Scientific Diary

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Chapter 1

Probability

Theorem 1.0.1 (Kolmogorov).
Пусть $\nu_{t_1 t_2 t_3 \ldots t_k}$ для $\forall t_1, t_2, t_3 \ldots t_k \in T, \quad k \in \mathrm{N}$ являются вероятностными мерами на R^{kn} такими, что:

$$\nu_{t_{\sigma(1)}, \cdots, t_{\sigma(k)}} \left(F_1 \times \cdots \times F_k \right) = \nu_{t_1, \cdots, t_k} \left(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)} \right) \tag{1.1}$$

для всех перестановок $\sigma \in S_k$

$$\begin{aligned}
\nu_{t_1, \ldots, t_k} & \left(F_1 \times \cdots \times \cdots \times F_k \right) \\
& = \nu_{t_1, \ldots, t_k, t_{k+1}, \ldots, t_{k+m}} \left(F_1 \times \cdots \times F_k \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n \right)
\end{aligned} \tag{1.2}$$

Тогда $\exists (\Omega, \mathcal{F}, \mathrm{P})$ и случайный процесс $\{X_t\}$ на Ω, $X_t : \Omega \to \mathrm{R}^n$

$$\nu_{t_1, \ldots, t_k} \left(F_1 \times \cdots \times F_k \right) = P \left[X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k \right] \tag{1.3}$$

Exercise 1.0.1 (Irwin-Hall distribution).
Try to derive Irwin-Hall distribution formula for pdf convolution formula.

Chapter 2

Monte-Carlo Methods

Code Base

Heuristic 2.0.1.
There is one useful thing that i found while making Monte-Carlo simulations. Sometimes it is computatationaly costly to generate random variable from normal distribution and very good approximation for it appeared to be Irwin-Hall distribution.

$$X_n = \sum_{i=0}^{n} U_k \quad where \quad U_k \quad are\ independent\ random\ variables\ drawn\ from\ uniform\ distribution \quad U(0,1) \tag{2.1}$$

The density function is given by:

$$f_X(x;n) = \frac{1}{2(n-1)!} \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) (x-k)^{n-1} \operatorname{sgn}(x-k) \tag{2.2}$$

This pdf is basically piecewise polynomial function with $\mu = \frac{n}{2}$ and $\sigma = \frac{n}{12}$.
For $n = 12$ it gives good approximation for normal distribution pdf.

$$\phi(x) \approx \sqrt{\frac{12}{n}} (f_X(x;n) - \frac{n}{2}) - 6 \tag{2.3}$$

$$\phi(x) \approx f_X(x;n) - 6 = \sum_{i=0}^{12} U_k \tag{2.4}$$

2.1 Financial Instrument Pricing

2.1.1 Black-Scholes Model

2.2 Metropolis-Hasting algorithm

Metropolis-Hasting (MH) algorithm arises in many contextes such as optimization, sampling from probability measure with unknown normalization. One of the benefits of this algorithm as it was said that it doesn't require measure to be normalized. The algorithm goes as follows:
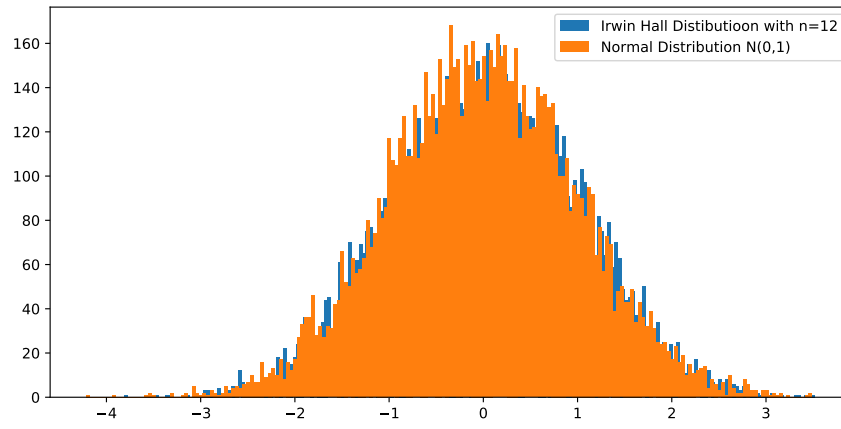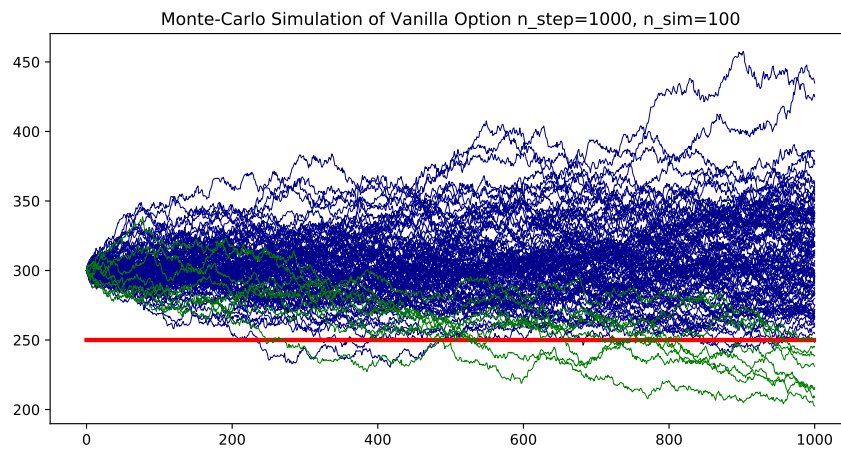
Figure 2.1:

Figure 2.2:

2.2.1 Discrete

We are given a matrix M_{ij} with positive entries s.t:

$$\forall i : 1 < i \leq d \sum_{1 < j \leq d} M_{ij} = 1 \tag{2.5}$$

In other words every row sums up to 1.

Such matrices are called stochastic matrices or Markov chain transitions.

Then we chooce initial random state X_0 and we define random variables X_n taking values in $E = \{1, \ldots, d\}$ by setting

$$X_n = F_n(X_{n_1}) \tag{2.6}$$

This sequence is called Markov chain taking values $E = \{1, \ldots, d\}$ with transition probabilities

$$\mathbb{P}(X_n = i | X_0 = j) = M^n(i, j) \tag{2.7}$$

Steps

1. We are given some line vector π such that $\wedge_{i \in E} \pi(i) > 0$ and $\sum_{1 < j \leq d} \pi(j) = 1$.

2. Choose the transition probablilty matrix $M(i, j)$ s.t $M(i, j) > 0$ and $M(j, i) > 0$

3. Set

$$a(i, j) = 1 \wedge \frac{\pi(i)M(i, j)}{\pi(j)M(j, i)} \in [0, 1] \tag{2.8}$$

4. On each step draw *i.i.d.* $(U_n)_{n \geq 0}$ on $[0, 1]$

5. Run accesptence-rejection procedure

$$X_{n-1} \rightsquigarrow Y_n = F_n(X_{n-1}) \rightsquigarrow X_n := \begin{cases} Y_n \text{ if } & U_n \leq a(X_{n-1}, Y_n) \\ X_{n-1} \text{ if } & U_n > a(X_{n-1}, Y_n) \end{cases} \tag{2.9}$$

The accepted sequence X_n is the sample we needed.

Chapter 3

Matrix Models

Chapter 4

Stochastic Processes

	$\mathrm{d}t$	$\mathrm{d}W_t$
$\mathrm{d}t$	0	0
$\mathrm{d}W_t$	0	$\mathrm{d}t$

Exercise 4.0.1 (Shreve 4.6).
Let $S_t = S(0) \exp((\alpha - \frac{\sigma^2}{2})t - \sigma W_t)$ be a geometric Brownian motion.
Let p be a positive constant. Compute $\mathrm{d}(S_t^p)$

Solution 4.0.1.
First of all lets apply Ito-Doeblin formula to the function $f(x) = x^p$

$$\mathrm{d}f(S_t) = \partial_x f(S_t)\, \mathrm{d}S + \frac{1}{2} \partial_{xx}^2 f(S_t)(\mathrm{d}S_t)^2 \tag{4.1}$$

For $\mathrm{d}S(t)$ we have

$$\mathrm{d}S_t = \alpha S_t\, \mathrm{d}t + \sigma S_t\, \mathrm{d}W_t \tag{4.2}$$

To compute the $(\mathrm{d}S_t)^2$ we apply mnemonic multiplication rule for stochastic differentials.

$$(\mathrm{d}S_t)^2 = (\sigma S_t\, \mathrm{d}W_t + \alpha S_t\, \mathrm{d}t)^2 = \sigma^2 S_t^2 (\mathrm{d}W_t)^2 + 2\sigma\alpha S_t^2\, \mathrm{d}t\, \mathrm{d}W_t + \alpha^2 S_t^2 (\mathrm{d}t)^2 = \sigma^2 S_t^2\, \mathrm{d}t \tag{4.3}$$

Substituting derivatives and differentials of all orders to the formula we get:

$$\mathrm{d}(S_t^p) = pS_t^p \left(\sigma\, \mathrm{d}W_t + \left(\alpha + \frac{p-1}{2}\sigma^2 \right) \mathrm{d}t \right) \tag{4.4}$$

Exercise 4.0.2 (Shreve 4.7).   *i Compute $\mathrm{d}W_t^4$ and then write $W^4(T)$ as the sum of the ordinary (Lebesgue) integral with respect to time and an Ito integral.*

ii Take expectations on both sides of the formula you obtained in i, use the fact that $\mathbb{E}\left[W_t^2\right] = t$, and derive formula $\mathbb{E}\left[W_T^4\right] = 3T^2$.

iii Use the method of i and ii to derive a formula for $\mathbb{E}\left[W_T^6\right]$.

Solution 4.0.2.   *i*

$$\mathrm{d}f(W_t) = \partial_x f(W_t)\,\mathrm{d}W_t + \frac{1}{2}\partial_{xx}^2 f(W_t)(\mathrm{d}W_t)^2,\ \text{where}\ f(W_t) = W_t^4.$$

$$\mathrm{d}W_t^4 = 4W_t^3\,\mathrm{d}W_t + \frac{4\cdot 3}{2}W_t^2(\mathrm{d}W_t)^2$$

$$\int_0^T \mathrm{d}W_t^4 = \int_0^T 4W_t^3\,\mathrm{d}W_t + \int_0^T 6W_t^2\,\mathrm{d}t \tag{4.5}$$

$$W_T^4 == \int_0^T 4W_t^3\,\mathrm{d}W_t + \int_0^T 6W_t^2\,\mathrm{d}t$$

ii

$$\mathbb{E}\left[W_T^4\right] = 4\cdot\mathbb{E}\left[\int_0^T W_t^3\,\mathrm{d}W_t\right] + 6\cdot\mathbb{E}\left[\int_0^T W_t^2\,\mathrm{d}t\right] \tag{4.6}$$

I will allow myself to be more detailed in this part because the following calculations are very useful and appear in many other places.
Firstly, lets handle the first summand and prove that it equals zero. One can recall theorem from stochastic calculus that states that integral of quandraticly integrable process w.t.r of Wiener process is martingale and if we prove that W_t^3 is quadraticly integrable the calculation of expectation becomes straightforward.
Lets show it for more general case W_t^n.

$$\mathbb{E}\left[X^n\right] = \begin{cases} 0 & n\ is\ odd \\ \sigma^n(n-1)!! & n\ is\ even \end{cases} \quad where\ X\sim\mathcal{N}(0,\sigma^2) \tag{4.7}$$

$$\mathbb{E}\left[\int_0^t W_s^n\,\mathrm{d}s\right] = \int_0^t \mathbb{E}\left[W_s^n\right]\mathrm{d}s = \begin{cases} 0 & n\ is\ odd \\ (n-1)!!\int_0^t s^{n/2}\,\mathrm{d}s & n\ is\ even \end{cases} \tag{4.8}$$

So the quadratica variation of W_t^3 : $\int_0^T W_s^6\,\mathrm{d}s < \infty$ From this follows that

$$\mathbb{E}\left[\int_0^T W_t^3\,\mathrm{d}W_t\,|\mathcal{F}_0\right] = 0 \tag{4.9}$$

$$\mathbb{E}\left[W_T^4\right] = 6\cdot\int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = 6\cdot(1)!!\cdot\frac{1}{2}T^2 = 3T^2 \tag{4.10}$$

iii

$$\mathrm{d}W_t^6 = 6W_t^5\,\mathrm{d}W_t + 15W_t^4\,\mathrm{d}t.$$

$$W_t^6 = 6\int_0^T W_t^5\,\mathrm{d}W_t + 15\int_0^T W_t^4\,\mathrm{d}t \tag{4.11}$$

$$\mathbb{E}\left[W_t^6\right] = 6\cdot\mathbb{E}\left[\int_0^T W_t^5\,\mathrm{d}W_t\right] + 15\cdot\mathbb{E}\left[\int_0^T W_t^4\,\mathrm{d}t\right] = 15\cdot\mathbb{E}\left[\int_0^T W_t^4\,\mathrm{d}t\right] = 15T^3$$

Exercise 4.0.3 (Shreve 4.8).
The Vasicek interest rate stochastic differential equation

$$\mathrm{d}R(t) = (\alpha - \beta R(t))\,\mathrm{d}t\sigma\,\mathrm{d}W_t \tag{4.12}$$

where α, β, σ are positive constants. Derive the solution.

Solution 4.0.3.

$$\mathrm{d}(e^{\beta t}R_t) = \beta e^{\beta t}R_t\,\mathrm{d}t + e^{\beta t}\,\mathrm{d}R_t = e^{\beta t}\left[\beta R_t\,\mathrm{d}t + (\alpha - \beta R_t)\,\sigma\,\mathrm{d}W_t\right] = e^{\beta t}\left(\alpha\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right)$$

$$\int_0^t \mathrm{d}(e^{\beta t}R_t) = \int_0^t e^{\beta t}\left(\alpha\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right)$$

$$R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right) + \sigma\int_0^t e^{-\beta(t-s)}\,\mathrm{d}W_s = R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right) + \frac{\sigma}{\sqrt{2\beta}}e^{-\beta t}W_{e^{2\beta}-1}$$

$$(4.13)$$

Problem 4.0.1 (1D Ising model with periodic boundary conditions)**.**

$$Z = \sum_{s_1=\pm 1}\cdots\sum_{s_N=\pm 1}\prod_i \exp\left(\beta J s_i s_{i+1} + \frac{\beta B}{2}\left(s_i + s_{i+1}\right)\right)$$

$$(4.14)$$

$$s_{N+1} = s_1$$

Solution 4.0.4.

$$T_{s_i,s_{i+1}} = \exp\left(\beta J s_i s_{i+1} + \frac{\beta B}{2}\left(s_i + s_{i+1}\right)\right)$$

$$Z = \sum_{s_1=\pm 1}\cdots\sum_{s_N=\pm 1}T_{s_i,s_{i+1}}$$

$$(4.15)$$

If we rewrite sum in terms of matrix multiplication we obtain:

$$V\left(\sigma,\sigma'\right) = \exp\left[k\sigma\sigma' + \frac{h}{2}\left(\sigma + \sigma'\right)\right]$$

$$Z_N = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N}V\left(\sigma_1,\sigma_2\right)V\left(\sigma_2,\sigma_3\right)\cdot\ldots\cdot V\left(\sigma_N,\sigma_1\right)$$

$$(4.16)$$

When summed each factor $V\left(\sigma_i,\sigma_{i+1}\right)V\left(\sigma_{i+2},\sigma_{i+3}\right)$ *can be viewed as the matrix product of* T

$$Z_N = \mathrm{Tr}\,T^N$$

$$(4.17)$$