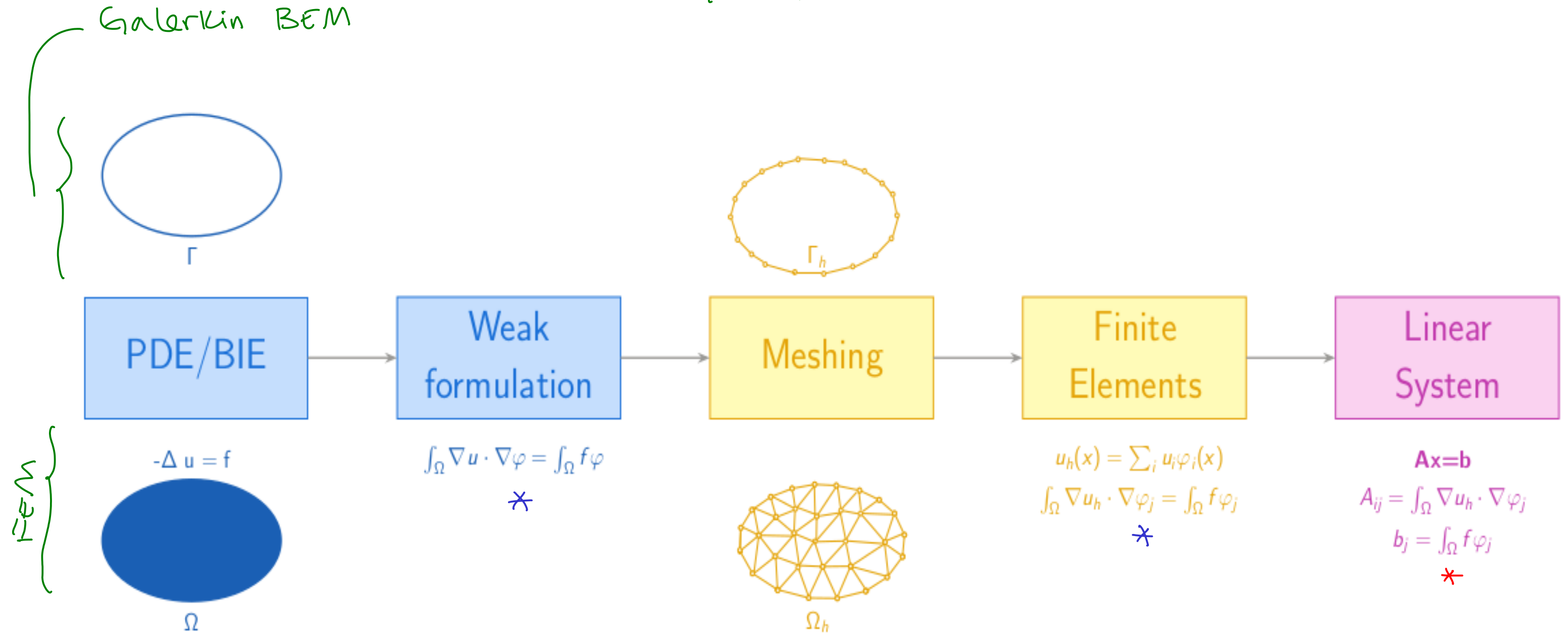


DISCLAIMER: This document contains the "whiteboard notes" taken during the "Introduction to FEM for Helmholtz" at the waves Summer School in August 2025.

Therefore, they are not complete and contain some simplifications for the sake of time, etc. Moreover, the document has not been proof-read and may therefore have some minor mistakes/typos that are typical during lectures

In the coming days I will add the corresponding references and sources for the pictures that are not my own.

FIGURE 1



INTRO TO FEM FOR HELMHOLTZ

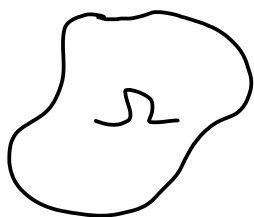
Three lectures:

- ① FEM in a nutshell
- ② Helmholtz problems: variational formulation & discretization
- ③ Numerical challenges

LECTURE 1: FEM IN A NUTSHELL

Let $\Omega \subset \mathbb{R}^n$ be a "regular enough" domain
Let us consider the BVP

$$\begin{cases} Lu = f & \text{in } \Omega \\ + \text{ boundary conditions on } \partial\Omega \end{cases}$$



$L \hat{=}$ operator corresponding to the PDE

$$\begin{aligned} L &= -\Delta, & L &= -\Delta - k^2 \text{Id} \\ L &= \partial_t - \Delta, & & \text{etc.} \end{aligned}$$

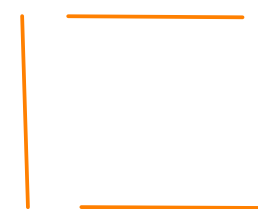
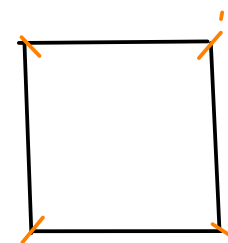
[Figure 1]

Let us illustrate this with the Laplacian.
First consider homogeneous Dirichlet problem

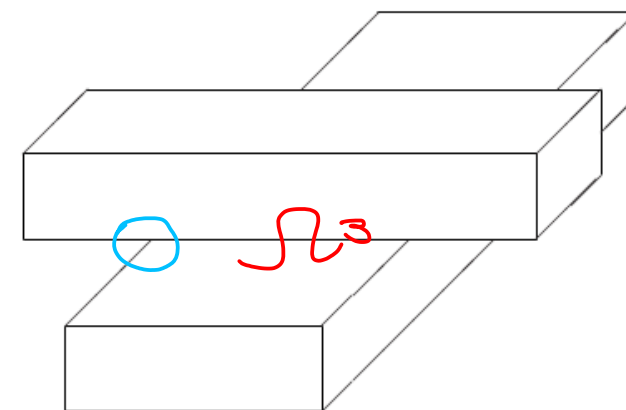
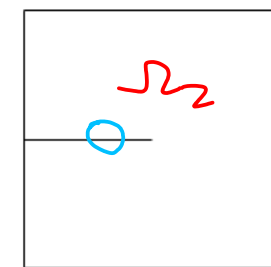
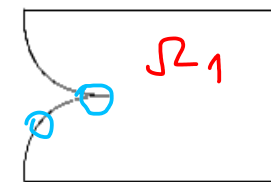
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

1.1 what are our assumptions on Ω ?

We assume Ω to be Lipschitz

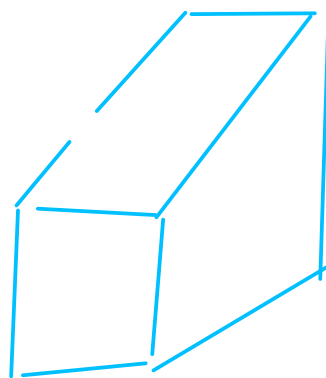
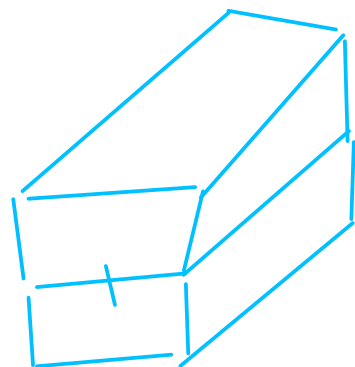


Lipschitz because we can split $\partial\Omega$ into pieces that can be parametrized by Lipschitz functions 😊



not Lipschitz!
!!

Discussion



1.2 Variational formulation

1.2.1. Main ingredients

▷ integration by parts

$$(1D) \quad \int_a^b u(x) v'(x) dx = u v \Big|_{x=a}^b - \int_a^b u'(x) v(x) dx \quad (E1)$$

$\forall u, v \in C^1[a, b]$

(2D-3D)

$$\int_{\Omega} \operatorname{div} \underline{u}(x) v(x) d\Omega = \int_{\partial\Omega} (\underline{u}(x) \cdot \underline{n}(x)) v(x) d\sigma \quad (E2)$$

$$- \int_{\Omega} \underline{u}(x) \cdot \nabla v(x) d\Omega$$

$v \in C^1(\Omega)$
 $\underline{u} \in [C^1(\bar{\Omega})]^n$

1.2.2. variational formulation for the Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (E3.1)$$

$$(E3.2)$$

i) Let $v \in H_0^1(\Omega) = \overline{C_0^\infty}^{H^1} \stackrel{(\underline{u})}{=} \{v \in H^1(\Omega) \text{ st } v|_{\partial\Omega} = 0\}$
 then

$$\int_{\Omega} -\Delta u v d\Omega = \int_{\Omega} f v d\Omega \quad \forall v \in H_0^1(\Omega)$$

ii) integration by parts $(-\Delta u = -\operatorname{div}(\nabla u))$

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x) d\Omega = \int_{\partial\Omega} (\nabla u \cdot \underline{n}) v d\sigma$$

$\xrightarrow{0}$
 $\approx v \in H_0^1(\Omega).$

$$= \int_{\Omega} f(x) v(x) d\Omega \quad \forall v \in H_0^1(\Omega)$$

1st step $-\Delta u = f$

Last step

Find $u \in H_0^1(\Omega)$ st

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f v \, d\Omega \quad \forall v \in H_0^1(\Omega).$$

Q where does f live?

What do we need for $\int_{\Omega} f v \, d\Omega$ to be well-defined.

1) f to be integrable.

$$2) \int_{\Omega} f v \, d\Omega < \infty$$

if we use L^2 duality, Cauchy-Schwarz gives you

$$\int_{\Omega} f v \, d\Omega \leq \|v\|_{H_0^1(\Omega)} \|f\|_{[H_0^1(\Omega)]'}$$

dual space with respect to $L^2(\Omega)$ (aka $H^{-1}(\Omega)$).

So now $f = \delta_0 \in H^{-1}(\Omega)$ is allowed.

Let us rewrite the variational formulation we got as follows:

Given $f \in [H_0^1(\Omega)]'$, find $u \in H_0^1(\Omega)$ st

$$\underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega}_{=: a(u, v)} = \underbrace{\int_{\Omega} f v \, d\Omega}_{=: l(v)} \quad \forall v \in H_0^1(\Omega).$$

1.3. Existence and uniqueness

1.3.1. The simplest case

is a continuous bilinear form

Let X Hilbert space, $a \in \mathcal{L}(X \times X, \mathbb{R})$, $l \in \mathcal{L}(X, \mathbb{R})$ [l is a bounded linear form].

the general variational problem can be written as:

$$\text{find } u \in X \text{ st} \\ a(u, v) = l(v) \quad \forall v \in X \quad (P_1)$$

Thm: Lax-Milgram

If $\exists \alpha, C > 0$ st

$$\alpha \|u\|_X^2 \leq |a(u, u)| \quad (\text{coercivity})$$

$$|a(u, u)| \leq C \|u\|_X^2 \quad (\text{continuity})$$

then $\exists! u \in X$ that solves (P_1)

1.3.2 Galerkin method Let $X_N \subset X$ st $\dim(X_N) < \infty$
 [finite dimensional]

$$\text{Find } u_N \in X_N \text{ st} \\ a(u_N, v_N) = l(v_N) \quad \forall v_N \in X_N \quad (P1N)$$

▷ Cea's lemma

$$\text{If } \exists \alpha_a, C_a > 0 \text{ st } \alpha_a \|u\|_X^2 \leq |a(u, u)| \\ |a(u, u)| \leq C_a \|u\|_X^2$$

then $\exists! \tilde{u} \in X$ that solves (P),
 $\exists! \tilde{u}_N \in X_N$ that solves (P1N),
 and they satisfy

$$\|\tilde{u} - \tilde{u}_N\|_X \leq C_{q0} \min_{w_N \in X_N} \|\tilde{u} - w_N\|_X \quad (E4)$$

$$\text{with } C_{q0} = C_a / \alpha_a.$$

Best approximation property.

▷ Approximation property The discrete space $X_N \subset X$ fulfills the approximation property if
 $\lim_{N \rightarrow \infty} \inf_{v_N \in X_N} \|v - v_N\| = 0 \quad \forall v \in X.$

* Cea's lemma + approx. property \Rightarrow convergence.
 i.e. $\lim_{N \rightarrow \infty} \tilde{u}_N \rightarrow \tilde{u}$

1.4 Discretization \rightarrow Finite elements

1.4.1 Concrete example

Recall the variational formulation for the Dirichlet Poisson problem (E3)

Given $f \in [H_0^1(\Omega)]'$, find $u \in H_0^1(\Omega)$ st

$$\underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega}_{a(u, v)} = \underbrace{\int_{\Omega} f v \, d\Omega}_{l(v)} \quad \forall v \in H_0^1(\Omega). \quad (E3')$$

Let $X_N \subset H_0^1(\Omega)$ st $\dim(X_N) = N$, $X_N = \text{span}\{\varphi_i\}_{i=1}^N$
 Then we can write any $u_N \in X_N$ as

$$u_N = \sum_{i=1}^N \beta_i \varphi_i$$

Then, the discretization of (LES) via the Galerkin method is

$$\begin{aligned} a(u_N, v_N) &= l(v_N) \quad \forall v_N \in X_N \\ \Leftrightarrow a(u_N, \varphi_j) &= l(\varphi_j) \quad j = 1, \dots, N \end{aligned}$$

$$\Leftrightarrow (\text{Plug } u_N = \sum_{i=1}^N \beta_i \varphi_i)$$

$$\sum_{i=1}^N \beta_i a(\varphi_i, \varphi_j) = l(\varphi_j) \quad j = 1, \dots, N$$

$$\Leftrightarrow \underline{S} \underline{\beta} = \underline{f}$$

where

$$S_{j,i} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega, \quad f_j = \int_{\Omega} f \varphi_j \, d\Omega$$

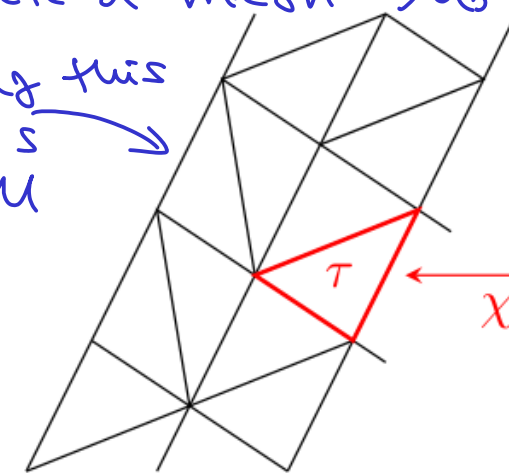
$\underline{\beta} \hat{=}$ coefficient vector for u_N .

1.4.2. Piecewise polynomial spaces X_N

i) Create a mesh \mathcal{M} of Ω

ii)

Say this
is
 \mathcal{M}



Reference element

Then, we construct our mesh \mathcal{M} as a collection of elements τ that are the image of a fixed reference element under a diffeomorphism
 $\chi_\tau : \hat{\tau} \rightarrow \tau$.

Let $p \in \mathbb{N}_0$ and

$\mathbb{P}_p = \{ \text{polynomials of degree } \leq p \text{ on } \hat{\tau} \}$

$X_N \hat{=}$ piecewise polynomial space st for each $\tau \in \mathcal{M}$
 and $u_N \in X_N$ we have

$$u_N|_{\tau} = \chi_\tau \circ P \quad \text{for some } P \in \mathbb{P}_p.$$

(i.e. that any element of X_N is the mapping of an element of TP_p).

You may also want to impose that all $w_N \in X_N$ are globally continuous for $X_N \subset X$.

Remark:

As $N \rightarrow \infty$, we require

- $h_M := \max_{T \in M} \text{diam}(T) \rightarrow 0$ (h-FEM)

- $p \rightarrow \infty$ (p-FEM)

- $h_M \rightarrow 0$ and $p \rightarrow \infty$ (hp-FEM)

△ Convergence rates: For a simplex $T \in \mathbb{R}^d$, we define

$$h_T = \text{diam}(T), \quad (h_M := \max_{T \in M} h_T)$$

shape regularity measure $\rho_T = h_T^d / |T|$

shape regularity of mesh M

$$\rho_M := \max_{T \in M} \rho_T$$

Theorem [Best approx. estimates for Lagrangian FEM]

Let $\Omega \subset \mathbb{R}^d$, $d=1,2,3$. be a bounded polygonal / polyhedral domain equipped with a mesh M consisting of simplices.

Then, $\forall r \in \mathbb{N} \quad \exists C > 0$ depending only r and ρ_M st

$$\inf_{v_N \in X_N} \|u - v_N\|_{H^1(\Omega)} \leq C \left(\frac{h_M}{p} \right)^{\min(p+1, r)-1} \|u\|_{H^{r+1}(\Omega)}$$

* This assumes M is a quasi-uniform mesh.

$p \triangleq$ polynomial degree.
 $r \triangleq$ "extra" regularity of u .

If $r=4$ and $p=3$, then $\left(\frac{h_M}{3}\right)^3$
 But $r=1$ and $p=3$, then $\left(\frac{h_M}{3}\right)^0$