{\partial \tau} \right) \\
\frac{\partial v}{\partial x} &= e^{\alpha x + \beta \tau} \left( \beta y + \frac{\partial y}{\partial \tau} \right) \\
\frac{\partial^2 v}{\partial x^2} &= e^{\alpha x + \beta \tau} \left( \alpha^2 y + 2\alpha \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} \right) \\
\frac{\partial y}{\partial \tau} &= \frac{\partial^2 y}{\partial x^2} + (k-1+2\alpha) \frac{\partial y}{\partial x} + (\alpha^2 - \alpha k - \alpha - k - \beta) y \\
\alpha &= \frac{1-k}{2} \beta = \alpha^2 - \alpha k - \alpha - k = \left( \frac{1-k}{2} \right) \left( \frac{1+k}{2} \right) - \left( \frac{1+k}{2} \right) = -\frac{1}{4}(1+k)^2 \\
\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

## 2.1 Boundary Conditions

Call

$$\begin{cases} V(T, S) = \max(S - E, 0) \\ V(t, 0) = 0 \\ V(t, S) \sim S, S \rightarrow \infty \end{cases} \quad \begin{cases} y(0, x) = e^{-\alpha x} \max(e^x - 1, 0) = \max(e^{(k+1)x/2} - e^{(k-1)x/2}, 0) \\ y(\tau, x) \rightarrow 0, x \rightarrow -\infty \\ y(\tau, S) \sim e^{(k+1)x/2}, x \rightarrow \infty \end{cases} \quad (13)$$

Put

$$\begin{cases} V(T, S) = \max(E - S, 0) \\ V(t, 0) = Ee^{-r(T-t)} \\ V(t, S) \rightarrow 0, S \rightarrow \infty \end{cases} \quad \begin{cases} y(0, x) = e^{-\alpha x} \max(e^x - 1, 0) = \max(e^{(k-1)x/2} - e^{(k+1)x/2}, 0) \\ y(\tau, x) \rightarrow 0, x \rightarrow \infty \\ y(\tau, S) \sim e^{(k-1)^2/4}, x \rightarrow -\infty \end{cases} \quad (14)$$