Free-surface waves using extended shallow water models part 3

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Schedule

Time	Monday	Tuesday	Wednesday	Thursday	Friday
8:50–9:00	*Opening*				
9:00–10:30	L3	L5	L2	L4	L6
10:30–11:00	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
11:00–12:30	L1	L1	L2	L4	L6
12:30–13:30	*Lunch*	*Lunch*	*Lunch*	*Lunch*	*Lunch*
13:30–15:00	L3	L5	L3	L5	
15:00–15:30	*Coffee break*	*Coffee break*			
15:30–17:00	Poster session	L1			
17:45–19:00			*Social event*		

Slides at: `https://github.com/scalaura/waves_summerschool`

L1: Mon 11-12:30
- overview
- motivation
- derivation

L2: Tue 11-12:30
- analysis

L3: Tue 15:30-17
- selected papers
- outlook

Content of this talk

1. Repetition
2. Analysis

1 Repetition

Desirable properties

Question: What are desirable model properties?

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N \partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

- high accuracy
- low complexity
- efficiency
- adaptivity
- extendability
- analytical form

($\checkmark$)

- conservation
- hyperbolicity
- stability
- equilibria
- steady states
- entropy

$\checkmark$

Hyperbolic SWME models

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N \partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

- Hyperbolic Shallow Water Moment Equations [JK, Rominger, 2020]
- Shallow Water Linearized Moment Equations [JK, Pimentel-Garcia, 2022]
- Primitive variable regularization [JK, submitted]
- axisymmetric quasi-2D [Verbiest, JK, 2025] and 2D [Bauerle et al., 2025]

2 selected papers

Source term

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N \partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

Different source/friction terms

- Newtonian slip flow [Kowalski, Torrilhon, 2019]
- Bedload Manning friction [Garres-Diaz et al., 2021]
- Sediment transport with erosion and deposition at bottom [Parvin et al., submitted]
- non-slip boundary conditions [Zhou et al., submitted]
- Savage-Hutter / varying viscosity [Huang, et al., in preparation]

Numerics: efficiency and structure

$$\partial_t \boldsymbol{u}_N + \partial_x \boldsymbol{F}(\boldsymbol{u}_N) = \boldsymbol{B}(\boldsymbol{u}_N)\partial_x \boldsymbol{u}_N + \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}(\boldsymbol{u}_N)\partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

efficiency

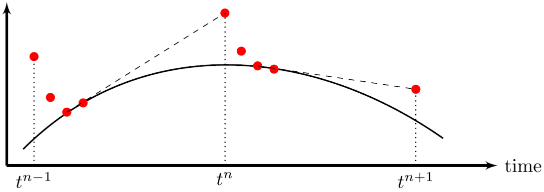
- (high-order) path-conservative finite volume schemes [JK, Castro, 2017]
- Roe method for large hyperbolic systems [Pimentel, et al., 2020]

structure

- asymptotic-preserving numerics [JK, Samaey, 2021],
- micro-macro decomposition [JK, Vandecasteele, 2023]
- well-balancing [JK, Pimentel, 2022], [Caballero, et al., 2025]

Projective Integration for stiff RHS

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}\left(\boldsymbol{u}_N\right) \partial_x \boldsymbol{u}_N = -\frac{1}{\tau} \boldsymbol{S}\left(\boldsymbol{u}_N\right)$$



Mitigate stiffness of RHS

- small inner steps for fast dynamics
- extrapolation step for slow dynamics

- for kinetic equations [JK, Samaey, 2021]
- for shallow flows [Amrita, JK, 2022]
- can be written as Runge-Kutta method [JK, Samaey, 2025]
- alternative is implicit splitting scheme [Huang, et al., 2022]

Further model reduction

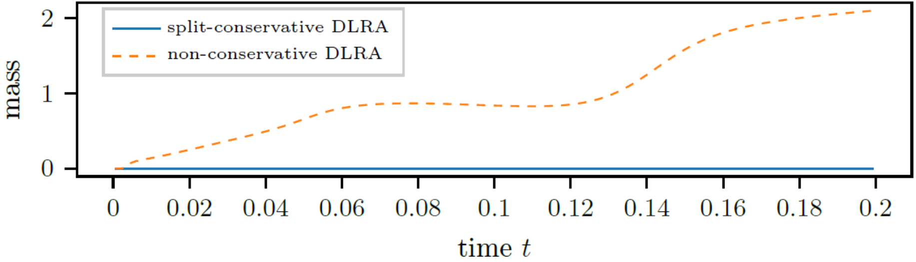
$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N\left(\boldsymbol{u}_N\right)\partial_x \boldsymbol{u}_N = \boldsymbol{S}\left(\boldsymbol{u}_N\right), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

$\boldsymbol{u}_N \in \mathbb{R}^{N+2}$ may be unnecessarily large $\Rightarrow$ model reduction

Further model reduction

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N(\boldsymbol{u}_N)\,\partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N \in \mathbb{R}^{N+2}$$

$\boldsymbol{u}_N \in \mathbb{R}^{N+2}$ may be unnecessarily large $\Rightarrow$ model reduction



Problem: standard model reduction (POD or DLRA) does not lead to mass conservation.

Hyperreduction [JK, KRAH, KUSCH, submitted]

Macro-micro decomposition:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{U} & \mathbf{V} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} h(x_j, t) & h(x_j, t)\, u_m(x_j, t) \end{bmatrix}_j \in \mathbb{R}^{N_x \times 2},$$

$$\mathbf{V} = \begin{bmatrix} h(x_j, t)\, \alpha_1(x_j, t) & \ldots & h(x_j, t)\, \alpha_N(x_j, t) \end{bmatrix}_j \in \mathbb{R}^{N_x \times N}.$$

Apply model reduction only to micro variables $\mathbf{V}$.
$\Rightarrow$ h and u_m are always included in the solution space, including conservation of mass.

Hyperreduction of SWME model: macro-micro decomposition

$$\partial_t \boldsymbol{q} + \mathbf{A}(\boldsymbol{q})\partial_x \boldsymbol{q} = \boldsymbol{g}(\boldsymbol{q}),$$

operator splitting:

$$\begin{aligned} &\text{Step 1: transport} \quad && \partial_t \boldsymbol{q} + \mathbf{A}(\boldsymbol{q})\partial_x \boldsymbol{q} = 0\,, && (1)\\ &\text{Step 2: friction} \quad && \partial_t \boldsymbol{q} \qquad\qquad\quad = \boldsymbol{g}(\boldsymbol{q}) && (2)\end{aligned}$$

macro-micro decomposition:

$$\begin{aligned} &\text{Step 1a: macro transport} \quad && \boldsymbol{q}^n = (\boldsymbol{u}^n, \boldsymbol{v}^n) && \overset{(1)}{\Rightarrow} (\widetilde{\boldsymbol{u}}^{n+1}, \boldsymbol{v}^n),\\ &\text{Step 1b: micro transport} \quad && (\widetilde{\boldsymbol{u}}^{n+1}, \boldsymbol{v}^n) && \overset{(1)}{\Rightarrow} (\widetilde{\boldsymbol{u}}^{n+1}, \widetilde{\boldsymbol{v}}^{n+1}),\\ &\text{Step 2a: macro friction} \quad && \widetilde{\boldsymbol{q}}^{n+1} = (\widetilde{\boldsymbol{u}}^{n+1}, \widetilde{\boldsymbol{v}}^{n+1}) && \overset{(2)}{\Rightarrow} (\boldsymbol{u}^{n+1}, \widetilde{\boldsymbol{v}}^{n+1}),\\ &\text{Step 2b: micro friction} \quad && (\boldsymbol{u}^{n+1}, \widetilde{\boldsymbol{v}}^{n+1}) && \overset{(2)}{\Rightarrow} (\boldsymbol{u}^{n+1}, \boldsymbol{v}^{n+1}).\end{aligned}$$

Micro-macro decomposition for transport step

$$\partial_t \boldsymbol{q} + \mathbf{A}(\boldsymbol{q})\partial_x \boldsymbol{q} = 0,$$

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix}, \quad \mathbf{A}(\boldsymbol{q}) = \begin{bmatrix} \mathbf{A}_{\boldsymbol{uu}} & \mathbf{A}_{\boldsymbol{uv}} \\ \mathbf{A}_{\boldsymbol{vu}} & \mathbf{A}_{\boldsymbol{vv}} \end{bmatrix}$$

$$\mathbf{A}_{\boldsymbol{uu}} = \begin{bmatrix} & 1 \\ gh - u_m^2 - \frac{1}{3}\alpha_1^2 & 2u_m \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \mathbf{A}_{\boldsymbol{uv}} = \begin{bmatrix} & & & \\ \frac{2}{3}\alpha_1 & & & \end{bmatrix} \in \mathbb{R}^{2\times N},$$

$$\mathbf{A}_{\boldsymbol{vu}} = \begin{bmatrix} -2u_m\alpha_1 & 2\alpha_1 \\ -\frac{2}{3}\alpha_1^2 & \\ & \\ & \end{bmatrix} \in \mathbb{R}^{N\times 2}, \; \mathbf{A}_{\boldsymbol{vv}} = \begin{bmatrix} u_m & \frac{3}{5}\alpha_1 & & \\ \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

Friction step

operator splitting:

$$\begin{aligned} &\text{Step 1: transport} \qquad && \partial_t \boldsymbol{q} + \mathbf{A}(\boldsymbol{q})\partial_x \boldsymbol{q} &&= 0\,, \\ &\text{Step 2: friction} \qquad && \partial_t \boldsymbol{q} &&= \boldsymbol{g}(\boldsymbol{q})\,, \end{aligned}$$

N=2:

$$\partial_t \boldsymbol{q} = \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

Friction step

operator splitting:

$$
\begin{aligned}
&\text{Step 1: transport} \qquad && \partial_t \boldsymbol{q} + \mathbf{A}(\boldsymbol{q})\partial_x \boldsymbol{q} = 0\,,\\
&\text{Step 2: friction} \qquad && \partial_t \boldsymbol{q} \qquad\qquad\quad\;\; = \boldsymbol{g}(\boldsymbol{q})\,,
\end{aligned}
$$

N=2:

$$
\partial_t \boldsymbol{q} = \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.
$$

Friction step

operator splitting:

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Two problems

1. small h or λ require implicit method
2. nonlinearity in h is difficult for model reduction

Friction step

$$\partial_t \boldsymbol{q} = \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

Two problems

1. small h or λ require implicit method
2. nonlinearity in h is difficult for model reduction

implicit scheme:

h is constant during the friction step $\Rightarrow$ explicit backward Euler [Pimentel, JK, 2022]

POD-Galerkin (offline phase)

singular value decomposition (SVD) of the *snapshot matrix*

$$\mathbf{V}^{\mathrm{POD}} = \begin{bmatrix} \mathbf{V}^0 \\ \vdots \\ \mathbf{V}^{N_{\mathrm{t}}-1} \end{bmatrix} \in \mathbb{R}^{(N_x N_{\mathrm{t}})\times N}, \tag{3}$$

where the $\mathbf{V}^n$ are solution snapshots.
Truncated SVD of $\mathbf{V}^{\mathrm{POD}}$:

$$\mathbf{V}^{\mathrm{POD}} = \mathbf{\Psi}\mathbf{\Sigma}\mathbf{W}^\top \tag{4}$$

$\mathbf{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $r \ll N$, is diagonal matrix containing the largest r singular values.
$\mathbf{\Psi} \in \mathbb{R}^{(N_x N_{\mathrm{t}})\times r}$, $\mathbf{W} \in \mathbb{R}^{N\times r}$ are left and right singular vectors.
POD approximates micro variables using orthonormal basis $\{\boldsymbol{w}\}_{k=1,\ldots,r}$

$$\boldsymbol{v}(x,t) \approx \widetilde{\boldsymbol{v}}(x,t) = \sum_{k=1}^{r} \hat{\alpha}_k(x,t)\boldsymbol{w}_k\,, \qquad r \ll N \leq N_{\mathrm{t}} \tag{5}$$

$$= \mathbf{W}\hat{\boldsymbol{v}}(x,t)\,, \tag{6}$$

POD-Galerkin (online phase)

Project discrete scheme onto pre-computed basis (intrusive).

1. Project micro transport step onto the pre-computed basis.
2. Project micro friction step onto the pre-computed basis.

POD-Galerkin (online phase)

Project discrete scheme onto pre-computed basis (intrusive).

1. Project micro transport step onto the pre-computed basis.
2. Project micro friction step onto the pre-computed basis.

$\Rightarrow$ Hyperreduction of the micro steps.

Dynamical Low-Rank Approximation (DLRA)

Apply DLRA only to microscopic correction terms $v_i := h\alpha_i$
$\mathbf{V}(t) \in \mathbb{R}^{N_x \times N}$, where $v_{ji} = h(t, x_j)\alpha_i(t, x_j)$,
dynamical low-rank approximation

$$\mathbf{V}(t) = \mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top$$

$\mathbf{X} \in \mathbb{R}^{N_x \times r}$ basis vectors in space
$\mathbf{W} \in \mathbb{R}^{N \times r}$ basis vectors in moments
$\mathbf{S} \in \mathbb{R}^{r \times r}$ coefficient matrix

$\Rightarrow$ Project discrete scheme onto dynamical basis (intrusive).

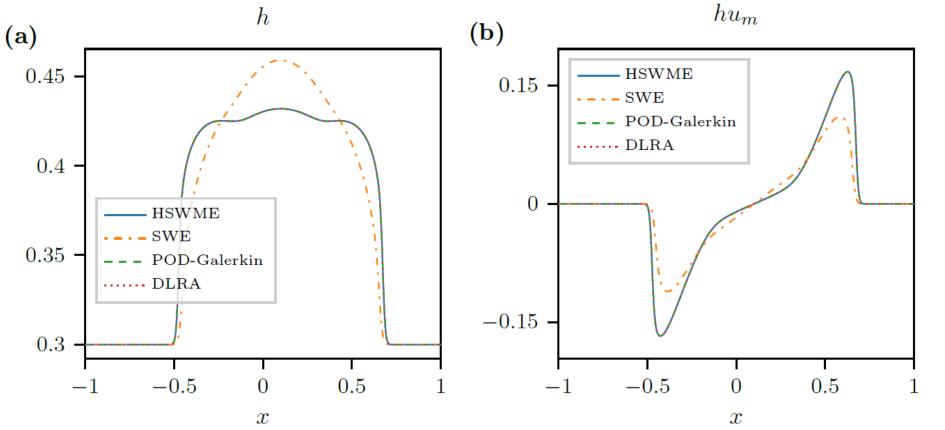
DLRA: BUG integrator

$$\mathbf{V}(t) = \mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top$$

$$\dot{\mathbf{V}}(t) \in \mathcal{T}_{\mathbf{V}(t)}\mathcal{N}_r \qquad \text{such that} \quad \|\dot{\mathbf{V}}(t) - R_v(\mathbf{U}(t), \mathbf{V}(t))\| \to \text{min!}$$

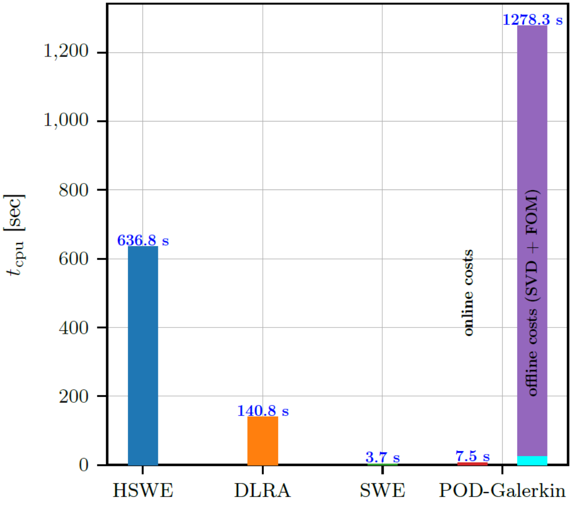
1. K-**step**: Update $\mathbf{X}^0$ to $\mathbf{X}^1$ via
$$\dot{\mathbf{K}}(t) = R_v(\mathbf{K}(t)\mathbf{W}^{0,\top})\mathbf{W}^0\,, \qquad \mathbf{K}(t_0) = \mathbf{X}^0\mathbf{S}^0\,.$$
Determine $\mathbf{X}^1$ with $\mathbf{K}(t_1) = \mathbf{X}^1\mathbf{R}$ and store $\mathbf{N} = \mathbf{X}^{1,\top}\mathbf{X}^0$.
2. L-**step**: Update $\mathbf{W}^0$ to $\mathbf{W}^1$ via
$$\dot{\mathbf{L}}(t) = R_v(\mathbf{X}^0\mathbf{L}(t)^T)^T\mathbf{X}^0\,, \qquad \mathbf{L}(t_0) = \mathbf{W}^0\mathbf{S}^\top\,.$$
Determine $\mathbf{W}^1$ with $\mathbf{L}(t_1) = \mathbf{W}^1\widetilde{\mathbf{R}}$ and store $\mathbf{N} = \mathbf{W}^{1,\top}\mathbf{W}^0$.
3. S-**step**: Update $\mathbf{S}^0$ to $\mathbf{S}^1$ via
$$\dot{\mathbf{S}}(t) = \mathbf{X}^{1,\top}R_v(\mathbf{X}^1\mathbf{S}(t)\mathbf{W}^{1,\top})\mathbf{W}^1\,, \qquad \mathbf{S}(t_0) = \mathbf{N}\mathbf{S}^0\mathbf{N}^\top$$
and set $\mathbf{S}^1 = \mathbf{S}(t_1)$.

Dam break accuracy [JK, KRAH, KUSCH, submitted]



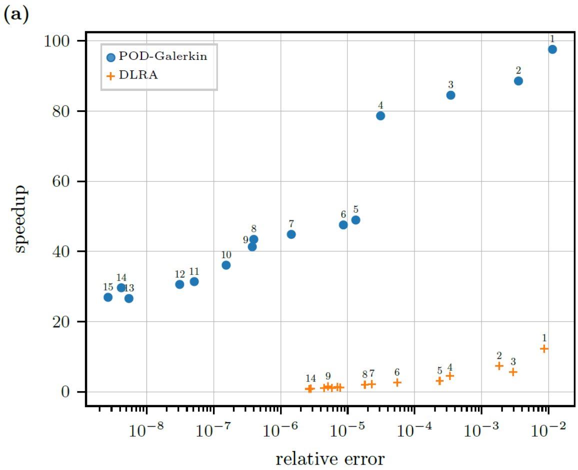
DLRA and POD solutions are as accurate as full model.

Dam break runtime ($r = 3, 4$)



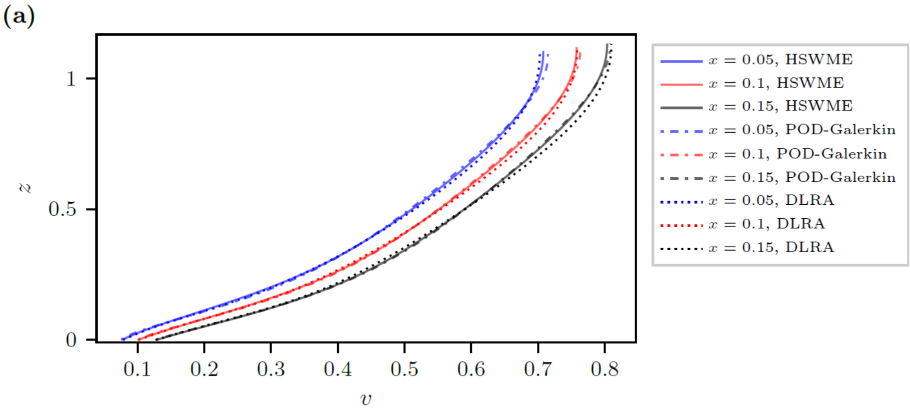
DLRA and POD both yield significant speedup

Dam break efficiency for changing r



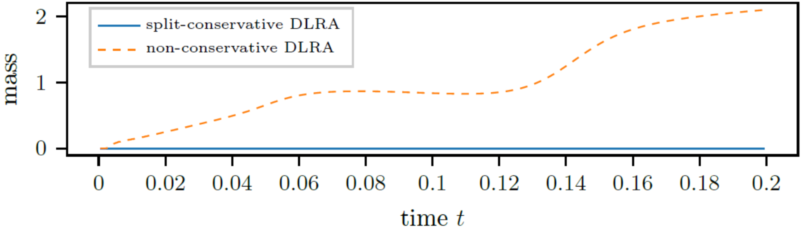
POD looks faster than DLRA because we do not count offline computation here.

Smooth wave velocity profile 1



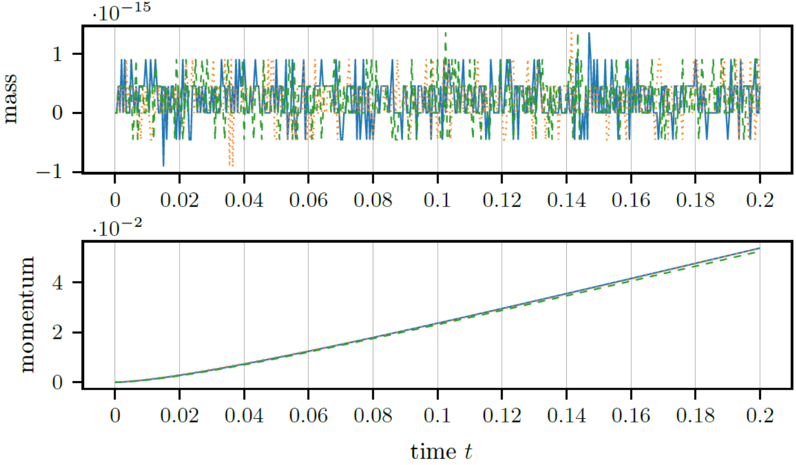
Accurate velocity profile for DLRA and POD.

Role of macro-micro splitting



Without the splitting in macro and micro variables, naive DLRA does not lead to mass conservation.

Smooth wave conservation properties

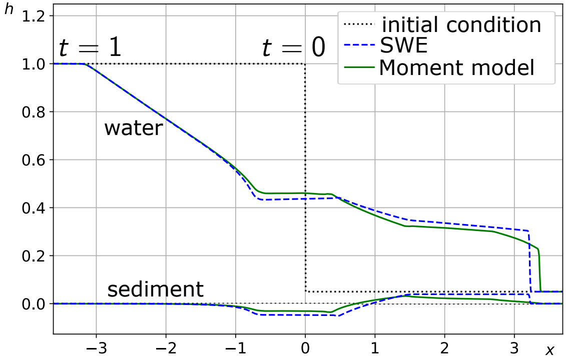


Mass is conserved up to machine precision.

Sediment transport [Garres-Diaz, et al., 2021]

Idea: Include moving bed and Manning friction

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}_N \partial_x \boldsymbol{u}_N = \boldsymbol{S_F}, \quad \boldsymbol{u}_N = (h, hu_m, h\alpha_1, \dots, h\alpha_N, h_b)^T \in \mathbb{R}^{N+3}$$



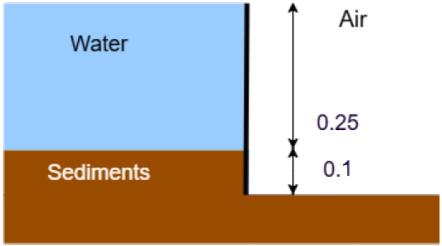
⇒ More realistic sediment transport

Idea: Include sediment concentration equation and erosion-deposition effects

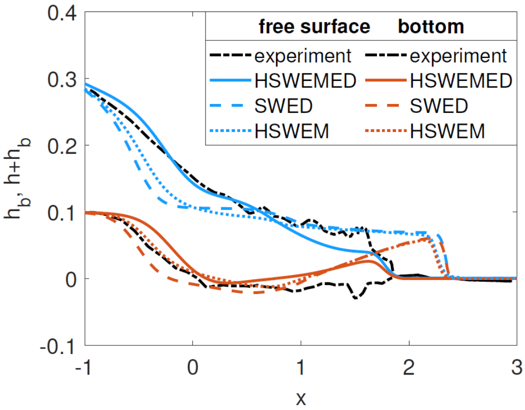
$$\partial_t \boldsymbol{u}_N + \left(\boldsymbol{A}_N + \boldsymbol{A}_s\right)\partial_x \boldsymbol{u}_N = \boldsymbol{S}_{\boldsymbol{ED}} + \boldsymbol{S}_{\boldsymbol{F}}$$

$$\boldsymbol{u}_N = \left(h, hu_m, h\alpha_1, \dots, h\alpha_N, hc, h_b\right)^T \in \mathbb{R}^{N+4}$$

Results: characteristic speed analysis and simulations



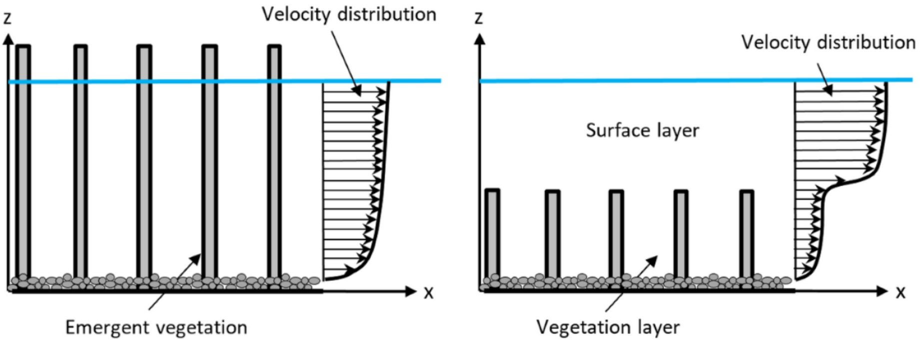
(a) test setup



(b) simulation result

⇒ Better accuracy of new models

Effect of Vegetation



$\Rightarrow$ Velocity variations can be captured by our models

Uncertainty Quantification [Kuijpers et al., in preparation]

$$h, hu \to h, q$$

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} + \partial_x \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{g}{2} h^2 \end{pmatrix} = -\frac{\nu}{\lambda} \begin{pmatrix} 0 \\ \frac{q}{h} \end{pmatrix}$$

$$h(t,x,\omega) = \sum_{k=0}^{K} \hat{h}_k(t,x)\psi_k(\xi(\omega))$$

$$q(t,x,\omega) = \sum_{k=0}^{K} \hat{q}_k(t,x)\psi_k(\xi(\omega))$$

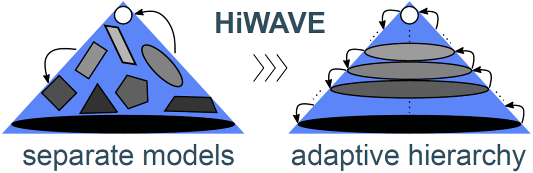
Idea: Polynomial chaos expansion of conservative variables

$$\frac{q(t,x)}{h(t,x)}(\omega) := \sum_{k=0}^{K} c_k(t,x)\psi_k(\xi(\omega))$$

Results: hyperbolicity proof for $N = 1$ $(h, hu_m, h\alpha_1 \to h, q, r)$

NWO Vidi project HiWAVE

Natural hazard prediction
with adaptive hierarchical wave models



(1) hierarchical model derivation
(2) hierarchical model reduction
(3) hierarchical model adaptivity

NWO Vidi project HiWAVE

Natural hazard prediction
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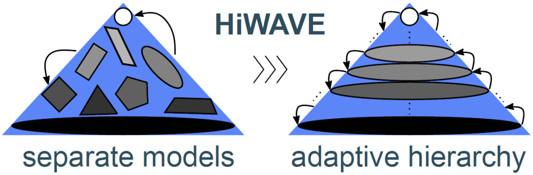

(1) hierarchical model derivation

(2) hierarchical model reduction

(3) hierarchical model adaptivity

Thank you for your attention!

summary

Part 1 Summary

1 repetition

- hyperbolic shallow water moment models

2 selected papers

- source term
- numerics
- POD and DLRA hyperreduction
- sediment transport
- steady states
- vegetation
- Uncertainty Quantification

Conclusion

part 1

- overview
- motivation
- derivation

part 2

- analysis

part 3

- numerics
- selected papers
- outlook