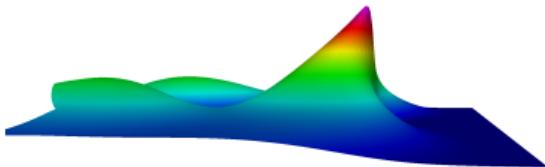


Mathematical Models in Nonlinear Acoustics

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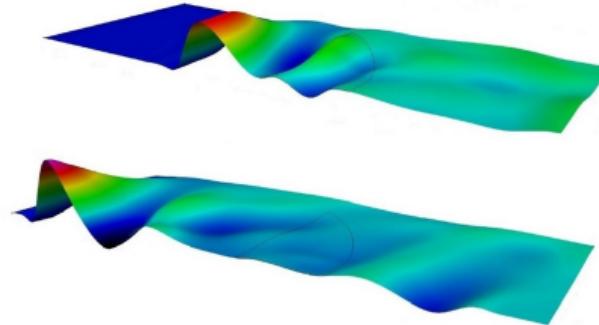
Nijmegen, August 2025

Organization

- ① 9–10:30 Lecture
- ② 11:00–12:30 Lecture + Problem session

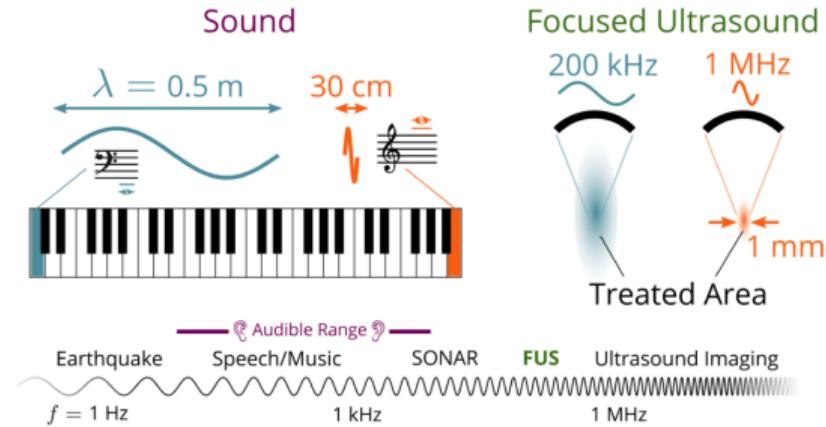
Acoustics

- Sound = Oscillatory disturbance that moves away from a source



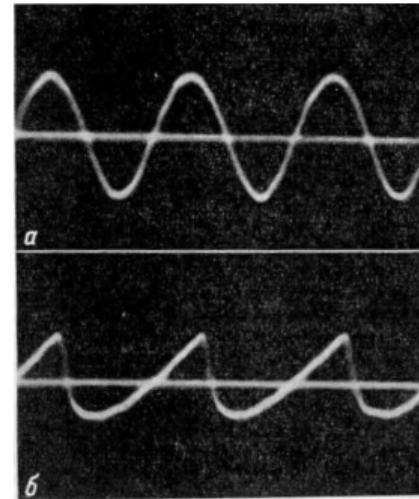
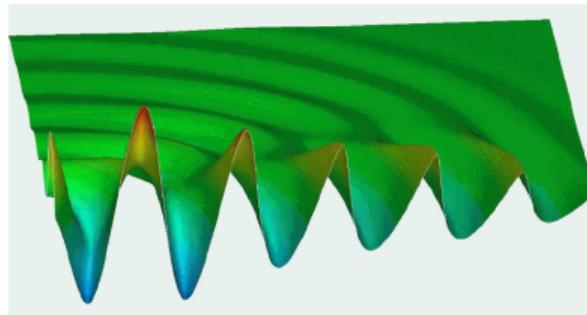
- Sound is a mechanical wave
 - it requires a medium to propagate through

Acoustics



[Schoen Jr. & Arvanitis, <https://acoustics.org/>]

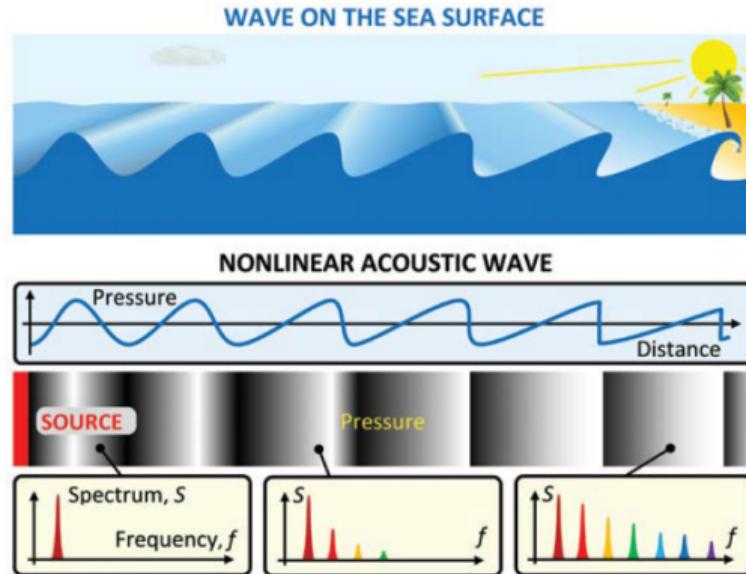
Nonlinear acoustics



[Naugol'nykh, Romanenko, 1958]

High amplitude-to-wavelength ratio \rightsquigarrow Nonlinear behavior

Nonlinear acoustic vs. water waves

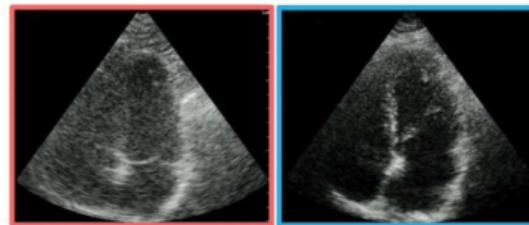


[Sapozhnikov et al., 2019]

Ultrasound applications

Diagnostic

Non-invasive imaging for real-time visualization of organs and tissues.



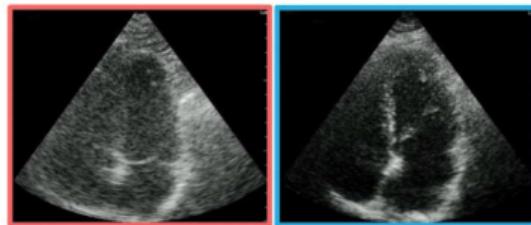
Linear vs. Nonlinear imaging

[Sapozhnikov et al., 2019]

Ultrasound applications

Diagnostic

Non-invasive imaging for real-time visualization of organs and tissues.

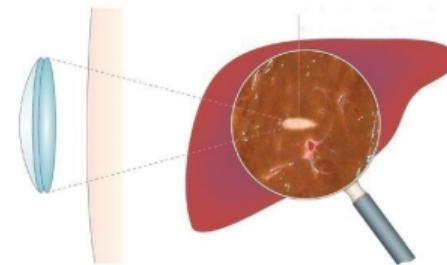


Linear vs. Nonlinear imaging

[Sapozhnikov et al., 2019]

Therapeutic

Includes cancer therapy, targeted drug delivery, and gene therapy.



Cancer therapy with HIFU

[Kennedy, 2005]

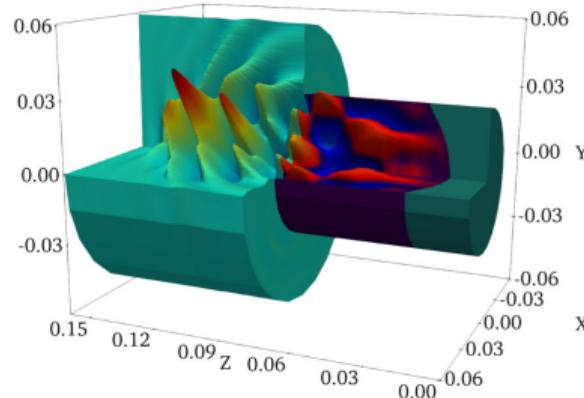
Ultrasound applications



Donders Institute for Brain, Cognition, and Behaviour, Radboud University

Modeling

- Modeling as a tool in advancing ultrasound applications



Simulation of an elasto-acoustic system
[Muhr, Nikolić, & Wohlmuth, 2022]

Modeling considerations

The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

if often not valid in practice.

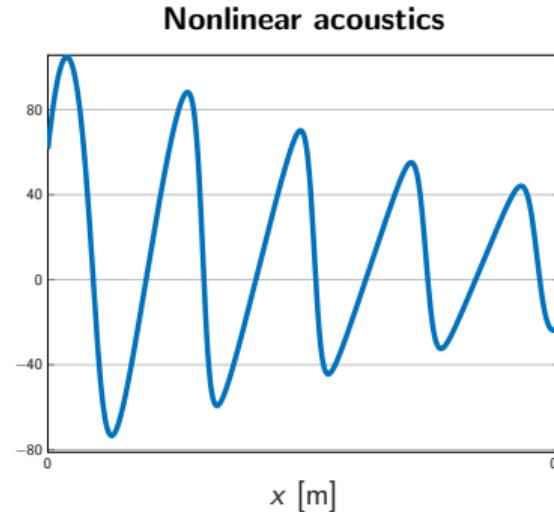
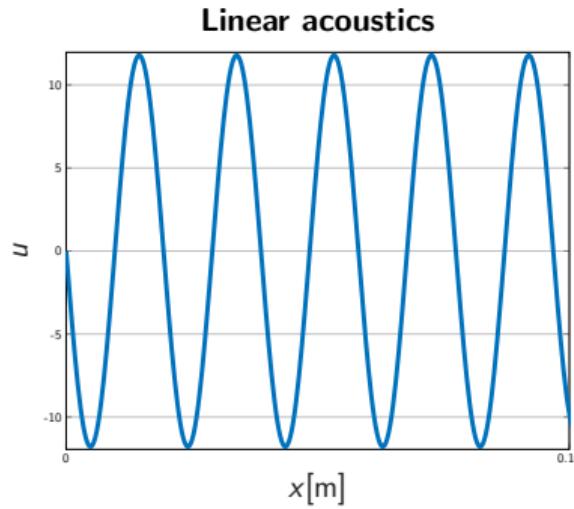
Modeling considerations

The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

if often not valid in practice.

- 1 At high frequencies/intensities, **nonlinear** effects are prominent



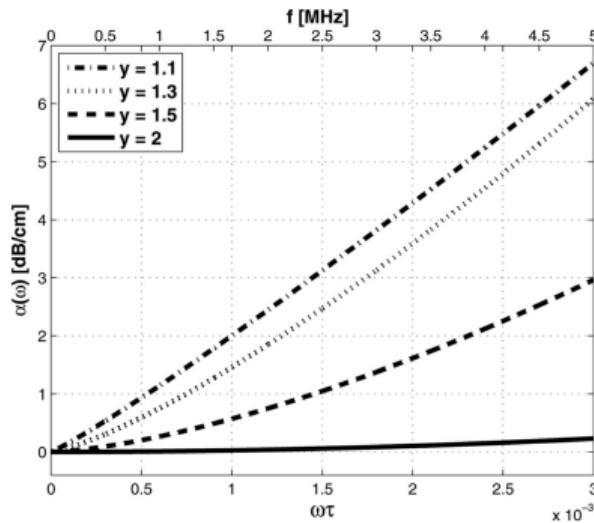
Modeling considerations

The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

if often not valid in practice.

- ② In biological media, observed **attenuation** obeys a **power law**



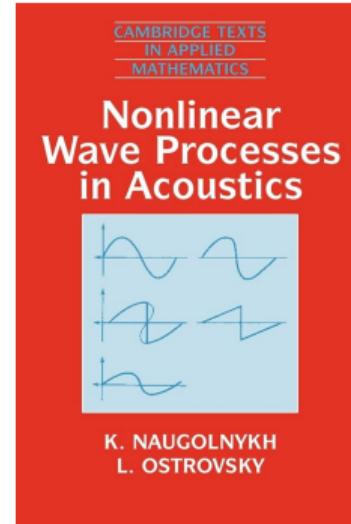
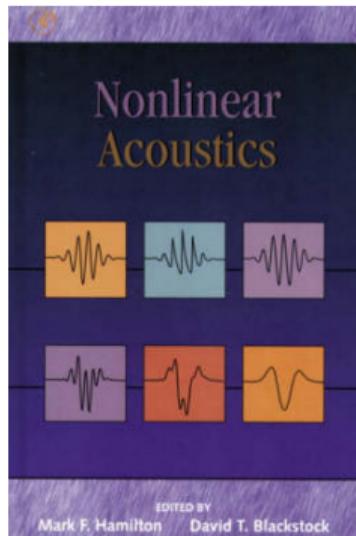
$$\text{attenuation} \sim |\omega|^{y-1}$$

source: [Prieur & Holm, 2011]

Next

- ① **Models** of nonlinear acoustic waves in fluids
- ② Energy methods in the **analysis**
- ③ **Time-fractional** attenuation in biological media

Modeling literature



papers: [Westervelt, 1963], [Blackstock, 1963], [Kuznetsov, 1973], ...

Acoustic field

- Sound wave perturbs background properties

$$\text{density: } \rho = \rho_0 + \rho',$$

$$\text{pressure: } p = p_0 + p',$$

$$\text{velocity: } \mathbf{v} = \mathbf{v}'$$

- Assumptions: $\mathbf{v}_0 = \mathbf{0}$, p_0 , $\rho_0 = \text{const.}$

Governing system of equations in viscous fluids

① Conservation of mass

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

② Momentum equation

$$\rho \mathbf{v}_t + \frac{\rho}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) + \nabla p' = \left(\frac{4}{3} \mu + \mu_b \right) \Delta \mathbf{v}$$

μ ... shear viscosity, μ_b ... bulk viscosity

Governing system of equations

- ③ Expanded pressure-density relation $p = p(\rho)$

$$p - p_0 \approx c^2(\rho - \rho_0) + \frac{c^2}{\rho_0} \frac{B}{2A} (\rho - \rho_0)^2$$

c ... speed of sound in the fluid, $A = \rho_0 \left(\frac{\partial p}{\partial \rho} \right) \approx \varrho_0 c^2$, $B = \rho_0^2 \left(\frac{\partial^2 p}{\partial \rho^2} \right)$

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- Nonlinearity parameter $\frac{B}{A}$ can be used for tissue characterization
 \rightsquigarrow Acoustic nonlinearity parameter tomography

Approximating the governing system

- ① Only first- and second-order terms are retained
 - first-order terms \sim linear with respect to alternating quantities
 - second-order terms \sim quadratic with respect to the alternating quantities and dissipative terms

Approximating the governing system

① Only first- and second-order terms are retained

- first-order terms \sim linear with respect to alternating quantities
- second-order terms \sim quadratic with respect to the alternating quantities and dissipative terms

For example, $\left(\frac{4}{3}\mu + \mu_b\right) \Delta v$ is a second-order term

Approximating the governing system

- ② **Substitution corollary:** any factor in a second-order term may be replaced by its first-order approximation

First-order approximations

$$\text{linear mass equation: } \nabla \cdot \mathbf{v} = -\frac{1}{\rho_0} \rho'_t = -\frac{1}{\rho_0} \frac{p'_t}{c^2},$$

$$\text{linear momentum equation: } \rho_0 \mathbf{v}'_t + \nabla p' = 0,$$

$$\text{linear state equation: } \rho' = \frac{p'}{c^2}$$

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$$\text{linear momentum equation: } \rho_0 \mathbf{v}'_t + \nabla p' = 0,$$

$$\text{linear state equation: } \rho' = \frac{p'}{c^2}$$

For example, $\rho' \mathbf{v}_t \approx \frac{p'}{c^2} \left(-\frac{1}{\rho_0} \nabla p' \right) = -\frac{1}{2} \frac{1}{\rho_0 c^2} \nabla p'^2$

What is the resulting acoustic wave equation?

Simplifying the conservation of momentum

$$\rho \mathbf{v}_t + \frac{\rho}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) + \nabla p' = \left(\frac{4}{3} \mu + \mu_b \right) \Delta \mathbf{v}$$

Or, equivalently,

$$(\rho_0 + \rho') \mathbf{v}_t + \frac{\rho_0 + \rho'}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) + \nabla p' = \left(\frac{4}{3} \mu + \mu_b \right) \Delta \mathbf{v}$$

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We use

$$\frac{\rho_0 + \rho'}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) \approx \frac{\rho_0}{2} \nabla (\mathbf{v} \cdot \mathbf{v})$$

because $\frac{\rho'}{2} \nabla (\mathbf{v} \cdot \mathbf{v})$ is of third order

Simplifying the momentum equation

$$(\rho_0 + \rho')\mathbf{v}_t + \frac{\rho_0}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) + \nabla p' = \left(\frac{4}{3}\mu + \mu_b\right)\Delta\mathbf{v}$$

Rewrite it as

Simplifying the momentum equation

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Rewrite it as

$$\rho_0\mathbf{v}_t + \nabla p' = \left(\frac{4}{3}\mu + \mu_b\right)\Delta\mathbf{v} - \rho'\mathbf{v}_t - \frac{\rho_0}{2}\nabla(\mathbf{v} \cdot \mathbf{v})$$

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Via the substitution corollary:

$$-\rho'\mathbf{v}_t \approx -\frac{p'}{c^2} \left(-\frac{1}{\rho_0}\nabla p'\right) = \frac{1}{2} \frac{1}{\rho_0 c^2} \nabla(p')^2$$

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$$\left(\frac{4}{3}\mu + \mu_b\right)\Delta\mathbf{v} \approx -\frac{1}{\rho_0 c^2} \left(\frac{4}{3}\mu + \mu_b\right) \nabla p'_t$$

Simplifying the conservation of momentum

Modified Euler equation

$$\rho_0 \mathbf{v}_t + \nabla p' = \frac{1}{2} \frac{1}{\rho c^2} \nabla(p')^2 - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \nabla p'_t - \frac{\rho_0}{2} \nabla (\mathbf{v} \cdot \mathbf{v})$$

Applying the divergence operator yields

Simplifying the conservation of momentum

Modified Euler equation

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Applying the divergence operator yields

$$\rho_0 \nabla \cdot \mathbf{v}_t + \Delta p' = \frac{1}{2} \frac{1}{\rho_0 c^2} \Delta(p')^2 - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t - \frac{\rho_0}{2} \Delta (\mathbf{v} \cdot \mathbf{v})$$

Simplifying the mass conservation

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

We can rewrite it as

$$\rho'_t + \rho_0 \nabla \cdot \mathbf{v} = -\rho' \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \rho'$$

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$$-\mathbf{v} \cdot \nabla \rho' \approx -\mathbf{v} \cdot \nabla \frac{p'}{c^2} = \frac{\rho_0}{c^2} \mathbf{v} \cdot \mathbf{v}_t = \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_t$$

Simplifying the mass conservation

$$\rho'_t + \rho_0 \nabla \cdot \mathbf{v} = \frac{1}{2} \frac{\rho_0}{c^4} (p'^2)_t + \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_t$$

We can express ρ'_t using the (differentiated) state equation and $(\rho')^2 \approx \frac{1}{c^4} (p')^2$

$$\frac{1}{c^2} p'_t - \frac{1}{\rho_0 c^4} \frac{B}{2A} (p'^2)_t + \rho_0 \nabla \cdot \mathbf{v} = \frac{1}{2} \frac{1}{\rho_0 c^4} (p'^2)_t + \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_t$$

Combining the two simplified equations

$$\frac{1}{c^2} p'_{tt} - \frac{1}{\rho_0 c^4} \frac{B}{2A} (p'^2)_{tt} + \rho_0 \nabla \cdot \mathbf{v}_t = \frac{1}{2} \frac{1}{\rho_0 c^4} (p'^2)_{tt} + \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_{tt}$$

—

$$\rho_0 \nabla \cdot \mathbf{v}_t + \Delta p' = \frac{1}{2} \frac{1}{\rho_0 c^2} \Delta (p')^2 - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t - \frac{\rho_0}{2} \Delta (\mathbf{v} \cdot \mathbf{v})$$

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$$\begin{aligned} & \frac{1}{c^2} p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t \\ &= \frac{1}{\rho_0 c^4} \frac{B}{2A} (p'^2)_{tt} + \frac{1}{2} \frac{1}{\rho_0 c^4} (p'^2)_{tt} - \frac{1}{2} \frac{1}{\rho_0 c^2} \Delta p'^2 + \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_{tt} + \frac{\rho_0}{2} \Delta (\mathbf{v} \cdot \mathbf{v}) \end{aligned}$$

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Using $(p'^2)_{tt} \approx c^2 \Delta p'^2$ yields

$$\begin{aligned} & \frac{1}{c^2} p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t \\ &= \frac{1}{\rho_0 c^4} \frac{B}{2A} (p'^2)_{tt} + \frac{1}{2} \frac{\rho_0}{c^2} (\mathbf{v}^2)_{tt} + \frac{\rho_0}{2} \Delta (\mathbf{v} \cdot \mathbf{v}) \end{aligned}$$

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Using $(\mathbf{v} \cdot \mathbf{v})_{tt} \approx c^2 \Delta(\mathbf{v} \cdot \mathbf{v})$ yields

$$\frac{1}{c^2} p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t = \frac{1}{\rho_0 c^4} \frac{B}{2A} ((p')^2)_{tt} + \frac{\rho_0}{c^2} (\mathbf{v}^2)_{tt}$$

The resulting pressure-velocity equation

$$\frac{1}{c^2} p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta p'_t = \frac{1}{\rho_0 c^4} \frac{B}{2A} ((p')^2)_{tt} + \frac{\rho_0}{c^2} (\mathbf{v}^2)_{tt}$$

To express the equation in terms of **one unknown**, we can work with the **acoustic velocity potential** ψ :

$$\mathbf{v}' = -\nabla \psi$$

$$p' = \rho_0 \psi_t$$

Kuznetsov equation

$$\frac{1}{c^2} \psi_{ttt} - \Delta \psi_t - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta \psi_{tt} = \frac{1}{c^4} \frac{B}{2A} (\psi_t^2)_{tt} + \frac{1}{c^2} (\nabla \psi \cdot \nabla \psi)_{tt}$$

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$$\frac{1}{c^2} \psi_{ttt} - \Delta \psi_t - \frac{1}{\rho_0 c^2} \left(\frac{4}{3} \mu + \mu_b \right) \Delta \psi_{tt} = \frac{1}{c^4} \frac{B}{2A} (\psi_t^2)_{tt} + \frac{1}{c^2} (\nabla \psi \cdot \nabla \psi)_{tt}$$

Integrating in time yields

Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left(\frac{1}{c^2} \frac{B}{2A} \psi_t^2 + \nabla \psi \cdot \nabla \psi \right)_t$$

Here $b = \frac{1}{\rho_0} \left(\frac{4}{3} \mu + \mu_b \right)$ is called the *sound diffusivity*.

Further simplification

Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left(\frac{1}{c^2} \frac{B}{2A} \psi_t^2 + \nabla \psi \cdot \nabla \psi \right)_t$$

If $c^2 |\nabla \psi|^2 \approx \psi_t^2$ holds (this will be the case sufficiently far from the source), then

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \frac{1}{c^2} \left(1 + \frac{B}{A} \right) (\psi_t^2)_t$$

Westervelt equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \frac{1}{c^2} \left(1 + \frac{B}{A}\right) (\psi_t^2)_t$$

Time-differentiating and using $\rho_0 \psi_t = p'$ yields

Westervelt equation

$$p'_{tt} - c^2 \Delta p' - b \Delta p'_t = \frac{1}{\rho_0 c^2} \left(1 + \frac{B}{2A}\right) (p'^2)_{tt}$$

Westervelt equation

$$p'_{tt} - c^2 \Delta p' - b \Delta p'_t = \frac{1}{\rho_0 c^2} \left(1 + \frac{B}{2A} \right) (p'^2)_{tt}$$



Peter Westervelt (1919–2015)

P. Westervelt, *Parametric acoustic array*, J. Acoust. Soc. Am., 1963. ($b = 0$)

Other wave equations

Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left(\frac{1}{c^2} \frac{B}{2A} \psi_t^2 + \nabla \psi \cdot \nabla \psi \right)_t$$

Other wave equations

Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left(\frac{1}{c^2} \frac{B}{2A} \psi_t^2 + \nabla \psi \cdot \nabla \psi \right)_t$$

Using $(\psi_t^2)_t \approx 2\psi_t c^2 \Delta \psi$ leads to

Blackstock equation

$$\psi_{tt} - \left(c^2 + \frac{B}{A} \psi_t \right) \Delta \psi - b \Delta \psi_t = (\nabla \psi \cdot \nabla \psi)_t$$

Other wave equations

Kuznetsov equation

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$$\psi_{tt} - \left(c^2 + \frac{B}{A} \psi_t \right) \Delta \psi - b \Delta \psi_t = (\nabla \psi \cdot \nabla \psi)_t$$

Effective speed is $\sqrt{c^2 + \frac{B}{A} \psi_t}$

Summary so far: Different models

Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left(\frac{1}{c^2} \frac{B}{2A} \psi_t^2 + \nabla \psi \cdot \nabla \psi \right)_t$$

Blackstock equation

$$\psi_{tt} - \left(c^2 + \frac{B}{A} \psi_t \right) \Delta \psi - b \Delta \psi_t = (\nabla \psi \cdot \nabla \psi)_t$$

Westervelt equation

$$p'_{tt} - c^2 \Delta p' - b \Delta p'_t = \frac{1}{\rho_0 c^2} \left(1 + \frac{B}{2A} \right) ((p')^2)_{tt}$$

Energy methods in the [analysis](#)

Westervelt equation

- Consider the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 \Delta u - b \Delta u_t &= k(u^2)_{tt} && \text{in } \Omega \times (0, T) \\ u &= 0 && \text{on } \partial\Omega \times (0, T) \\ (u, u_t) &= (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}$$

u ... acoustic pressure ($= p'$)

$\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$... “nice” bounded domain

$T > 0$... final time, $k = \frac{1}{\rho_0 c^2} \left(1 + \frac{B}{2A}\right)$

(other BCs are also of interest but more technical to analyze)

Literature on the analysis of nonlinear acoustic waves

- Well-posedness & qualitative behavior

[Hughes, Kato, Marsden 1977] ($b = 0$)

[Kawashima, Shibata 1992]

[Mizohata, Ukai 1993] (Kuznetsov eq.)

[Kaltenbacher, Lasiecka 2009, 2011] (Westervelt eq., Kuznetsov eq.)

[Meyer, Wilke 2011] ($\max L^p$ regularity approach)

[Dörfler, Gerner, Schnaubelt 2016] ($b = 0$)

[Kaltenbacher, Rundell 2021] (Westervelt eq. with fractional damping)

[Kaltenbacher, Meliani, Nikolić 2023] (acoustic eq. with fractional damping)

...

Well-posedness analysis in a linear setting

- Consider first the **linear** initial boundary-value problem:

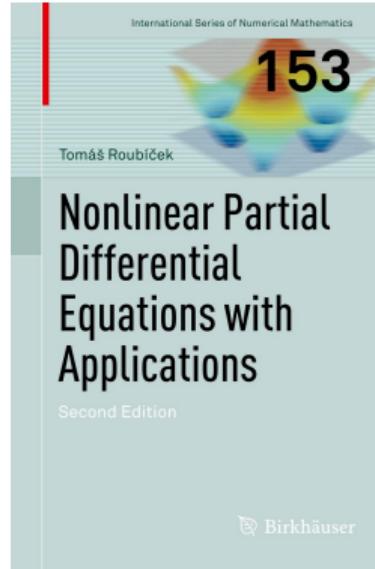
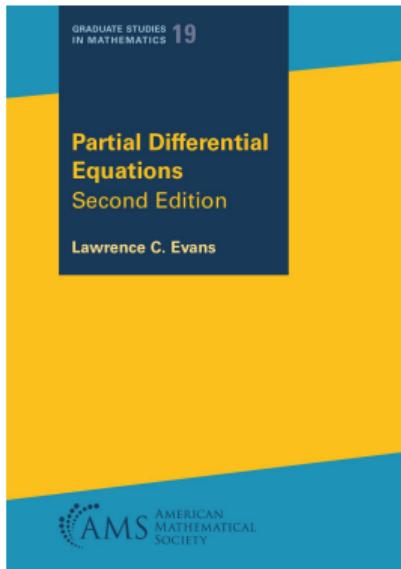
$$u_{tt} - c^2 \Delta u - b \Delta u_t = 0 \quad \text{in } \Omega \times (0, T)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T)$$

$$(u, u_t) = (u_0, u_1) \quad \text{in } \Omega \times \{0\}$$

(we have set $k = 0$)

Literature on the Faedo–Galerkin method



named after Boris Galerkin (1871–1945) and Alessandro Faedo (1913–2001)

Steps of the Faedo–Galerkin method

① Discretize the problem in space

- Semi-discrete problem is a system of n ODEs
- We can exploit existence theory for ODEs

Steps of the Faedo–Galerkin method

② Uniformly bound the energy of the approximate solution u^n

- Canonical testing function is u_t^n
- Canonical energy functional is

$$E(u^n) = \frac{1}{2} \|u_t^n(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u^n(t)\|_{L^2(\Omega)}^2$$

- Here we also have dissipation

$$D(u^n) = b \int_0^t \|\nabla u_t^n(s)\|_{L^2(\Omega)}^2 ds$$

and we will bound $E(u^n) + D(u^n)$

Steps of the Faedo–Galerkin method

③ Extract a (weakly) convergent subsequence $\{u^{n_l}\}_{l \geq 1}$

- Check if the limit u is a solution of the original problem
- Check if this is the only solution



Notation: Bochner spaces

- $L^p(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \rightarrow X$, such that

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

- $L^\infty(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \rightarrow X$, such that

$$\|u\|_{L^\infty(0, T; X)} = \underset{t \in (0, T)}{\text{ess sup}} \|u(t)\|_X < \infty$$

- $W^{1,p}(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \rightarrow X$, such that

$$u \in L^p(0, T; X), \quad u_t \in L^p(0, T; X), \quad 1 \leq p \leq \infty$$

Weak form of the problem

- Multiply the equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = 0$$

with $\phi \in L^2(0, T; H_0^1(\Omega))$ and integrate over space and time

Weak form of the problem

- Multiply the equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = 0$$

with $\phi \in L^2(0, T; H_0^1(\Omega))$ and integrate over space and time

- Weak form: find $u \in \mathcal{U}$, such that

$$\int_0^T \left\{ \langle u_{tt}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + c^2 (\nabla u, \nabla \phi)_{L^2(\Omega)} + b(\nabla u_t, \nabla \phi)_{L^2(\Omega)} \right\} dt = 0$$

for all test functions $\phi \in L^2(0, T; H_0^1(\Omega))$ with $(u, u_t)|_{t=0} = (u_0, u_1)$

Step 1: Discretization in space

- $\{\mathbf{w}_i\}_{i=1}^{\infty}$ eigenfunctions for the Dirichlet–Laplacian

$$\begin{cases} -\Delta \mathbf{w}_i = \lambda_i \mathbf{w}_i \\ \mathbf{w}_i|_{\partial\Omega} = 0 \end{cases}$$

- *an orthogonal basis of $H_0^1(\Omega)$*
- *an orthonormal basis of $L^2(\Omega)$*

so that

$$(w_i, w_j)_{L^2(\Omega)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad (\nabla w_i, \nabla w_j)_{L^2(\Omega)} = 0, \quad i \neq j$$

Step 1: Discretization in space

- Fix $n \in \mathbb{N}$ and let $V_n = \text{span}\{w_1, \dots, w_n\}$

$$u^n(t) = \sum_{k=1}^n \xi_i^n(t) w_i(x) \in V_n$$

- The coefficients $\xi_i^n : [0, T] \rightarrow \mathbb{R}$ are unknown

Step 1: Discretization in space

- Approximate solution $\textcolor{blue}{u}^n(t) = \sum_{i=1}^{\textcolor{green}{n}} \xi_i^n(t) w_i(x)$ satisfies

$$(\textcolor{blue}{u}_{tt}^n, \phi^n)_{L^2(\Omega)} + c^2 (\nabla \textcolor{blue}{u}^n, \nabla \phi^n)_{L^2(\Omega)} + b(\nabla \textcolor{blue}{u}_t^n, \phi^n)_{L^2(\Omega)} = 0$$

for all $\phi^n \in V_n$, a.e. in time, and we set

$$\xi_i^n(0) = (u_0, w_i)_{L^2(\Omega)}, \quad \xi_{i,t}^n(0) = (u_1, w_i)_{L^2(\Omega)}$$

Step 1: Discretization in space

- For each $n \in \mathbb{N}$, the semi-discrete problem reduces to

$$M\xi_{tt}^n + K\xi^n + C\xi_t^n = 0 \quad \text{in } [0, T]$$

supplemented by initial conditions, where $\xi^n = [\xi_1^n \dots \xi_n^n]^T$

- Existence theorems for ODEs yield a unique $\xi_i^n \in C^2[0, T]$ for each $i \in [1, n]$ and thus a unique

$$u^n \in C^2([0, T]; V_n)$$

Step 2: Uniform bounds in n

Testing with $\phi^n = \mathbf{u}_t^n \in V_n$ leads to

$$\int_0^t \left\{ (u_{tt}^n, \mathbf{u}_t^n)_{L^2(\Omega)} + c^2 (\nabla u^n, \nabla \mathbf{u}_t^n)_{L^2(\Omega)} + b(\nabla u_t^n, \nabla u_t^n)_{L^2(\Omega)} \right\} ds = 0$$

Step 2: Uniform bounds in n

Testing with $\phi^n = \textcolor{blue}{u_t^n} \in V_n$ leads to

$$\int_0^t \left\{ (u_{tt}^n, \textcolor{blue}{u_t^n})_{L^2(\Omega)} + c^2 (\nabla u^n, \nabla \textcolor{blue}{u_t^n})_{L^2(\Omega)} + b(\nabla u_t^n, \nabla u_t^n)_{L^2(\Omega)} \right\} ds = 0$$

- Integration by parts in time yields

$$\int_0^t (u_{tt}^n, u_t^n)_{L^2(\Omega)} ds = \frac{1}{2} \|u_t^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t$$

$$c^2 \int_0^t (\nabla u^n, \nabla u_t^n)_{L^2(\Omega)} ds = \frac{c^2}{2} \|\nabla u^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t$$

Step 2: Uniform bounds in n

Energy identity

$$\frac{1}{2} \| \mathbf{u}_t^n(s) \|_{L^2(\Omega)}^2 \Big|_0^t + \frac{c^2}{2} \| \nabla \mathbf{u}^n(s) \|_{L^2(\Omega)}^2 \Big|_0^t + b \int_0^t \| \nabla \mathbf{u}_t^n(s) \|_{L^2(\Omega)}^2 ds = 0$$

Step 2: Uniform bounds in n

Energy identity

$$\frac{1}{2} \| \mathbf{u}_t^n(s) \|_{L^2(\Omega)}^2 \Big|_0^t + \frac{c^2}{2} \| \nabla \mathbf{u}^n(s) \|_{L^2(\Omega)}^2 \Big|_0^t + b \int_0^t \| \nabla \mathbf{u}_t^n(s) \|_{L^2(\Omega)}^2 ds = 0$$

- Equivalently,

$$\begin{aligned} \frac{1}{2} \| \mathbf{u}_t^n(t) \|_{L^2(\Omega)}^2 + \frac{c^2}{2} \| \nabla \mathbf{u}^n(t) \|_{L^2(\Omega)}^2 + b \int_0^t \| \nabla \mathbf{u}_t^n(s) \|_{L^2(\Omega)}^2 ds \\ = \frac{1}{2} \| \mathbf{u}_t^n(0) \|_{L^2(\Omega)}^2 + \frac{c^2}{2} \| \nabla \mathbf{u}^n(0) \|_{L^2(\Omega)}^2 \end{aligned}$$

Step 2: Uniform bounds in n

- The derived identity implies that

$$E[u^n](\textcolor{blue}{t}) + D[u^n](t) = E[u^n](\textcolor{blue}{0}), \quad t \in (0, T)$$

and thus we have a **uniform bound** with respect to n :

$$\frac{1}{2} \|u_t^n(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u^n(t)\|_{L^2(\Omega)}^2 + b \int_0^t \|\nabla u_t^n(s)\|_{L^2(\Omega)}^2 ds \leq C, \quad t \in (0, T)$$

- We can additionally estimate $\|u_{tt}^n\|_{L^2(0, T; H^{-1}(\Omega))}$ via the semi-discrete PDE

Step 3: Passing to the limit

- The Faedo–Galerkin method is combined with a compactness argument

*Any bounded set in a reflexive Banach space is weakly sequentially compact:
any sequence in a bounded set has a weakly converging subsequence.*

- Weak convergence in a Banach space B :

Def. Let $u^n, u \in B$. If for any $f \in B'$ (dual space)

$$f(u^n) \rightarrow f(u)$$

as $n \rightarrow \infty$, then we say that

$$u^n \rightharpoonup u \text{ weakly in } B.$$

Step 3: Passing to the limit

- Thus $\{u^n\}_{n \geq 1}$ has a convergent subsequence

$$u^n \rightharpoonup u \text{ weakly in } L^2(0, T; H_0^1(\Omega))$$

and

$$u_t^n \rightharpoonup u_t \text{ weakly in } L^2(0, T; H_0^1(\Omega))$$

and

$$u_{tt}^n \rightharpoonup u_{tt} \text{ weakly in } L^2(0, T; H^{-1}(\Omega))$$

Step 3: Passing to the limit

Passing to the limit shows that u satisfies

$$\int_0^T \left\{ \langle u_{tt}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + c^2 (\nabla u, \nabla \phi)_{L^2(\Omega)} + b(\nabla u_t, \nabla \phi)_{L^2(\Omega)} \right\} dt = 0$$

for all $\phi \in L^2(0, T; H_0^1(\Omega))$

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for all $\phi \in L^2(0, T; H_0^1(\Omega))$

It remains to prove that

- $(u, u_t)|_{t=0} = (u_0, u_1)$
- u satisfies the same energy bound
- the constructed solution u is the **only solution**

□

What changes in the Faedo–Galerkin method for [nonlinear acoustic](#) equations?

Westervelt equation

- Consider the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 \Delta u - b \Delta u_t &= k(u^2)_{tt} && \text{in } \Omega \times (0, T) \\ u &= 0 && \text{on } \partial\Omega \times (0, T) \\ (u, u_t) &= (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}$$

for simplicity, let $k = \text{const.} > 0$

(the proof can be easily adapted to $k \in L^\infty(\Omega)$)

Westervelt equation

- Rewrite the equation:

$$\begin{aligned} ((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t &= 0 && \text{in } \Omega \times (0, T) \\ u &= 0 && \text{on } \partial\Omega \times (0, T) \\ (u, u_t) &= (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}$$

Approach in the analysis

Consider a linearization

$$((1 - 2k\textcolor{blue}{u}^*)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Approach in the analysis

Consider a linearization

$$((1 - 2k\mathbf{u}^*)\mathbf{u}_t)_t - c^2 \Delta \mathbf{u} - b \Delta \mathbf{u}_t = 0$$

Define the mapping

$$\mathcal{T} : \mathbf{u}^* \mapsto \mathbf{u}$$

with

- \mathbf{u}^* in a ball $\mathbb{B} \subset \mathcal{U}$ (solution space)
- \mathbf{u} solves the linearized problem

The **fixed point** $\mathbf{u}^* = \mathbf{u}$ solves the nonlinear problem.

Things to keep in mind

Quasilinear evolution

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Needed to **avoid degeneracy**:

$$1 - 2ku > 0 \quad \text{in } \Omega \times (0, T)$$

Things to keep in mind

Quasilinear evolution

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Needed to [avoid degeneracy](#):

$$u < \frac{1}{2k} \quad \text{in } \Omega \times (0, T)$$

Things to keep in mind

Quasilinear evolution

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Needed to avoid degeneracy:

$$\|u(t)\|_{L^\infty(\Omega)} < \frac{1}{2k}, \quad t \in (0, T)$$

Things to keep in mind

Quasilinear evolution

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Approach via an [embedding](#):

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \|u(t)\|_{H^2(\Omega)} < \frac{1}{2k}, \quad t \in (0, T)$$

Things to keep in mind

Quasilinear evolution

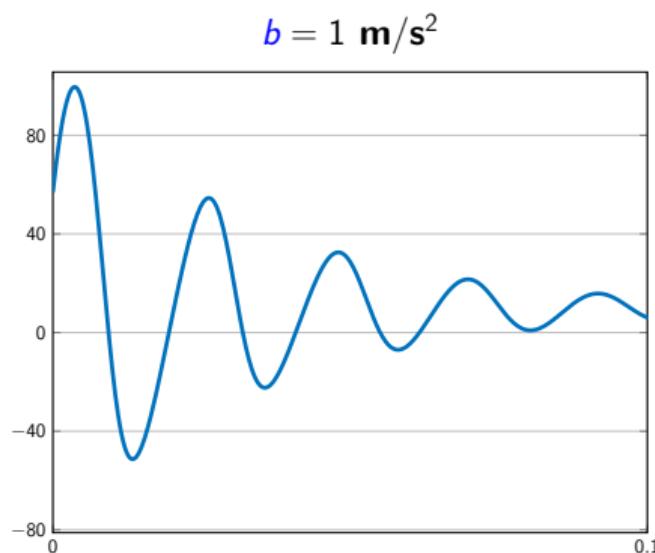
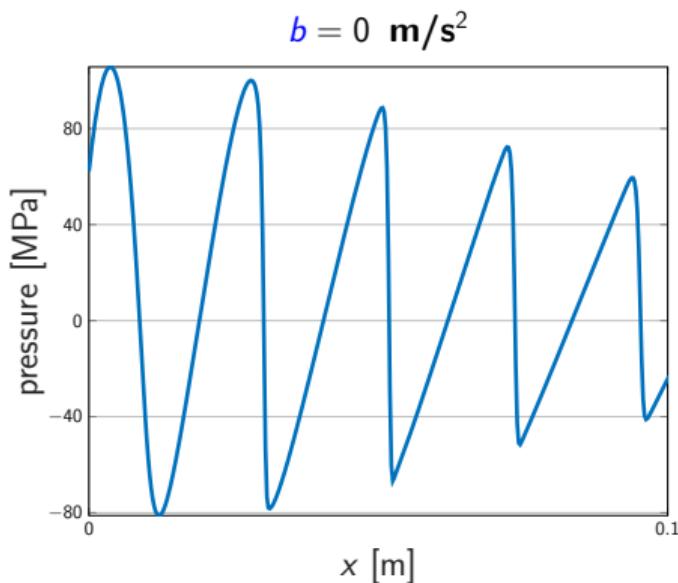
$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

Approach via an embedding combined with smallness of data:

$$\|u(t)\|_{H^2(\Omega)} \leq C(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}) < \frac{1}{2k}, \quad t \in (0, T)$$

Things to keep in mind: Strong damping

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$



Notion of the solution

To avoid degeneracy, we need $u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$

we require that u satisfies

$$\int_0^T \int_{\Omega} \{((1 - 2ku)u_t)_t - c^2 \Delta u - b \Delta u_t\} \phi \, dx dt = 0$$

for all $\phi \in L^2(0, T; L^2(\Omega))$ with $(u, u_t)|_{t=0} = (u_0, u_1)$

The fixed-point argument

Apply Banach's fixed-point theorem on

$$\mathcal{T} : \textcolor{blue}{u^*} \mapsto u$$

mapping u^* from the ball

$$\mathbb{B} = \{\textcolor{blue}{u^*} \in \mathcal{U} : \| \textcolor{blue}{u^*} \|_{\mathcal{U}} \leq R, (\textcolor{blue}{u^*}, \textcolor{blue}{u_t^*})|_{t=0} = (u_0, u_1)\}$$

to the solution u of

$$((1 - 2k\textcolor{blue}{u^*})u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

We need that $\mathcal{U} \hookrightarrow L^\infty(0, T; \textcolor{green}{H}_0^1(\Omega) \cap H^2(\Omega))$.

The fixed-point argument

The mapping $\mathcal{T} : \textcolor{blue}{u^*} \mapsto u$ is **well-defined** if the linear problem is well-posed:

$$((1 - 2k\textcolor{blue}{u^*})u_t)_t - c^2 \Delta u - b \Delta u_t = 0$$

- We can rely on the **Galerkin approach** to prove it
- We need a bound on $\|u(t)\|_{H^2(\Omega)}$

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- We can rely on the **Galerkin approach** to prove it
- We need a bound on $\|u(t)\|_{H^2(\Omega)}$
- Is the linearized problem **non-degenerate**? Yes, for small R :

$$\|u^*\|_{L^\infty(0, T; \textcolor{red}{L^\infty(\Omega)})} \leq C \|u^*\|_{\mathcal{U}} \leq \textcolor{red}{R}$$

Galerkin method for the linearization

We set up the semi-discretization as before and consider

$$\int_{\Omega} \left\{ ((1 - 2ku^*)u_t^n)_t - c^2 \Delta u^n - b \Delta u_t^n \right\} \phi^n \, dx = 0$$

for all $\phi^n \in V_n$ a.e. in time, with $(u^n, u_t^n)|_{t=0} = (u_0^n, u_1^n)$

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Testing strategy: $\phi^n = -\Delta u^n \in V_n$

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Testing strategy: $\phi^n = -\Delta u^n \in V_n$

$$b \int_0^t (\Delta u_t^n, \Delta u^n)_{L^2(\Omega)} \, ds = \frac{b}{2} \|\Delta u^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t$$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = -\Delta u^n$ leads to

$$\begin{aligned} & \frac{b}{2} \|\Delta u^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t + c^2 \int_0^t \|\Delta u^n\|_{L^2(\Omega)}^2 \, ds \\ & \leq \int_0^t \|(1 - 2ku^*) \color{red}{u_{tt}^n} - 2ku_t^* u_t^n\|_{L^2(\Omega)} \|\Delta u^n\|_{L^2(\Omega)} \, ds \end{aligned}$$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = -\Delta u^n$ leads to

$$\begin{aligned} & \frac{b}{2} \|\Delta u^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t + c^2 \int_0^t \|\Delta u^n\|_{L^2(\Omega)}^2 \, ds \\ & \leq \int_0^t \|(1 - 2ku^*) \textcolor{red}{u_{tt}^n} - 2ku_t^* u_t^n\|_{L^2(\Omega)} \|\Delta u^n\|_{L^2(\Omega)} \, ds \end{aligned}$$

Additional testing needed: $\phi^n = \textcolor{red}{u_{tt}^n} \in V_n$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = \mathbf{u}_{tt}^n$ and using

$$\begin{aligned}\int_0^t ((1 - 2ku^*) u_{tt}^n, \mathbf{u}_{tt}^n)_{L^2(\Omega)} ds &= \int_0^t \|\sqrt{1 - 2ku^*} \mathbf{u}_{tt}^n(s)\|_{L^2(\Omega)}^2 ds \\ &\geq \gamma \int_0^t \|\mathbf{u}_{tt}^n(s)\|_{L^2(\Omega)}^2 ds\end{aligned}$$

Faedo–Galerkin method for the linearization

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leads to

$$\begin{aligned}& \gamma \int_0^t \|\underline{u}_{tt}^n(s)\|_{L^2(\Omega)}^2 ds + \frac{b}{2} \|\nabla u_t^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t \\ & \leq \frac{c^4}{2\varepsilon} \int_0^t \|\Delta u^n(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{\varepsilon} k^2 \int_0^t \|u_t^*\|_{L^3(\Omega)}^2 \|u_t^n\|_{L^6(\Omega)}^2 ds \\ & \quad + \varepsilon \int_0^t \|\underline{u}_{tt}^n(s)\|_{L^2(\Omega)}^2 ds\end{aligned}$$

for any $\varepsilon > 0$

Faedo–Galerkin method for the linearization

Adding the two estimates leads to

$$\begin{aligned} & \frac{\gamma}{4} \int_0^t \|u_{tt}^n(s)\|_{L^2(\Omega)}^2 ds + \frac{b}{2} \|\nabla u_t^n(t)\|_{L^2(\Omega)}^2 + \frac{b}{2} \|\Delta u^n(t)\|_{L^2(\Omega)}^2 \\ & \leq C(\textcolor{green}{u}^*, T) \left(\|u_1\|_{H^1(\Omega)}^2 + \|u_0\|_{H^2(\Omega)}^2 \right) \end{aligned}$$

- The ***n*-uniform bound** is the key step in the Galerkin procedure
- Solution space

$$\begin{aligned} \mathcal{U} = \{u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) : & u_t \in L^\infty(0, T; H_0^1(\Omega)) \\ & u_{tt} \in L^2(0, T; L^2(\Omega))\} \end{aligned}$$

Faedo–Galerkin method for the linearization

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- Solution space

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- We can **bootstrap** regularity to $u_t \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega))$

The fixed-point argument

To apply the **Banach fixed-point theorem**, we need to prove that

- $\mathcal{T}(B) \subset B$
- \mathcal{T} is strictly contractive

We can guarantee this by decreasing the radius of the ball

$$\mathbb{B} = \{u^* \in \mathcal{U} : \|u^*\|_{\mathcal{U}} \leq R, (u^*, u_t^*)|_{t=0} = (u_0, u_1)\}$$

which reduces the size of initial data



Local well-posedness

Theorem. Let $b > 0$ and $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$. There exists data size $r = r(T) > 0$ such that if

$$\|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 \leq r^2,$$

then there is a unique solution $u \in \mathcal{U}$. The solution satisfies the estimate

$$\begin{aligned} & \|u_{tt}\|_{L^2(L^2(\Omega))}^2 + \|u\|_{L^\infty(0, T; H^2(\Omega))}^2 + \|u_t\|_{L^\infty(0, T; H^1(\Omega))}^2 + \|u_t\|_{L^2(H^2(\Omega))}^2 \\ & \leq C(T) \left(\|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 \right). \end{aligned}$$

details: [Kaltenbacher & Lasiecka, 2009], try in the problem session

On the Kuznetsov equation

$$(1 - 2ku_t)u_{tt} - c^2\Delta u - b\Delta u_t = (\nabla u \cdot \nabla u)_t$$

- Here the non-degeneracy condition is

$$1 - 2ku_t \geq \gamma > 0$$

On the Kuznetsov equation

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- Here the non-degeneracy condition is

$$1 - 2k u_t \geq \gamma > 0$$

⇝ a bound on $\|u_t(t)\|_{L^\infty(\Omega)}$ is needed

On the Kuznetsov equation

$$(1 - 2ku_t)u_{tt} - c^2\Delta u - b\Delta u_t = (\nabla u \cdot \nabla u)_t$$

- Here the non-degeneracy condition is

$$1 - 2ku_t \geq \gamma > 0$$

⇝ a bound on $\|u_t(t)\|_{L^\infty(\Omega)}$ is needed

Smoother data are needed to ensure non-degeneracy and treat the gradient terms.

details: [Mizohata, Ukai, 1993]

On the Blackstock equation

$$u_{tt} - c^2(1 + 2ku_t)\Delta u - b\Delta u_t = (\nabla u \cdot \nabla u)_t$$

- Here $1 + 2ku_t$ is allowed to degenerate
- A bound on $\|u_t\|_{L^2(L^\infty(\Omega))}$ is still needed to handle the nonlinear terms
- Regularity-wise it is in between Westervelt and Kuznetsov equation

details: [Fritz, Nikolić, Wohlmuth, 2019], [Nikolić, Said-Houari, , 2023]

An open question

Well-posedness in heterogeneous media:

$$((1 - 2ku)u_t)_t - \operatorname{div}(c^2(x)\nabla u) - \operatorname{div}(b(x)\nabla u_t) = 0$$

where $c, b \in L^\infty(\Omega)$ with $c \geq c_0 > 0, b \geq b_0 > 0$ a.e.

Issue: Insufficient global spacial regularity because $u(t) \notin H^2(\Omega)$

Attenuation in biological media

Attenuation in biological media

- In biological media, attenuation follows a **power law** in terms of **frequency**

$$\text{attenuation} \sim \omega^\alpha, \quad \alpha \in (0, 1)$$

How to model this in the time-domain acoustic equation? We know that

Attenuation in biological media

- In biological media, attenuation follows a **power law** in terms of **frequency**

$$\text{attenuation} \sim \omega^\alpha, \quad \alpha \in (0, 1)$$

How to model this in the time-domain acoustic equation? We know that

$$u_t \xrightarrow{\mathcal{F}} (\imath\omega) \hat{u}$$

$$\partial_t^n u \xrightarrow{\mathcal{F}} (\imath\omega)^n \hat{u}$$

$$\partial_t^\alpha u \xrightarrow{\mathcal{F}} (\imath\omega)^\alpha \hat{u} \quad (\text{assuming } u(0) = 0)$$

Westervelt eq. with time-fractional attenuation

$$u_{tt} - \Delta u - b \Delta \partial_t^\alpha u = \frac{\beta_a}{\rho_0 c^2} ((u)^2)_{tt}, \quad \alpha \in (0, 1)$$

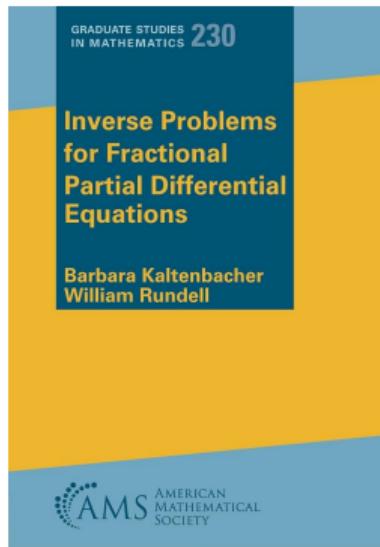
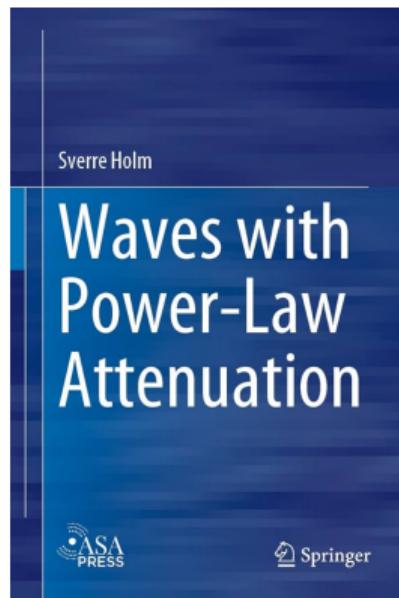
where

$$b = \frac{1}{\rho_0} \tau^{\alpha-1} \left(\frac{4}{3} \mu + \mu_b \right) \quad \beta_a = 1 + \frac{B}{2A}$$

$\tau > 0 \dots$ relaxation parameter

modeling details: [Prieur & Holm, 2011]

Literature on fractional modeling



A glimpse into **fractional calculus**:

How to define $\partial_t^\alpha(\cdot)$?

Fractional calculus



Leibniz (1646–1716)



l'Hôpital (1661–1704)

Letter, September 30th 1695

*Can the meaning of derivatives with integer order be generalized
to derivatives with non-integer order?*

Motivation behind fract. integration

- **Cauchy formula** for repeated integration

$$I^n u(t) = \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{n-1}} u(\sigma_n) d\sigma_n \dots d\sigma_1$$

Motivation behind fract. integration

- **Cauchy formula** for repeated integration

$$\begin{aligned} I^n u(t) &= \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{n-1}} u(\sigma_n) d\sigma_n \dots d\sigma_1 \\ &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s) ds \end{aligned}$$

Fractional integration

- The Riemann–Liouville integral

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \alpha > 0$$

- Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N}$$

$$\Gamma(z+1) = z\Gamma(z)$$

Fractional integration

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \alpha > 0$$

- Well-defined a.e. for $u \in L^1(0, T)$
- Can be seen using Young's convolution inequality

$$\|\mathfrak{K} * u\|_{L^1(0, T)} \leq \|\mathfrak{K}\|_{L^1(0, T)} \|u\|_{L^1(0, T)}$$

Example: Fractional integral of a constant function

$$I^\alpha 1 =$$

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$$I^\alpha \mathbf{1} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot \mathbf{1} \, ds$$

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$$I^\alpha \mathbf{1} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot \mathbf{1} \, ds$$

$$= \frac{1}{\alpha \Gamma(\alpha)} t^\alpha$$

$$= \frac{1}{\Gamma(\alpha + 1)} t^\alpha$$

Fractional derivatives

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- Idea: Differentiation as inverse of integration

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or

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or

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Not the same! unless $u(0) = 0$

Djrbashian–Caputo derivative



Mkhitar Djrbashian (1918–1994)



Michele Caputo (1927–)

$$\partial_t^\alpha u = I^{1-\alpha} u_t, \quad \alpha \in (0, 1)$$

[Djrbashian, Nauka, (1966)], [Caputo, Geophys. J. Int., (1967)]

Fractional derivative

- Written out

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds, \quad \alpha \in (0,1)$$

- Well-defined for $u \in W^{1,1}(0, T)$

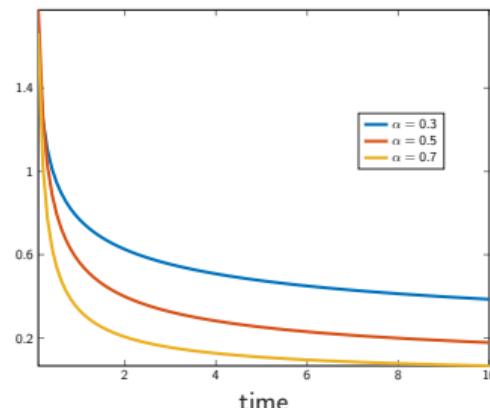
Fractional derivative

- Can be seen as a **temporal convolution**

$$\partial_t^\alpha u = \mathcal{R} * u_t$$

using the memory kernel

$$\mathcal{R}(s) = \frac{1}{\Gamma(1-\alpha)} s^{-\alpha}$$



$$\lim_{s \rightarrow 0^+} \mathcal{R}(s) = +\infty$$

$$\mathcal{R} \in L^1(0, T)$$

Example: Fractional derivative of a constant

$$\partial_t^\alpha \mathbf{1} =$$

Example: Fractional derivative of a constant

$$\partial_t^\alpha 1 = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \cdot 0 \, ds$$

$$= 0$$

Example: Fractional derivative of a constant

$$\begin{aligned}\partial_t^\alpha 1 &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \cdot 0 \, ds \\ &= 0\end{aligned}$$

Side note: The Riemann–Liouville notion of the fractional derivative does not give zero

$$D_t^\alpha 1 = D_t (t^{1-\alpha} 1) = D_t \left(\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \right) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$$

Good and not so good properties

- We can impose initial conditions in the usual way

$$\partial_t^\alpha u = f, \quad u(0) = u_0$$

~~ Useful when studying fractional ODEs/PDEs

Good and not so good properties

- We can impose initial conditions in the usual way

$$\partial_t^\alpha u = f, \quad u(0) = u_0$$

~~ Useful when studying fractional ODEs/PDEs

- We know that

$$\partial_t^\alpha \partial_t u = \partial_t^{1+\alpha} u$$

but

$$\partial_t \partial_t^\alpha u = \partial_t^{\alpha+1} u + \mathfrak{K}(t)u_t(0)$$

Good and not so good properties

- No commutative or semigroup properties

$$\partial_t^\alpha \partial_t^\beta u \neq \partial_t^\beta \partial_t^\alpha u$$

$$\partial_t^\alpha \partial_t^\beta u \neq \partial_t^{\alpha+\beta} u$$

- No Leibniz rule

$$\partial_t^\alpha(uv) \neq \partial_t^\alpha u \cdot v + u \cdot \partial_t^\alpha v$$

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$$\partial_t^\alpha(uv) \neq \partial_t^\alpha u \cdot v + u \cdot \partial_t^\alpha v$$

~~~

No integration by parts formula

## Instead of integration by parts: Useful inequalities

$$\int_0^t \int_{\Omega} \partial_t^\alpha y(s) \partial_t y(s) dx ds \geq C_\alpha \int_0^t \|\partial_t^\alpha y(s)\|_{L^2(\Omega)}^2 ds$$

$$\int_0^t \int_{\Omega} \partial_t^\alpha y(s) y(s) dx ds \geq -C_\alpha \|y(0)\|_{L^2(\Omega)}^2$$

## Instead of integration by parts: Useful inequalities

$$\int_0^t \int_{\Omega} \partial_t^\alpha y(s) \partial_t y(s) dx ds \geq C_\alpha \int_0^t \|\partial_t^\alpha y(s)\|_{L^2(\Omega)}^2 ds$$

$$\int_0^t \int_{\Omega} \partial_t^\alpha y(s) y(s) dx ds \geq -C_\alpha \|y(0)\|_{L^2(\Omega)}^2$$

⚠ We do not have a similar lower bound for

$$\int_0^t \int_{\Omega} \partial_t^\alpha y(s) y_{tt}(s) dx ds$$

## What changes in the PDE analysis?

- Consider the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 \Delta u - b \Delta \partial_t^\alpha u &= k(u^2)_{tt} && \text{in } \Omega \times (0, T) \\ u &= 0 && \text{on } \partial\Omega \times (0, T) \\ (u, u_t) &= (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}$$

$\partial_t^\alpha$  ... Caputo partial derivative of order  $\alpha \in (0, 1)$

$\Omega \subset \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$  ... smooth bounded domain

$T > 0$  ... final time

# Approach in the well-posedness analysis

As before, consider a linearization

$$((1 - 2k\textcolor{blue}{u}^*)u_t)_t - c^2\Delta t - b\Delta\partial_t^\alpha u = 0$$

Define the mapping

$$\mathcal{T} : \textcolor{blue}{u}^* \mapsto u,$$

with

- $\textcolor{blue}{u}^*$  in a ball  $\mathbb{B} \subset \mathcal{U}$  (solution space)
- $u$  solves the linearized problem

The **fixed point**  $\textcolor{blue}{u}^* = u$  solves the nonlinear problem.

# Differences in the Galerkin approach

- The semi-discrete problem is now a system of **integro-differential equations**
- Previous testing strategy does not work
  - If we test with  $u_{tt}^n$ , we cannot integrate by parts in time to bound from below

$$b \int_0^t (\nabla \partial_t^\alpha u^n, \nabla u_{tt}^n)_{L^2(\Omega)} \, ds$$

## Differences in energy estimates

- We need to use test functions of higher order

$$\phi^n = \Delta^2 u_t^n$$

- After testing and integrating over  $\Omega$ ,

$$(\alpha(u^*) u_{tt}^n - c^2 \Delta u^n - b \Delta \partial_t^\alpha u^n - 2k u_t^* u_t^n, \Delta^2 u_t^n)_{L^2(\Omega)} = 0$$

where

$$\alpha(u^*) = 1 - 2k u^*$$

## Energy estimates

- After testing with  $-\Delta u_t^n$ , we can use the lower bound

$$\begin{aligned} & -b \int_0^t (\Delta \partial_t^\alpha u^n, \Delta^2 u_t^n)_{L^2(\Omega)} ds \\ &= b \int_0^t (\nabla \Delta \partial_t^\alpha u^n, \nabla \Delta u_t^n)_{L^2(\Omega)} ds \\ &\geq b C_\alpha \int_0^t \|\nabla \Delta \partial_t^\alpha u^n\|_{L^2(\Omega)}^2 ds \end{aligned}$$

## Energy estimates

- In this manner, we arrive at

$$\begin{aligned} & \frac{1}{2} \left\| \sqrt{\alpha} \Delta u_t^n \right\|_{L^2}^2 \Big|_0^t + \frac{c^2}{2} \left\| \nabla \Delta u^n(s) \right\|_{L^2}^2 \Big|_0^t + C_\alpha \int_0^t \|(\nabla \Delta \partial_t^\alpha u^n)(s)\|_{L^2(\Omega)}^2 ds \\ & \leq \frac{1}{2} \int_0^t (\alpha_t \Delta u_t^n, \Delta u_t^n)_{L^2(\Omega)} ds + \int_0^t (\textcolor{red}{u_{tt}^n} \Delta \alpha + 2 \nabla \textcolor{red}{u_{tt}^n} \cdot \nabla \alpha, \Delta u_t^n)_{L^2(\Omega)} ds \end{aligned}$$

- Additional estimate of  $\|u_{tt}^n\|_{H^1(\Omega)}$  follows by using the semi-discrete PDE

# Energy estimates

- Energy functionals are higher-order in space

$$E[u](t) = \|u_t(t)\|_{H^2(\Omega)}^2 + \|u(t)\|_{H^3(\Omega)}^2$$

- Solution space now is

$$\begin{aligned}\mathcal{U} = \{u : u &\in L^\infty(0, T; H_\diamondsuit^3(\Omega) \cap W^{1,\infty}(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \\ &\partial_t^\alpha u \in L^2(0, T; H_\diamondsuit^3(\Omega)))\}\end{aligned}$$

where

$$H_\diamondsuit^3(\Omega) = \{u \in H^3(\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}$$

# Local well-posedness

**Theorem.** Let the assumptions on  $\mathcal{K}$  hold and let

$$(u_0, u_1) \in H_{\diamond}^3(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega)).$$

There exists data size  $r = r(T) > 0$ , such that if

$$\|u_0\|_{H^3(\Omega)}^2 + \|u_1\|_{H^2(\Omega)}^2 \leq r^2,$$

then there is a unique solution in  $\mathcal{U}$ , which satisfies

$$\|u\|_{\mathcal{U}}^2 \leq C(T)(\|u_0\|_{H^3(\Omega)}^2 + \|u_1\|_{H^2(\Omega)}^2).$$

details: [Kaltenbacher, Meliani, & Nikolić, 2023]

## An open question

Analysis is uniform in  $b$  (we are not exploiting damping)

- Can we exploit damping to reduce the regularity assumptions on data?

# On numerical analysis

# On numerical analysis

A finite element **semi-discretization** of the strongly damped equation:

Find  $u_h : [0, T] \rightarrow V_h$ , such that

$$\begin{cases} (((1 - 2ku_h)u_{h,t})_t, \phi_h)_{L^2} + (c^2 \nabla u_h + b \nabla u_{h,t}, \nabla \phi_h)_{L^2} = 0 \\ \text{for all } \phi_h \in V_h \text{ a.e. in time, with} \\ (u_h, u_{h,t})|_{t=0} = (u_{0h}, u_{1h})|_{t=0} \end{cases}$$

**Delicate point:** In general, discrete solutions are not smooth enough

- Inverse inequalities or discrete embeddings have to be employed

$$\|u_h(t)\|_{L^\infty(\Omega)} \leq C h^{-d/2} \|u_h(t)\|_{L^2(\Omega)} \quad \text{or} \quad \|u_h(t)\|_{L^\infty(\Omega)} \leq C \|\Delta_h u_h(t)\|_{L^2(\Omega)}$$

# On numerical analysis

- Still many open questions in this area
- Some recent advances:
  - **FEM** semi-discretization  
[Nikolić, Wohlmuth SIMA, 2019], [Hochbruck, Maier, IMAJNA, 2022], ...
  - **Space-time** finite element methods  
[Gómez & Nikolić, IMAJNA, 2025]
  - **Multiharmonic approaches** ↪ connection to Helmholtz problems  
[Kaltenbacher & Rainer, ESAIM: M2AN, 2025]
  - Efficient methods for **time-fractional** models  
[Baker, Banjai, & Ptashnyk, Math. Comput., 2022]

## Summary: Nonlinear acoustic models

- Nonlinear acoustic modeling by retaining second-order terms

Westervelt equation

$$((1 - 2ku)u_t)_t - c^2 \Delta u - b\Delta u_t = 0$$

- Analysis via energy arguments  $\rightsquigarrow$  small-data well-posedness
- Time-fractional attenuation invokes higher smoothness arguments

Next  $\Rightarrow$  Problem sheet with Westervelt equation