Waves: Modeling, Analysis, and Numerics
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4TU.AMI

Linear Waves: From Physics to Numerics

Part II

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Mathematics of Computational Science
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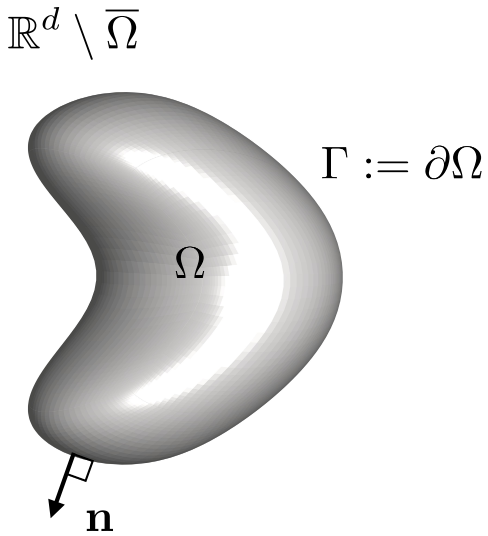
Scattering Problem Setup

Scattering by Bounded Obstacles



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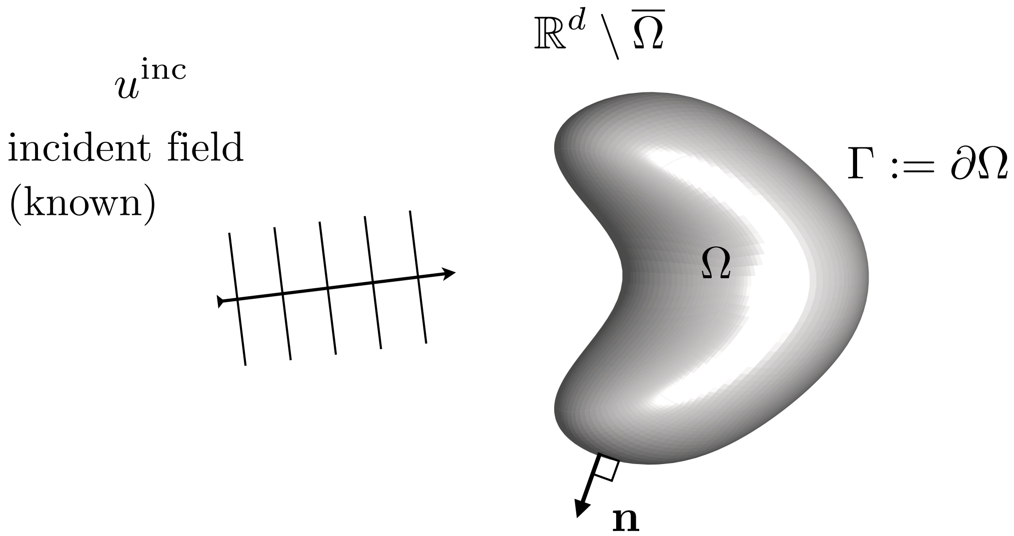

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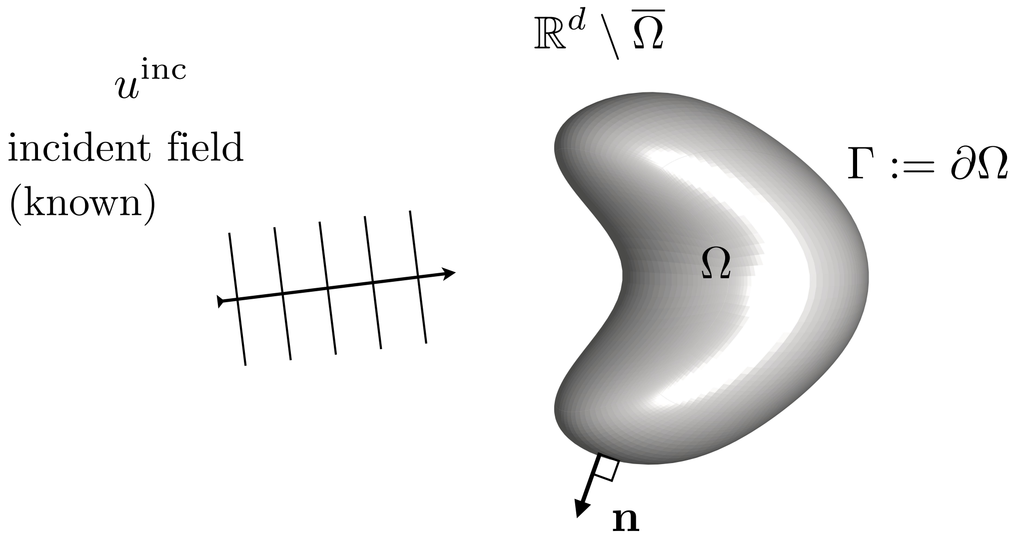

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$$\Delta u^{\text{inc}} + k^2 u^{\text{inc}} = 0 \quad \text{in} \quad D \supset \Gamma$$

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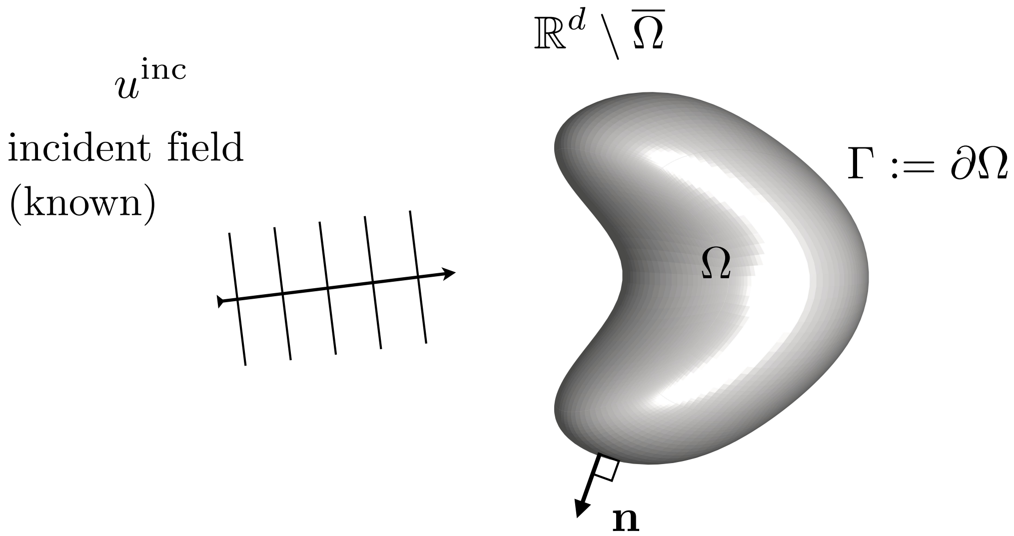

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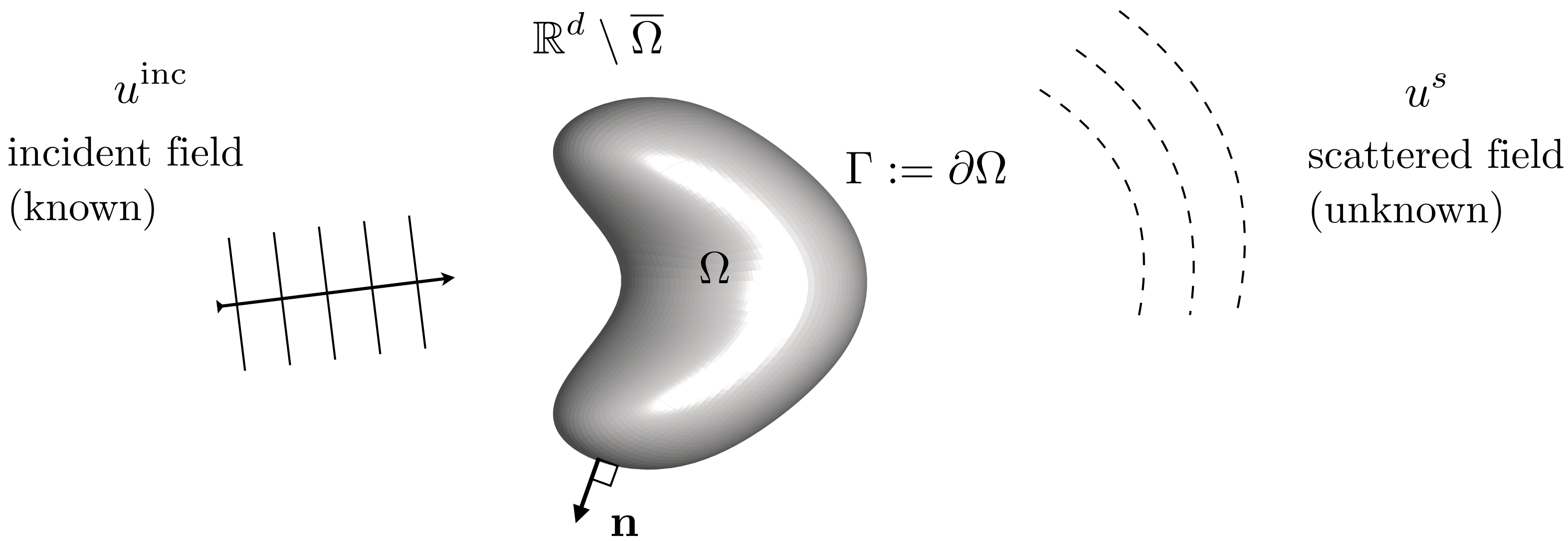

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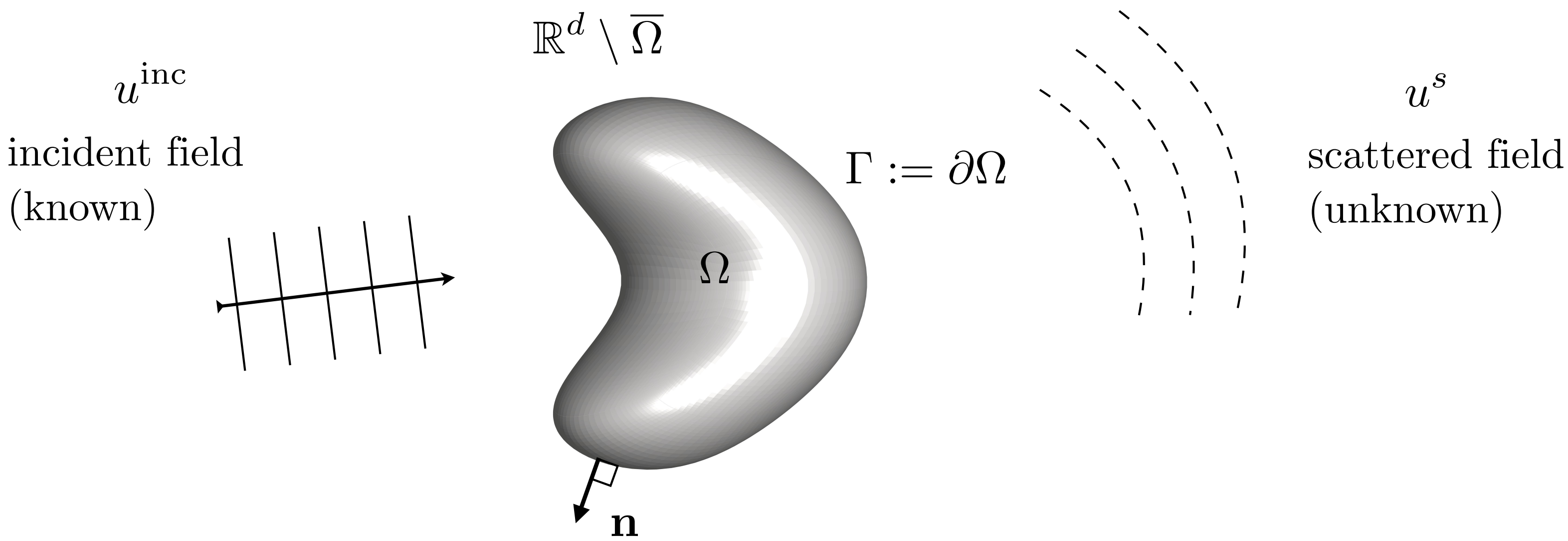

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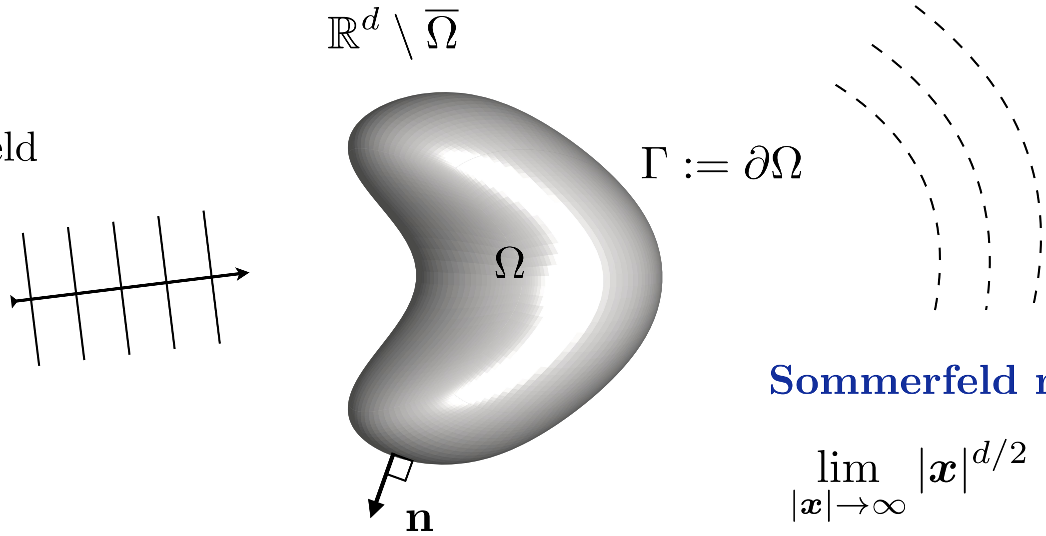

Sommerfeld radiation condition:

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}|^{d/2} \left(\frac{\partial u^s}{\partial |\boldsymbol{x}|} - iku^s \right) = 0$$

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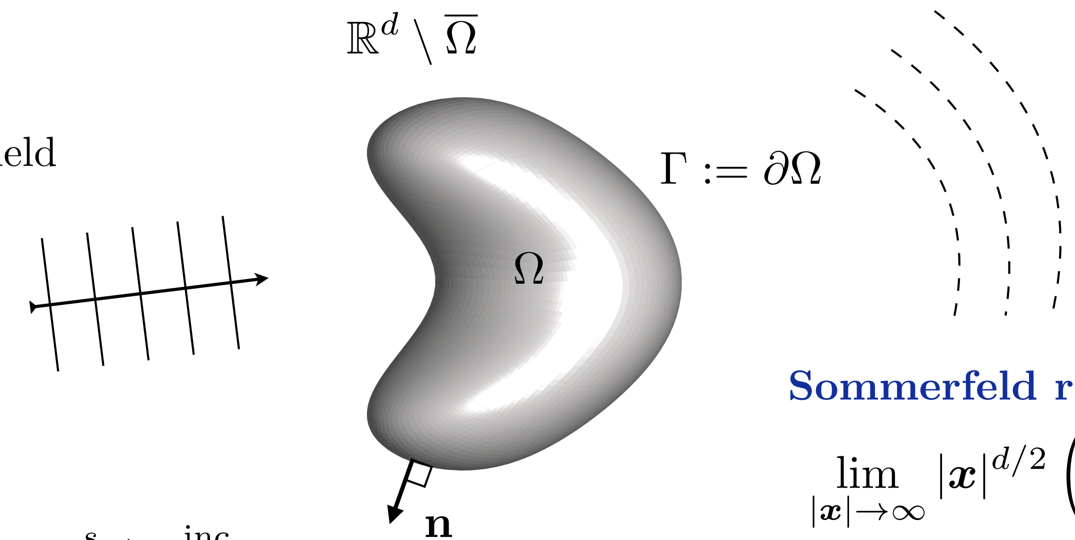
Scattering by Bounded Obstacles

u^{inc}
incident field
(known)



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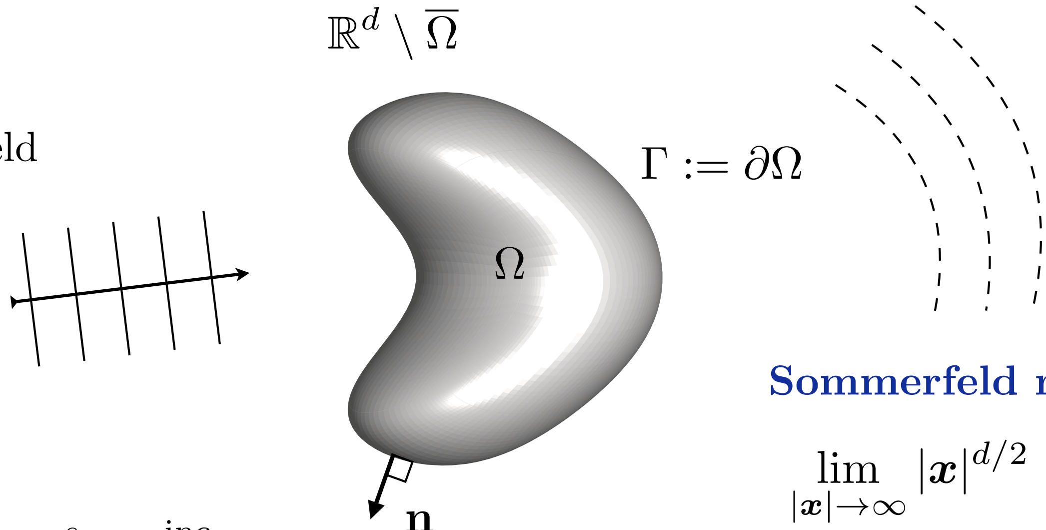

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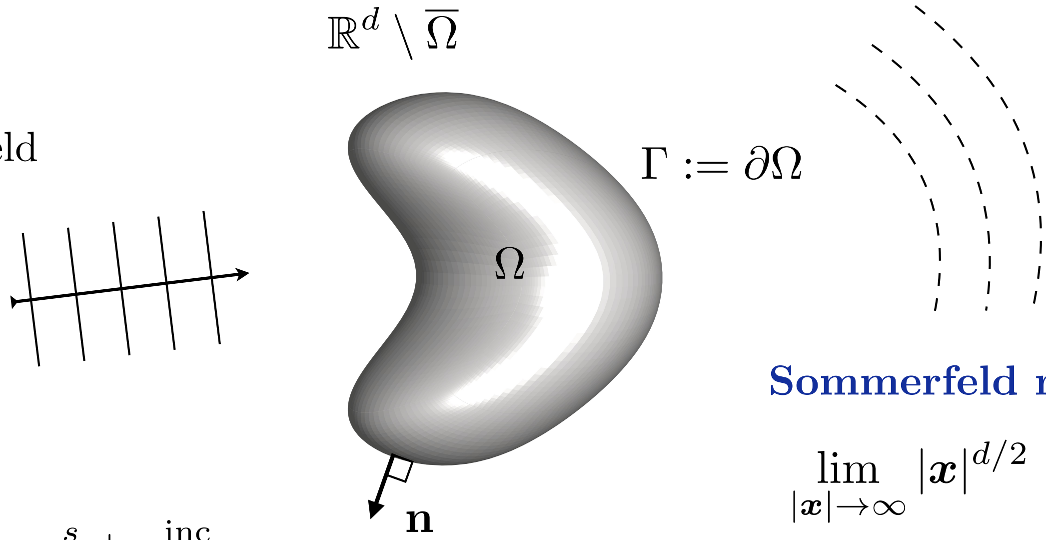

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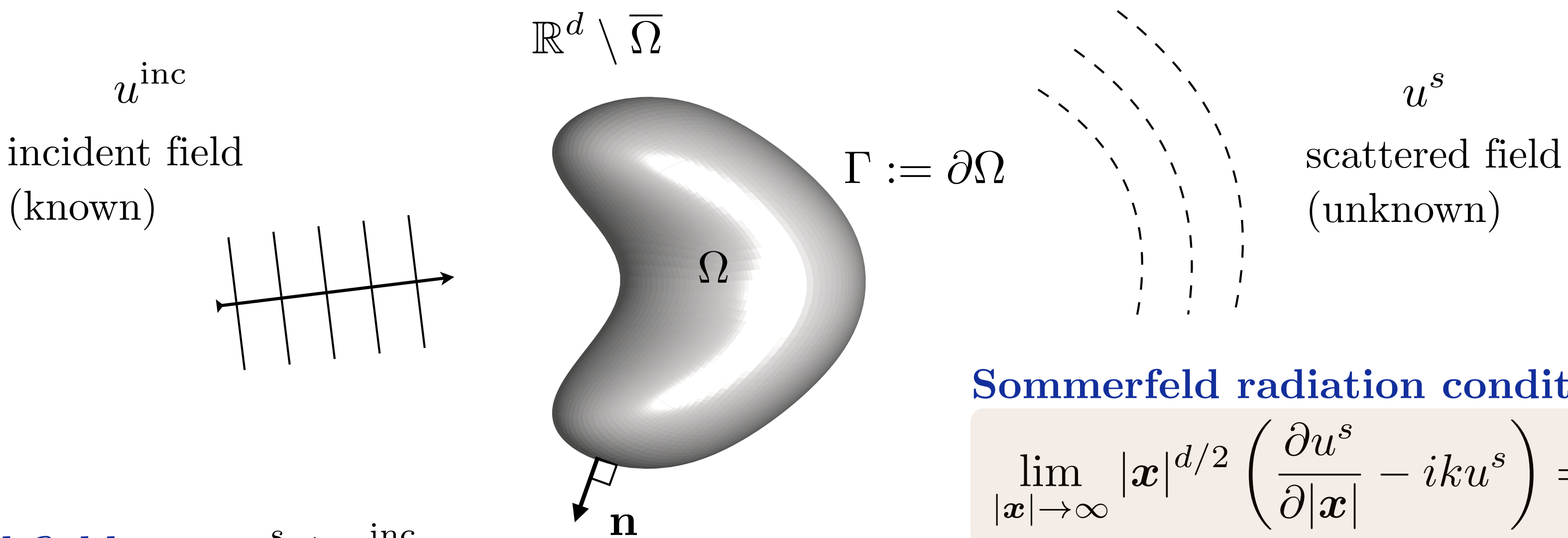

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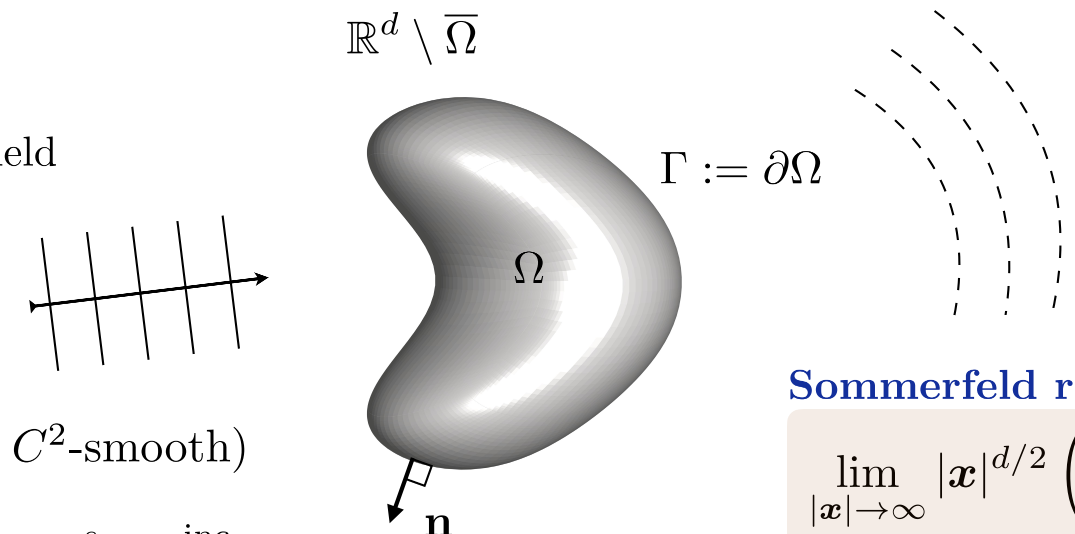

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(Γ is assumed C^2-smooth)

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Fundamental Solution of the Helmholtz Equation

Fundamental Solution

Formally, the fundamental solution satisfies: $\Delta E + k^2 E = -\delta_0 \quad \text{in} \quad \mathbb{R}^d$

(Dirac delta: $\delta_0 \in \mathcal{D}'(\mathbb{R}^d),\ \delta_0 v = v(\mathbf{0})\ \forall v \in C_0^\infty(\mathbb{R}^d)$)

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For $d = 3$ $(\mathbb{R}^3)$: We look for radial solutions: $E = E(r), \quad r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dE}{dr}\right) + k^2 E = 0, \quad r > 0$$

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$$\begin{cases} E(r) = c_1 \dfrac{\cos(kr)}{r} + c_2 \dfrac{\sin(kr)}{r} \\ \quad\;\; = a_1 \dfrac{\exp(ikr)}{r} + a_2 \dfrac{\exp(-ikr)}{r} \end{cases}$$

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We have several options, in particular:

$$E_1(r) = \frac{\cos(kr)}{4\pi r}, \quad E_2(r) = \frac{\exp(ikr)}{4\pi r}, \quad \text{and} \quad E_3(r) = \frac{\exp(-ikr)}{4\pi r}$$

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Which one of them is "physically correct" ?

Free-Space Green's Function

In the "real" world, there is always some damping (energy dissipation):

$$\text{Acoustics:} \quad k = \frac{\omega}{c} + i\beta, \quad \beta > 0$$

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Limiting Absorption Principle: Solve the problem with damping, and define the undamped solution as the limit as the damping parameter β or $\sigma \to 0^+$.

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For $d = 2\left(\mathbb{R}^2\right)$: Bessel's equation: $\dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}E}{\mathrm{d}r}\right) + k^2 E = 0, \quad r = \sqrt{x^2 + y^2} > 0$

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Hankel function of the first kind and order zero

- Since:

$$H_0^{(1)}(r) = \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{i(r-\frac{\pi}{4})}}{\sqrt{r}} + O(r^{-\frac{3}{2}}) \quad \text{and} \quad H_0^{(1)\prime}(r) = -H_1^{(1)}(r) = i\sqrt{\frac{2}{\pi r}}\,\mathrm{e}^{i(r-\frac{\pi}{4})} + O(r^{-\frac{3}{2}}) \quad \text{as } r \to \infty$$

$$\frac{\mathrm{d}}{\mathrm{d}r}H_0^{(1)}(kr) - ikH_0^{(1)}(kr) = O(r^{-\frac{3}{2}}) \quad \text{as } r \to \infty \implies E \text{ satisfies the radiation condition}$$

Free-Space Green's Function

For $d = 2\ (\mathbb{R}^2)$: Bessel's equation: $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}E}{\mathrm{d}r}\right) + k^2 E = 0, \quad r = \sqrt{x^2+y^2} > 0$

A solution is given by: $E(r) = \underline{\frac{i}{4}H_0^{(1)}(kr)} = \frac{i}{4}\left\{J_0(kr) + iY_0(kr)\right\}$

Hankel function of the first kind and order zero

- Since:

$$H_0^{(1)}(r) = \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{i(r-\frac{\pi}{4})}}{\sqrt{r}} + O(r^{-\frac{3}{2}}) \quad \text{and} \quad {H_0^{(1)}}'(r) = -H_1^{(1)}(r) = i\sqrt{\frac{2}{\pi r}}\,\mathrm{e}^{i(r-\frac{\pi}{4})} + O(r^{-\frac{3}{2}}) \quad \text{as } r \to \infty$$

$$\frac{\mathrm{d}}{\mathrm{d}r}H_0^{(1)}(kr) - ikH_0^{(1)}(kr) = O(r^{-\frac{3}{2}}) \quad \text{as } r \to \infty \implies E \text{ satisfies the radiation condition}$$

- Since: $\frac{i}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}\log(kr) + (\text{more regular terms})$ it follows that

$$\left(\Delta + k^2\right)\frac{i}{4}H_0^{(1)}(kr) = -\delta_0$$

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fundamental solution of the Laplace equation

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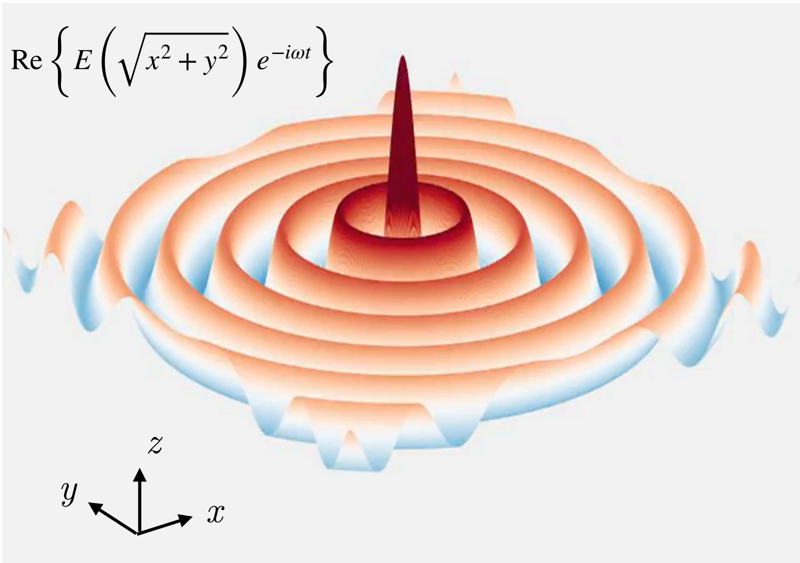
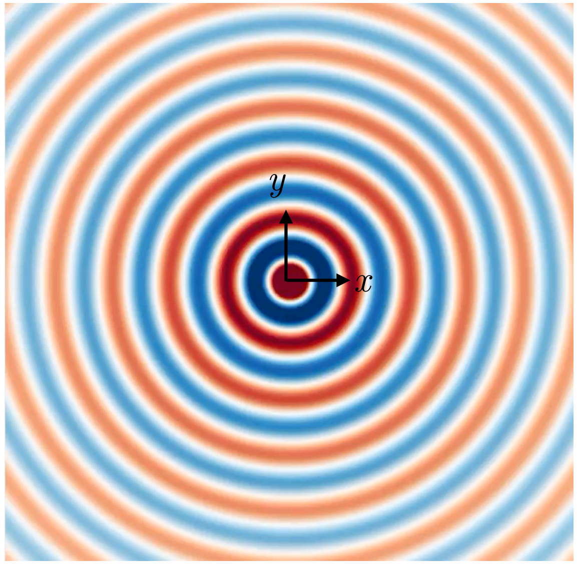
Hankel function of the first kind and order zero





We refer to $G(x, y) := E(|x - y|)$ as the **free-space Green's function**.

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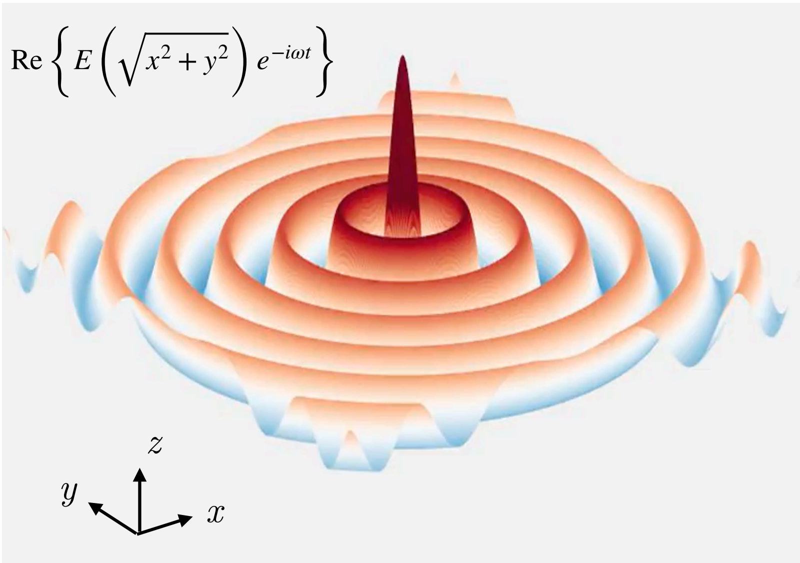

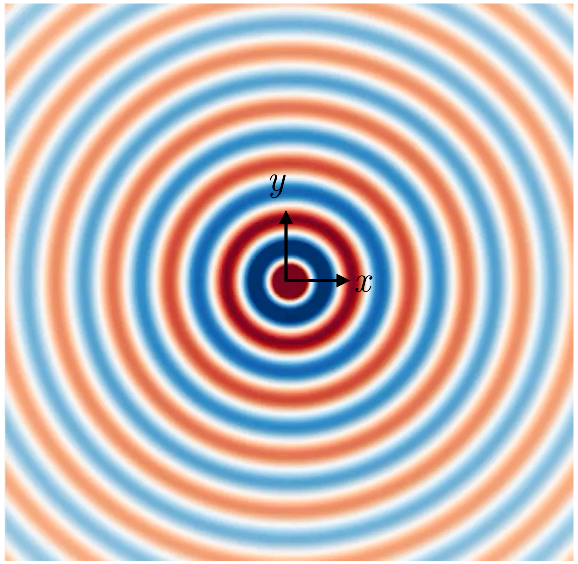

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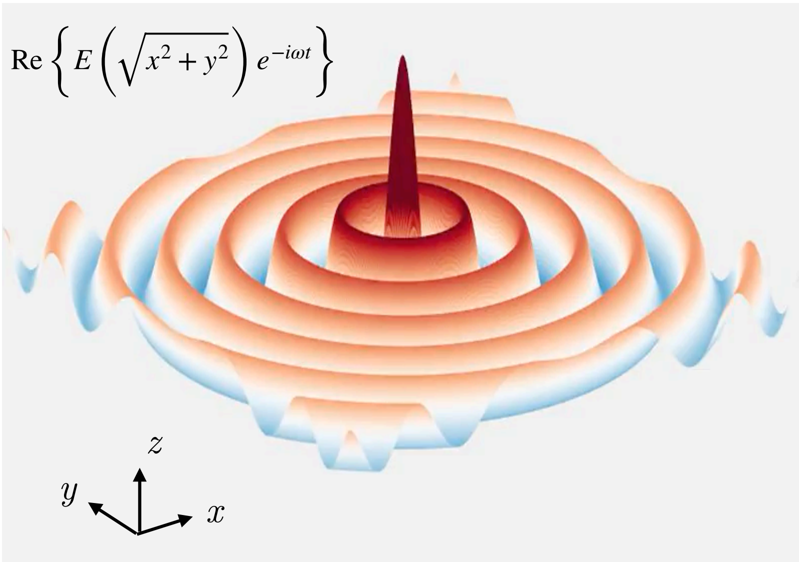

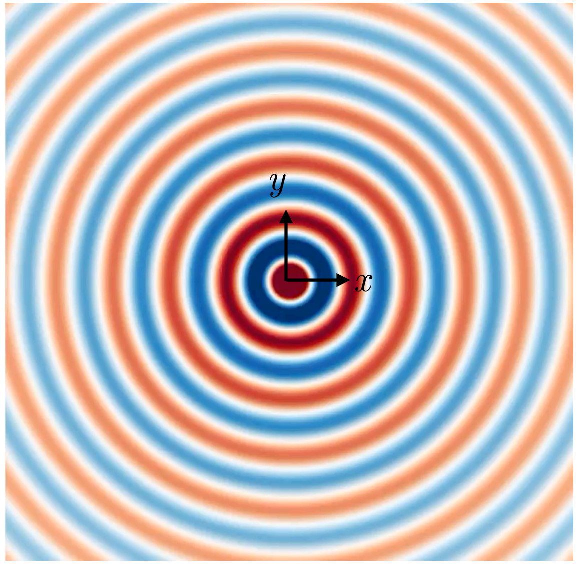

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Green's Representation Formula

Green's Representation Theorem

Theorem. (**exterior Green's Formula**) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary. Suppose that $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ satisfies:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}, \qquad (\mathrm{Im}(k) \geq 0)$$

and the *Sommerfeld radiation condition* at infinity. Then:

- For all $\boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}$,

$$u(\boldsymbol{x}) = \int_\Gamma \left\{ \frac{\partial u}{\partial n}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) - u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} \right\} \mathrm{d}s(\boldsymbol{y});$$

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- for all $\boldsymbol{x} \in \Gamma$,

$$\frac{1}{2} u(\boldsymbol{x}) = \int_\Gamma \left\{ \frac{\partial u}{\partial n}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) - u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} \right\} \mathrm{d}s(\boldsymbol{y});$$

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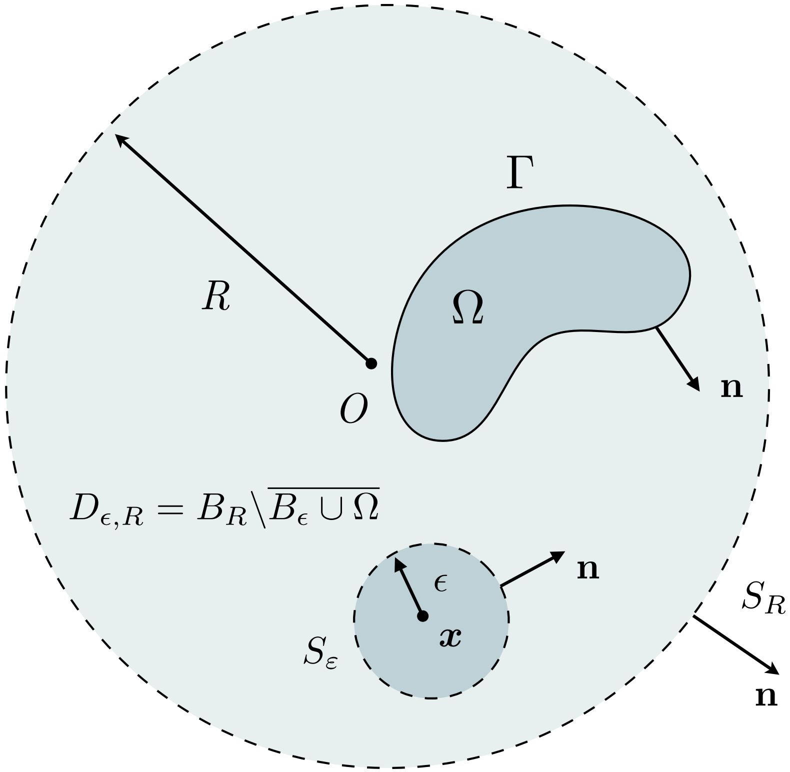
Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)



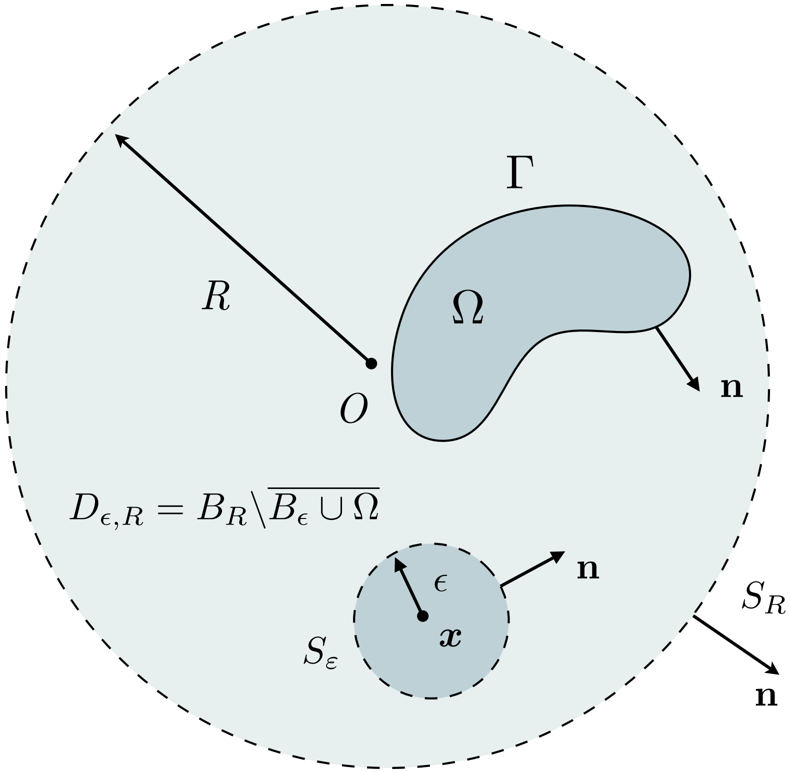
Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

Second Green's identity:

$$-\int_{D_{\epsilon,R}} \{u(y)\Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y})\}\,\mathrm{d}\boldsymbol{y} =$$

$$\int_{\Gamma + S_\epsilon - S_R} \left\{u(\boldsymbol{y})\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - \frac{\partial u(\boldsymbol{y})}{\partial n} G(\boldsymbol{x},\boldsymbol{y})\right\}\,\mathrm{d}s(\boldsymbol{y}) = 0$$



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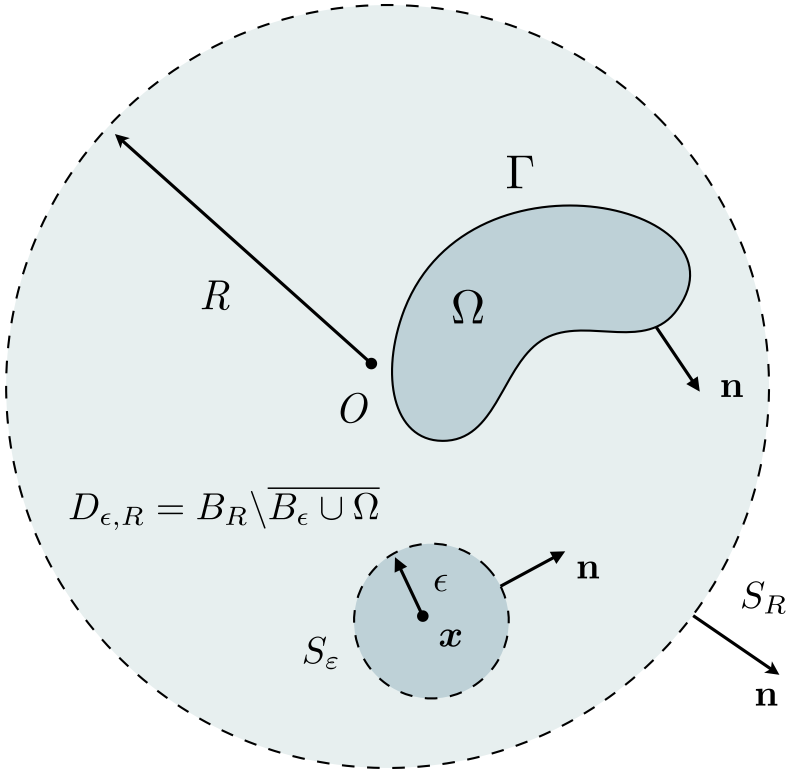
Taking the limit as $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{\partial u(\boldsymbol{y})}{\partial n} G(\boldsymbol{x},\boldsymbol{y})\, \mathrm{d}s(\boldsymbol{y}) = 0$$

$$\lim_{\epsilon \to 0} \int_{S_\epsilon} u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})}\, \mathrm{d}s(\boldsymbol{y}) = -u(\boldsymbol{x})$$



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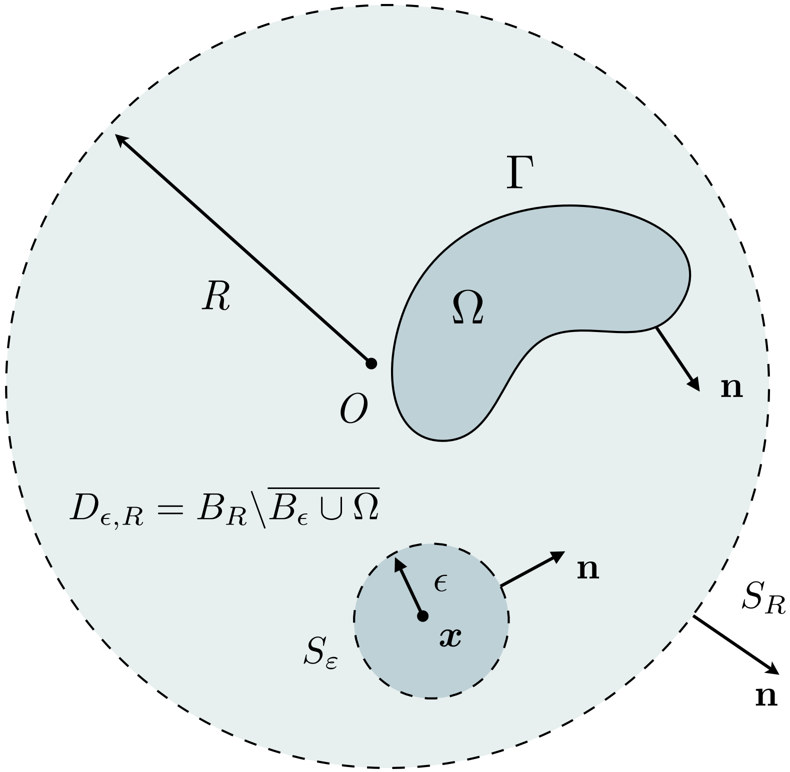

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use that $\dfrac{\mathrm{e}^{ikr}}{4\pi r} = \dfrac{1}{4\pi r} + \dfrac{\cos(kr) - 1}{4\pi r} + \dfrac{i \sin(kr)}{4\pi r}$

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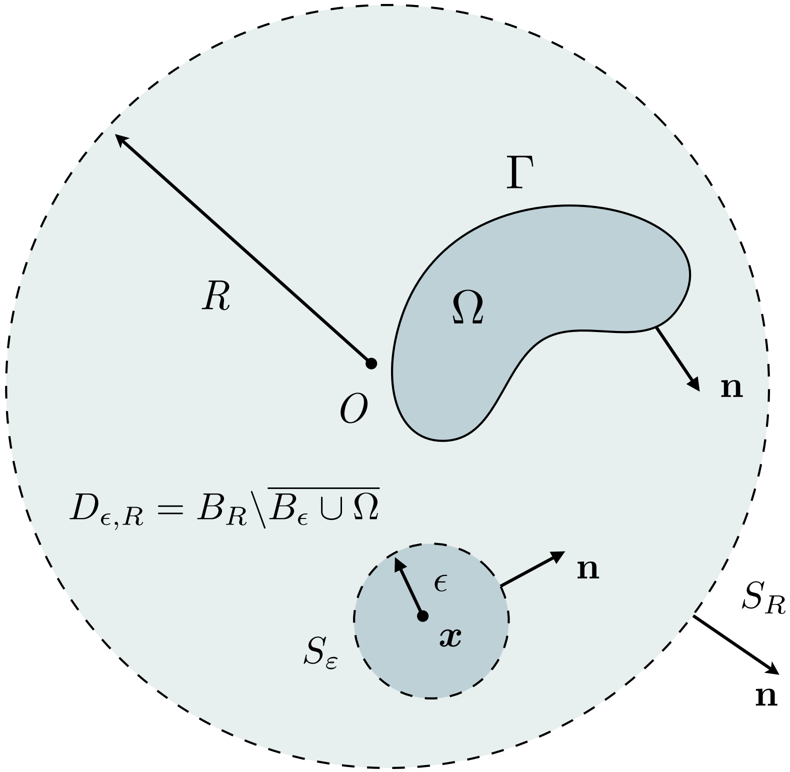

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use that $\quad \dfrac{\mathrm{d}}{\mathrm{d}r}\dfrac{\mathrm{e}^{ikr}}{4\pi r} = -\dfrac{1}{4\pi r^2} + \dfrac{-k\sin(kr)r + (\cos(kr)-1)}{4\pi r^2} + \dfrac{\mathrm{d}}{\mathrm{d}r}\dfrac{i\sin(kr)}{4\pi r}$

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

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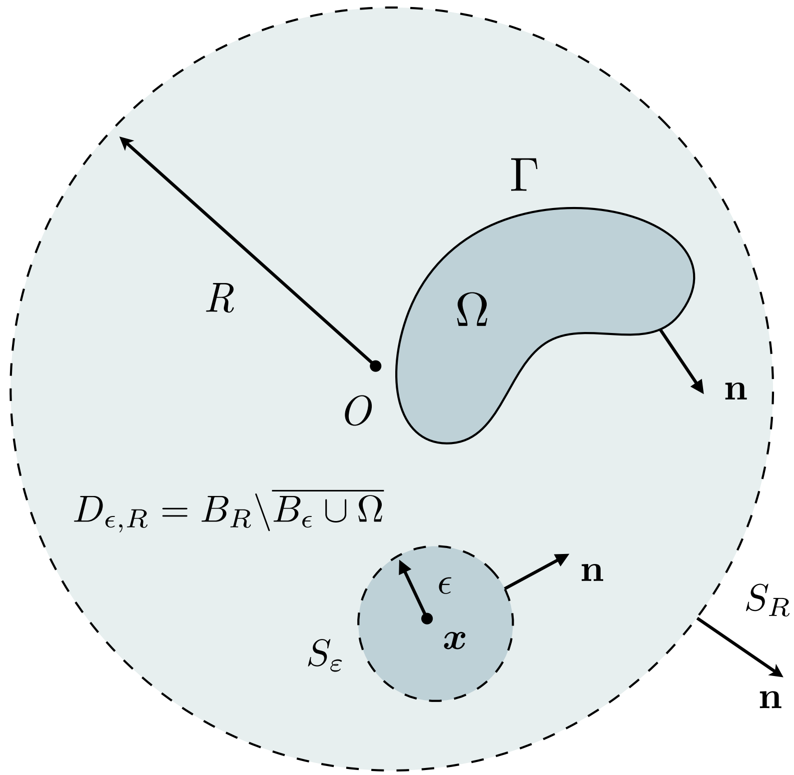

Green's Representation Theorem

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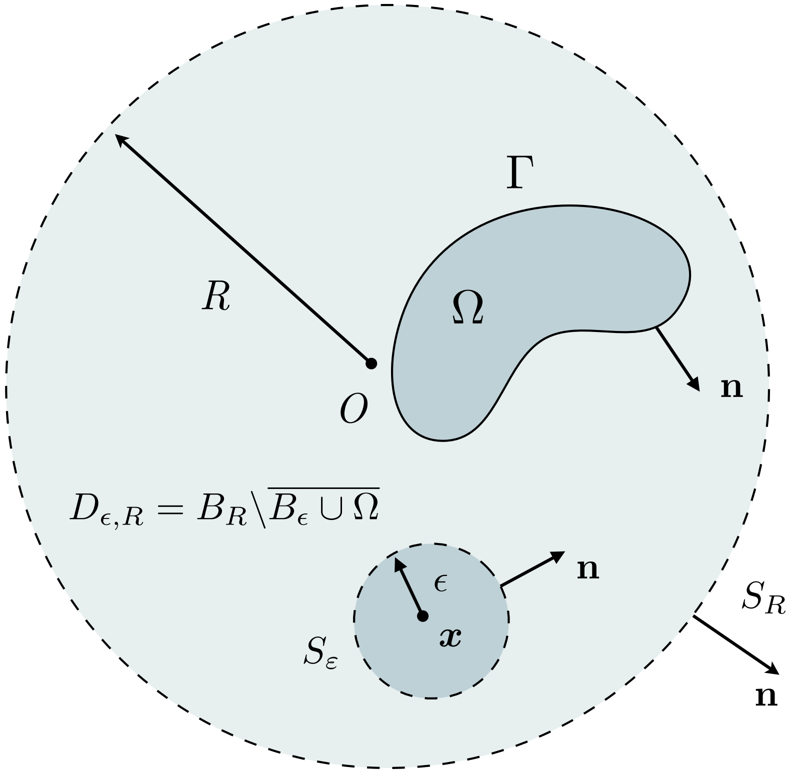

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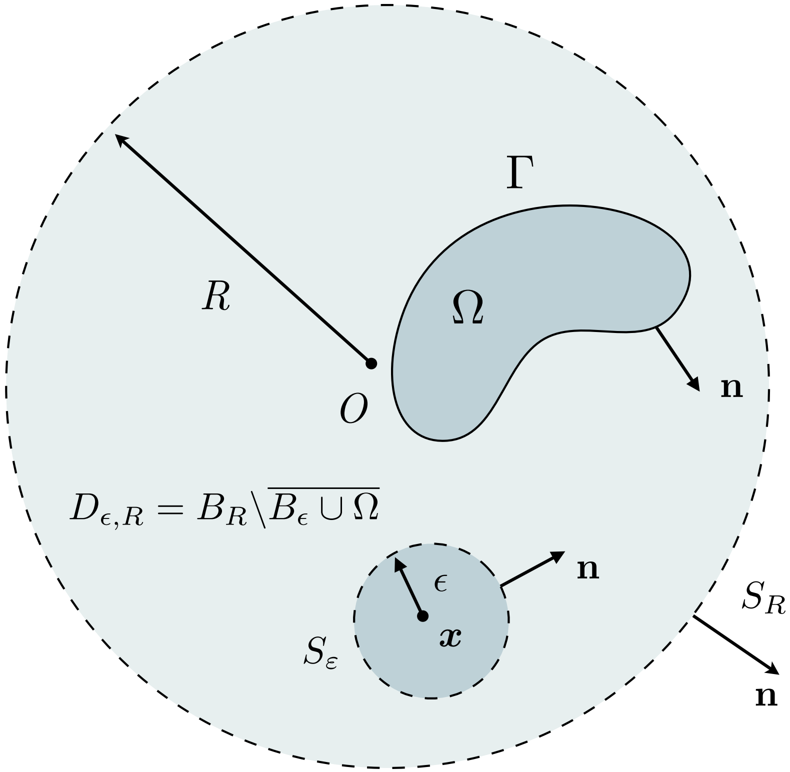

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$$\lim_{R\to\infty} \int_{S_R} \left\{ u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - \frac{\partial u(\boldsymbol{y})}{\partial n} G(\boldsymbol{x},\boldsymbol{y}) \right\} \mathrm{d}s(\boldsymbol{y}) =$$

$$\lim_{R\to\infty} \int_{S_R} \left\{ u(\boldsymbol{y}) \left(\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - ikG(\boldsymbol{x},\boldsymbol{y}) \right) - \left(\frac{\partial u(\boldsymbol{y})}{\partial n} - iku(\boldsymbol{y}) \right) G(\boldsymbol{x},\boldsymbol{y}) \right\} \mathrm{d}s(\boldsymbol{y}) = \lim_{R\to\infty} (I_1 + I_2)$$

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

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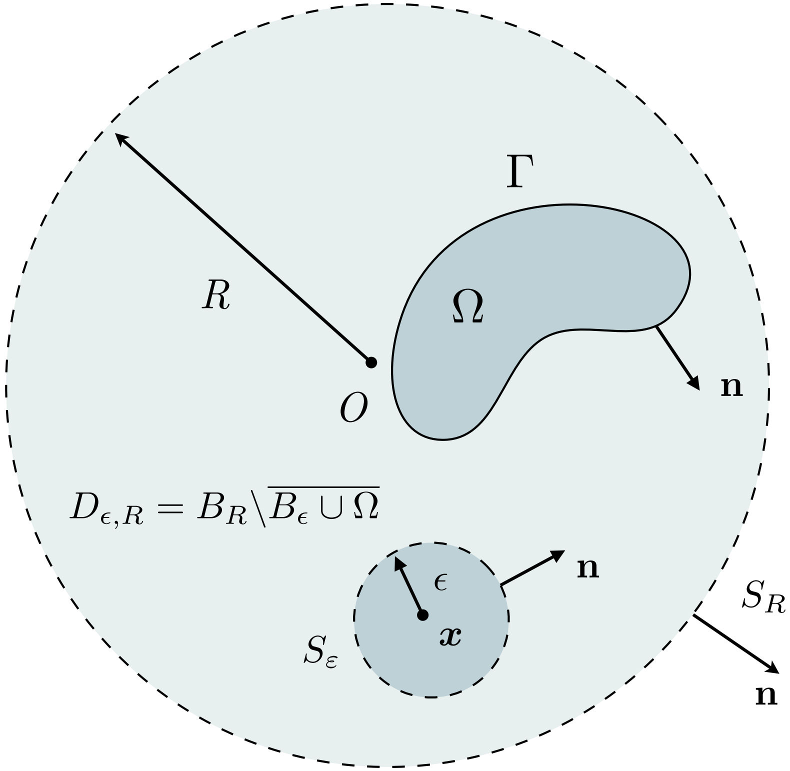

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use the radiation condition

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

... taking the limit as $R \to \infty$:

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

... taking the limit as $R \to \infty$:

- $$|I_1| = \left| \int_{S_R} u(\boldsymbol{y}) \left(\frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} - ikG(\boldsymbol{x}, \boldsymbol{y}) \right) \mathrm{d}s(\boldsymbol{y}) \right| \leq \left(\int_{S_R} |u|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{S_R} \left| \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} - ikG(\boldsymbol{x}, \boldsymbol{y}) \right|^2 \mathrm{d}s(\boldsymbol{y}) \right)^{1/2}$$

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

... taking the limit as $R \to \infty$:

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Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

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to be shown

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

... taking the limit as $R \to \infty$:

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$$\Longrightarrow \ I_1 = O\left(R^{-1}\right) \to 0 \quad \text{as } R \to \infty$$

to be shown

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$)

... taking the limit as $R \to \infty$:

- $$|I_1| = \left| \int_{S_R} u(\boldsymbol{y}) \underbrace{\left(\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - ikG(\boldsymbol{x},\boldsymbol{y}) \right)}_{O(R^{-2})} \underbrace{\mathrm{d}s(\boldsymbol{y})}_{O(R^2)} \right| \leq \underbrace{\left(\int_{S_R} |u|^2 \, \mathrm{d}s \right)^{1/2}}_{O(1)} \left(\int_{S_R} \underbrace{\left| \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - ikG(\boldsymbol{x},\boldsymbol{y}) \right|^2}_{O(R^{-4})} \underbrace{\mathrm{d}s(\boldsymbol{y})}_{O(R^2)} \right)^{1/2}$$

  $$\Longrightarrow \; I_1 = O\left(R^{-1}\right) \to 0 \quad \text{as } R \to \infty$$

  to be shown

- $$I_2 = \int_{S_R} \left(\frac{\partial u(\boldsymbol{y})}{\partial n} - iku(\boldsymbol{y}) \right) G(\boldsymbol{x},\boldsymbol{y}) \, \mathrm{d}s(\boldsymbol{y})$$

Green's Representation Theorem

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  $$\implies I_1 = O\left(R^{-1}\right) \to 0 \quad \text{as } R \to \infty$$

- $$I_2 = \int_{S_R} \underbrace{\left(\frac{\partial u(\boldsymbol{y})}{\partial n} - iku(\boldsymbol{y}) \right)}_{o\left(R^{-1}\right)} G(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}s(\boldsymbol{y})$$

  Sommerfeld R.C.: $o\left(R^{-1}\right)$

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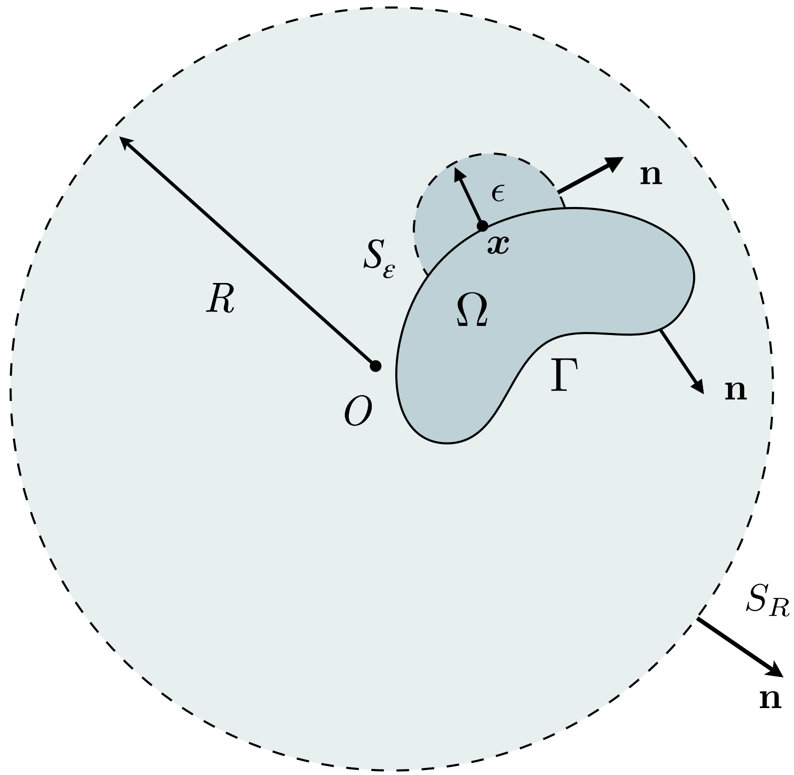
We then conclude that:

$$u(\boldsymbol{x}) = \int_\Gamma \left\{ u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - \frac{\partial u(\boldsymbol{y})}{\partial n} G(\boldsymbol{x},\boldsymbol{y}) \right\} \mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$$

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \Gamma$)



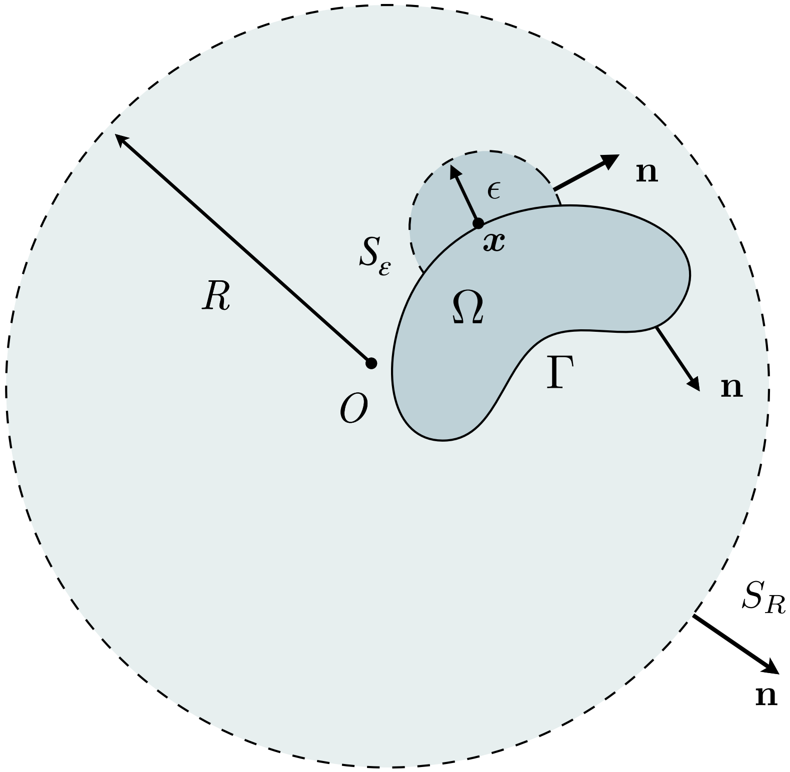
Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ (case $\boldsymbol{x} \in \Gamma$)

Taking the limit as $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{\partial u(\boldsymbol{y})}{\partial n} G(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}s(\boldsymbol{y}) = 0$$

$$\lim_{\epsilon \to 0} \int_{S_\epsilon} u(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} \, \mathrm{d}s(\boldsymbol{y}) = -\frac{u(\boldsymbol{x})}{2}$$



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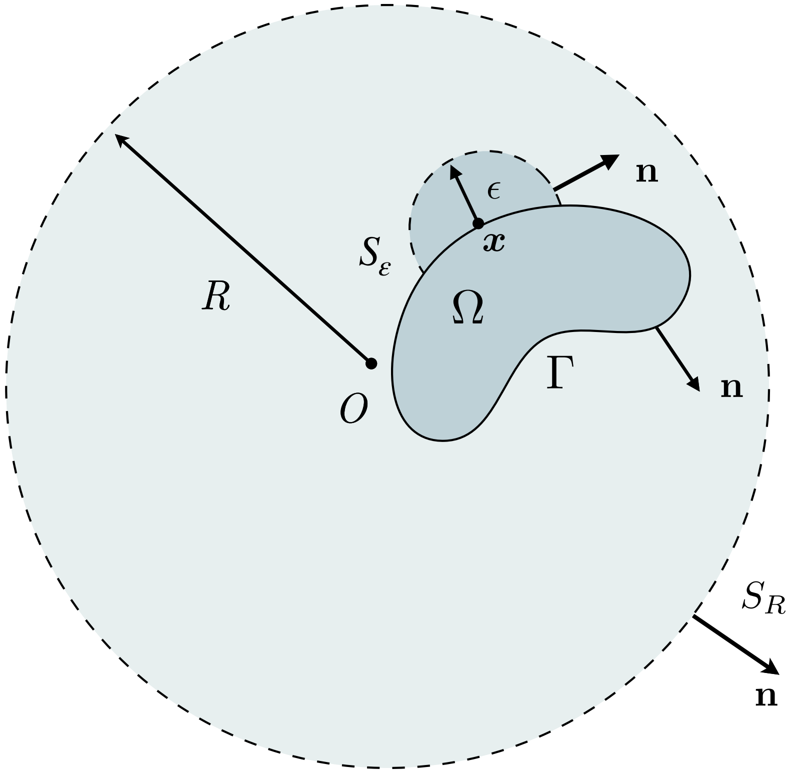
... integration over half a sphere
because Γ is C^2-smooth



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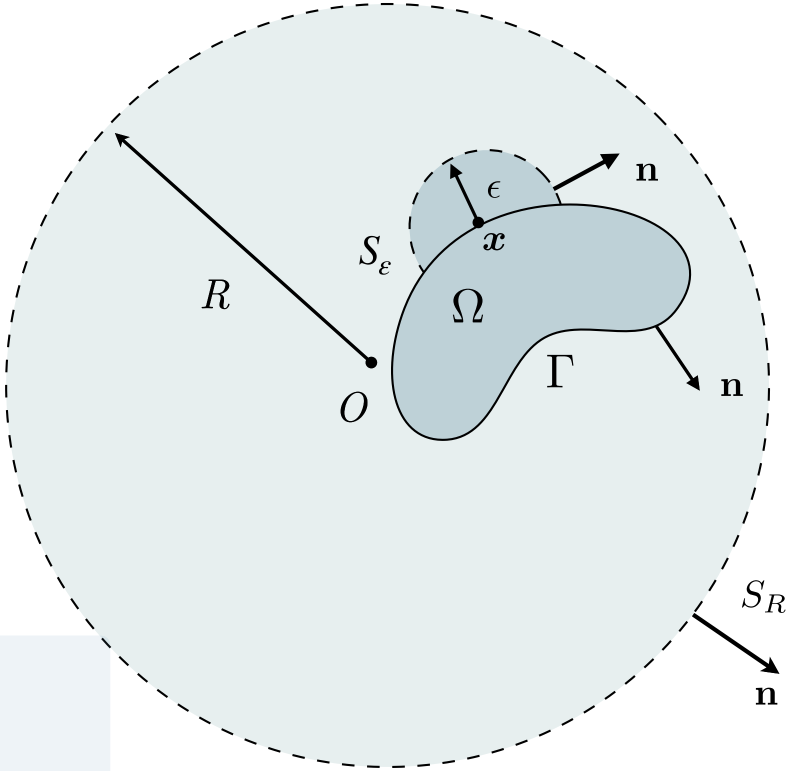

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Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$

To show: $$\left(\int_{S_R} |u|^2 \, \mathrm{d}s\right)^{1/2} = O(1) \quad \text{as } R \to \infty.$$

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To show: $$\left(\int_{S_R} |u|^2 \,\mathrm{d}s\right)^{1/2} = O(1) \quad \text{as } R \to \infty.$$

From Sommerfeld's radiation condition
$$0 = \lim_{R\to\infty} \int_{S_R} \left|\frac{\partial u}{\partial n} - iku\right|^2 \mathrm{d}s$$
$$= \lim_{R\to\infty} \int_{S_R} \left\{ \left|\frac{\partial u}{\partial n}\right|^2 + |k|^2|u|^2 + 2\,\mathrm{Im}\left(ku\frac{\partial \overline{u}}{\partial n}\right)\right\} \mathrm{d}s$$

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From Green's first identity:

$$0 = -\int_{D_{0,R}} u\overline{(\Delta u + k^2 u)}\,\mathrm{d}\boldsymbol{y} = -\int_{S_R} u\frac{\partial \bar{u}}{\partial n}\,\mathrm{d}s + \int_{\Gamma} u\frac{\partial \bar{u}}{\partial n}\,\mathrm{d}s - \bar{k}^2 \int_{D_{0,R}} |u|^2\,\mathrm{d}\boldsymbol{y} + \int_{D_{0,R}} |\nabla u|^2\,\mathrm{d}\boldsymbol{y}$$

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$$\Longrightarrow \lim_{R\to\infty}\left\{\int_{S_R}\left\{\left|\frac{\partial u}{\partial n}\right|^2 + |k|^2|u|^2\right\}\mathrm{d}s + 2\,\mathrm{Im}(k)\int_{D_{0,R}}\left\{|k|^2|u|^2 + |\nabla u|^2\right\}\mathrm{d}\boldsymbol{y}\right\} = -2\,\mathrm{Im}\left(k\int_\Gamma u\frac{\partial \bar{u}}{\partial n}\,\mathrm{d}s\right)$$

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$$\implies \lim_{R\to\infty}\left\{\int_{S_R}\left\{\underbrace{\left|\frac{\partial u}{\partial n}\right|^2}_{\geq 0} + \underbrace{|k|^2|u|^2}_{\geq 0}\right\}\mathrm{d}s + 2\,\underbrace{\mathrm{Im}(k)}_{\geq 0}\int_{D_{0,R}}\left\{\underbrace{|k|^2|u|^2}_{\geq 0} + \underbrace{|\nabla u|^2}_{\geq 0}\right\}\mathrm{d}\boldsymbol{y}\right\} = \underbrace{-2\,\mathrm{Im}\left(k\int_{\Gamma} u\frac{\partial \bar{u}}{\partial n}\,\mathrm{d}s\right)}_{O(1)}$$

Green's Representation Theorem

Proof sketch in $\mathbb{R}^3$ To show: $\displaystyle\left(\int_{S_R} |u|^2 \,\mathrm{d}s\right)^{1/2} = O(1) \quad \text{as } R \to \infty.$

From Sommerfeld's radiation condition
$$0 = \lim_{R\to\infty} \int_{S_R} \left|\frac{\partial u}{\partial n} - iku\right|^2 \mathrm{d}s$$
$$= \lim_{R\to\infty} \int_{S_R} \left\{\left|\frac{\partial u}{\partial n}\right|^2 + |k|^2|u|^2 + 2\,\underline{\mathrm{Im}\left(ku\frac{\partial \bar u}{\partial n}\right)}\right\} \mathrm{d}s$$

From Green's first identity:
$$0 = -\int_{D_{0,R}} u\overline{(\Delta u + k^2 u)}\,\mathrm{d}\boldsymbol{y} = \boxed{-\int_{S_R} u\frac{\partial \bar u}{\partial n}\,\mathrm{d}s} + \int_\Gamma u\frac{\partial \bar u}{\partial n}\,\mathrm{d}s - \bar k^2 \int_{D_{0,R}} |u|^2\,\mathrm{d}\boldsymbol{y} + \int_{D_{0,R}} |\nabla u|^2\,\mathrm{d}\boldsymbol{y}$$
$$\Longrightarrow k\int_{S_R} u\frac{\partial \bar u}{\partial n}\,\mathrm{d}s = k\int_\Gamma u\frac{\partial \bar u}{\partial n}\,\mathrm{d}s - \bar k|k|^2 \int_{D_{0,R}} |u|^2\,\mathrm{d}\boldsymbol{y} + k\int_{D_{0,R}} |\nabla u|^2\,\mathrm{d}\boldsymbol{y}$$
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Corollary. Any $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ solution to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}$$

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Proof (idea). For each fixed $\boldsymbol{y} \in \Gamma$, the mappings

$$\boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{y}) \quad \text{and} \quad \boldsymbol{x} \mapsto \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})}$$

are analytic in $\mathbb{R}^d \setminus \overline{\Omega}$, meaning they extend holomorphically to a neighborhood of this set in the complexified space and satisfy the Cauchy–Riemann equations with respect to the complexified variables x_1, x_2, x_3.

Then, from Green's representation formula

$$u(\boldsymbol{x}) = \int_\Gamma \left\{ \frac{\partial u}{\partial n}(\boldsymbol{y})\, G(\boldsymbol{x}, \boldsymbol{y}) - u(\boldsymbol{y})\, \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} \right\} \mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega},$$

we get that u inherits analyticity in $\boldsymbol{x}$ from the integrand. This can be justified by differentiating under the integral sign and verifying that u satisfies the Cauchy–Riemann equations.

Uniqueness Theorem

(via Rellich lemma)

Uniqueness Theorem

Lemma (Rellich). Let $k > 0$ and let $u \in C\left(\mathbb{R}^d \setminus \overline{\Omega}\right)$ be a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition and

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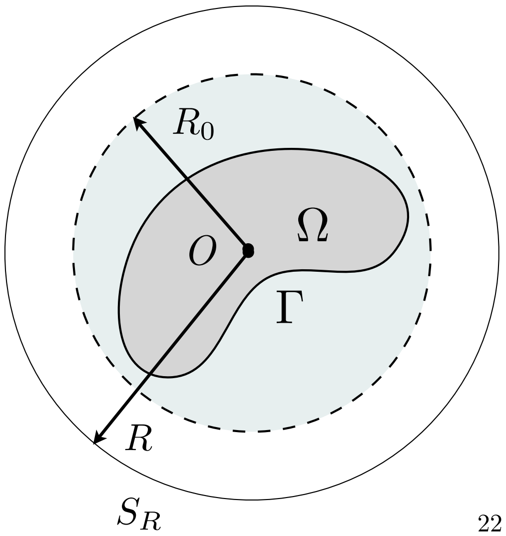
Proof (sketch in $\mathbb{R}^2$).

The Helmholtz equation in polar coordinates takes the form

$$\frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0, \quad R > R_0, \ \theta \in (0, \pi].$$



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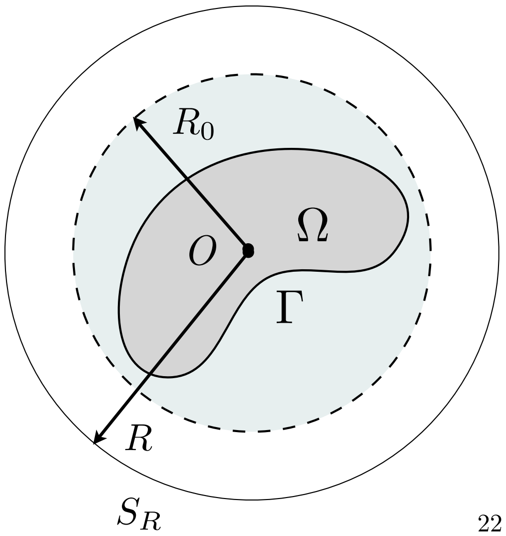
By separation of variables, for $R_0 > 0$ sufficiently large,

$$u(R,\theta) = \sum_{n\in\mathbb{Z}} \left[a_n H_n^{(1)}(kR) + b_n H_n^{(2)}(kR)\right] \mathrm{e}^{in\theta}, \quad R > R_0, \theta \in (0, 2\pi]$$



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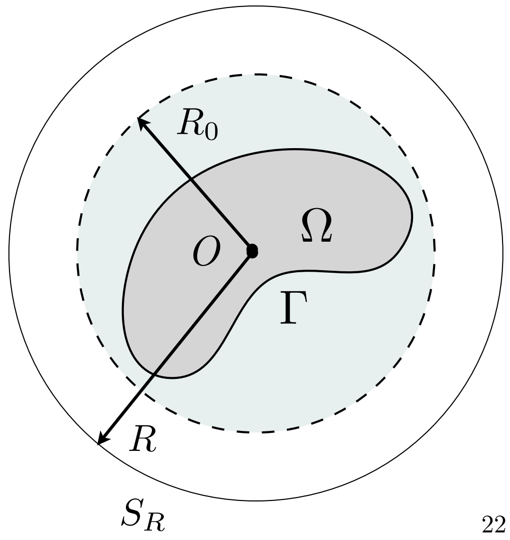
Sommerfeld R.C.



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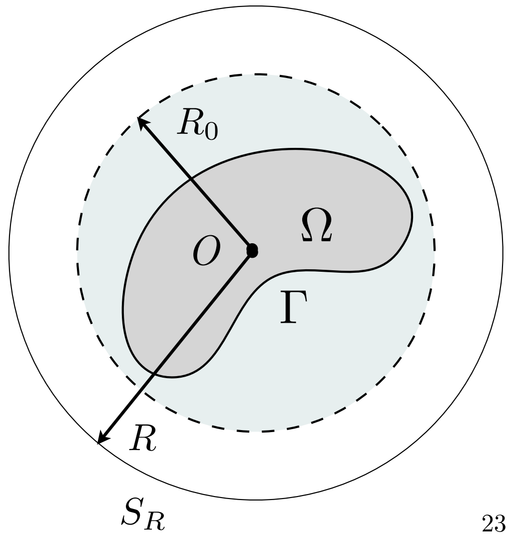

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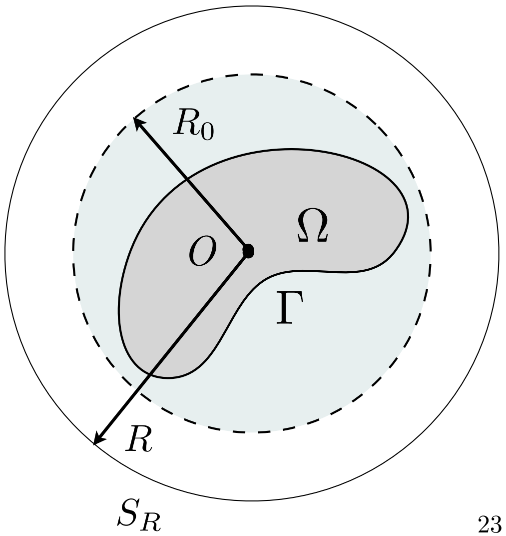

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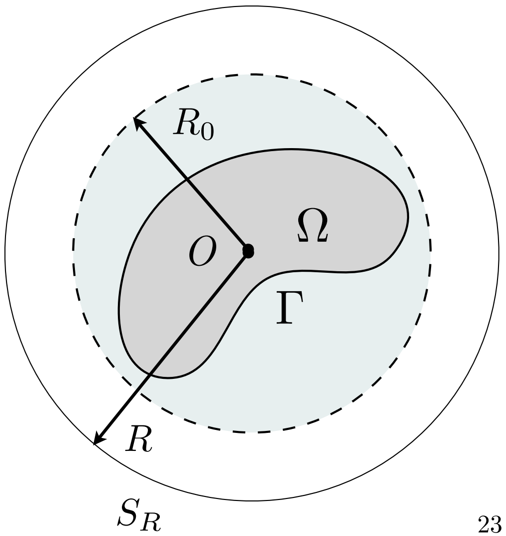

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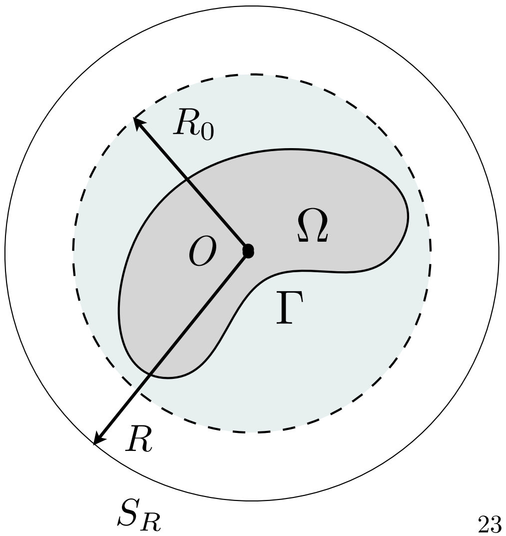

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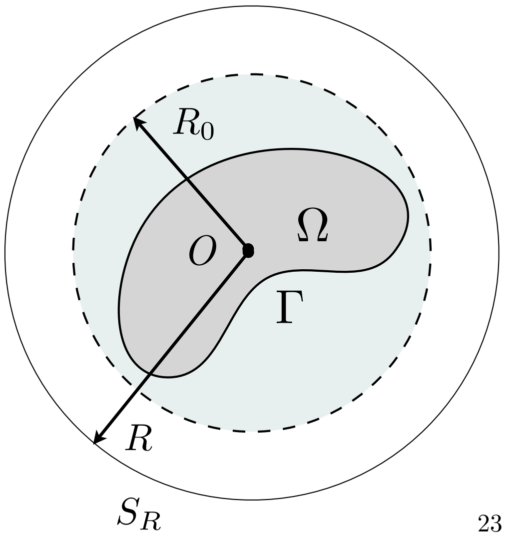
Since u is analytic in $\mathbb{R}^2 \setminus \overline{\Omega}$ and vanishes on the open set $\mathbb{R}^2 \setminus \overline{B_{R_0}(0)}$, the **Unique Continuation Principle** implies that $u \equiv 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$.



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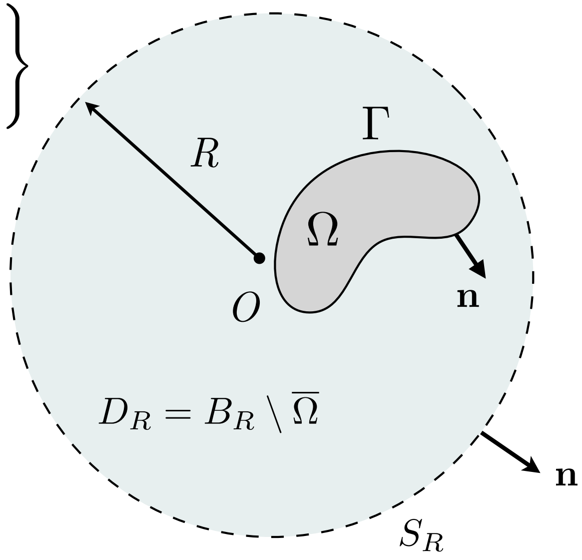

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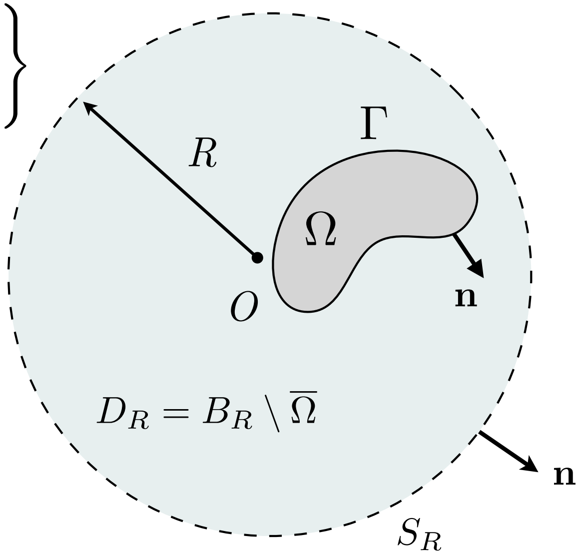

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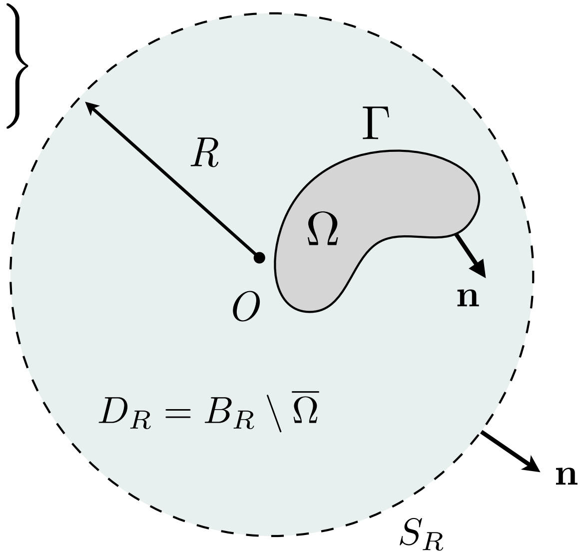

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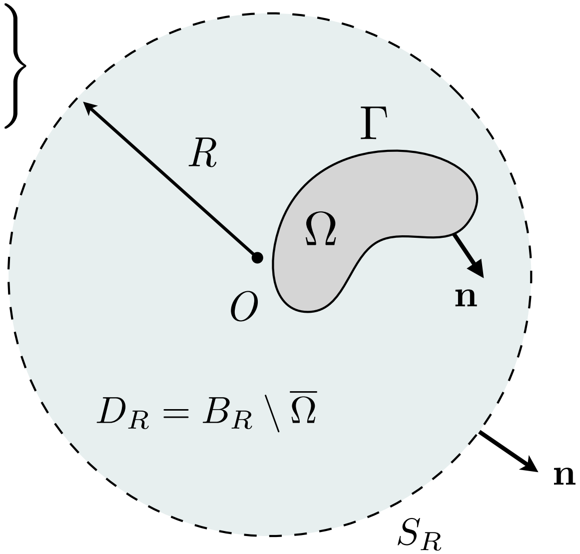

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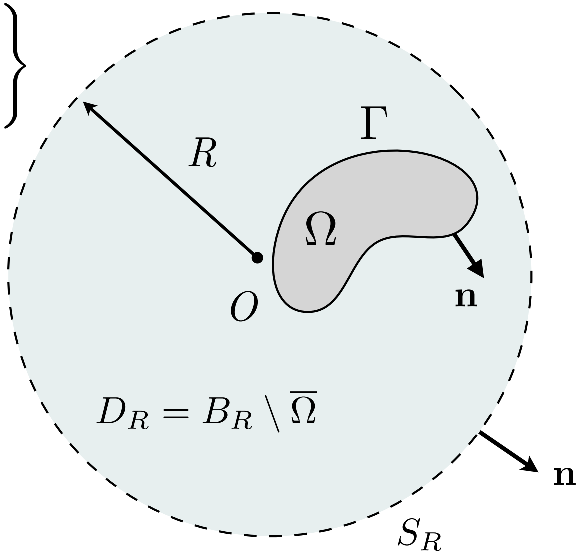

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Existence Theorem

(via Boundary Integral Equations)

Key Facts About Integral Operators

Definition. Let X and Y be Banach spaces. A linear operator $A : X \to Y$ is said to be **compact** if, for every bounded sequence $\{\varphi_n\} \subset X$, the sequence $\{A\varphi_n\} \subset Y$ contains a convergent subsequence.

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Definition. Let A be the integral operator defined by

$$(A\varphi)(\boldsymbol{x}) = \int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma,$$

for $\varphi \in C(\Gamma)$, where $\Gamma = \partial\Omega \subset \mathbb{R}^d$. The operator A is said to be **weakly singular** if its kernel Φ is continuous for all $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$ with $\boldsymbol{x} \neq \boldsymbol{y}$, and there exist constants $\alpha \in (0, 2]$ and $M > 0$ such that

$$|\Phi(\boldsymbol{x}, \boldsymbol{y})| \leq M|\boldsymbol{x} - \boldsymbol{y}|^{\alpha-d+1}.$$

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Proof (idea).

(i) If A is bounded and of finite rank, then A is compact:

Assume $\|\varphi_n\| \leq C$ for all $n \in \mathbb{N}$. Then, $\|A\varphi_n\| \leq C\|A\|$. Therefore, $\{A\varphi_n\}$ is a bounded sequence in the finite-dimensional space $A(C(\Gamma))$. By the Bolzano–Weierstrass theorem, it contains a convergent subsequence.

(ii) If Φ is continuous, then A can be approximated uniformly by a sequence of finite-rank operators. Hence, A is compact.

(iii) If Φ is weakly singular, then A can be approximated uniformly by a sequence of operators with continuous kernels. Hence, A is compact.

Key Facts About Integral Operators

Theorem. An integral operator A with a weakly singular kernel Φ is **compact** on $C(\Gamma)$.

Proof (idea).

(i) If A is bounded and of finite rank, then A is compact:

Assume $\|\varphi_n\| \le C$ for all $n \in \mathbb{N}$. Then, $\|A\varphi_n\| \le C\|A\|$. Therefore, $\{A\varphi_n\}$ is a bounded sequence in the finite-dimensional space $A(C(\Gamma))$. By the Bolzano–Weierstrass theorem, it contains a convergent subsequence.

(ii) If Φ is continuous, then A can be approximated uniformly by a sequence of finite-rank operators. Hence, A is compact.

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Corollary. An integral operator A with weakly singular kernel Φ is bounded/continuous on $C(\Gamma)$.

Single- and Double-Layer Potentials

Single-layer potential: $\mathcal{S} : C(\Gamma) \to C^2\left(\mathbb{R}^d \setminus \Gamma\right)$

$$(\mathcal{S}\varphi)(\boldsymbol{x}) := \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma.$$

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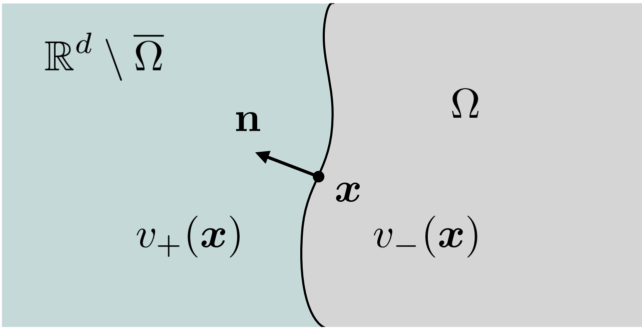
Properties of the Layer Potential

For sufficiently regular functions, we define the exterior and interior Dirichlet traces as

$$v_+(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ \boldsymbol{y}\in\mathbb{R}^d\setminus\overline{\Omega}}} v(\boldsymbol{y}), \qquad v_-(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ \boldsymbol{y}\in\Omega}} v(\boldsymbol{y}), \qquad \boldsymbol{x}\in\partial\Omega.$$



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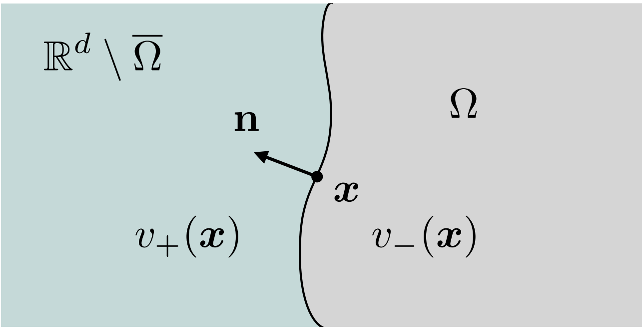

It can be shown that for $\varphi\in C(\Gamma)$, the following hold:

$$(\mathcal{S}\varphi)_\pm = S\varphi, \qquad (\mathcal{D}\varphi)_\pm = \pm\frac{1}{2}\varphi + K\varphi,$$

$$\left(\frac{\partial}{\partial n}\mathcal{S}\varphi\right)_\pm = \mp\frac{1}{2}\varphi + K^\top\varphi \qquad \text{where}$$

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Adjoint double-layer operator: $$\left(K^\top\varphi\right)(\boldsymbol{x}) := \int_\Gamma \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})}\varphi(\boldsymbol{y})\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x}\in\Gamma$$

Weakly Singular Integral Operators

Theorem. Let Γ be a closed C^2 curve or surface. Then, there exists a constant $L > 0$ such that

$$|\mathbf{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|^2 \quad \text{and} \quad |\mathbf{n}(\boldsymbol{x}) - \mathbf{n}(\boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|, \quad \boldsymbol{x}, \boldsymbol{y} \in \Gamma.$$

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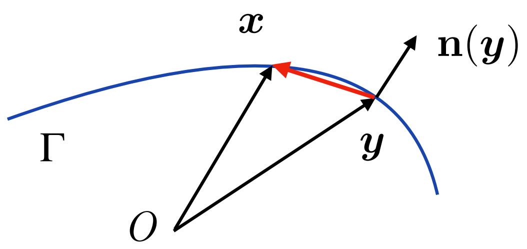

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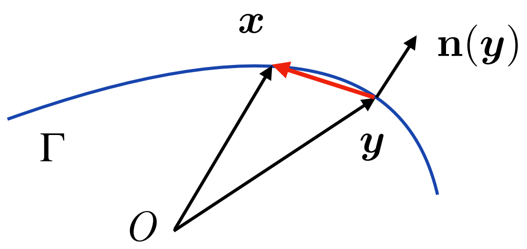

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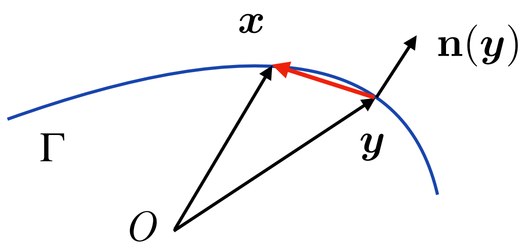

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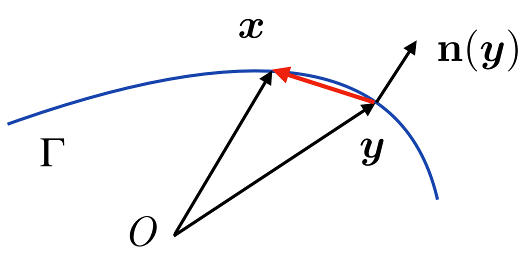

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Proof:

Single-layer operator: For the Helmholtz fundamental solution $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \dfrac{\mathrm{e}^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}$, it holds that $|\Phi(\boldsymbol{x}, \boldsymbol{y})| \leq \dfrac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}$, so Φ is weakly singular with $\alpha = 1$.

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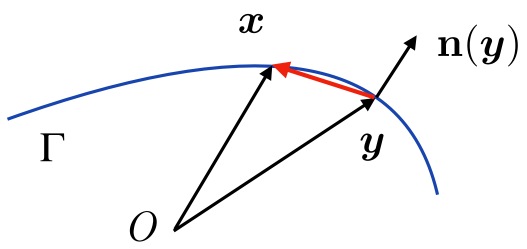

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Double-layer operator: Let $r = |\boldsymbol{x} - \boldsymbol{y}|$. The kernel $\Phi(\boldsymbol{x}, \boldsymbol{y}) = (ikr - 1)\dfrac{\mathrm{e}^{ikr}}{r} \cdot \dfrac{(\boldsymbol{y} - \boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{y})}{4\pi r^2}$ satisfies the estimate $|\Phi(\boldsymbol{x}, \boldsymbol{y})| \leq \dfrac{|k|L}{4\pi} + \dfrac{L}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}$, so Φ is weakly singular with $\alpha = 1$.

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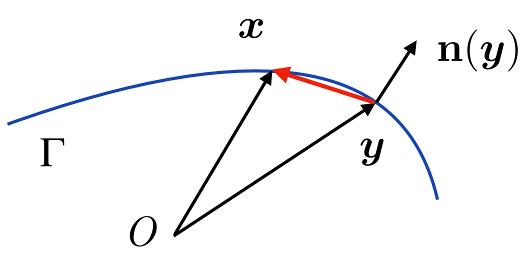

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Adjoint double-layer operator: Let $r = |\boldsymbol{x} - \boldsymbol{y}|$. The kernel $\Phi(\boldsymbol{x}, \boldsymbol{y}) = (1 - ikr)\frac{e^{ikr}}{r} \cdot \frac{(\boldsymbol{y} - \boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x})}{4\pi r^2}$ satisfies the estimate $|\Phi(\boldsymbol{x}, \boldsymbol{y})| \leq \frac{|k|L}{4\pi} + \frac{L}{4\pi r} + \frac{|(\mathbf{n}(\boldsymbol{x}) - \mathbf{n}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y})|}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^2}$, so Φ is weakly singular with $\alpha = 1$.

Hyper-Singular Operator

We have not yet discussed the operator:

$$(T\varphi)(\boldsymbol{x}) := \frac{\partial}{\partial n(\boldsymbol{x})}(\mathcal{D}\varphi)(\boldsymbol{x}) = \frac{\partial}{\partial n(\boldsymbol{x})}\int_\Gamma \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})}\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) = \mathrm{f.\,p.}\int_\Gamma \frac{\partial^2 G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})\,\partial n(\boldsymbol{y})}\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})$$

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The operator T is **hypersingular**: the surface integral that defines it must be interpreted in the sense of the Hadamard finite part. For a sufficiently smooth density φ, one has

$$(T\varphi)(\boldsymbol{x}) = k^2\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\,\mathbf{n}(\boldsymbol{x})\cdot\mathbf{n}(\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) + \mathrm{p.v.}\int_\Gamma \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y})\cdot\nabla_\Gamma\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x}\in\Gamma.$$

It follows that $T : C^{1,\alpha}(\Gamma) \to C(\Gamma)$ is continuous (but not compact). However, $T - T_0 : C(\Gamma) \to C(\Gamma)$ is compact, where T_0 denotes the operator corresponding to $k = 0$.

Hyper-Singular Operator

Cauchy principal value (p.v.). Let φ be Hölder continuous on (a, b). The principal value of

$$\int_a^b \frac{\varphi(t)}{t - x_0}\,\mathrm{d}t, \quad x_0 \in (a, b),$$

is defined by

$$\text{p.v.}\int_a^b \frac{\varphi(t)}{t - x_0}\,\mathrm{d}t := \lim_{\varepsilon \to 0^+}\left(\int_a^{x_0-\varepsilon} \frac{\varphi(t)}{t - x_0}\,\mathrm{d}t + \int_{x_0+\varepsilon}^b \frac{\varphi(t)}{t - x_0}\,\mathrm{d}t\right).$$

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Hadamard finite part (f.p.). Let $\varphi \in C^{1,\alpha}(a, b)$, $\alpha \in (0, 1]$. The finite part of

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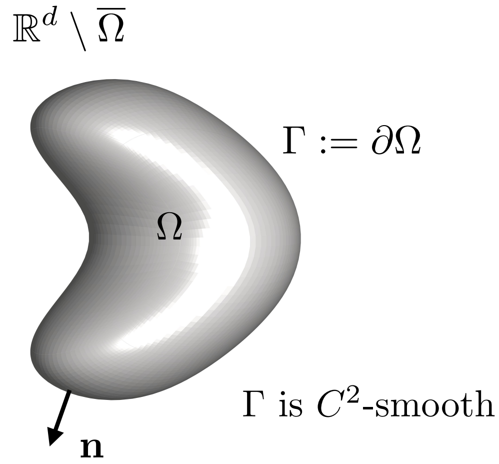
Boundary Integral Equation Formulations

Exterior Helmholtz Problem

Returning to the problem we are studying:

Given $f \in C^{1,\alpha}(\Gamma)$, find $u \in C^2\left(\mathbb{R}^d \setminus \overline{\Omega}\right) \cap C^{1,\alpha}\left(\mathbb{R}^d \setminus \Omega\right)$ such that:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega},$$
$$u = f \quad \text{on} \quad \Gamma,$$
$$\lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^{(d-1)/2} \left\{ \frac{\partial u}{\partial |\boldsymbol{x}|} - iku \right\} = 0.$$



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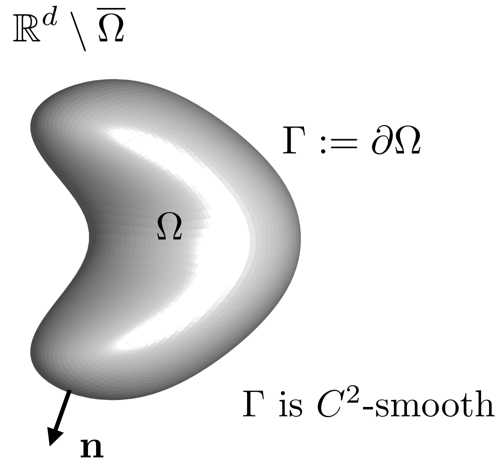

From Green's representation theorem, we know that in order to find u, it suffices to determine the Neumann data $g = \dfrac{\partial u}{\partial n}$. Once g is known, the solution of the problem is given by

$$u(\boldsymbol{x}) = (\mathcal{D}f)(\boldsymbol{x}) - (\mathcal{S}g)(\boldsymbol{x}) = \int_\Gamma \left\{ \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})} f(\boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y}) \right\} \mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}.$$

Direct Boundary Integral Equation Formulations

From Green's representation formula:

$$\frac{f(\boldsymbol{x})}{2} = -\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) + \int_\Gamma \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} f(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})$$

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  - We seek a solution of the form: $u(\boldsymbol{x}) = (\mathcal{S}\varphi)(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma$
  - Imposing the boundary condition, we obtain:

$$S\varphi = f \quad \text{on } \Gamma$$

Issues:
- badly conditioned
- lack of uniqueness

- **Second-kind indirect** boundary integral equation:
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Uniqueness Issues

Let $\lambda > 0$ and $0 \neq v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta v &= \lambda v \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \Gamma. \end{aligned}$$

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Applying Green's representation theorem (interior), we have

$$v(\boldsymbol{x}) = \left(\mathcal{S}\frac{\partial v}{\partial n}\right)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \left(\frac{\partial v}{\partial n} \neq 0\right),$$

and since the single-layer potential is continuous up to the boundary,

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Conclusion: If $k^2 = \lambda$, the single-layer operator S has a nontrivial kernel and, therefore, the equation

$$S\varphi = f$$

admits infinitely many solutions.

Combined-Field Integral Equation (CFIE)

We seek a solution of the form:

$$u(\boldsymbol{x}) = (\mathcal{D} - i\eta\mathcal{S})\varphi(\boldsymbol{x}) = \int_\Gamma \left\{ \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - i\eta G(\boldsymbol{x},\boldsymbol{y}) \right\} \varphi(\boldsymbol{y}) \, \mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma, \quad \eta > 0$$

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Proof: Suppose there exists $0 \neq \varphi \in C(\Gamma)$ such that

$$\frac{\varphi}{2} + K\varphi - i\eta S\varphi = 0 \quad \text{on } \Gamma.$$

Then

$$u(\boldsymbol{x}) = (\mathcal{D} - i\eta\mathcal{S})\varphi(\boldsymbol{x})$$

satisfies $\Delta u + k^2 u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$, $u = 0$ on Γ, and the radiation condition. By the uniqueness theorem for exterior Helmholtz equation, it follows that $u = 0$ in $\mathbb{R}^d \setminus \Omega$.

Combined-Field Integral Equation (CFIE)

Proof (cont.)

On the other hand, $\Delta u + k^2 u = 0$ in Ω, and its interior traces satisfy

$$u_- - u_+ = u_- = -\varphi, \qquad \left(\frac{\partial u}{\partial n}\right)_- - \left(\frac{\partial u}{\partial n}\right)_+ = \left(\frac{\partial u}{\partial n}\right)_- = -i\eta\varphi.$$

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Then $\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} - i\eta u = 0 & \text{on } \Gamma. \end{cases}$

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Multiplying by $\bar{u}$ and integrating by part, we get:

$$-k^2 \int_\Omega |u|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega u\,\Delta\bar{u}\,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\frac{\partial \bar{u}}{\partial n}\,\mathrm{d}s - \int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x} = -i\eta\int_\Gamma |u|^2\,\mathrm{d}s - \int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x}$$

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Taking the imaginary part: $\displaystyle\int_\Gamma |u|^2\,\mathrm{d}s = 0 \implies u = 0 \;\text{ on } \Gamma.$

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Thus, since $\varphi = -u_- = 0$, we conclude that the integral equation admits at most one solution.

Fredholm Alternative

Theorem (Fredholm Alternative). Let $A : X \to X$ be a **compact operator** on a Banach space X, and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then exactly one of the following holds:

1. The homogeneous equation $(I - \lambda A)x = 0$ has only the trivial solution $x = 0$, and the inhomogeneous equation $(I - \lambda A)x = y$ has a unique solution for every $y \in X$.

2. The homogeneous equation $(I - \lambda A)x = 0$ has nontrivial solutions, in which case the inhomogeneous equation $(I - \lambda A)x = y$ is solvable if and only if y is orthogonal to all solutions of the adjoint homogeneous equation $(I - \overline{\lambda} A^*)x^* = 0$.

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Proof (first part, finite-dimensional case). Let X be a finite-dimensional Banach space and $M := I - \lambda A : X \to X$. If $\ker M = \{0\}$, the rank–nullity theorem gives

$$\dim \mathcal{R}(M) = \dim X,$$

so M is surjective. Hence M is bijective, and in finite dimensions any bijective linear map is bounded with a bounded inverse. Therefore M is invertible, and for every $y \in X$ the equation $(I - \lambda A)x = y$ admits the unique solution $x = (I - \lambda A)^{-1}y$.

Existence (and Uniqueness)

Let $A := -K + i\eta S : C(\Gamma) \to C(\Gamma)$ and $\lambda := 1$. Since $\ker(I - \lambda A) = \{0\}$, the **Fredholm alternative** implies that the second-kind boundary integral equation

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has a unique solution $\varphi \in C(\Gamma)$.

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Returning to the problem of interest, we conclude that

$$u(\boldsymbol{x}) = (\mathcal{D} - i\eta\mathcal{S})\varphi(\boldsymbol{x}) = \int_\Gamma \left\{\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} - i\eta G(\boldsymbol{x},\boldsymbol{y})\right\}\varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega},$$

where $\varphi \in C(\Gamma)$ is the solution of the **CFIE**, is the unique solution of the problem of scattering:

$$\begin{aligned} \Delta u + k^2 u = 0 \quad &\text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}, \\ u = f \quad &\text{on} \quad \Gamma, \\ \lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}|^{(d-1)/2} &\left\{\frac{\partial u}{\partial |\boldsymbol{x}|} - iku\right\} = 0. \end{aligned}$$

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