Problem Sheet

Mathematical models in nonlinear acoustics

Westervelt equation is a classical model used for describing the propagation of nonlinear sound waves. In thermoviscous fluid media, it is given by

$$u_{tt} - c^2\Delta u - b\Delta u_t = k(u^2)_{tt} + f. \tag{1}$$

Here u denotes the acoustic pressure, $c > 0$ is the speed of sound in the medium, $b > 0$ the sound diffusivity, and $k \in \mathbb{R}$ the nonlinearity coefficient. The function $f = f(x, t)$ acts as the source of sound.

Consider a linearized version of (1) with $k = 0$:

$$u_{tt} - c^2\Delta u - b\Delta u_t = f, \tag{2}$$

on $\Omega \times (0, T)$, and supplement it with homogeneous initial data $(u, u_t)|_{t=0} = (0, 0)$ and homogeneous Dirichlet boundary conditions. Let $f \in L^2(0, T; L^2(\Omega))$. Assuming the solution of this problem exists and is sufficiently smooth, by using suitable test functions, show that the following (higher-order) energy inequality holds:

$$\begin{aligned}&\int_0^t \|u_{tt}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \|\Delta u(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{H^1(\Omega)} + \int_0^t \|\Delta u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\ &\leq C(T)\int_0^t \|f(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\end{aligned}$$

for all $t \in (0, T)$.

These energy arguments can be transferred to the study of Westervelt equation (1) by first carrying out the estimates on a linearization and then tying them to the original equation via Banach's fixed-point theorem on a ball of a sufficiently small radius. If you have time, you can try to work out the details.

You might need to rely on Grönwall's inequality: *Let w, $v \in L^\infty(0, T)$ be almost everywhere non-negative functions that satisfy*

$$w(t) + v(t) \leq a_1 + \int_0^t a_2(s)w(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in [0, T],$$

where $a_1 \geq 0$ and $a_2 \in L^1(0, T)$ is an almost everywhere non-negative function. Then the following Grönwall inequality holds:

$$w(t) + v(t) \leq a_1 e^{\int_0^t a_2(s)\,\mathrm{d}s} \quad \textit{for a.e. } t \in [0, T].$$

1 Uniform energy bounds

The estimate can be obtained through two testing steps.

- $\boxed{\text{Testing with } u_{tt}}$

We take $\phi = u_{tt}$ as the test function and integrate over $(0, t)$:

$$\int_0^t (u_{tt} - c^2\Delta u - b\Delta u_t, u_{tt})_{L^2(\Omega)} \,\mathrm{d}s = \int_0^t (f, u_{tt})_{L^2(\Omega)} \,\mathrm{d}s.$$

We can use integration by parts in time and space to treat the b term on the left-hand side:

$$\begin{aligned} -b\int_0^t (\Delta u_t, u_{tt})_{L^2(\Omega)} \,\mathrm{d}s &= b\int_0^t (\nabla u_t, \nabla u_{tt})_{L^2(\Omega)} \,\mathrm{d}s \\ &= b(\nabla u_t, \nabla u_t)_{L^2(\Omega)}\big|_0^t - b\int_0^t (\nabla u_{tt}, \nabla u_{tt})_{L^2(\Omega)} \,\mathrm{d}s. \end{aligned}$$

From here we have

$$-b\int_0^t (\Delta u_t, u_{tt})_{L^2(\Omega)} = b\int_0^t (\nabla u_t, \nabla u_{tt})_{L^2(\Omega)} = \frac{b}{2}\|\nabla u_t(t)\|^2_{L^2(\Omega)}$$

The f term on the right-hand side can be estimated by using Cauchy–Schwarz inequality combined with Young's ε inequality:

$$xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon}y^2, \quad \varepsilon > 0. \tag{3}$$

In this manner, we obtain

$$\int_0^t (f, u_{tt})_{L^2(\Omega)} \,\mathrm{d}s \leq \varepsilon \int_0^t \|u_{tt}\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{4\varepsilon}\int_0^t \|f\|^2_{L^2(\Omega)} \,\mathrm{d}s.$$

Similarly,

$$c^2\int_0^t (\Delta u, u_{tt})_{L^2(\Omega)} \,\mathrm{d}s \leq \varepsilon \int_0^t \|u_{tt}\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{4\varepsilon}c^4\int_0^t \|\Delta u\|^2_{L^2(\Omega)} \,\mathrm{d}s$$

for any $\varepsilon > 0$. By picking ε to be small enough, we arrive at

$$\begin{aligned} &\int_0^t \|u_{tt}\|^2_{L^2(\Omega)} + \frac{b}{2}\|\nabla u_t(t)\|^2_{L^2(\Omega)} \\ \leq &\frac{1}{4\varepsilon}\int_0^t \|f\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{4\varepsilon}c^4\int_0^t \|\Delta u\|^2_{L^2(\Omega)} \,\mathrm{d}s. \end{aligned} \tag{4}$$

- $\boxed{\text{Testing with } -\Delta u}$

With this choice of the test function, we have, after integration in time,

$$\int_0^t (u_{tt} - c^2\Delta u - b\Delta u_t, -\Delta u)_{L^2(\Omega)} \,\mathrm{d}s = \int_0^t (f, -\Delta u)_{L^2(\Omega)} \,\mathrm{d}s.$$

Similarly to before, we can estimate

$$\int_0^t (f, -\Delta u)_{L^2(\Omega)} \,\mathrm{d}s \leq \frac{1}{2}\int_0^t \|\Delta u\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{2}\int_0^t \|f\|^2_{L^2(\Omega)} \,\mathrm{d}s$$

and (after transferring it to the right-hand side of the equation)

$$\int_0^t (u_{tt}, \Delta u)_{L^2(\Omega)} \,\mathrm{d}s \leq \varepsilon \int_0^t \|u_{tt}\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{4\varepsilon}\int_0^t \|\Delta u\|^2_{L^2(\Omega)} \,\mathrm{d}s.$$

From here we have

$$\begin{aligned}
&\frac{c^2}{2}\|\Delta u(t)\|^2_{L^2(\Omega)} + b\int_0^t \|\Delta u_t\|^2_{L^2(\Omega)} \,\mathrm{d}s \\
\leq\, &\varepsilon \int_0^t \|u_{tt}\|^2_{L^2(\Omega)} \,\mathrm{d}s + \int_0^t \left(1 + \frac{1}{4\varepsilon}\right) \|\Delta u\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{1}{2}\int_0^t \|f\|^2_{L^2(\Omega)} \,\mathrm{d}s.
\end{aligned}$$

for any $\varepsilon > 0$ (different than the one we had before).

We can now add this bound to (4), choose $\varepsilon > 0$ small so that we can absorb the u_{tt} term on the right and then employ Grönwall's inequality. This leads to the desired estimate.