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WAVES.NL Summer School | Nijmegen
Assignment 2 | 26.8.2025
Due Date: -

Exercise 1. (Shallow Water Equations)
The shallow water equations for water height $h(t,x)$ and vertical velocity $u(t,x)$ are

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix} = -\frac{1}{\lambda}\begin{pmatrix} 0 \\ u \end{pmatrix},$$

where $h(t,x)$ and $u(t,x)$ are the unknowns and g, λ are parameters.

a) What physical interpretations do g and λ have?

b) Write the system in the following form:

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + A(h,u) \cdot \partial_x \begin{pmatrix} h \\ hu \end{pmatrix} = -\frac{1}{\lambda}\begin{pmatrix} 0 \\ u \end{pmatrix},$$

for a matrix $A(h,u) \in \mathbb{R}^{2\times 2}$ depending on h, u, g.

c) Check that the eigenvalues of A are given by

$$\sigma(A) = \{u(t,x) - \sqrt{g \cdot h(t,x)}, u(t,x) + \sqrt{g \cdot h(t,x)}\}.$$

Exercise 2. (Euler equations and hyperbolicity)
We consider the Euler equations of fluid dynamics

$$\partial_t \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute the propagation speeds of the model (the eigenvalues of the matrix).
Subsequently show that the model is conditionally hyperbolic.

Exercise 3. (Eigenvalues of non-orthonormal, linearized Grad model)

Recall from the previous assignment that the non-orthonormal, linearized Grad model for $M + 1$ equations can be written in matrix form as

$$\partial_t \boldsymbol{f} + A_M \cdot \partial_x \boldsymbol{f} = 0,$$

with

$$A_M = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & M \\ & & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{M+1 \times M+1},$$

where all other entries are zero.
Compute a recursion formula for the characteristic polynomial $\chi_{A_M}(\lambda) = \det(A_M - \lambda \cdot I)$.
Based on that, what are the eigenvalues of A_M?

Exercise 4. (Eigenvalues of part of the HME model)

Compute the eigenvalues of the matrix

$$A_M = \begin{pmatrix} u & 1 & & & \\ T & u & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & T & u & M \\ & & & T & u \end{pmatrix} \in \mathbb{R}^{M+1 \times M+1}.$$

Note that this is a part of the system matrix of the HME model, see lecture notes.
Hint: Use a variable transformation for the variable of the characteristic polynomial λ, based on the variable transformation from c to ξ.

Exercise 5. (Linear stability analysis of the Burgers equation)

We consider the scalar, linearized, viscous Burgers equation

$$\frac{\partial}{\partial t} u + u_0^2 \frac{\partial}{\partial x} u = D \frac{\partial^2}{\partial x^2} u, \quad u_0 \in \mathbb{R}, D \in \mathbb{R}, \tag{1}$$

with viscosity constant D and advection velocity u_0.
We want to perform a linear stability analysis of equation (1) using the wave ansatz

$$u(t, x) = c \cdot e^{i(kx - \omega t)}, \tag{2}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $c \in \mathbb{R}$.

Insert the wave ansatz (2) into the Burgers equation (1) to derive a stability condition for the Burgers equation.