Free-surface waves using extended shallow water models part 1

Julian Koellermeier
University of Groningen and Ghent University

WAVES.NL Summer school, Nijmegen, 25 August 2025

1 Overview

Schedule

Time	Monday	Tuesday	Wednesday	Thursday	Friday
8:50–9:00	*Opening*				
9:00–10:30	L3	L5	L2	L4	L6
10:30–11:00	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
11:00–12:30	L1	L1	L2	L4	L6
12:30–13:30	*Lunch*	*Lunch*	*Lunch*	*Lunch*	*Lunch*
13:30–15:00	L3	L5	L3	L5	
15:00–15:30	*Coffee break*	*Coffee break*			
15:30–17:00	Poster session	L1			
17:45–19:00			*Social event*		

L1: Mon 11-12:30

- overview
- motivation
- derivation

L2: Tue 11-12:30

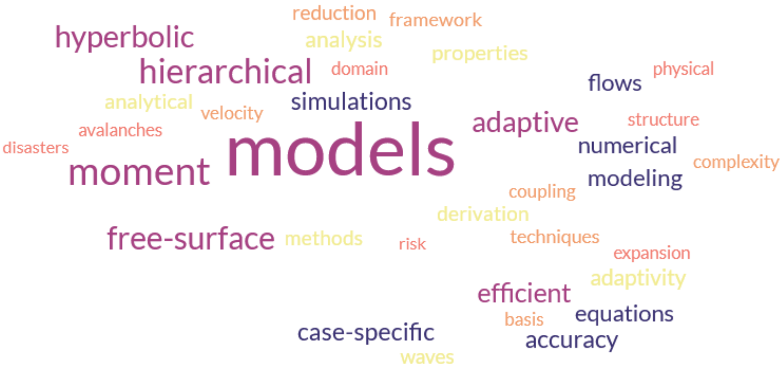
- analysis

L3: Tue 15:30-17

- selected papers
- outlook

Free-surface waves using extended shallow water models

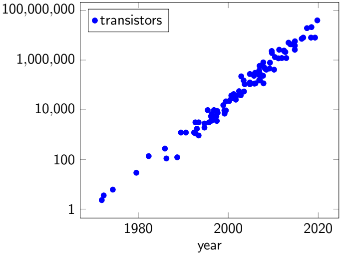
Free-surface waves using extended shallow water models



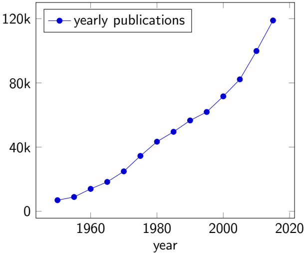
Content of this talk

Progress in mathematical modeling and numerical simulation

Computing power increases (Moore's law)

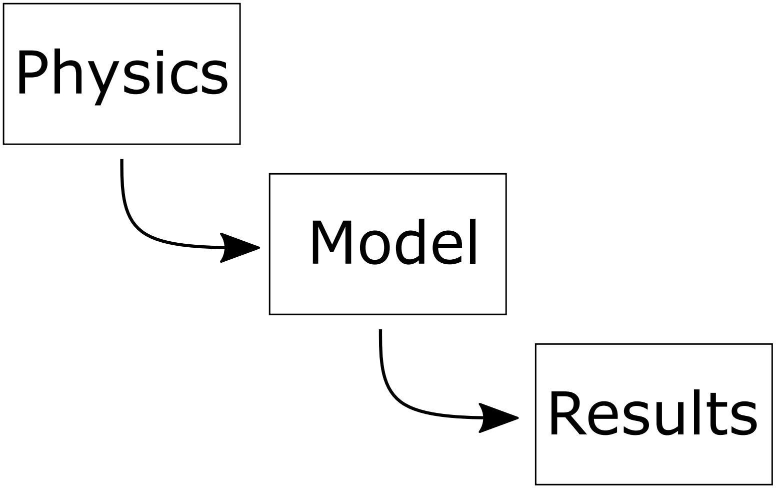


Number of papers increases (mathscinet)



Status quo: diversity of complex models $\Rightarrow$ repeated derivation, analysis, implementation

Question: How to use our resources efficiently?


Physics
Model
Results

Modeling and model reduction

Physics

- first principles
- laws of physics

Model

- PDEs
- particle methods
- neural networks

Results

- applications
- optimization

Physics

- first principles
- laws of physics

Model

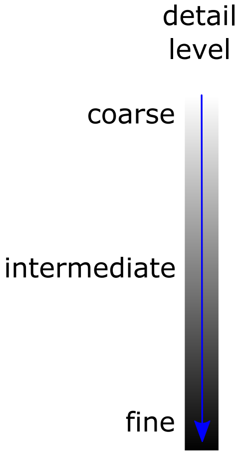
- PDEs
- particle methods
- neural networks

Results

- applications
- optimization

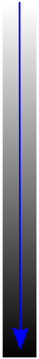
Question: What are desirable model properties?

Mathematical Modeling



detail
level

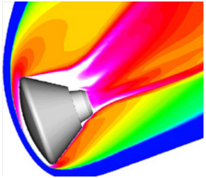
coarse



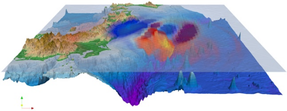
intermediate

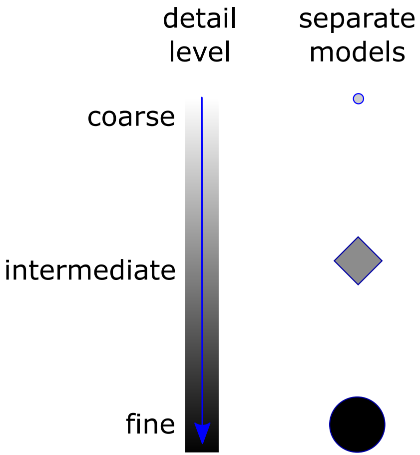
fine

a) rarefied gases



b) free-surface flows





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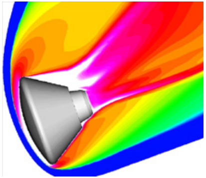

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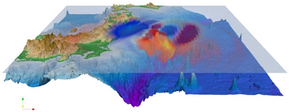

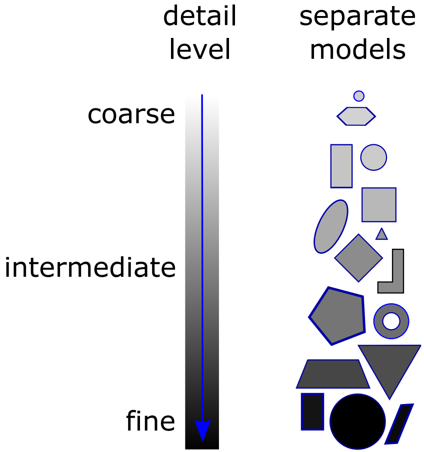

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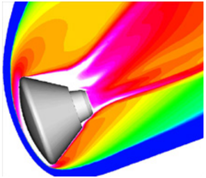

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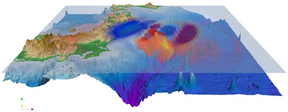
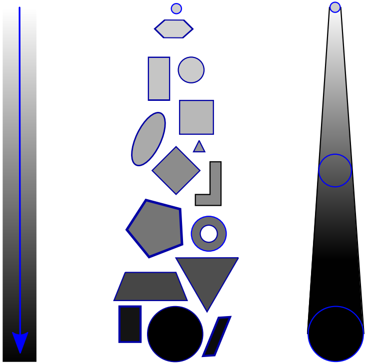


detail level | separate models | model hierarchy

coarse

intermediate

fine



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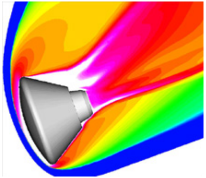
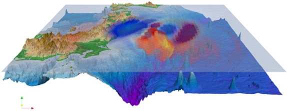


b) free-surface flows



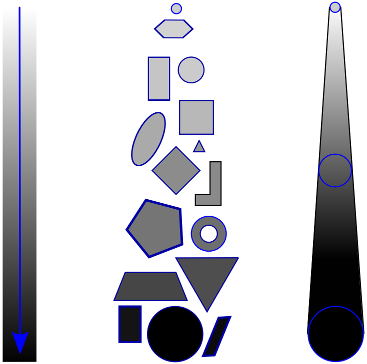
Mathematical Modeling

detail level	separate models	model hierarchy

coarse
$M \approx 1$

intermediate
$1 < M < \infty$

fine
$M \to \infty$



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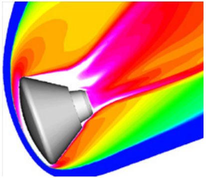

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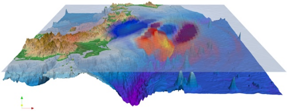
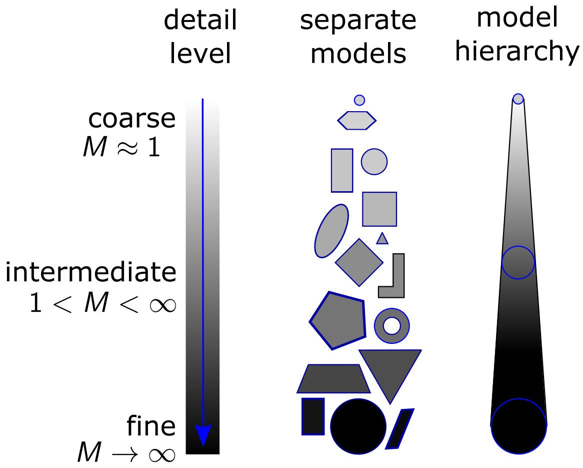


Hierarchical mathematical modeling



Hierarchical moment models

Advantages

1. general derivation
2. structure preserving
3. accurate results

$\Rightarrow$ adaptive simulations

My research field

Hierarchical Simulation Using Moment Models

Model derivation

- rarefied gases
- shallow flows

Model analysis

- hyperbolicity
- further analysis

Numerics & Applications

- numerical schemes
- numerical results

My research field

Hierarchical Simulation Using Moment Models

Model derivation

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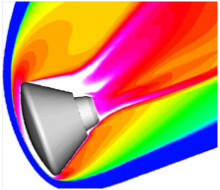
2 Motivation

Motivation: Rarefied gases and shallow flows

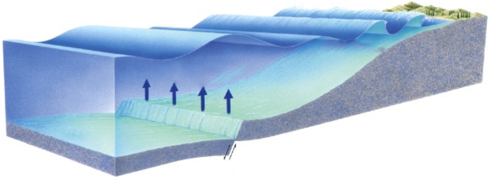
a) rarefied gases



Scale is the *Knudsen number*

$$\mathrm{Kn} = \frac{\text{mean free path length}}{\text{reference length}} = \frac{l}{L}$$

b) shallow flows



Scale is the *shallowness*

$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

Motivation: Rarefied gases and shallow flows

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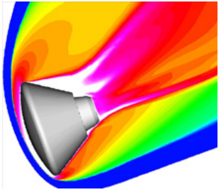
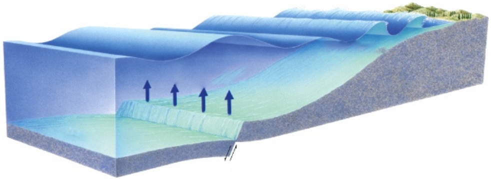


b) shallow flows



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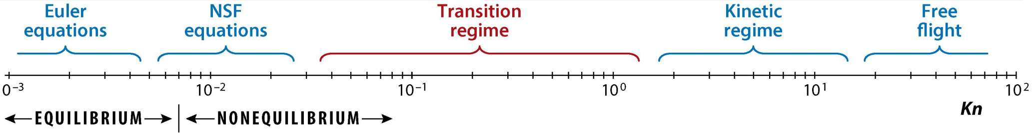

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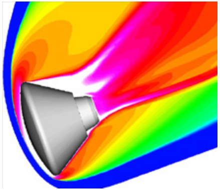
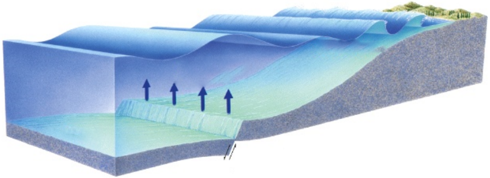


b) shallow flows



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b)

10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}

shallow → | ← transitional → | ← deep S

Model equation: Rarefied gases and shallow flows

a) rarefied gases

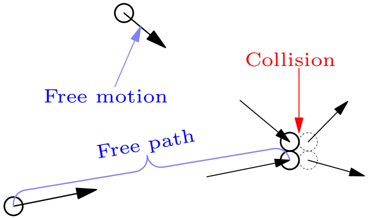
b) shallow flows

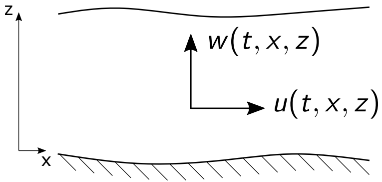
Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \boldsymbol{x}, \boldsymbol{c}) + c_i \frac{\partial}{\partial x_i} f(t, \boldsymbol{x}, \boldsymbol{c}) = S(f)$$

Incompressible Navier-Stokes Equations

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + g$$





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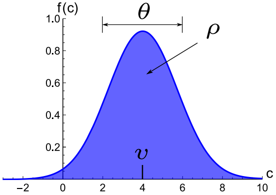

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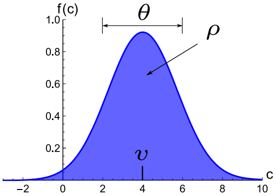

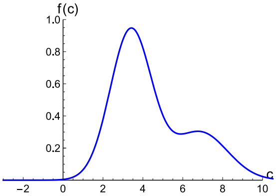

Euler equations $\rightarrow$?

a) rarefied gases

b) shallow flows

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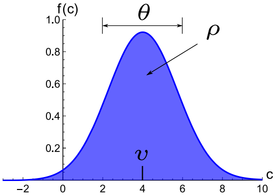

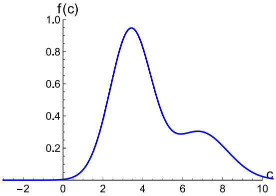

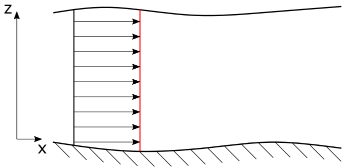

Euler equations $\rightarrow$?

Model equation: Rarefied gases and shallow flows

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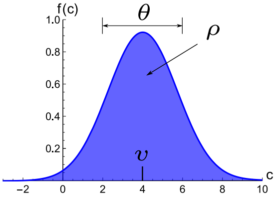
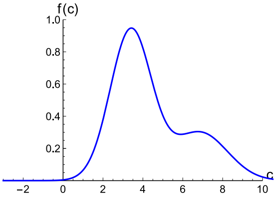
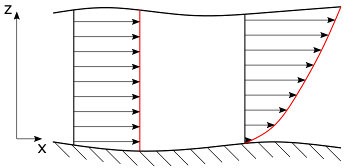






Euler equations $\rightarrow$?

SWE $\rightarrow$?

Moment models

1. underlying model equation

$$\mathcal{D}\left(\boldsymbol{U}(t, \boldsymbol{x}, \boldsymbol{y})\right) = 0$$

2. expansion with ansatz

$$\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i \in \mathbb{M}} \boldsymbol{U}_i(t, \boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

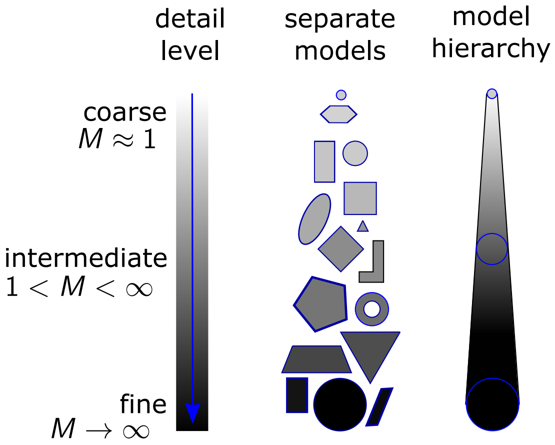
3. *moment* projection

$$\int_\Omega \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, \boldsymbol{dy} \text{ for } j \in \mathbb{M}$$

Moment model

Hierarchical system of lower-dimensional PDEs for $\boldsymbol{U}_i(t, \boldsymbol{x})$

General derivation of hierarchical moment models



Ansatz:

$$\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i \in \mathbb{M}} \boldsymbol{U}_i(t, \boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

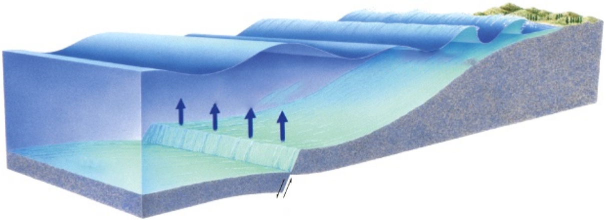
Projection:

$$\int_{\Omega} \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, d\boldsymbol{y} \text{ for } j \in \mathbb{M}$$

Other models

- uncertainty quantification
- traffic flow

3 Derivation



Scale is the *shallowness*

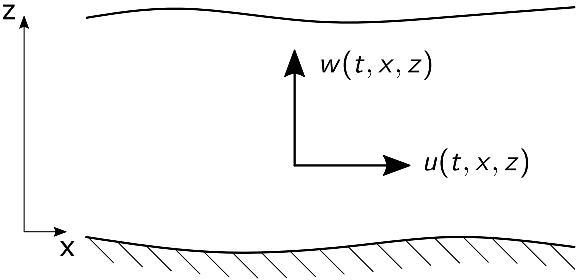
$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}

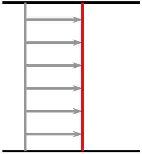
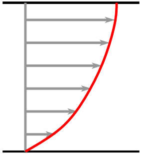
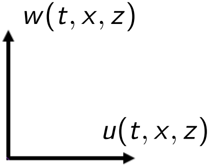
shallow $\rightarrow | \leftarrow$ transitional $\rightarrow | \leftarrow$ deep S

Incompressible Navier-Stokes Equations

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + g$$



Shallow flows: Micro-, Macro- and Meso-scale

macroscopic	mesoscopic	microscopic
		
shallow water equations	moment equations	incompressible Navier–Stokes

Shallow water equations:

$$\partial_t h + \partial_x (h u_m) = 0$$

$$\partial_t (h u_m) + \partial_x \left(h u_m^2 + \frac{1}{2} g h^2 \right) = -hg\partial_x h_b$$

Incompressible Navier–Stokes:

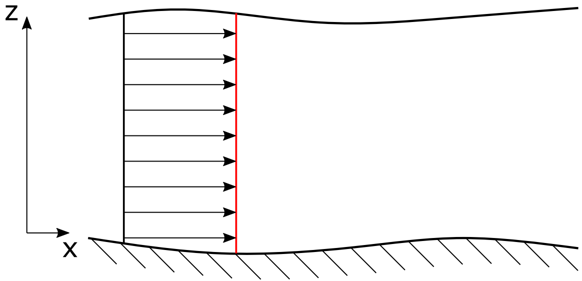
$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + g$$

Model equation: shallow flows

Shallow Water Equations

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix}$$



Model equation: shallow flows

Shallow Water Equations

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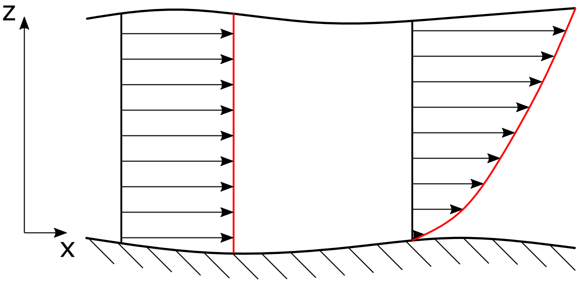
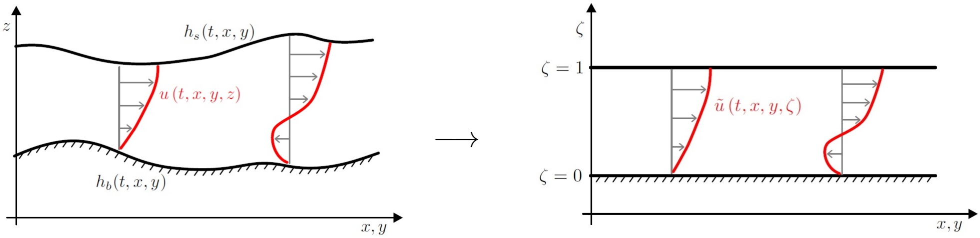


SWE $\rightarrow$?

Transformation [Torrilhon, Kowalski, 2018]



$$z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h}$$

$$z \in [h_b(t,x), h_s(t,x)] \quad \Rightarrow \quad \zeta \in [0,1]$$

From NSE to transformed system

NSE

$$\partial_t u + \partial_x(u^2) + \partial_z(uw) = -\frac{1}{\rho}\partial_x p + \frac{1}{\rho}\partial_x \sigma,$$
$$\partial_x u + \partial_z w = 0,$$

The pressure is assumed hydrostatic $p = (h_s - z)g$.
transformed system

$$\partial_t h + \partial_x\left(hu_m\right) + \partial_y\left(hv_m\right) = 0,$$
$$\partial_t(hu + \frac{g}{2}h^2) + \partial_\zeta\left(hu\omega - \frac{1}{\rho}\sigma_{xz}\right) = gh\partial_x h_b.$$

The vertical coupling ω is defined as

$$\omega = \frac{1}{h}\int_0^\zeta \left(\int_0^1 \partial_x(hu(\check{\zeta}))d\check{\zeta} - \partial_x(hu(\hat{\zeta}))\right) d\hat{\zeta}.$$

Boundary conditions example

- no slip boundary condition at the bottom
- zero Neumann boundary condition at the top

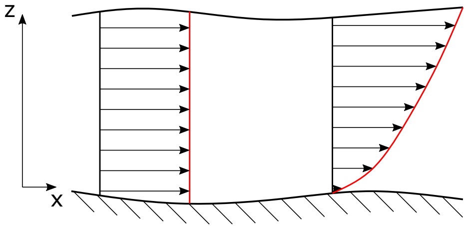
$$
\begin{aligned}
\partial_\zeta u \,|_{\zeta=1} &= 0 \\
u \,|_{\zeta=0} &= 0
\end{aligned}
$$

- slip boundary condition or friction at the bottom are possible
- for mud flows or land slides, Mohr-Coloumb friction law has to be used

Polynomial ansatz [Kowalski, Torrilhon, 2018]

Represent variations over depth with polynomials

$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} + \sum_{i=1}^{M} \alpha_i(t,x)\, \underbrace{\phi_i\left(\frac{z-h_b}{h_s-h_b}\right)}_{\phi_i(\zeta)}$$



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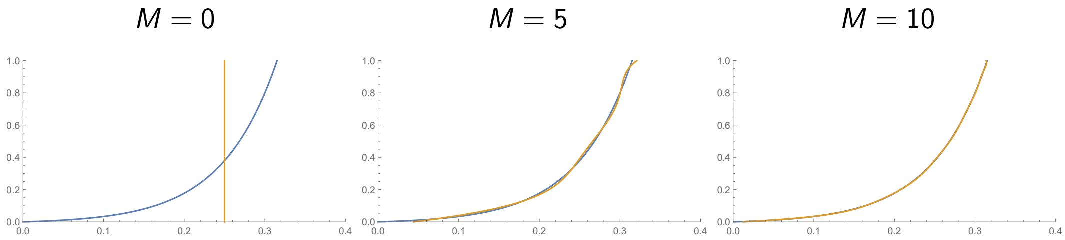

Example/Exercise 1: Polynomial velocity expansions

We expand the velocity profile $u(t,x,\zeta)$ in Legendre polynomials as follows:

$$u(t,x,\zeta) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x) \cdot \phi_i(\zeta),$$

where the first three Legendre polynomials are given by:

$$\phi_1(\zeta) = 1 - 2\zeta, \qquad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2, \qquad \phi_3(\zeta) = 1 - 12\zeta + 30\zeta^2 - 20\zeta^3,$$

with normalization $\phi_i(0) = 1$ and orthogonality on $[0,1]$

$$\int_0^1 \phi_i(\zeta)\phi_j(\zeta)d\zeta = \frac{\delta_{ij}}{2i+1}.$$

Compute the values of the variables $u_m, \alpha_1, \alpha_2, \alpha_3$ for the following velocity profiles:

1. Constant profile: $u(t,x,\zeta) = 0.25$.
2. Linear profile: $u(t,x,\zeta) = 0.5\zeta$.
3. Quadratic profile: $u(t,x,\zeta) = 1.5\zeta(1-\zeta)$.

Moment models

1. underlying model equation

$$\mathcal{D}\left(\boldsymbol{U}(t,\boldsymbol{x},\boldsymbol{y})\right) = 0$$

2. expansion with ansatz

$$\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in\mathbb{M}} \boldsymbol{U}_i(t,\boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

3. *moment* projection

$$\int_\Omega \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, \boldsymbol{dy} \text{ for } j \in \mathbb{M}$$

Moment models [Grad, 1949], [Kowalski, Torrilhon, 2018]

1a) rarefied gases: BTE

$$\frac{\partial}{\partial t} f(t,x,c) + c\frac{\partial}{\partial x} f(t,x,c) = S(f)$$

1b) shallow flows: NSE

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + g$$

2. expansion with ansatz

$$\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in\mathbb{M}} \boldsymbol{U}_i(t,\boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

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2a) Hermite ansatz

$$f(t,x,c) = \sum_{i=0}^{M} f_i(t,x)\, \phi_i\left(\frac{c-v}{\sqrt{\theta}}\right)$$

2b) Legendre ansatz

$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} + \sum_{i=1}^{M} \alpha_i(t,x)\phi_i\left(\frac{z-h_b}{h}\right)$$

3. *moment* projection

$$\int_\Omega \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, \boldsymbol{dy} \text{ for } j \in \mathbb{M}$$

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2b) Legendre ansatz

$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} + \sum_{i=1}^{M} \alpha_i(t,x) \phi_i\left(\frac{z-h_b}{h}\right)$$

3a) *moment* projection

$$\int_{\mathbb{R}} \cdot\, \psi_j(c)\, dc$$

3b) *moment* projection

$$\int_{h_b}^{h_s} \cdot\, \psi_j(z)\, dz$$

Moment models [Grad, 1949], [Kowalski, Torrilhon, 2018]

1a) rarefied gases: BTE

$$\frac{\partial}{\partial t} f(t,x,c) + c \frac{\partial}{\partial x} f(t,x,c) = S(f)$$

1b) shallow flows: NSE

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + g$$

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2b) Legendre ansatz

$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} + \sum_{i=1}^{M} \alpha_i(t,x) \phi_i\left(\frac{z-h_b}{h}\right)$$

3a) *moment* projection

$$\int_{\mathbb{R}} \cdot\, \psi_j(c)\, dc$$

3b) *moment* projection

$$\int_{h_b}^{h_s} \cdot\, \psi_j(z)\, dz$$

Moment model

Hierarchical system of lower-dimensional PDEs

SWME system

$$
\begin{cases}
\partial_t h + \partial_x (h u_m) = 0, \\
\partial_t (h u_m) + \partial_x \left(h u_m^2 + h \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} \right) + g h \partial_x (b + h) = -\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^{N} \alpha_j \right), \\
\partial_t (h \alpha_i) + \partial_x \left(h \left(2 u_m \alpha_i + \sum_{j,k=1}^{N} A_{ijk} \alpha_j \alpha_k \right) \right) = u_m \partial_x (h \alpha_i) - \sum_{j,k=1}^{N} B_{ijk} \alpha_k \partial_x (h \alpha_j) \\
\qquad -(2i+1) \left(-\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^{N} \alpha_j \right) + \frac{\nu}{h} \sum_{j=1}^{N} C_{ij} \alpha_j \right)
\end{cases}
$$

A_{ijk}, B_{ijk}, C_{ij} are constant coefficients:

$$
\frac{A_{ijk}}{2i+1} = \int_0^1 \phi_i \phi_j \phi_k d\xi, \quad \frac{B_{ijk}}{2i+1} = \int_0^1 \phi_i' \left(\int_0^\xi \phi_j d\xi \right) \phi_k d\xi, \quad \text{and} \quad C_{ij} = \int_0^1 \phi_i' \phi_j' d\xi.
$$

Example/Exercise 2: free-surface flow friction

The transformed equation includes a friction term $-\frac{1}{\rho}\partial_\zeta \tilde{\sigma}_{xz}$.
Compute the final term in the moment equations obtained by projection with test function $\psi_j = \phi_j$ using

- Expansion $u(t,x,\zeta) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x) \cdot \phi_i(\zeta)$.
- Orthogonal Legendre basis with normalization $\phi_i(\zeta)|_{\zeta=0} = 1$.
- Newtonian friction law in the bulk: $\zeta \in [0,1] : \frac{1}{\rho}\tilde{\sigma}_{xz} = \frac{\nu}{h} \cdot \partial_\zeta u(\zeta)$ and:
  1. No slip boundary condition at the top: $\zeta = 1 \Rightarrow \tilde{\sigma}_{xz}(1) = 0$.
  2. Slip boundary condition at the bottom: $\zeta = 0 \Rightarrow \frac{1}{\rho}\tilde{\sigma}_{xz}(0) = \frac{\nu}{\lambda} \cdot u(0)$ with slip length λ and viscosity coefficient ν

Shallow Water Equations [Kowalski, Torrilhon, 2019]

$(M = 0)$

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} \end{pmatrix} = -\begin{pmatrix} 0 \\ gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda}\begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν.

Shallow Water Moment Equations [Kowalski, Torrilhon, 2019]

$M = 1$

First order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta)$, $\quad \phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{pmatrix}$$

$M = 2$

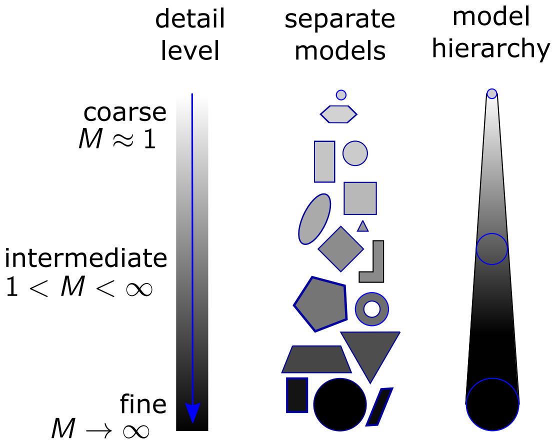
Second order model: $u(\zeta) = u_m + \alpha_1\phi_1(\zeta) + \alpha_2\phi_2(\zeta), \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda}P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

General derivation of hierarchical moment models



Ansatz:

$$\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i \in \mathbb{M}} \boldsymbol{U}_i(t, \boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

Projection:

$$\int_{\Omega} \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t, \boldsymbol{x}, \boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y}) \, \boldsymbol{dy} \text{ for } j \in \mathbb{M}$$

Other models

- uncertainty quantification
- traffic flow

4 Exercises/Examples

Example/Exercise 3: kinetic equation 1

Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t,x,c) + c\partial_x f(t,x,c) = 0,$$

with $c \in \mathbb{R}$ and the following ansatz:

$$f(t,x,c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^{M} \alpha_i(t,x) \cdot He_i(c),$$

for orthonormal Hermite polynomials $He_i(c)$ following the recursions:

$$cHe_i(c) = \sqrt{i+1}He_{i+1}(c) + \sqrt{i}He_{i-1}(c),$$

$$\int He_i(c)He_j(c) \cdot w(c)dc = \delta_{ij},$$

for weight function $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$.

Example/Exercise 4: kinetic equation 2

Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t,x,c) + c\partial_x f(t,x,c) = 0,$$

with $c \in \mathbb{R}$ and the following ansatz:

$$f(t,x,c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^{M} f_i(t,x) \cdot He_i(c)$$

with orthogonal, but non-orthonormal Hermite basis polynomials $He_i(c)$:

$$\int_{\mathbb{R}} He_i(c) \cdot He_j(c) \cdot w(c) dc = j! \delta_{i,j},$$

$$cHe_i(c) = He_{i+1}(c) + i \cdot He_{i-1}(c),$$

for weight function $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$.

Example/Exercise 5: Uncertainty quantification and stochastic Galerkin

The hot shower model is given by the delay differential equation:

$$\dot{x}(t) = -(K + w) \cdot x(t - \tau),$$

with

$$\begin{aligned} x &: \quad \text{target temperature difference} \\ w &: \quad \text{uniformly distributed uncertainty } w \sim U(-0.1, 0.1) \\ K &: \quad \text{reaction parameter} \\ \tau &: \quad \text{delay.} \end{aligned}$$

1. Rewrite the model with normalized uncertainty $w \sim U(-1, 1)$.
2. Use the polynomial chaos expansion (PCE) $x(t, w) = \sum_{i=0}^{N} x_i(t)\phi_i(w)$, with ϕ_i Legendre polynomials, orthonormal on $[-1, 1]$, to derive a stochastic Galerkin model for the evolution of the coefficients x_i in matrix-vector form.

summary

Part 1 Summary

1 Overview

- efficient models
- model reduction
- model hierarchy

2 Motivation

- (rarefied gases)
- shallow flows
- moment models

3 Derivation

- transformation
- Legendre ansatz
- Shallow Water Moment Equations