Scientific Diary

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Chapter 1

Probability

Theorem 1.0.1 (Kolmogorov)**.**
Пусть $\nu_{t_1 t_2 t_3 \dots t_k}$ *для* $\forall t_1, t_2, t_3 \dots t_k \in T, \quad k \in \mathrm{N}$ *являются вероятностными мерами на* R^{kn} *такими, что:*

$$\nu_{t_{\sigma(1)}}, \cdots, t_{\sigma(k)} \left(F_1 \times \cdots \times F_k\right) = \nu_{t_1, \cdots, t_k} \left(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}\right) \tag{1.1}$$

для всех перестановок $\sigma \in S_k$

$$\begin{aligned} &\nu_{t_1, \dots, t_k} \left(F_1 \times \cdots \times \cdots \times F_k\right) \\ &\quad = \nu_{t_1, \dots, t_k, t_{k+1}, \dots, t_{k+m}} \left(F_1 \times \cdots \times F_k \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n\right) \end{aligned} \tag{1.2}$$

Тогда $\exists (\Omega, \mathcal{F}, \mathrm{P})$ *и случайный процесс* $\{X_t\}$ *на* Ω, $X_t : \Omega \to \mathrm{R}^n$

$$\nu_{t_1, \dots, t_k} \left(F_1 \times \cdots \times F_k\right) = P\left[X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k\right] \tag{1.3}$$

Exercise 1.0.1 (Irwin-Hall distribution)**.**
Try to derive Irwin-Hall distribution formula for pdf convolution formula.

Chapter 2

Monte-Carlo Methods

Code Base

Heuristic 2.0.1.
There is one useful thing that i found while making Monte-Carlo simulations. Sometimes it is computatationaly costly to generate random variable from normal distribution and very good approximation for it appeared to be Irwin-Hall distribution.

$$X_n = \sum_{i=0}^{n} U_k \quad \text{where} \quad U_k \quad \text{are independent random variables drawn from uniform distribution} \quad U(0,1) \tag{2.1}$$

The density function is given by:

$$f_X(x;n) = \frac{1}{2(n-1)!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} (x-k)^{n-1} \operatorname{sgn}(x-k) \tag{2.2}$$

This pdf is basically piecewise polynomial function with $\mu = \frac{n}{2}$ and $\sigma = \frac{n}{12}$.
For $n = 12$ it gives good approximation for normal distribution pdf.

$$\phi(x) \approx \sqrt{\frac{12}{n}}(f_X(x;n) - \frac{n}{2}) - 6 \tag{2.3}$$

$$\phi(x) \approx f_X(x;n) - 6 = \sum_{i=0}^{12} U_k \tag{2.4}$$

2.1 Financial Instrument Pricing

2.1.1 Black-Scholes Model

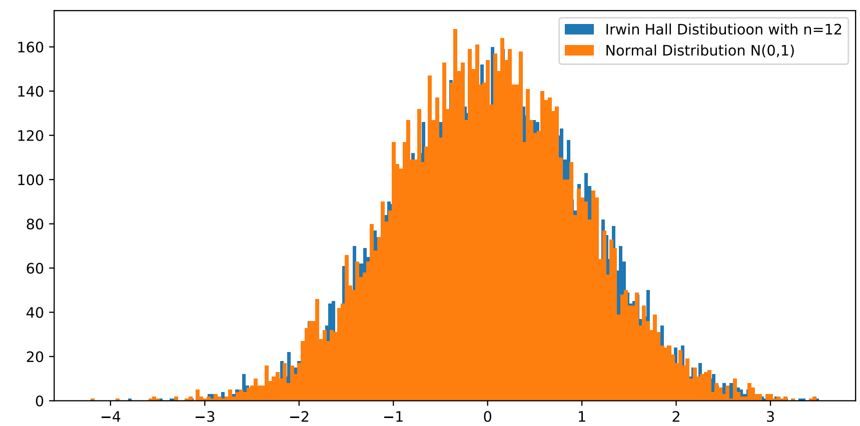

Figure 2.1:

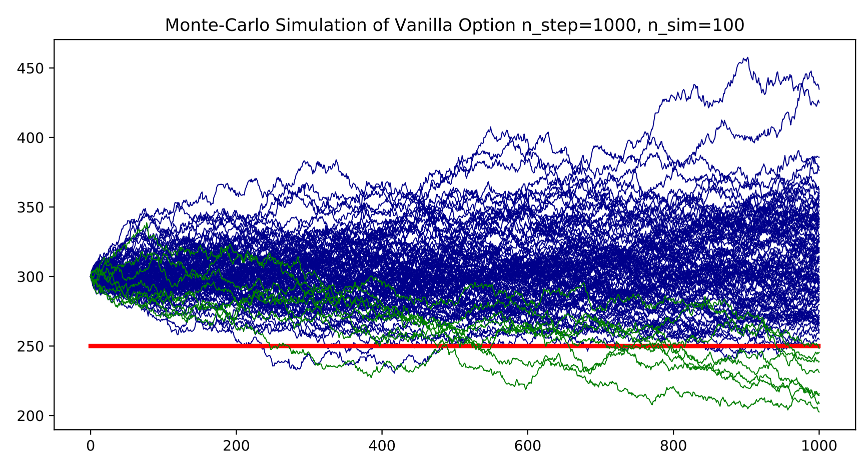

Figure 2.2:

Chapter 3

Matrix Models

Chapter 4

Stochastic Processes

	$\mathrm{d}t$	$\mathrm{d}W_t$
$\mathrm{d}t$	0	0
$\mathrm{d}W_t$	0	$\mathrm{d}t$

Exercise 4.0.1 (Shreve 4.6).
Let $S_t = S(0)\exp((\alpha - \frac{\sigma^2}{2})t - \sigma W_t)$ be a geometric Brownian motion.
Let p be a positive constant. Compute $\mathrm{d}(S_t^p)$

Solution 4.0.1.
First of all lets apply Ito-Doeblin formula to the function $f(x) = x^p$

$$\mathrm{d}f(S_t) = \partial_x f(S_t)\,\mathrm{d}S + \frac{1}{2}\partial_{xx}^2 f(S_t)(\mathrm{d}S_t)^2 \tag{4.1}$$

For $\mathrm{d}S(t)$ we have

$$\mathrm{d}S_t = \alpha S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \tag{4.2}$$

To compute the $(\mathrm{d}S_t)^2$ we apply mnemonic multiplication rule for stochastic differentials.

$$(\mathrm{d}S_t)^2 = (\sigma S_t\,\mathrm{d}W_t + \alpha S_t\,\mathrm{d}t)^2 = \sigma^2 S_t^2(\mathrm{d}W_t)^2 + 2\sigma\alpha S_t^2\,\mathrm{d}t\,\mathrm{d}W_t + \alpha^2 S_t^2(\mathrm{d}t)^2 = \sigma^2 S_t^2\,\mathrm{d}t \tag{4.3}$$

Substituting derivatives and differentials of all orders to the formula we get:

$$\mathrm{d}(S_t^p) = pS_t^p\left(\sigma\,\mathrm{d}W_t + \left(\alpha + \frac{p-1}{2}\sigma^2\right)\mathrm{d}t\right) \tag{4.4}$$

Exercise 4.0.2 (Shreve 4.7).

i Compute $\mathrm{d}W_t^4$ and then write $W^4(T)$ as the sum of the ordinary (Lebesgue) integral with respect to time and an Ito integral.

ii Take expectations on both sides of the formula you obtained in i, use the fact that $\mathbb{E}\left[W_t^2\right] = t$, and derive formula $\mathbb{E}\left[W_T^4\right] = 3T^2$.

iii Use the method of i and ii to derive a formula for $\mathbb{E}\left[W_T^6\right]$.

Solution 4.0.2. *i*

$$
\begin{aligned}
\mathrm{d}f(W_t) = \partial_x f(W_t)\,\mathrm{d}W_t + \frac{1}{2}\partial^2_{xx} f(W_t)(\mathrm{d}W_t)^2,\ \text{where}\, f(W_t) = W_t^4.\\
\mathrm{d}W_t^4 = 4W_t^3\,\mathrm{d}W_t + \frac{4\cdot 3}{2}W_t^2(\mathrm{d}W_t)^2\\
\int_0^T \mathrm{d}W_t^4 = \int_0^T 4W_t^3\,\mathrm{d}W_t + \int_0^T 6W_t^2\,\mathrm{d}t\\
W_T^4 == \int_0^T 4W_t^3\,\mathrm{d}W_t + \int_0^T 6W_t^2\,\mathrm{d}t
\end{aligned}
\tag{4.5}
$$

ii

$$
\mathbb{E}\left[W_T^4\right] = 4\cdot\mathbb{E}\left[\int_0^T W_t^3\,\mathrm{d}W_t\right] + 6\cdot\mathbb{E}\left[\int_0^T W_t^2\,\mathrm{d}t\right] \tag{4.6}
$$

I will allow myself to be more detailed in this part because the following calculations are very useful and appear in many other places.
Firstly, lets handle the first summand and prove that it equals zero. One can recall theorem from stochastic calculus that states that integral of quandraticly integrable process w.t.r of Wiener process is martingale and if we prove that W_t^3 is quadraticly integrable the calculation of expectation becomes straightforward.
Lets show it for more general case W_t^n.

$$
\mathbb{E}\left[X^n\right] = \begin{cases} 0 & n \text{ is odd} \\ \sigma^n(n-1)!! & n \text{ is even} \end{cases} \quad \text{where}\, X\sim\mathcal{N}(0,\sigma^2) \tag{4.7}
$$

$$
\mathbb{E}\left[\int_0^t W_s^n\,\mathrm{d}s\right] = \int_0^t \mathbb{E}\left[W_s^n\right]\mathrm{d}s = \begin{cases} 0 & n \text{ is odd} \\ (n-1)!!\int_0^t s^{n/2}\,\mathrm{d}s & n \text{ is even} \end{cases} \tag{4.8}
$$

So the quadratica variation of W_t^3 : $\int_0^T W_s^6\,\mathrm{d}s < \infty$ From this follows that

$$
\mathbb{E}\left[\int_0^T W_t^3\,\mathrm{d}W_t|\mathcal{F}_0\right] = 0 \tag{4.9}
$$

$$
\mathbb{E}\left[W_T^4\right] = 6\cdot\int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = 6\cdot(1)!!\cdot\frac{1}{2}T^2 = 3T^2 \tag{4.10}
$$

iii

$$
\begin{aligned}
\mathrm{d}W_t^6 = 6W_t^5\,\mathrm{d}W_t + 15W_t^4\,\mathrm{d}t.\\
W_t^6 = 6\int_0^T W_t^5\,\mathrm{d}W_t + 15\int_0^T W_t^4\,\mathrm{d}t\\
\mathbb{E}\left[W_t^6\right] = 6\cdot\mathbb{E}\left[\int_0^T W_t^5\,\mathrm{d}W_t\right] + 15\cdot\mathbb{E}\left[\int_0^T W_t^4\,\mathrm{d}t\right] = 15\cdot\mathbb{E}\left[\int_0^T W_t^4\,\mathrm{d}t\right] = 15T^3
\end{aligned}
\tag{4.11}
$$