

# A GUIDED TOUR OF AI: FROM FOUNDATIONS TO LATEST APPLICATION

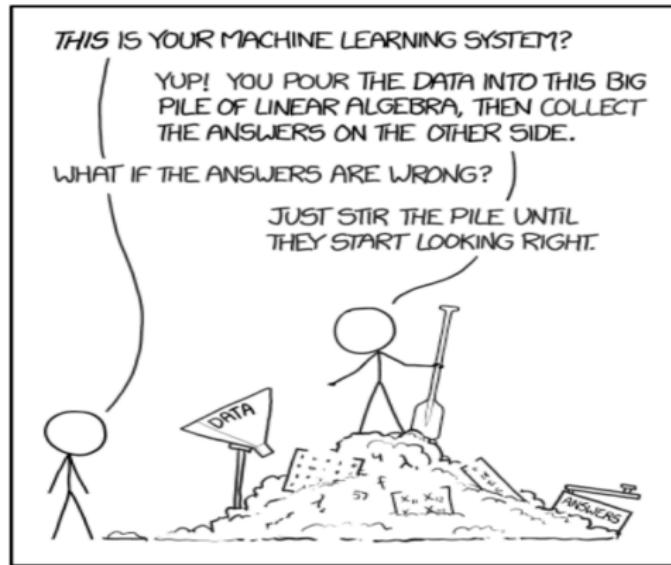
## INTRODUCTION TO LINEAR ALGEBRA

SCAI, Sorbonne University Abu Dhabi, Abu Dhabi, UAE



# IMPORTANCE OF LINEAR ALGEBRA IN ML

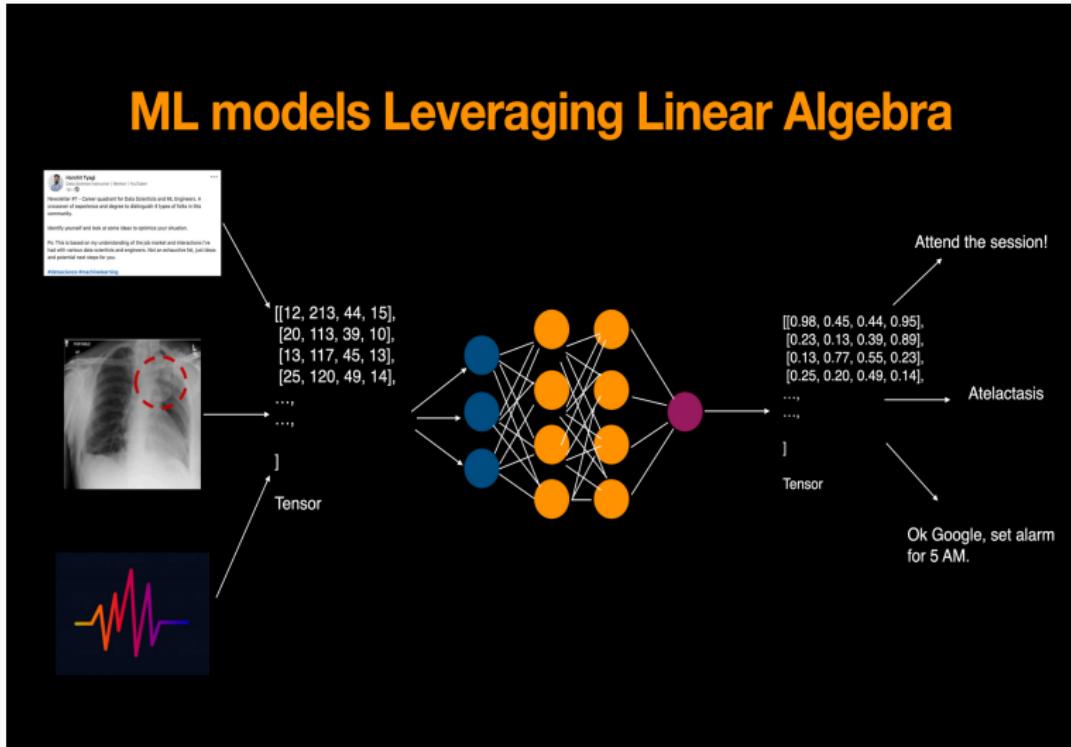
Importance in ML: We convert input vectors  $(x_1, \dots, x_n)$  into outputs by a series of linear transformations.



But what is RIGHT? And is that enough? (Image: [Machine Learning, XKCD](#))

# ML MODELS LEVERAGING LINEAR ALGEBRA

## ML models Leveraging Linear Algebra



# OUTLINE

## 1. INTRODUCTION TO MATRICES

Model a real-life situation using a system of linear equations

Introduction to matrices

Matrix algebra

## 2. EIGENVALUES, EIGENVECTORS

# OUTLINE

## 1. INTRODUCTION TO MATRICES

Model a real-life situation using a system of linear equations

Introduction to matrices

Matrix algebra

## 2. EIGENVALUES, EIGENVECTORS

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

We can model this real-life situation using an equation with one unknown value, represented by one variable, to say  $x$ .

① Identify the unknown and define your variable:

- the unknown value is the number of cows,
- set  $x$  to be the number of cows.

② Analyse the problem and write your equation accordingly:

- every cow has four legs,
- $4 \times \text{the number of cows} = 124$  legs
- $4x = 124$

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

$$4x = 124 \iff x = 124/4 = 31$$

There are 31 cows in the pasture 😊

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Let's review some important vocabulary:

- A system of equations is a set of two or more equations with the same set of unknown values, which are represented by the same variables.
- They are also called **simultaneous equations**

For example:

$$\left\{ \begin{array}{l} y - \frac{1}{2}x = 2 \\ y + x = -1 \end{array} \right.$$

- Notice that both equations have the same two variables  $x$  and  $y$
- The brace on the left is written to show that the equations are simultaneous, that is, the variables  $x$  and  $y$  represent the same unknowns in both equations

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} y - \frac{1}{2}x = 2 \\ y + x = -1 \end{cases}$$

is the ordered pair  $(-2, 1)$ .

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 1 - \frac{1}{2}(-2) = 2 \\ 1 + (-2) = -1 \end{cases}$$

is the ordered pair  $(-2, 1)$ .

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 1 + 1 = 2 \\ 1 - 2 = -1 \end{cases}$$

is the ordered pair  $(-2, 1)$ .

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 2 = 2 & \text{TRUE!} \\ -1 = -1 & \text{TRUE!} \end{cases}$$

is the ordered pair  $(-2, 1)$ .

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:



"What unknown values will I need to find for this system?"

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

## Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:



"The number of cows and the number of chickens "

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

## Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set  $x = \text{number of cows}$  and  $y = \text{number of chickens}$

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

## Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set  $x = \text{number of cows}$  and  $y = \text{number of chickens}$
- ① Heads equation:  $x + y = 35$
  - ② Legs equation:  $4x + 2y = 110$

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set  $x = \text{number of cows}$  and  $y = \text{number of chickens}$
- ① Heads equation:  $x + y = 35$
- ② Legs equation:  $4x + 2y = 110$
- And here we go!

$$\begin{cases} x + y = 35 \\ 4x + 2y = 110 \end{cases}$$

# MODEL A REAL-LIFE SITUATION USING A SYSTEM OF LINEAR EQUATIONS

## Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4x + 2y = 110 \end{cases}$$



"How do we solve it?"

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

Since we are familiar with solving an equation with one variable, we can proceed as follows:

- ① **Eliminate** one variable, i.e., using **eq. 1** and **eq. 2**, find a new equation depending only on 1 variable (to say  $y$ ).
- ② Solve the obtained equation (for  $y$ ).
- ③ Substitute the obtained value for its corresponding variable in either **eq. 1** or **eq. 2** and then solve for the other variable ( $x$ ).

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say  $y$ )

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say  $y$ )

$$\begin{cases} \boxed{-4x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say  $y$ )

$$\begin{cases} \boxed{-4x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y + \boxed{(-4)x} + (-4)y = 110 + (-4)35 & \text{eq. 2} + (-4) \times \text{eq. 1} \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say  $y$ )

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ \boxed{4x} + 2y + \boxed{(-4)x} + (-4)y = 110 + (-4)35 & \text{eq. 2} + (-4) \times \text{eq. 1} \end{cases}$$

$$\begin{cases} x + y = 35 \\ \boxed{2y - 4y = 110 - 4 \times 35} \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for  $y$ ):

$$\left\{ \begin{array}{l} x + y = 35 \\ 2y - 4y = 110 - 4 \times 35 \end{array} \right. \iff \left\{ \begin{array}{l} x + y = 35 \\ -2y = -30 \end{array} \right.$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for  $y$ ):

$$\begin{cases} x + y = 35 \\ -2y = -30 \end{cases} \iff \begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

- Substitute the obtained value for its corresponding variable in either eq. 1 or eq. 2 and then solve for the other variable.

$$\begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases} \iff \begin{cases} x + \boxed{15} = 35 \\ \boxed{y = 15} \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} x = 20 \\ y = 15 \end{cases}$$

# SOLVING A SYSTEM OF LINEAR EQUATIONS

## ELIMINATION

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4x + 2y = 110 \end{cases} \iff \begin{cases} x = 20 \\ y = 15 \end{cases}$$



"There are 20 cows in the pasture"

# INTRODUCTION TO MATRICES

- We notice while solving this system that what really matter in the operations done ( $-4 \times$  equ. 1 and equ. 2  $-4 \times$  equ. 1) (during the elimination) are the numbers attached to the variables (the coefficients).
- We can represent the same system in an equivalent way, but without any letters (without the variables  $x$  and  $y$ )
- In this case we obtain the following **rectangular table of numbers** (let's call it **matrix**)

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

# INTRODUCTION TO MATRICES

- Let  $R_1$  and  $R_2$  be respectively the first and second row of our matrix.
- By following the same operations done in the elimination step previously, that is replacing equ.2 by equ.2  $-4 \times$  equ.1, we can, in an equivalent way, replace  $R_2$  by  $R_2 - 4 \times R_1$  in the matrix (that is an **elementary matrix row operation**) to obtain

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \quad \rightarrow \quad \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array} \right)$$

# INTRODUCTION TO MATRICES

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \quad \rightarrow \quad \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array} \right)$$

- The number  $0$  represents the variable  $x$  being eliminated in a new equation of 1 variable  $y$ :  $-2y = -4$ .
- In this case, we can immediately see again the new equivalent system

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \quad \rightarrow \quad \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 0x + -2y = -30 \end{array} \right. \quad \leftarrow \quad \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array} \right)$$

# INTRODUCTION TO MATRICES

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

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# INTRODUCTION TO MATRICES

$$\left\{ \begin{array}{l} 1x + 1y = 35 \\ 4x + 2y = 110 \end{array} \right. \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array} \right)$$

$$\left\{ \begin{array}{l} x + y = 35 \\ -2y = -30 \end{array} \right. \leftarrow \left( \begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array} \right)$$

$$\left\{ \begin{array}{l} x = 20 \\ y = 15 \end{array} \right.$$

More precisely:

- A **matrix** (plural matrices) is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to represent a mathematical object or a property of such an object.
- The **dimensions** of a matrix tells its size: the number of rows and columns of the matrix, in that order.

For example:

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 3 & 2 \end{pmatrix}$$

is a matrix with two rows and three columns. This is often referred to as a "two by three matrix", a " $2 \times 3$ -matrix", or a matrix of dimension  $2 \times 3$ .

# INTRODUCTION TO MATRICES

## Matrix Elements:

- A **matrix element** is simply a matrix entry. Each element in a matrix is identified by naming the row and column in which it appears.

For example, consider the matrix  $M$ :

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & -5 & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

The element  $m_{2,1}$  is the entry in the second row and the first column.

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & \textcolor{purple}{-5} & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

In this case,  $m_{2,1} = 31$ .

- A system of equations can be represented by an **augmented matrix**.
- In an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.
- Augmented matrices are a shorthand way of writing systems of equations. The organization of the numbers into the matrix makes it unnecessary to write various symbols like  $x$ ,  $y$ , and  $=$ , yet all of the information is still there!

For example:

$$\left\{ \begin{array}{l} -4x + 5y = -1 \\ 35x + 7y = 123 \end{array} \right. \rightarrow \left( \begin{array}{cc|c} -4 & 5 & -1 \\ 35 & 7 & 123 \end{array} \right)$$

## EXAMPLE

Write the following system of equations as an augmented matrix.

$$\left\{ \begin{array}{rcl} 3x - 2y & = & 4 \\ x + 5z & = & 3 \\ -4x - y + 3z & = & 0 \end{array} \right.$$

## EXAMPLE

Write the following system of equations as an augmented matrix.

$$\left\{ \begin{array}{rcl} 3x - 2y & = & 4 \\ x + 5z & = & 3 \\ -4x - y + 3z & = & 0 \end{array} \right.$$

### Solution:

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

**EXAMPLE**

Write the following system of equations as an augmented matrix.

$$\left\{ \begin{array}{l} 3x + (-2)y + 0z = 4 \\ 1x + 0y + 5z = 3 \\ -4x + (-1)y + 3z = 0 \end{array} \right.$$

**Solution:**

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

## EXAMPLE

Write the following system of equations as an augmented matrix.

$$\left\{ \begin{array}{l} 3x + (-2)y + 0z = 4 \\ 1x + 0y + 5z = 3 \\ -4x + (-1)y + 3z = 0 \end{array} \right.$$

Solution:

$$\left( \begin{array}{ccc|c} 3 & -2 & 0 & 4 \\ 1 & 0 & 5 & 3 \\ -4 & -1 & 3 & 0 \end{array} \right)$$

## REMARK

In general, before converting a system into an augmented matrix, be sure that the variables appear in the same order in each equation, and that the constant terms are isolated on one side.

# ELEMENTARY MATRIX ROW OPERATIONS

There are three elementary matrix row operations:

- ① Switch any two rows: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 7 & 7 & 0 \\ 1 & 3 & 5 \end{pmatrix}$$

- ② Multiply a row by a nonzero constant: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow 2 \times R_1} \begin{pmatrix} 2 \times 1 & 2 \times 3 & 2 \times 5 \\ 7 & 7 & 0 \end{pmatrix}$$

- ③ Add a multiple of one row to another: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & 5 \\ 7 + 3 \times 1 & 7 + 3 \times 3 & 0 + 3 \times 5 \end{pmatrix}$$

# ELEMENTARY MATRIX ROW OPERATIONS

## Systems of equations and matrix row operations:

- Recall that in an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.
- For example, the system on the left corresponds to the augmented matrix on the right.

$$\text{System: } \begin{cases} 3x + y = 5 \\ x + 2y = 6 \end{cases} \quad \text{Matrix: } \left( \begin{array}{cc|c} 3 & 1 & 5 \\ 1 & 2 & 6 \end{array} \right)$$

When working with augmented matrices, we can perform any of the **matrix row operations** to create a new augmented matrix that produces an equivalent system of equations. Why?

# ROW-ECHELON FORM & GAUSSIAN ELIMINATION

We say that a matrix is in **row-echelon form** if it meets the following two requirements:

- ① For each row, the first (leftmost) nonzero entry (called a **leading coefficient** or **pivot**) is to the right of the one above it.
- ② Any non-zero rows are always above rows with all zeros.

Example of row-echelon form matrices:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{3} \end{pmatrix}$$

Example of matrices that are not in row echelon form:

$$\begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & \boxed{1} & 0 & 4 \end{pmatrix}$$

# ROW-ECHELON FORM & GAUSSIAN ELIMINATION

- Any matrix can be transformed to reduced row echelon form using one or more of the row operations
  - ① Interchange one row with another.
  - ② Multiply one row by a non-zero constant.
  - ③ Replace one row with: one row, plus a constant, times another row.
- In addition, it isn't enough just to know the rules, you have to be able to look at the matrix and make a logical decision about which rule you're going to use and when.

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \rightarrow \text{Not echelon form}$$

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2.R_2} \begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix} \rightarrow \text{Echelon form}$$

## GAUSSIAN ELIMINATION

- **Gaussian Elimination** is a set of well-defined instructions to solve a system of linear equations.
- It consists of a sequence of **elementary row operations** performed on the corresponding **augmented matrix**, in order to get in a **row-echelon form**.
- Then the system of linear equations corresponding to the row-echelon form is said to be **triangular**.

## EXAMPLE

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + y + 4z = 5 \\ x + 3y + 2z = 1 \end{cases}$$

## EXAMPLE

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + y + 4z = 5 \\ x + 3y + 2z = 1 \end{cases}$$

- Solution:

The table below is the **Gaussian Elimination** process applied simultaneously to the system of equations and its associated augmented matrix.

# ROW-ECHELON FORM & GAUSSIAN ELIMINATION

- The Gaussian Elimination procedure may be summarized as follows:
  - eliminate  $x$  from all equations below  $R_1$ , and then eliminate  $y$  from all equations below  $R_2$ . This will put the system into **triangular form**,
  - then, using back-substitution, each unknown can be solved for.

System of equations	Row operations	Augmented matrix
$\begin{cases} x + 2y + 3z = 2 \\ 2x + y + 4z = 5 \\ x + 3y + 2z = 1 \end{cases}$		$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x + y - z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x + 0y - 3z = 0 \end{cases}$	$R_3 \leftarrow R_3 + R_2$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -3 & 0 \end{array} \right)$
The matrix now in echelon form and the system is triangular		

# ROW-ECHELON FORM & GAUSSIAN ELIMINATION

System of equations	Row operations	Augmented matrix
$\begin{cases} x + 2y + 3z = 2 \\ 2x + y + 4z = 5 \\ x + 3y + 2z = 1 \end{cases}$		$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x + y - z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x + 0y - 3z = 0 \end{cases}$	$R_3 \leftarrow R_3 + R_2$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -3 & 0 \end{array} \right)$

The matrix now in echelon form and the system is triangular

- ① From  $R_3$ :  $-3z = 0 \iff z = 0$ .
- ② From  $R_2$ :  $-y - 2z = 1 \stackrel{z=0}{\iff} -y = 1 \iff y = -1$ .
- ③ From  $R_1$ :  $x + 2y + 3z = 2 \stackrel{z=0, y=-1}{\iff} x - 2 = 2 \iff x = 4$ .

## REDUCED ROW ECHELON FORM

**Reduced row echelon** form is a type of matrix used to solve systems of linear equations. Reduced row echelon form has four requirements:

- The first non-zero number in the first row (the leading coefficient) is the number 1.
- The second row also starts with the number 1, which is further to the right than the leading coefficient in the first row. For every subsequent row, the number 1 must be further to the right.
- The leading coefficient in each row must be the only non-zero number in its column.
- Any non-zero rows are placed at the bottom of the matrix.

## REDUCED ROW ECHELON FORM

**Reduced row echelon** form is a type of matrix used to solve systems of linear equations. For example

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & -3 & 0 & 7 \\ 0 & 0 & 0 & 1 & 12 \end{pmatrix}$$

# ROW-ECHELON FORM & GAUSSIAN ELIMINATION

System of equations	Row operations	Augmented matrix
$\begin{cases} x + 2y + 3z = 2 \\ 2x + y + 4z = 5 \\ x + 3y + 2z = 1 \end{cases}$		$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x + y - z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array} \right)$
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x+0y - 3z = 0 \end{cases}$	$R_3 \leftarrow R_3 + R_2$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -3 & 0 \end{array} \right)$
The matrix now in echelon form and the system is triangular		
$\begin{cases} x + 2y + 3z = 2 \\ 0x - y - 2z = 1 \\ 0x+0y+z = 0 \end{cases}$	$R_3 \leftarrow -\frac{1}{3}R_3$	$\left( \begin{array}{ccc c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right)$
$\begin{cases} x + 2y + 0z = 2 \\ 0x - y + 0z = 1 \\ 0x+0y+z = 0 \end{cases}$	$R_2 \leftarrow R_2 + 2R_3$ $R_1 \leftarrow R_1 - 3R_3$	$\left( \begin{array}{ccc c} 1 & 2 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right)$
$\begin{cases} x + 2y + 0z = 2 \\ 0x+y+0z = -1 \\ 0x+0y+z = 0 \end{cases}$	$R_2 \leftarrow -R_2$	$\left( \begin{array}{ccc c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$
$\begin{cases} x+0y+0z = 4 \\ 0x+y+0z = -1 \\ 0x+0y+z = 0 \end{cases}$	$R_1 \leftarrow R_1 - 2.R_2$	$\left( \begin{array}{ccc c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$

# ADDING AND SUBTRACTING MATRICES

We recall that

- a matrix is a rectangular arrangement of numbers into rows and columns,
- each number in a matrix is referred to as a **matrix element** or **entry**.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix} \quad \rightarrow \quad \text{2 rows, 3 columns,}$$

- the **dimensions** of a matrix give the number of rows and columns of the matrix in that order. Since matrix  $A$  has 2 rows and 3 columns, it is called a  $2 \times 3$  matrix.

# ADDING AND SUBTRACTING MATRICES

As long as the dimensions of two matrices are the same, we can add and subtract them much like we add and subtract numbers.

Let's take a closer look!

# ADDING AND SUBTRACTING MATRICES

## Adding matrices:

- Given  $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$ , let's find  $A + B$ .
- We can find the sum simply by adding the corresponding entries in matrices  $A$  and  $B$ :

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1+3 & 2+4 \\ 3+5 & 0+6 \\ 4+7 & 3+8 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 8 & 6 \\ 11 & 11 \end{pmatrix} \end{aligned}$$

# ADDING AND SUBTRACTING MATRICES

## Subtracting matrices:

- Given  $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$ , let's find  $A - B$ .
- Similarly, we can find the  $A - B$  simply by subtracting the corresponding entries in matrices  $A$  and  $B$ :

$$\begin{aligned} A - B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 3 & 2 - 4 \\ 3 - 5 & 0 - 6 \\ 4 - 7 & 3 - 8 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -2 \\ -2 & -6 \\ -3 & -5 \end{pmatrix} \end{aligned}$$

# MULTIPLYING MATRICES BY SCALARS

## Scalar multiplication:

- Given  $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$ , consider the scalar 3 and let's find  $3A$ .
- This scalar multiplication can be seen as repeated addition:  
 $3A = A + A + A$
- In this case, we have

$$\begin{aligned} 3A = A + A + A &= \begin{pmatrix} 1+1+1 & 2+2+2 \\ 3+3+3 & 0+0+0 \\ 4+4+4 & 3+3+3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 0 \\ 3 \cdot 4 & 3 \cdot 3 \end{pmatrix} \end{aligned}$$

- In general, in scalar multiplication, each entry in the matrix is multiplied by the given scalar.

# MULTIPLYING MATRICES BY MATRICES

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$$

We recall that

- $A$  is a  $2 \times 3$  matrix.
- the element  $a_{2,1}$  is the entry in the second row and the first column of matrix  $A$ , that is  $a_{2,1} = 2$ .

How to find the product of two matrices? For example, find

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

# MULTIPLYING MATRICES BY MATRICES



- Up until now, you may have found operations with matrices fairly intuitive. For example
  - when you add two matrices, you add the corresponding entries,
  - in scalar multiplication, each entry in the matrix is multiplied by the given scalar.
- But things do not work as you'd expect them to work with multiplication. To multiply two matrices, we **cannot** simply multiply the corresponding entries.

# MULTIPLYING MATRICES BY MATRICES

- Before studying **matrix multiplication**, let's first understand how to find the **dot product** of two ordered lists of numbers, which can help us tremendously in this quest!

**n-tuples and the dot product:**

- We are familiar with ordered pairs, for example  $(1, 2)$ ,  $(-1, 0), \dots$  and perhaps even ordered triples, for example  $(1, 1, 2)$ ,  $(-1, 0, 2), \dots$
- An  **$n$ -tuple** is a generalization of this. It is an ordered list of  $n$  numbers.
- We can find the **dot product** of two  $n$ -tuples of equal length by summing the products of corresponding entries.
- For example, to find the dot product of two ordered pairs, we multiply the first coordinates and the second coordinates and add the results.

$$\begin{aligned}(2, 3) \cdot (1, 4) &= 2 \cdot 1 + 3 \cdot 4 \\ &= 2 + 12 \\ &= 14\end{aligned}$$

# MULTIPLYING MATRICES BY MATRICES

- Ordered  $n$ -tuples are often indicated by a variable with an arrow on top. For example, we can let  $\vec{a} = (1, 2, -1)$  and  $\vec{b} = (3, 4, 1)$ . The expression  $\vec{a} \cdot \vec{b}$  indicates the dot product of these two ordered triples and can be found as follows:

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (1, 2, -1) \cdot (3, 4, 1) \\ &= 1 \cdot 3 + 2 \cdot 4 + (-1) \cdot 1 \\ &= 3 + 8 - 1 \\ &= 10\end{aligned}$$



Notice that the dot product of two  $n$ -tuples of equal length is always a single real number (called scalar).

## Matrices and $n$ -tuples:

- When multiplying matrices, it's useful to think of each matrix row and column as an  $n$ -tuple.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- In this matrix, denote
  - row 1 by  $\vec{r}_1 = (1, 2)$
  - row 2 by  $\vec{r}_2 = (3, 4)$
  - column 1 by  $\vec{c}_1 = (1, 3)$
  - column 2 by  $\vec{c}_2 = (2, 4)$

# MULTIPLYING MATRICES BY MATRICES

**Matrix multiplication:** Now we are ready to answer our question

Given  $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$ . Let's find  $C = AB$

- denote
  - ① row 1 of  $A$  by  $\vec{a}_1$ ,
  - ② row 2 of  $A$  by  $\vec{a}_2$ ,
  - ③ column 1 of  $A$  by  $\vec{b}_1$ ,
  - ④ column 2 of  $A$  by  $\vec{b}_2$ .
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 \end{pmatrix}$$

# MULTIPLYING MATRICES BY MATRICES

Generally speaking, in matrix multiplication, the entry in the product matrix located in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column, is the dot product of the  $i^{\text{th}}$  row in the first matrix and the  $j^{\text{th}}$  column in the second matrix.



"But when are we allowed to multiply two matrices?"

"What are the properties of this operation?"

# PROPERTIES OF MATRIX MULTIPLICATION

- ① **When is matrix multiplication defined?** In order for matrix multiplication to be defined, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$(m \times \underbrace{n}_{\text{product is defined}}) \cdot (n \times k)$$

- ② **What about dimensions the obtained matrix product?**

When the matrix multiplication is defined, then the resulting matrix product has the number of lines of the first matrix and the number of columns of the second matrix.

$$(m \times \underbrace{n}_{\text{product is defined}}) \cdot (n \times k) = (m \times k)$$

# PROPERTIES OF MATRIX MULTIPLICATION

- A matrix that has the same number of rows and columns is called **square matrix**.
- The entries of a matrix that lie on the  $i^{th}$  row and the  $i^{th}$  column form the so-called **diagonal** of a matrix. For example, the diagonal of the following matrix is given by the blue entries

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- The **square matrix** where the entries on the diagonal from the upper left to the bottom right are all 1's, and all other entries are 0 is called **identity matrix**, and is denoted by  $I_n$  where  $n$  is the number of rows (and columns) of the matrix. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# PROPERTIES OF MATRIX MULTIPLICATION

- The product of any square matrix and the appropriate identity matrix is always the original matrix, regardless of the order in which the multiplication was performed!
- In other words, for a square matrix  $A$  we have

$$A \cdot I = I \cdot A = A.$$

- Let  $A$ ,  $B$ , and  $C$  be  $(n \times n)$  matrices and  $I$  the  $(n \times n)$  *identity matrix*. then we have:
  - ①  $AB \neq BA$  (Check it!)
  - ②  $(AB)C = A(BC)$
  - ③  $A(B + C) = AB + AC$
  - ④  $(B + C)A = BA + CA$
- If  $AB = BA = I$ , then we say that  $A$  is the **inverse** of  $B$  (or even  $B$  is the **inverse** of  $A$ )

## DETERMINANT OF A $2 \times 2$ MATRIX

- The determinant of a  $(2 \times 2)$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $|A| = ad - cb$ . It is simply obtained by cross multiplying the elements starting from the top left, then subtracting the products.
- For example, if  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ , then the determinant  $|A| = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$ .

# INTRODUCTION TO MATRIX INVERSES

- We know that the inverse of a square matrix  $A$  is a square matrix  $B$  that verifies  $AB = BA = I$ , where  $I$  is the identity matrix.
- When such a matrix  $B$  exists, we say  $A$  is **invertible**.
- So let's state a very important rule!

A matrix is **invertible** if and only if its determinant is  
**nonzero**

# INTRODUCTION TO MATRIX INVERSES

## EXAMPLE

Are the following matrices invertible?

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 6 \\ -5 & 1 \end{pmatrix}$$

# OUTLINE

## 1. INTRODUCTION TO MATRICES

Model a real-life situation using a system of linear equations

Introduction to matrices

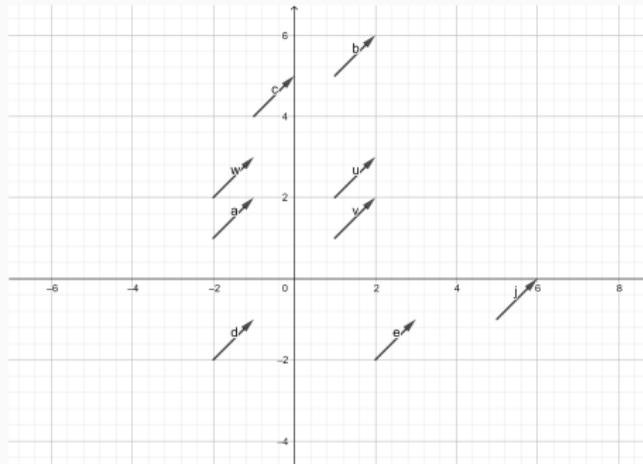
Matrix algebra

## 2. EIGENVALUES, EIGENVECTORS

- ① A **vector** is an  $n$ -tuple, or simply an ordered list of numbers. For example, a vector of the flat plane is an ordered pair, for instance  $(9, 2)$ ,  $(-1, 2)$ , etc..
- ② A vector of the flat plane can also be written like  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ,  
 $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ , etc..
- ③ In general, a vector of  $n$  components can be seen as an  $n \times 1$  or  $1 \times n$  matrix.

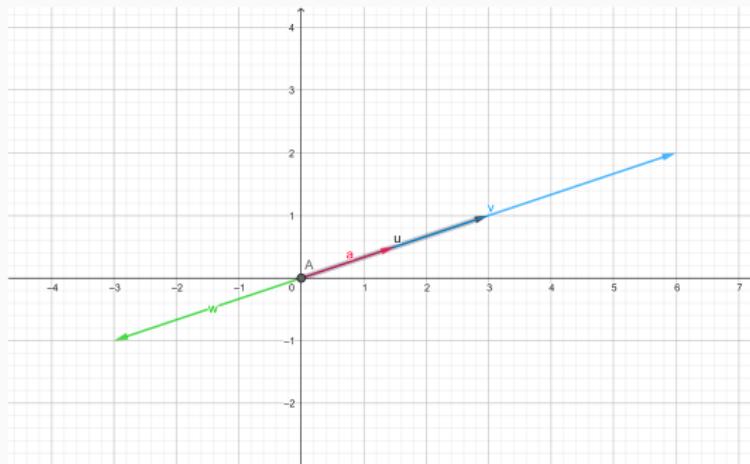
# VECTORS

- ① A vector can also be seen geometrically as an arrow pointing in space. In this case, what define a given vector are its **length** and the **direction** it's pointing in.
- ② Two vectors that have same length and direction are considered to be the same!



# VECTORS

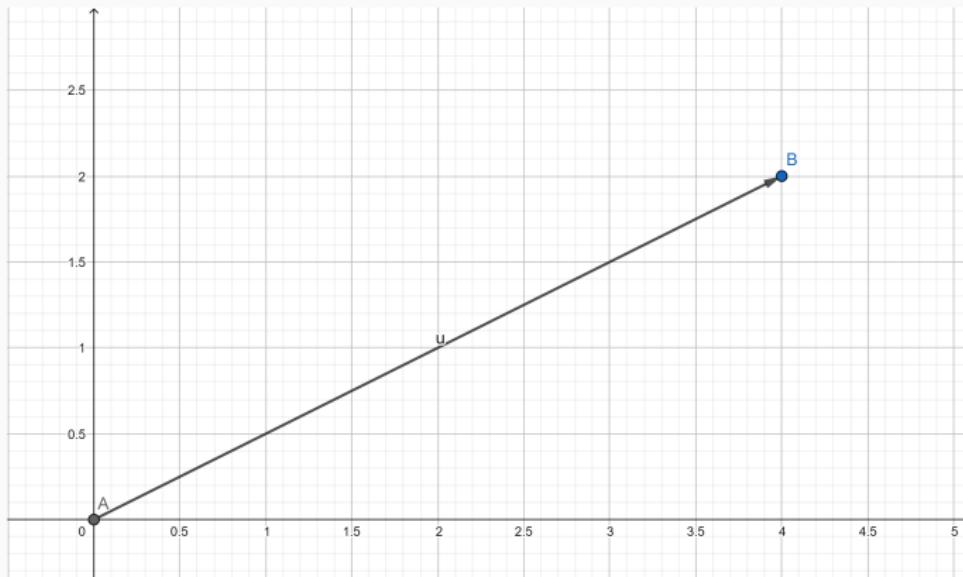
- Multiplying a vector by a number (**scalar**) would scale this vector, i.e., either stretching it or squishing it.
- Consider for example the vector  $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$  then the vectors  
 $2\begin{pmatrix} 1 \\ 3 \end{pmatrix}, -1\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



- In general, vectors such that one is the scale of the other are called **colinear vectors**.
- Hereafter, we focus on vectors in the flat coordinate plane (the  $(x, y)$ -plane), where a vector is given by  $\begin{pmatrix} a \\ b \end{pmatrix}$
- In this case, we think of a vector as an arrow in the  $(x, y)$ -plane, having its tail at the origin and its tip at the point  $(a, b)$ .

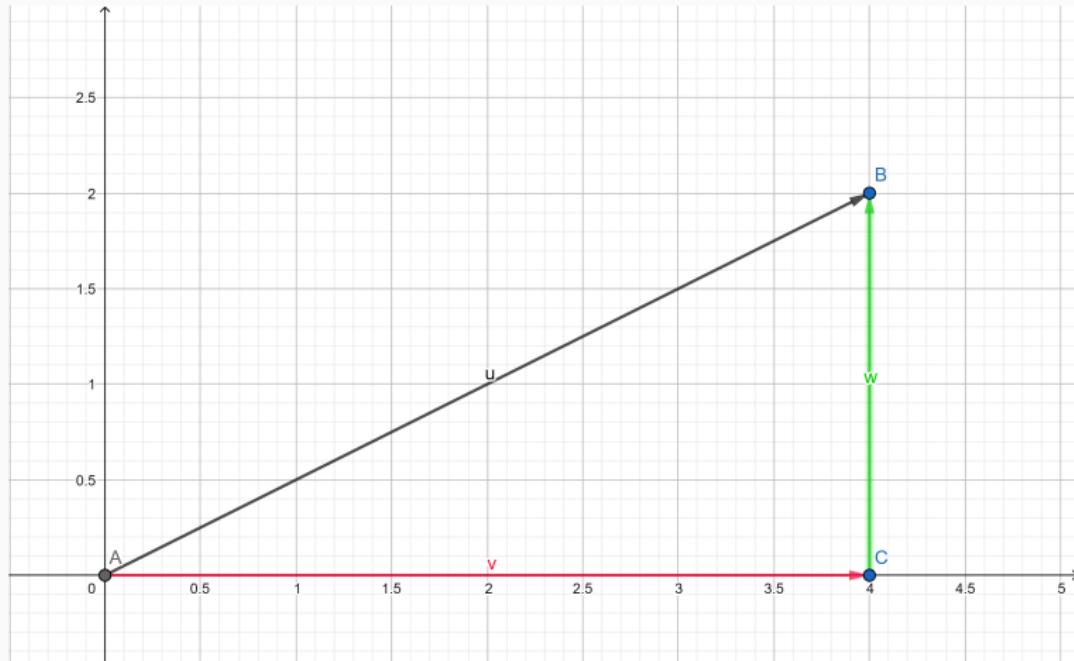
# VECTORS

- See for instance the vector  $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$  of tail  $A$  at the origin, and tip  $B$  of coordinates  $(4, 2)$ .
- And let's simply denote this vector  $\overrightarrow{AB}$  (symbolizing a trajectory from  $A$  to  $B$ ).



# VECTORS

- But as we can see, going from  $A$  to  $B$  is equivalent to going from  $A$  to  $C$  and then from  $C$  to  $B$ .



# VECTORS

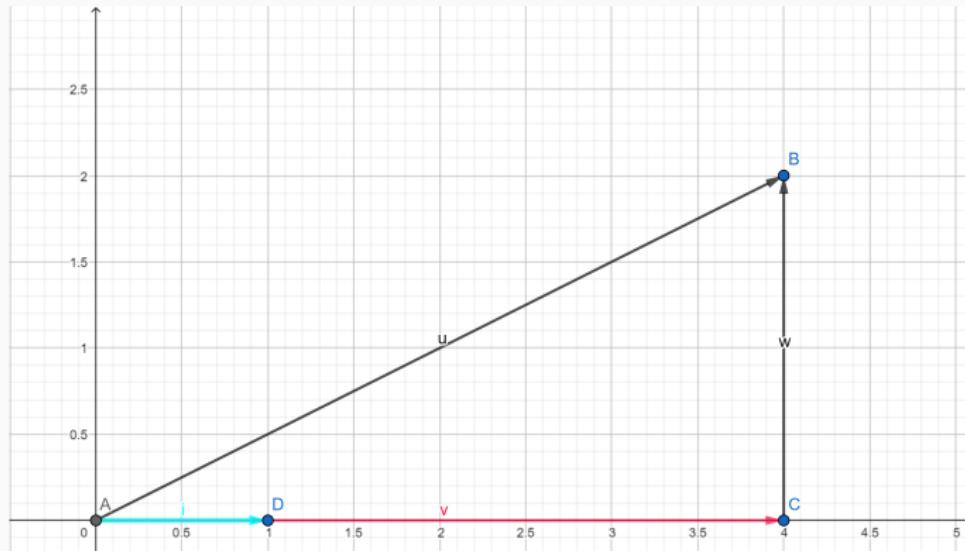
- This means that  $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$ .



# VECTORS

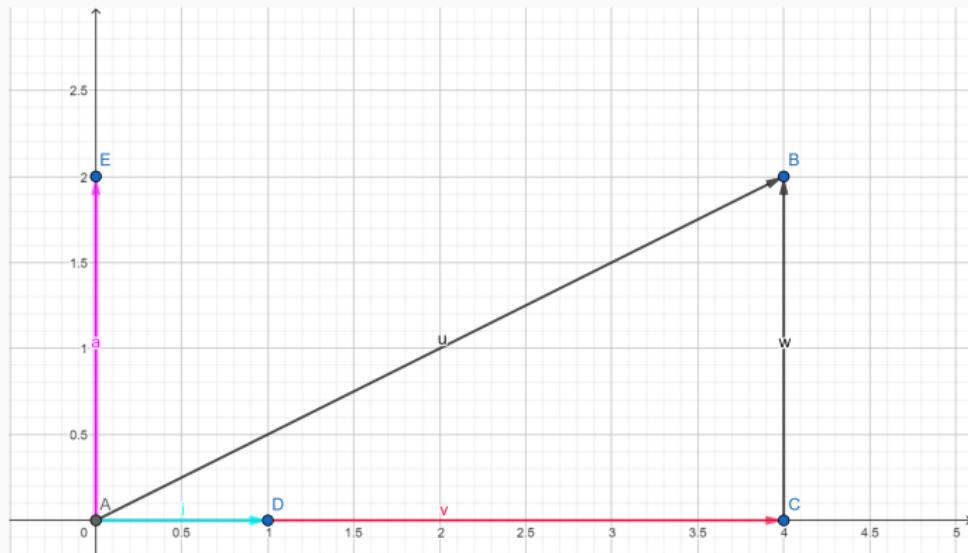
- The vector  $\overrightarrow{AC}$  starts from the origin and ends at the point  $C$  of coordinates  $(4, 0)$ , thus

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4\vec{i}$$



# VECTORS

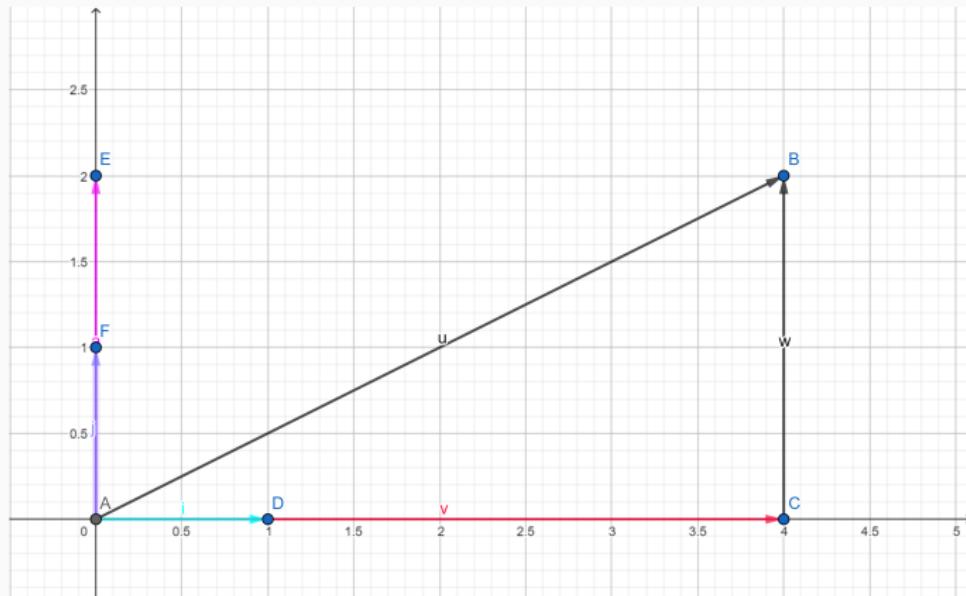
- The vector  $\overrightarrow{CB}$  doesn't start from the origin. But having the same length and direction,  $\overrightarrow{CB}$  is the same vector as  $\overrightarrow{AE}$ , that starts at the origin, and ends at the point  $E$  of coordinates  $(0, 2)$ .



# VECTORS

- Thus

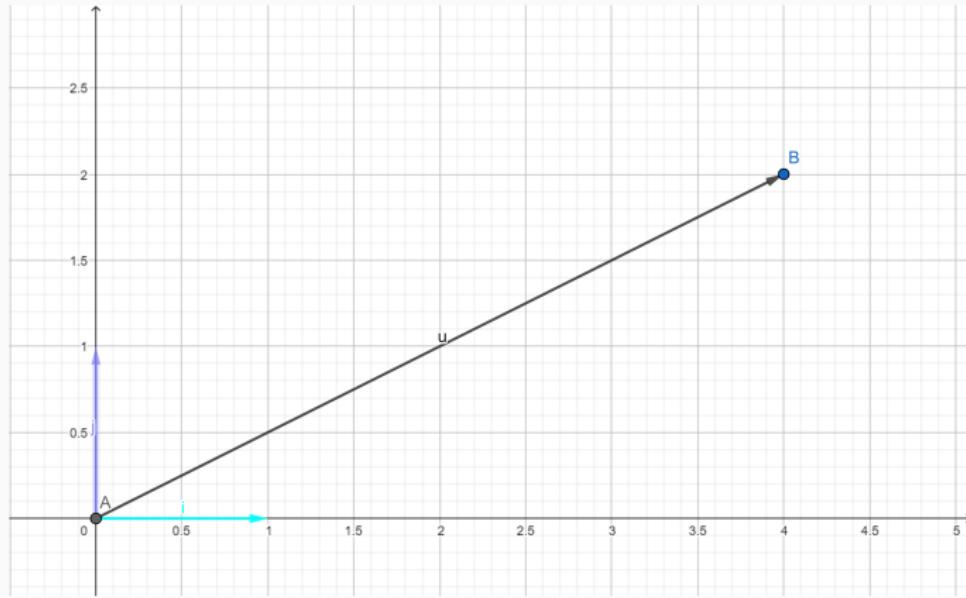
$$\overrightarrow{CB} = \overrightarrow{AE} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\vec{j}$$



# VECTORS

- Finally,

$$\vec{AB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\vec{i} + 2\vec{j}$$



- In general, any vector  $\overrightarrow{AB} = \begin{pmatrix} \textcolor{green}{a} \\ \textcolor{violet}{b} \end{pmatrix}$  in the coordinate plane can be written as

$$\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\vec{i} + b\vec{j}$$

- In this case, we say that  $\overrightarrow{AB}$  is a **linear combination** of the vectors  $\vec{i}$  and  $\vec{j}$ .
- This property, along with the fact that  $\vec{i}$  and  $\vec{j}$  are **not colinear**, make of the couple  $(\vec{i}, \vec{j})$  a so-called **basis** of the coordinate flat plane, but more precisely, its **the canonical basis**.
- In this case, we say that  $a$  and  $b$  are the coordinates of the vector  $\overrightarrow{AB}$  in the canonical basis  $(\vec{i}, \vec{j})$ .

- In general, if we chose any two vectors in the coordinate flat plane that are not colinear, these two vectors form a basis.
- This means, in the  $(x, y)$ -plane, if two vectors  $\vec{u}$  and  $\vec{v}$  are not colinear, then they form a basis, that is, any vector  $\overrightarrow{AB}$  can be written as a linear combination of  $\vec{u}$  and  $\vec{v}$ .
- This means, that we can find two scalars  $c$  and  $d$  such that  $\overrightarrow{AB} = c\vec{u} + d\vec{v}$ .
- In this case, we say  $c$  and  $d$  are the coordinates of  $\overrightarrow{AB}$  in the basis  $(\vec{u}, \vec{v})$ .

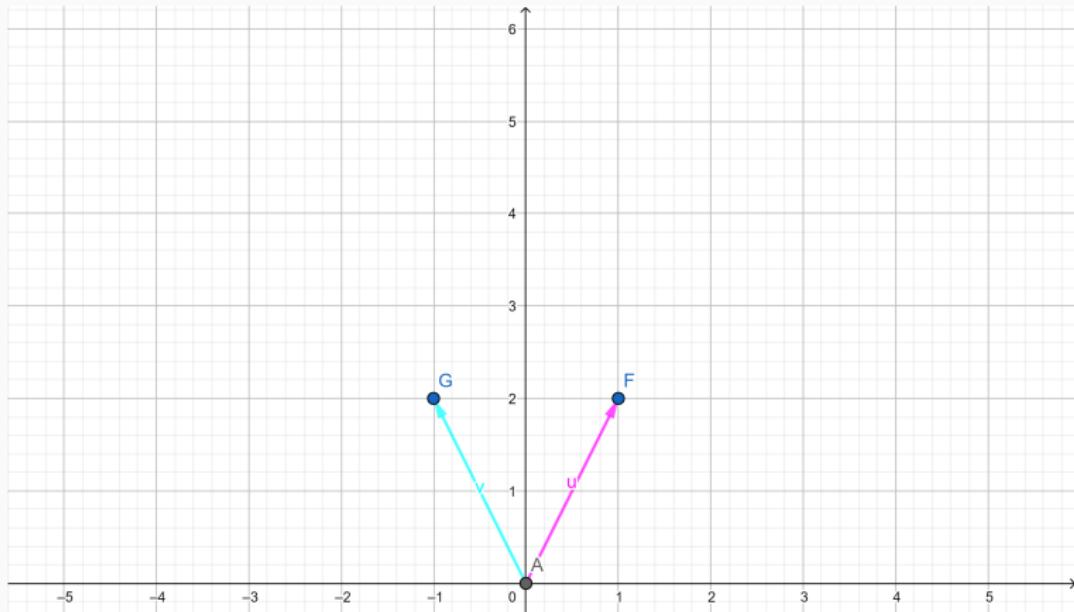
- For example, if we consider the non-colinear vectors  $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ , we know they form a basis.
- In this case,

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\vec{u} + \vec{v}$$

- In this case, we can say that the vector  $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ :
  - ① Is of coordinates  $(1, 6)$  in the canonical basis
  - ② Is of coordinates  $(2, 1)$  in the basis  $(\vec{u}, \vec{v})$ .

# VECTORS

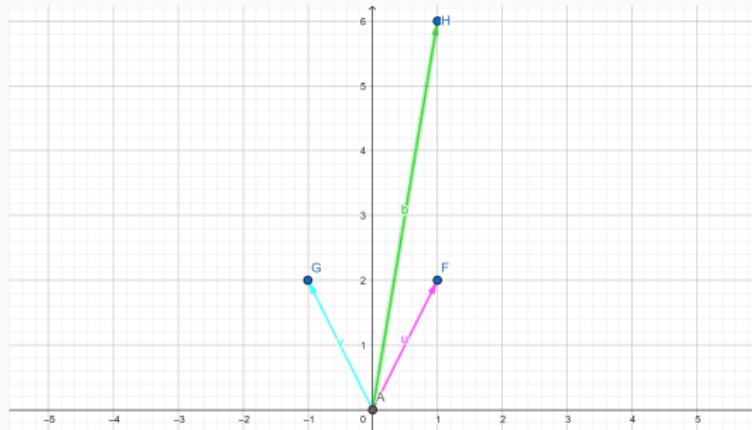
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# VECTORS

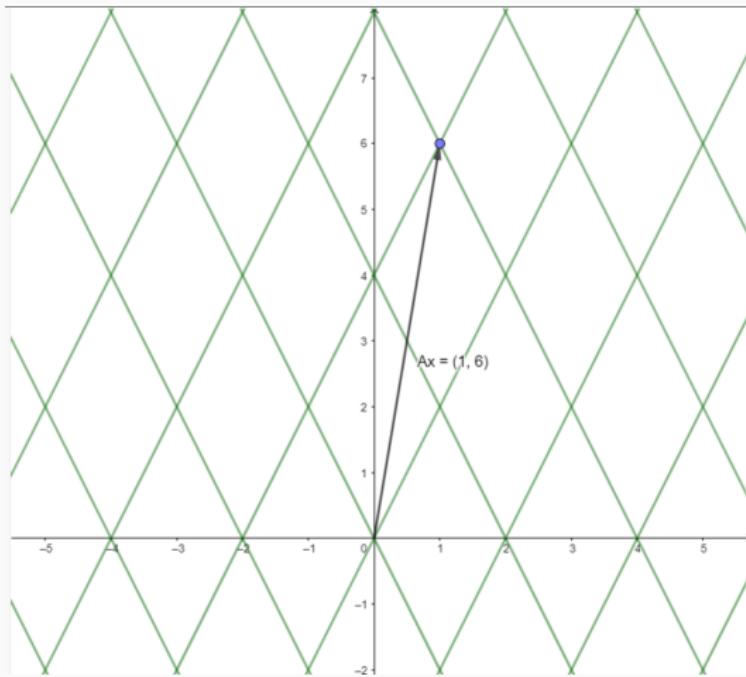
- For example, if we consider the non-colinear vectors  $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ , we know they form a basis.
- In this case,

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\vec{u} + \vec{v}$$



# VECTORS

- In this case, we can say that the vector  $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ :
  - Is of coordinates  $(1, 6)$  in the canonical basis
  - Is of coordinates  $(2, 1)$  in the basis  $(\vec{u}, \vec{v})$ .



# MATRIX AS TRANSFORMATION

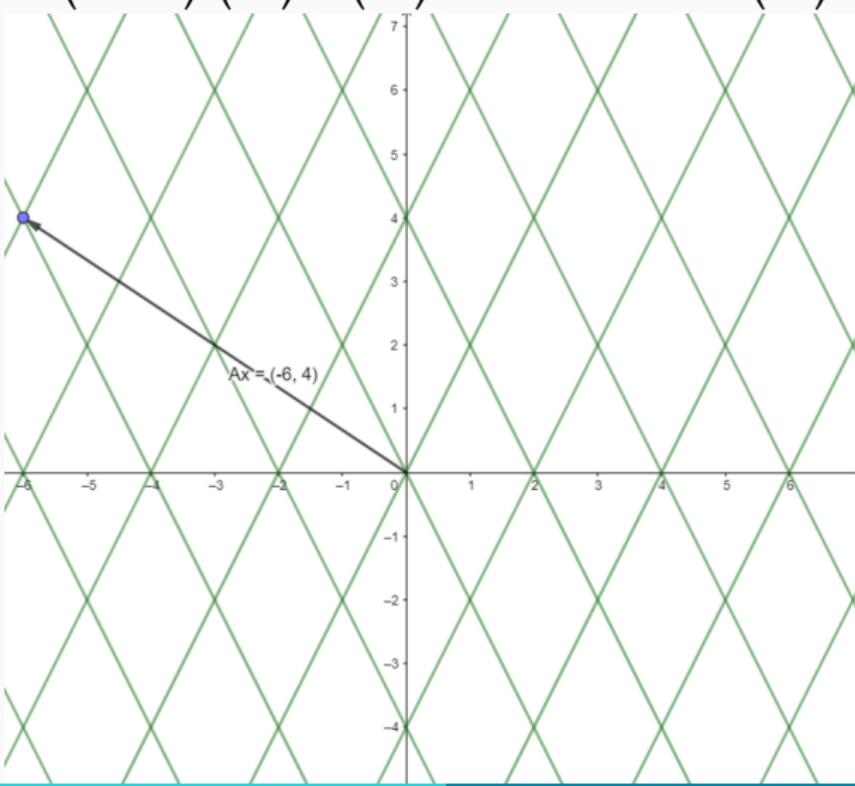
- Now take again the non-colinear vectors  $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and then consider the matrix  $M$  having as columns the vectors  $\vec{u}$  and  $\vec{v}$

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

- Can you imagine how the vector of coordinates  $(-2, 4)$  in the basis  $(\vec{u}, \vec{v})$  would look like?
- Of course you can, but you don't have to, you can simply **multiply**  $M$  by the vector  $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and it will take straight to the vector you are searching for, that is the vector of coordinates  $(-2, 4)$  in the basis  $(\vec{u}, \vec{v})$ .

# MATRIX AS TRANSFORMATION

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \iff -2\vec{u} + 4\vec{v} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$



# MATRIX AS TRANSFORMATION

This means that the matrix  $M = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$  transforms the vector  $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$  to the vector  $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ :

- $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$  is of coordinates  $(-6, 4)$  in the canonical basis.

- $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$  is of coordinates  $(-2, 4)$  in the basis  $(\vec{u}, \vec{v})$ .

# EIGENVALUES, EIGENVECTORS

- For some matrices, there are some special vectors such that after the transformation, the new obtained vector in the new basis, is simply a scale of the original one.
- In other words, we can have a matrix  $M$  and a vector  $\vec{v}$  such that

$$M \cdot \vec{v} = \text{a number} \cdot \vec{v}, \text{ that is, } M \cdot \vec{v} = \lambda \cdot \vec{v}$$

## EXAMPLE

- Consider the matrix  $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$
- Compute  $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- Compute  $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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## Solution:

- Consider the matrix  $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$
- $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 + 2 \times 2 \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 + 2 \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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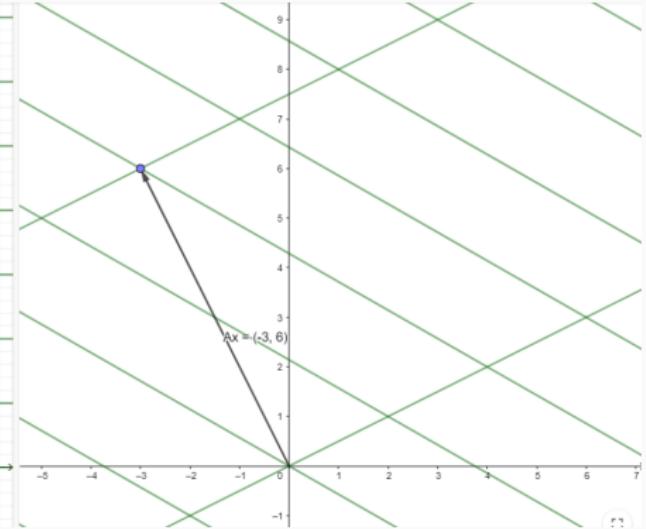
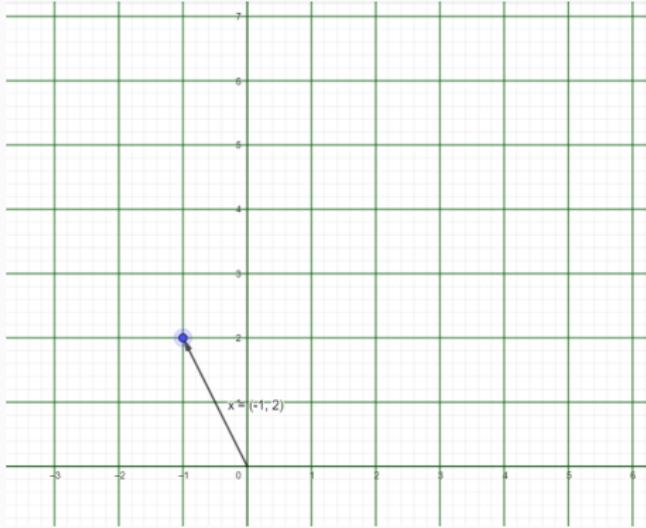
For instance, the matrix  $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$  has

- an eigenvector  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$  associated to the eigenvalue 3,
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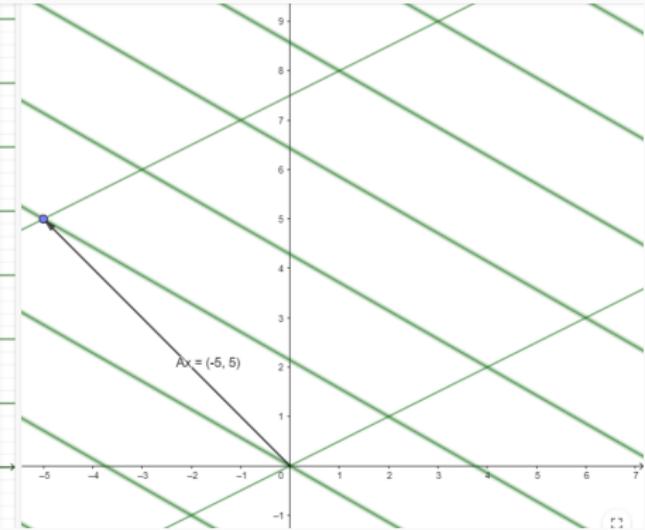
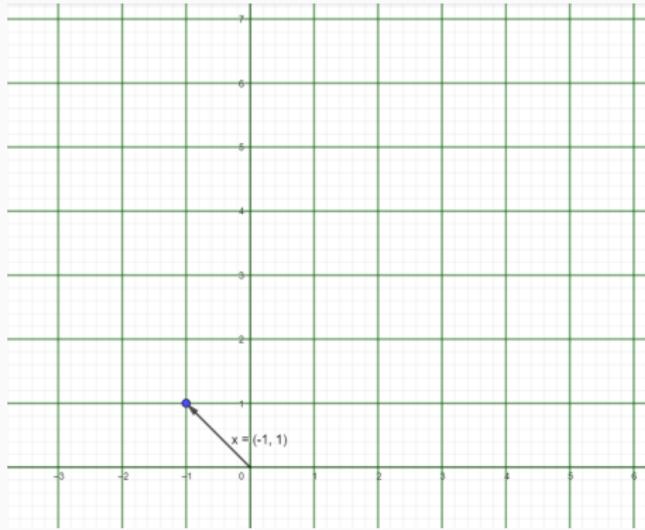
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# EIGENVALUES, EIGENVECTORS

We can visualize more examples on [Geogebra](#)

# THANK YOU!