# Computational Finance

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## 1 Question 1: The Binomial Method

Problem statement

**Question 1.1.** *What is the proce of the zero-coupon bond $P(t_i, t_m)$?*

**Solution 1.1.**

$$P(t_i, t_m) = Fe^{-r(m-i)\Delta t} \tag{1}$$

**Question 1.2.** *For what values of $u, d$ and $p$ do we obtain the risk-neutral dynamics?*

**Solution 1.2.** *Parameteres must satisfy following equation*

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2}$$

**Question 1.3.** *Price a European call at $t_0 = 0$ with maturity $t_M = 1$, strike $K = 70$, interest rate $r = 0.01$, upward move $u = 1.05$ and current spot $S_0 = 50$. Derive numerically the convergence rate as $M \to \infty$*

**Solution 1.3.**

$$C(M) = e^{-rT}\mathbb{E}\left[(S_{t_M} - K)^+\right]$$

$$\mathbb{P}\left(S_{t_M} = S_0 u^k d^{M-k}\right) = \binom{n}{k} p^k (1-p)^{M-k}$$

$$C(M) = e^{-rT}\sum_{k}^{M}\binom{n}{k} p^k (1-p)^{M-k}\left(S_{t_M} - K\right)^+ = e^{-rT}\sum_{a}^{M}\binom{n}{k} p^k (1-p)^{M-k}\left(S_{t_M} - K\right)$$

$$\text{where } a = \min\left(j \in \overline{1,n} | j \geq b\right)\ b = \frac{\ln\left(K/S_0\right) - n\ln d}{\ln\left(u/d\right)}$$

**Question 1.4.** *Construct a delta hedging strategy. Experiment with Monte-Carlo simulations how well do you replicate the option as $M \to \infty$.*

**Solution 1.4.**

$$d\Pi_t = dV_t - \Delta_t dS_t \tag{3}$$

## 2 Question 2: PDEs Method

Assume a constant continuously compounded interest rate $r > 0$ and GBM mode for the stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4}$$

**Question 2.1.** *Derive the Black-Scholes PDE for the present value $V(t, S_t)$ of a European call or put option:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV \tag{5}$$

**Solution 2.1.** *The solution can be derived from very useful statement of Feynman-Kac theorem which connects stochastic differential equations with PDE. The BS PDE can be directly obtained by this theorem, but it is useful to repeat that calculation once more.*

**Theorem 2.1** (Feynman-Kac)**.**

$$dX_t = \beta(u, X_t)du + \gamma(u, X(u))dW_u \tag{6}$$

*Let $h(y)$ be a Borel-measurable function. Fix $T > 0$ and let $t \in [0, T]$.*

$$V(t, X(t)) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h(X(T))\right] \tag{7}$$

*Then $V(t, X(t))$ satisfies the partial differential equation*

$$V_t(t, x) + \beta V_x(t, x) + \frac{1}{2}\gamma^2 V_{xx}(t, x = r)V(t, x) \tag{8}$$

*and the terminal condition $f(T, x) = h(x) \quad \forall x$*

$$V(t, X(t)) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h((T))|\mathcal{F}(t)\right] \tag{9}$$

$$V(t, X(t)) \text{ is not a martingale} \tag{10}$$

$$\mathbb{E}^{t,x}\left[e^{-r(T-t)}h((T))|\mathcal{F}(s)\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-r(T-t)}h((T))|\mathcal{F}(t)\right]|\mathcal{F}(s) \neq \mathbb{E}\left[e^{-r(T-s)}h((T))|\mathcal{F}(t)\right]|\mathcal{F}(s)\right] \tag{11}$$

*To get rid of the dependency from the conditioning we multiply both side of the equation by $e^{-rt}$*

$$d\left(e^{-rt}V(t, X(t))\right) = e^{-rt}\left[-rVdt + f_t dt + f_x dX + \frac{1}{2}V_{xx}dXdX\right] \tag{12}$$

$$= e^{-rt}\left[-rV + V_t + V_x + \frac{1}{2}\gamma^2 V_{xx}\right]dt + e^{-rt}\gamma V_x dW \tag{13}$$

*Setting the $dt$ term to zero we obtain the equation $V_t + V_x + \frac{1}{2}\gamma^2 V_{xx} = -rV$*

$$\begin{aligned} X(t) &= S(t) \\ \gamma &= \sigma S(t) \\ \beta &= \mu \end{aligned}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

**Question 2.2.** *Transform to heat equation*

**Solution 2.2.**

$$\begin{aligned} \tau &= T - \frac{2t}{\sigma^2} \\ S &= Ee^x \\ V(t, S) &= Ev(\tau, x) \end{aligned}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S/E}\frac{1}{E} = \frac{1}{S}, \qquad \frac{\partial x}{\partial t} = 0$$
$$\frac{\partial \tau}{\partial t} = -\tfrac{1}{2}\sigma^2 \qquad \frac{\partial \tau}{\partial S} = 0$$
$$\frac{\partial V}{\partial t} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}\right) = -\tfrac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau},$$
$$\frac{\partial V}{\partial S} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial S}\right) = \frac{E}{S}\frac{\partial v}{\partial x}$$
$$\frac{\partial^2 V}{\partial S^2} = E\frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right)$$
$$E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} + \frac{1}{S}\frac{\partial^2 v}{\partial \tau \partial x}\frac{\partial \tau}{\partial S}\right) = E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right)$$

*Black Scholes PDE*

$$-\frac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\frac{E}{S}\frac{\partial v}{\partial x} - rEv = 0$$
$$k = \frac{2r}{\sigma^2}$$
$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

$$v(\tau, x) = e^{\alpha x + \beta\tau} y(\tau, x)$$
$$\frac{\partial v}{\partial \tau} = e^{\alpha x + \beta\tau}\left(\beta y + \frac{\partial y}{\partial \tau}\right)$$
$$\frac{\partial v}{\partial x} = e^{\alpha x + \beta\tau}\left(\beta y + \frac{\partial y}{\partial \tau}\right)$$
$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x + \beta\tau}\left(\alpha^2 y + 2\alpha\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right)$$
$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} + (k - 1 + 2\alpha)\frac{\partial y}{\partial x} + \left(\alpha^2 - \alpha k - \alpha - k - \beta\right) y$$
$$\alpha = \frac{1-k}{2}\beta = \alpha^2 - \alpha k - \alpha - k = \left(\frac{1-k}{2}\right)\left(\frac{1+k}{2}\right) - \left(\frac{1+k}{2}\right) = -\frac{1}{4}(1+k)^2$$
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

## 2.1 Boundary Conditions

*Call*

$$\begin{cases} V(T,S) = \max(S-E, 0) \\ V(t,0) = 0 \\ V(t,S) \sim S, S \to \infty \end{cases} \qquad \begin{cases} y(0,x) = e^{-\alpha x}\max(e^x - 1, 0) = \max\left(e^{(k+1)x/2} - e^{(k-1)x/2}, 0\right) \\ y(\tau, x) \to 0, x \to -\infty \\ y(\tau, S) \sim e^{(k+1)x/2}, x \to \infty \end{cases} \tag{14}$$

*Put*

$$\begin{cases} V(T,S) = \max(E-S, 0) \\ V(t,0) = Ee^{-r(T-t)} \\ V(t,S) \to 0, S \to \infty \end{cases} \qquad \begin{cases} y(0,x) = e^{-\alpha x}\max(e^x - 1, 0) = \max\left(e^{(k-1)x/2} - e^{(k+1)x/2}, 0\right) \\ y(\tau, x) \to 0, x \to \infty \\ y(\tau, S) \sim e^{(k-1)^2/4}, x \to -\infty \end{cases} \tag{15}$$

**Question 2.3.** *For the heat equation above, derive the explicit, the implicit and the Crank-Nicolson finite-difference schemes with time, space steps $\Delta\tau, \Delta x$ respectively. For each of those schemes*

- *For the heat equation above, derive the explicit, the implicit and the Crank-Nicolson finite-difference schemes with time, space steps $\Delta\tau, \Delta x$ respectively. For each of those schemes $x_{min} \leq x \leq x_{max}$*

- *For what values of $\Delta\tau, \Delta x$ we have numerical stability.*

- *What is the order of convergence $\Delta\tau, \Delta x \to 0$*

**Solution 2.3.** ***Explicit***

$$\frac{\partial y_{i,\nu}}{\partial \tau} := \frac{\partial y\left(x_i, \tau_\nu\right)}{\partial \tau} = \frac{y_{i,\nu+1} - y_{i,\nu}}{\Delta\tau} + O(\Delta\tau) \tag{16}$$

$$\frac{\partial^2 y_{i,\nu}}{\partial x^2} = \frac{y_{i+1,\nu} - 2y_{i,\nu} + y_{i-1,\nu}}{\Delta x^2} + O\left(\Delta x^2\right) \tag{17}$$

$$\frac{w_{i,\nu+1} - w_{i,\nu}}{\Delta\tau} = \frac{w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu}}{\Delta x^2} \tag{18}$$

$$w_{i,\nu+1} = w_{i,\nu} + \frac{\Delta\tau}{\Delta x^2}\left(w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu}\right) \tag{19}$$

$$\lambda \leq \frac{1}{2} \tag{20}$$

***Implicit***

$$\frac{\partial y_{i,\nu}}{\partial \tau} = \frac{y_{i,\nu} - y_{i,\nu-1}}{\Delta\tau} + O(\Delta\tau) \tag{21}$$

$$-\lambda w_{i+1,\nu} + (1+2\lambda)w_{i,\nu} - \lambda w_{i-1,\nu} = w_{i,\nu-1} \tag{22}$$

$$\lambda > 0 \tag{23}$$

***Crank-Nicolson***

$$\frac{w_{i,\nu+1} - w_{i,\nu}}{\Delta\tau} = \frac{w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu}}{\Delta x^2} \tag{24}$$

$$\frac{w_{i,\nu+1} - w_{i,\nu}}{\Delta\tau} = \frac{w_{i+1,\nu+1} - 2w_{i,\nu+1} + w_{i-1,\nu+1}}{\Delta x^2} \tag{25}$$

$$\frac{w_{i,\nu+1} - w_{i,\nu}}{\Delta\tau} = \frac{1}{2\Delta x^2}\left(w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu} + w_{i+1,\nu+1} - 2w_{i,\nu+1} + w_{i-1,\nu+1}\right). \tag{26}$$

$$\lambda > 0 \tag{27}$$