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# Free-surface waves using extended shallow water models part 1

Julian Koellermeier  
University of Groningen and Ghent University

WAVES.NL Summer school, Nijmegen, 25 August 2025

# 1 Overview

# Schedule

Time	Monday	Tuesday	Wednesday	Thursday	Friday
8:50–9:00	<i>Opening</i>				
9:00–10:30	L3	L5	L2	L4	L6
10:30–11:00	<i>Coffee break</i>	<i>Coffee break</i>	<i>Coffee break</i>	<i>Coffee break</i>	<i>Coffee break</i>
11:00–12:30	L1	L1	L2	L4	L6
12:30–13:30	<i>Lunch</i>	<i>Lunch</i>	<i>Lunch</i>	<i>Lunch</i>	<i>Lunch</i>
13:30–15:00	L3	L5	L3	L5	
15:00–15:30	<i>Coffee break</i>	<i>Coffee break</i>			
15:30–17:00	Poster session	L1			
17:45–19:00			<i>Social event</i>		

## L1: Mon 11-12:30

- overview
- motivation
- derivation

## L2: Tue 11-12:30

- analysis

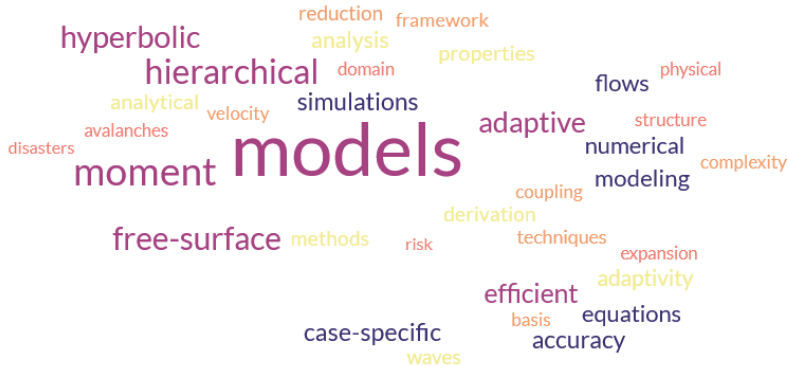
## L3: Tue 15:30-17

- selected papers
- outlook

Slides at: [https://github.com/scalaura/waves\\_summerschool](https://github.com/scalaura/waves_summerschool)

# Free-surface waves using extended shallow water models

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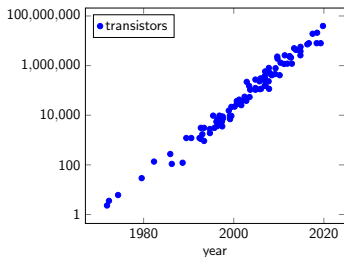


# Content of this talk

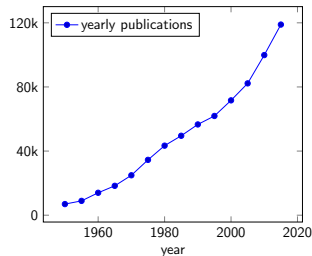
- 1 Overview
- 2 Motivation
- 3 Derivation
- 4 Examples/Exercises

# Progress in mathematical modeling and numerical simulation

Computing power increases (Moore's law)

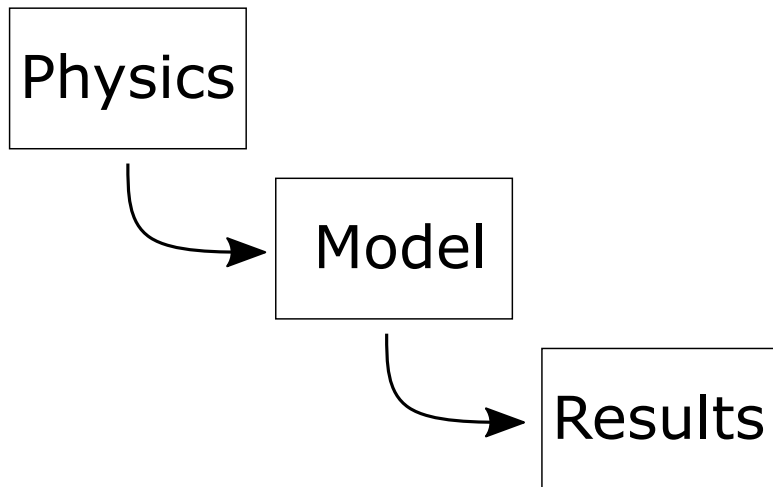


Number of papers increases (mathscinet)

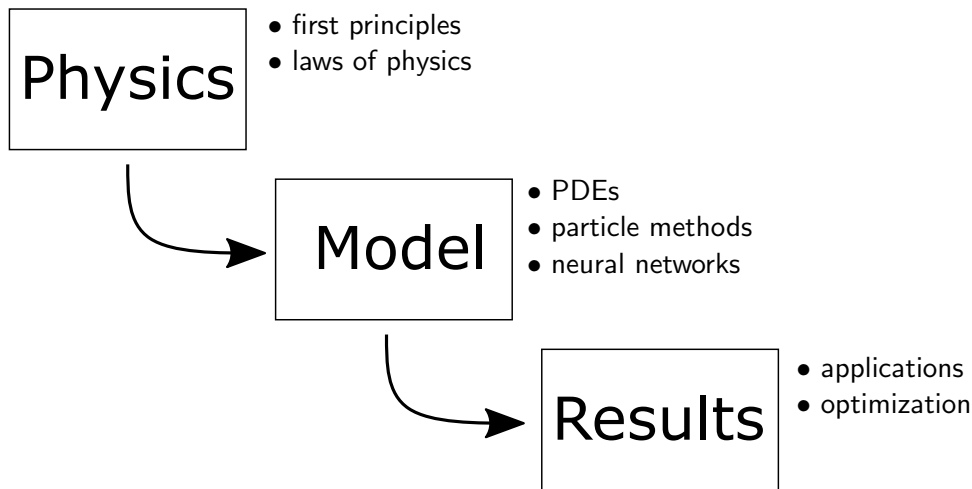


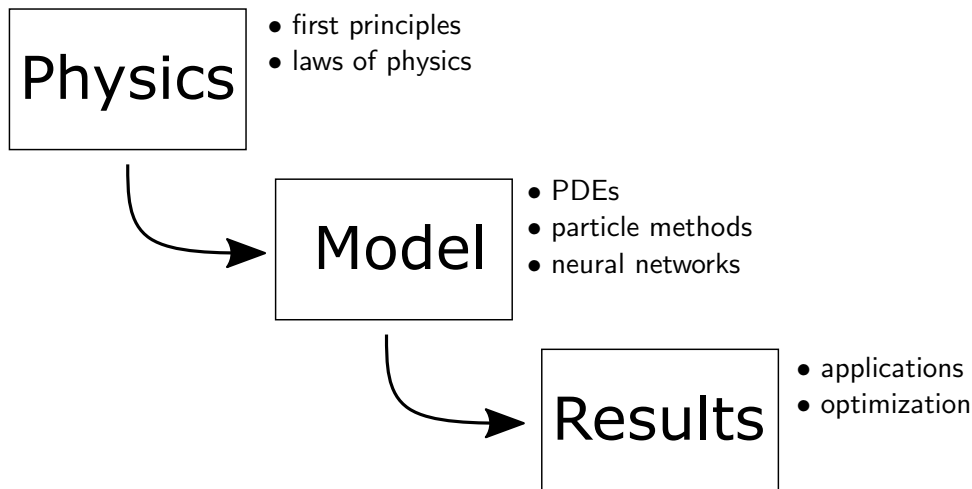
**Status quo:** diversity of complex models  $\Rightarrow$  repeated derivation, analysis, implementation

**Question:** How to use our resources efficiently?

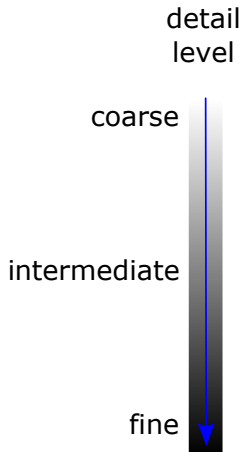








**Question: What are desirable model properties?**



detail  
level

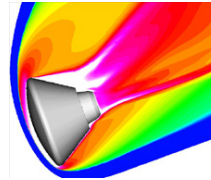
coarse

intermediate

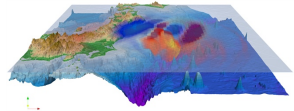
fine

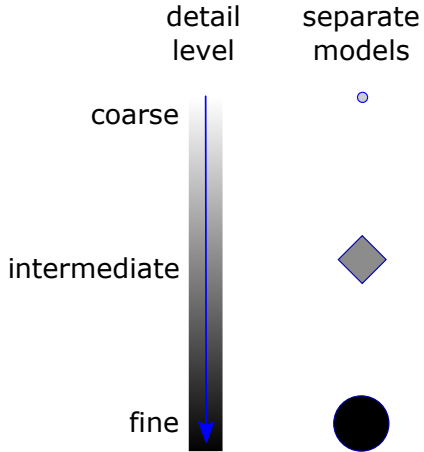


a) rarefied gases

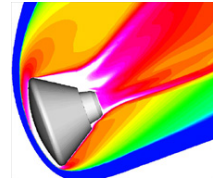


b) free-surface flows

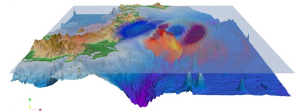


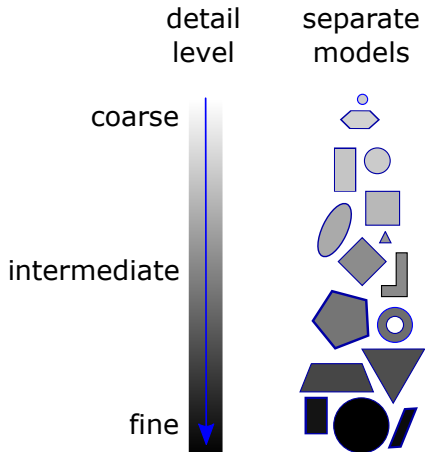


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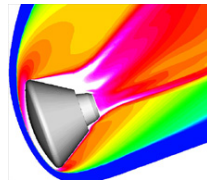


b) free-surface flows

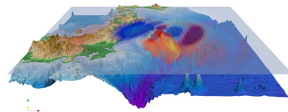


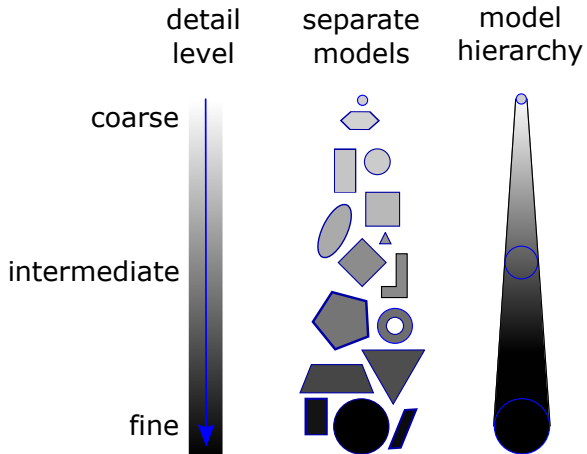


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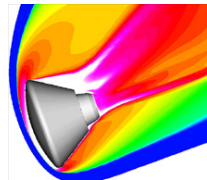


b) free-surface flows

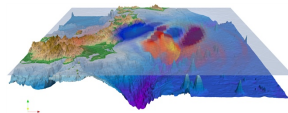


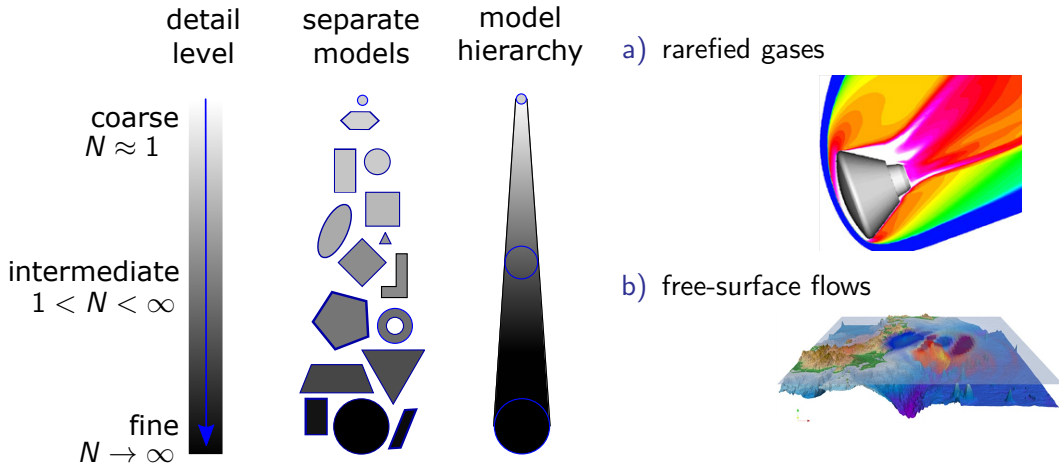


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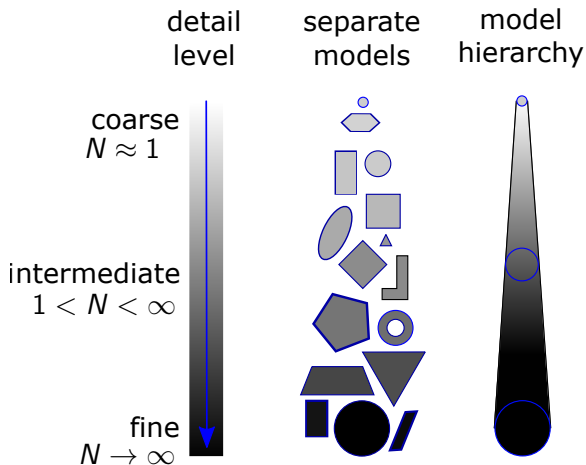
b) free-surface flows







# Hierarchical mathematical modeling



## Hierarchical moment models

### Advantages

1. general derivation
  2. structure preserving
  3. accurate results
- ⇒ adaptive simulations

## Hierarchical Simulation Using Moment Models

### Model derivation

- rarefied gases
- shallow flows

### Model analysis

- hyperbolicity
- further analysis

### Numerics & Applications

- numerical schemes
- numerical results

## Hierarchical Simulation Using Moment Models

### Model derivation

- rarefied gases
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### Model analysis

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- further analysis

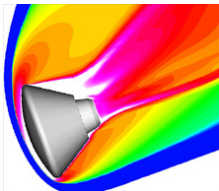
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## 2 Motivation

# Motivation: Rarefied gases and shallow flows

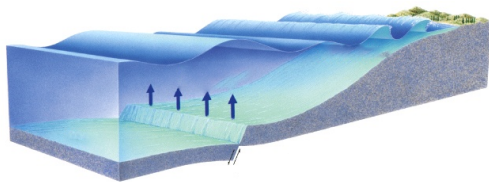
## a) rarefied gases



Scale is the *Knudsen number*

$$Kn = \frac{\text{mean free path length}}{\text{reference length}} = \frac{l}{L}$$

## b) shallow flows

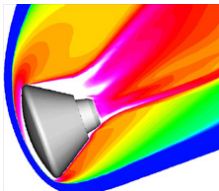


Scale is the *shallowness*

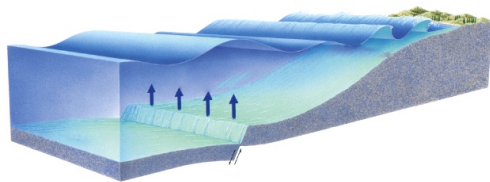
$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

# Motivation: Rarefied gases and shallow flows

a) rarefied gases



b) shallow flows



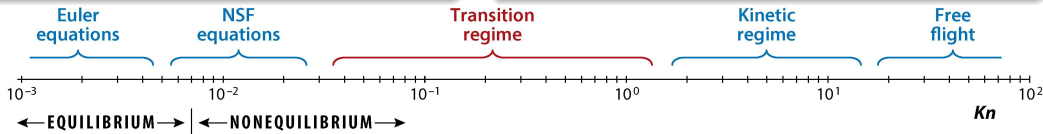
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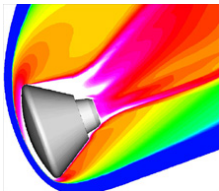
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# Motivation: Rarefied gases and shallow flows

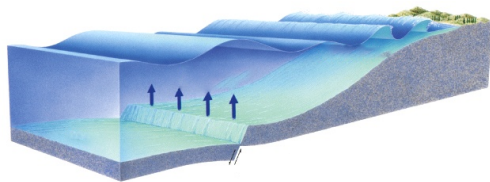
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b)

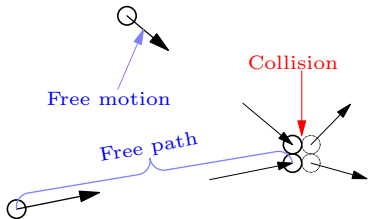


# Model equation: Rarefied gases and shallow flows

a) rarefied gases

Boltzmann Transport Equation

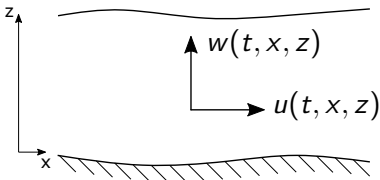
$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$



b) shallow flows

Incompressible Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$



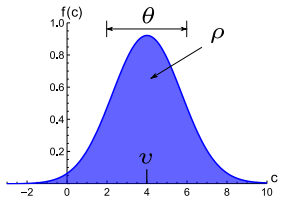


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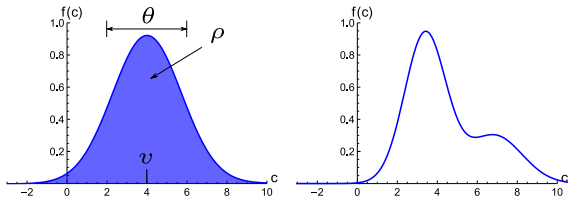
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Euler equations



?

b) shallow flows

Incompressible Navier-Stokes Equations

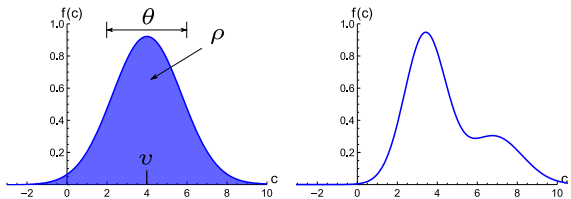
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Euler equations

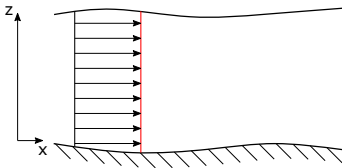


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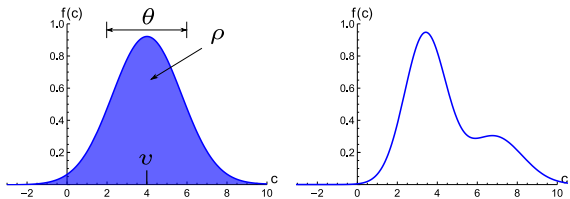


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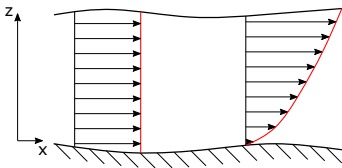
→

?

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SWE

→

?

# Moment models

## 1. underlying model equation

$$\mathcal{D}(\mathbf{U}(t, \mathbf{x}, \mathbf{y})) = 0$$

## 2. expansion with ansatz

$$\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{N}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^{\mathbf{U}}(\mathbf{y})$$

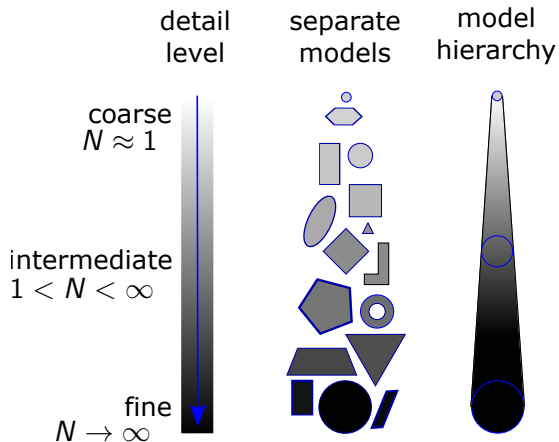
## 3. *moment* projection

$$\int_{\Omega} \mathcal{D}(\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y})) \cdot \Psi_j^{\mathbf{U}}(\mathbf{y}) \, d\mathbf{y} \text{ for } j \in \mathbb{N}$$

## Moment model

Hierarchical system of lower-dimensional PDEs for  $\mathbf{U}_i(t, \mathbf{x})$

# General derivation of hierarchical moment models



Ansatz:

$$\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{N}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^{\mathbf{U}}(\mathbf{y})$$

Projection:

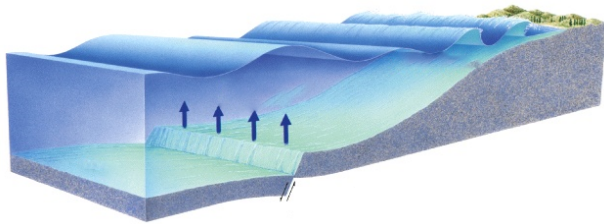
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Other models

- uncertainty quantification
- traffic flow

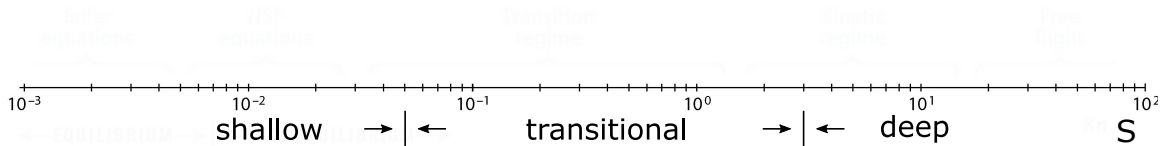
### 3 Derivation

# Motivation: shallow flows



Scale is the *shallowness*

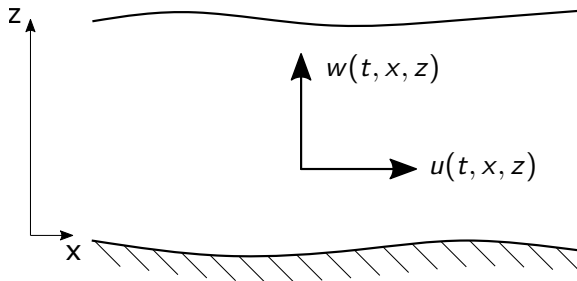
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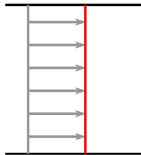
## Incompressible Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$



# Shallow flows: Micro-, Macro- and Meso-scale

macroscopic

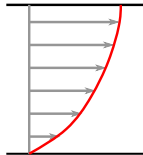


shallow water equations

$$\partial_t h + \partial_x(hu_m) = 0$$

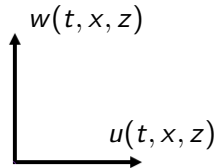
$$\partial_t(hu_m) + \partial_x\left(hu_m^2 + \frac{1}{2}gh^2\right) = -hg\partial_x h_b$$

mesoscopic



moment equations

microscopic



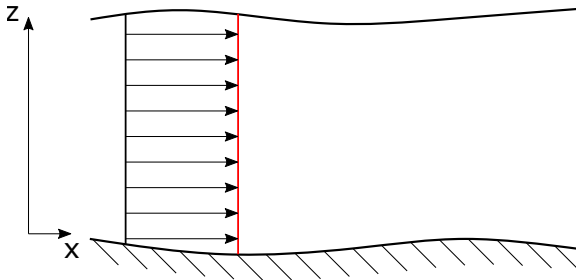
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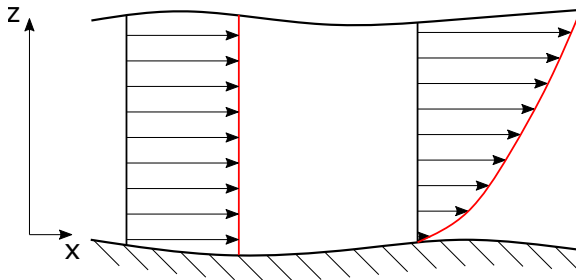
## Shallow Water Equations

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix}$$

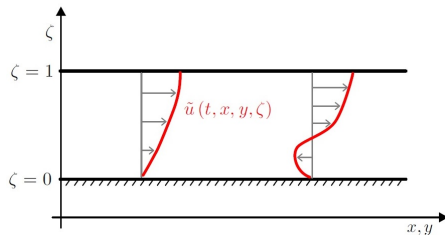
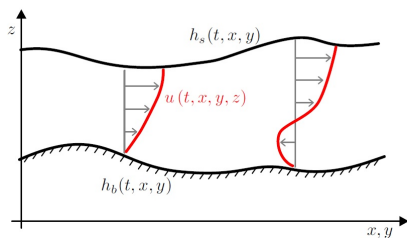


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SVE  $\rightarrow$  ?



$$z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h}$$

$$z \in [h_b(t, x), h_s(t, x)] \quad \Rightarrow \quad \zeta \in [0, 1]$$

# From NSE to transformed system

## NSE

$$\begin{aligned}\partial_x u + \partial_z w &= 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) &= -\frac{1}{\rho}\partial_x p + \frac{1}{\rho}\partial_x \sigma_{xz}\end{aligned}$$

The pressure is assumed hydrostatic  $p = (h_s - z)g$ .

transformed system

$$\begin{aligned}\partial_t h + \partial_x(hu_m) &= 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_\zeta\left(hu\omega - \frac{1}{\rho}\sigma_{xz}\right) &= -gh\partial_x h_b.\end{aligned}$$

The vertical coupling  $\omega$  is defined as

$$\omega = \frac{1}{h} \int_0^\zeta \left( \int_0^1 \partial_x(hu(\check{\zeta})) d\check{\zeta} - \partial_x(hu(\hat{\zeta})) \right) d\hat{\zeta}.$$

# Boundary conditions example

- no slip boundary condition at the bottom
- zero Neumann boundary condition at the top

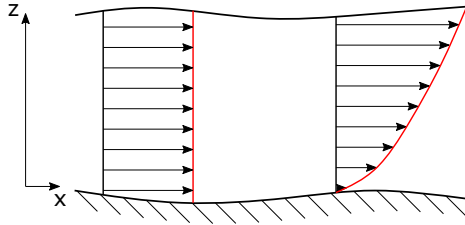
$$\partial_{\zeta} u \mid_{\zeta=1} = 0$$

$$u \mid_{\zeta=0} = 0$$

- slip boundary condition or friction at the bottom are possible
- for mud flows or land slides, Mohr-Coloumb friction law has to be used

Represent variations over depth with polynomials

$$u(t, x, z) = \underbrace{u_m(t, x)}_{\text{mean of } u} + \sum_{i=1}^N \alpha_i(t, x) \underbrace{\phi_i\left(\frac{z - h_b}{h_s - h_b}\right)}_{\phi_i(\zeta)}$$

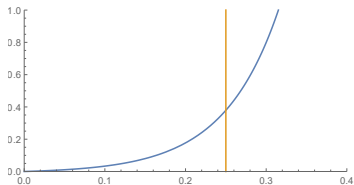




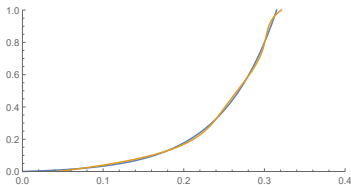
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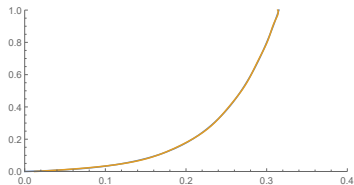
$N = 0$



$N = 5$



$N = 10$



## Example/Exercise 1: Polynomial velocity expansions

We expand the velocity profile  $u(t, x, \zeta)$  in Legendre polynomials as follows:

$$u(t, x, \zeta) = u_m(t, x) + \sum_{i=1}^N \alpha_i(t, x) \cdot \phi_i(\zeta),$$

where the first three Legendre polynomials are given by:

$$\phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2, \quad \phi_3(\zeta) = 1 - 12\zeta + 30\zeta^2 - 20\zeta^3,$$

with normalization  $\phi_i(0) = 1$  and orthogonality on  $[0, 1]$

$$\int_0^1 \phi_i(\zeta) \phi_j(\zeta) d\zeta = \frac{\delta_{ij}}{2i + 1}.$$

Compute the values of the variables  $u_m, \alpha_1, \alpha_2, \alpha_3$  for the following velocity profiles:

- ❶ Constant profile:  $u(t, x, \zeta) = 0.25$ .
- ❷ Linear profile:  $u(t, x, \zeta) = 0.5\zeta$ .
- ❸ Quadratic profile:  $u(t, x, \zeta) = 1.5\zeta(1 - \zeta)$ .

# Moment models

## 1. underlying model equation

$$\mathcal{D}(\mathbf{U}(t, \mathbf{x}, \mathbf{y})) = 0$$

## 2. expansion with ansatz

$$\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{N}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^U(\mathbf{y})$$

## 3. *moment* projection

$$\int_{\Omega} \mathcal{D}(\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y})) \cdot \Psi_j^U(\mathbf{y}) \, d\mathbf{y} \text{ for } j \in \mathbb{N}$$

# Moment models [GRAD, 1949], [KOWALSKI, TORRILHON, 2018]

1a) rarefied gases: BTE

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, c) + c \frac{\partial}{\partial \mathbf{x}} f(t, \mathbf{x}, c) = S(f)$$

1b) shallow flows: NSE

$$\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$

2. expansion with ansatz

$$\mathbf{u}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{N}} \mathbf{u}_i(t, \mathbf{x}) \cdot \Phi_i^{\mathbf{u}}(\mathbf{y})$$

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2a) Hermite ansatz

$$f(t, x, c) = \sum_{i=0}^N f_i(t, x) \phi_i\left(\frac{c - v}{\sqrt{\theta}}\right)$$

2b) Legendre ansatz

$$u(t, x, z) = \underbrace{u_m(t, x)}_{\text{mean of } u} + \sum_{i=1}^N \alpha_i(t, x) \phi_i\left(\frac{z - h_b}{h}\right)$$

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$$\int_{\mathbb{R}} \cdot \psi_j(c) dc$$

3b) *moment* projection

$$\int_{h_b}^{h_s} \cdot \psi_j(z) dz$$

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Moment model

Hierarchical system of lower-dimensional PDEs

# Moment method for free-surface waves

1. transformed system:

$$\begin{aligned}\partial_t h + \partial_x (hu_m) &= 0, \\ \partial_t (hu) + \partial_x \left( hu^2 + \frac{g}{2} h^2 \right) + \partial_\zeta \left( hu\omega - \frac{1}{\rho} \sigma_{xz} \right) &= -gh\partial_x h_b.\end{aligned}$$

2. Legendre ansatz:

$$u(t, x, \zeta) = \underbrace{u_m(t, x)}_{\text{mean of } u} + \sum_{i=1}^N \alpha_i(t, x) \phi_i(\zeta)$$

3. projection:

$$\int_0^1 \cdot \phi_j(\zeta) d\zeta$$



$$\left\{ \begin{array}{l} \partial_t h + \partial_x (hu_m) = 0, \\ \partial_t (hu_m) + \partial_x \left( hu_m^2 + h \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} \right) + gh \partial_x (b+h) = -\frac{\nu}{\lambda} \left( u_m + \sum_{j=1}^N \alpha_j \right), \\ \partial_t (h\alpha_i) + \partial_x \left( h \left( 2u_m \alpha_i + \sum_{j,k=1}^N A_{ijk} \alpha_j \alpha_k \right) \right) = u_m \partial_x (h\alpha_i) - \sum_{j,k=1}^N B_{ijk} \alpha_k \partial_x (h\alpha_j) \\ \quad - (2i+1) \left( -\frac{\nu}{\lambda} \left( u_m + \sum_{j=1}^N \alpha_j \right) + \frac{\nu}{h} \sum_{j=1}^N C_{ij} \alpha_j \right) \end{array} \right.$$

$A_{ijk}, B_{ijk}, C_{ij}$  are constant coefficients:

$$\frac{A_{ijk}}{2i+1} = \int_0^1 \phi_i \phi_j \phi_k d\xi, \quad \frac{B_{ijk}}{2i+1} = \int_0^1 \phi'_i \left( \int_0^\xi \phi_j d\xi \right) \phi_k d\xi, \quad \text{and} \quad C_{ij} = \int_0^1 \phi'_i \phi'_j d\xi.$$

## Example/Exercise 2: free-surface flow friction

The transformed equation includes a friction term  $-\frac{1}{\rho}\partial_{\zeta}\tilde{\sigma}_{xz}$ .

Compute the final term in the moment equations obtained by projection with test function  $\psi_j = \phi_j$  using

- Expansion  $u(t, x, \zeta) = u_m(t, x) + \sum_{i=1}^N \alpha_i(t, x) \cdot \phi_i(\zeta)$ .
- Orthogonal Legendre basis with normalization  $\phi_i(\zeta)|_{\zeta=0} = 1$ .
- Newtonian friction law in the bulk:  $\zeta \in [0, 1] : \frac{1}{\rho}\tilde{\sigma}_{xz} = \frac{\nu}{h} \cdot \partial_{\zeta}u(\zeta)$  and:
  - 1 No slip boundary condition at the top:  $\zeta = 1 \Rightarrow \tilde{\sigma}_{xz}(1) = 0$ .
  - 2 Slip boundary condition at the bottom:  $\zeta = 0 \Rightarrow \frac{1}{\rho}\tilde{\sigma}_{xz}(0) = \frac{\nu}{\lambda} \cdot u(0)$  with slip length  $\lambda$  and viscosity coefficient  $\nu$

( $N = 0$ )

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} \end{pmatrix} = - \begin{pmatrix} 0 \\ gh \partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length  $\lambda$  and viscosity  $\nu$ .

$N = 1$

First order model:  $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta)$ ,  $\phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{pmatrix}$$

# Shallow Water Moment Equations [KOWALSKI, TORRILHON, 2019]

$N = 2$

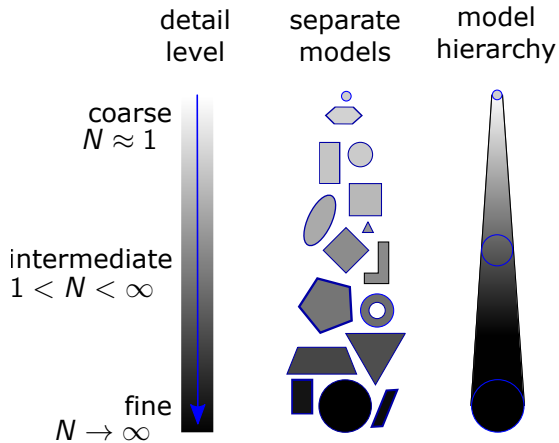
Second order model:  $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta) + \alpha_2 \phi_2(\zeta)$ ,  $\phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} + \frac{1}{3} h \alpha_1^2 + \frac{1}{5} h \alpha_2^2 \\ 2hu_m \alpha_1 + \frac{4}{5} h \alpha_1 \alpha_2 \\ 2hu_m \alpha_2 + \frac{2}{3} h \alpha_1^2 + \frac{2}{7} h \alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3 \left( u_m + \alpha_1 + \alpha_2 + 4 \frac{\lambda}{h} \alpha_1 \right) \\ 5 \left( u_m + \alpha_1 + \alpha_2 + 12 \frac{\lambda}{h} \alpha_2 \right) \end{pmatrix}.$$

# General derivation of hierarchical moment models



Ansatz:

$$\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{N}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^{\mathbf{U}}(\mathbf{y})$$

Projection:

$$\int_{\Omega} \mathcal{D}(\mathbf{U}_{\mathbb{N}}(t, \mathbf{x}, \mathbf{y})) \cdot \Psi_j^{\mathbf{U}}(\mathbf{y}) \, d\mathbf{y} \text{ for } j \in \mathbb{N}$$

Other models

- uncertainty quantification
- traffic flow

## 4 Exercises/Examples

## Example/Exercise 3: kinetic equation 1

Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t, x, c) + c \partial_x f(t, x, c) = 0,$$

with  $c \in \mathbb{R}$  and the following ansatz:

$$f(t, x, c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^N \alpha_i(t, x) \cdot He_i(c),$$

for orthonormal Hermite polynomials  $He_i(c)$  following the recursions:

$$c He_i(c) = \sqrt{i+1} He_{i+1}(c) + \sqrt{i} He_{i-1}(c),$$

$$\int He_i(c) He_j(c) \cdot w(c) dc = \delta_{ij},$$

for weight function  $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$ .



## Example/Exercise 4: kinetic equation 2

Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t, x, c) + c \partial_x f(t, x, c) = 0,$$

with  $c \in \mathbb{R}$  and the following ansatz:

$$f(t, x, c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^N f_i(t, x) \cdot He_i(c)$$

with orthogonal, but non-orthonormal Hermite basis polynomials  $He_i(c)$ :

$$\int_{\mathbb{R}} He_i(c) \cdot He_j(c) \cdot w(c) dc = j! \delta_{i,j},$$

$$c He_i(c) = He_{i+1}(c) + i \cdot He_{i-1}(c),$$

for weight function  $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$ .

## Example/Exercise 5: Uncertainty quantification and stochastic Galerkin

The hot shower model is given by the delay differential equation:

$$\dot{x}(t) = -(K + w) \cdot x(t - \tau),$$

with

$x$  : target temperature difference

$w$  : uniformly distributed uncertainty  $w \sim U(-0.1, 0.1)$

$K$  : reaction parameter

$\tau$  : delay.

❶ Rewrite the model with normalized uncertainty  $w \sim U(-1, 1)$ .

❷ Use the polynomial chaos expansion (PCE)  $x(t, w) = \sum_{i=0}^N x_i(t) \phi_i(w)$ , with  $\phi_i$  Legendre polynomials, orthonormal on  $[-1, 1]$ , to derive a stochastic Galerkin model for the evolution of the coefficients  $x_i$  in matrix-vector form.

summary

# Part 1 Summary

## 1 Overview

- efficient models
- model reduction
- model hierarchy

## 2 Motivation

- (rarefied gases)
- shallow flows
- moment models

## 3 Derivation

- transformation
- Legendre ansatz
- Shallow Water Moment Equations