Problem Sheet

Mathematical models in nonlinear acoustics

Westervelt equation is a classical model used for describing the propagation of nonlinear sound waves. In thermoviscous fluid media, it is given by

$$u_{tt} - c^2 \Delta u - b \Delta u_t = k(u^2)_{tt} + f. \tag{1}$$

Here u denotes the acoustic pressure, $c > 0$ is the speed of sound in the medium, $b > 0$ the sound diffusivity, and $k \in \mathbb{R}$ the nonlinearity coefficient. The function $f = f(x, t)$ acts as the source of sound.

Consider a linearized version of (1) with $k = 0$:

$$u_{tt} - c^2 \Delta u - b \Delta u_t = f, \tag{2}$$

on $\Omega \times (0, T)$, and supplement it with homogeneous initial data $(u, u_t)|_{t=0} = (0, 0)$ and homogeneous Dirichlet boundary conditions. Let $f \in L^2(0, T; L^2(\Omega))$. Assuming the solution of this problem exists and is sufficiently smooth, by using suitable test functions, show that the following (higher-order) energy inequality holds:

$$\begin{aligned} &\int_0^t \|u_{tt}(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \|\Delta u(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{H^1(\Omega)} + \int_0^t \|\Delta u(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \\ &\leq C(T) \int_0^t \|f(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \end{aligned}$$

for all $t \in (0, T)$.

These energy arguments can be transferred to the study of Westervelt equation (1) by first carrying out the estimates on a linearization and then tying them to the original equation via Banach's fixed-point theorem on a ball of a sufficiently small radius. If you have time, you can try to work out the details.

You might need to rely on Grönwall's inequality: *Let $w,\ v \in L^\infty(0, T)$ be almost everywhere non-negative functions that satisfy*

$$w(t) + v(t) \leq a_1 + \int_0^t a_2(s) w(s) \,\mathrm{d}s \quad \textit{for a.e. } t \in [0, T],$$

where $a_1 \geq 0$ and $a_2 \in L^1(0, T)$ is an almost everywhere non-negative function. Then the following Grönwall inequality holds:

$$w(t) + v(t) \leq a_1 e^{\int_0^t a_2(s) \,\mathrm{d}s} \quad \textit{for a.e. } t \in [0, T].$$