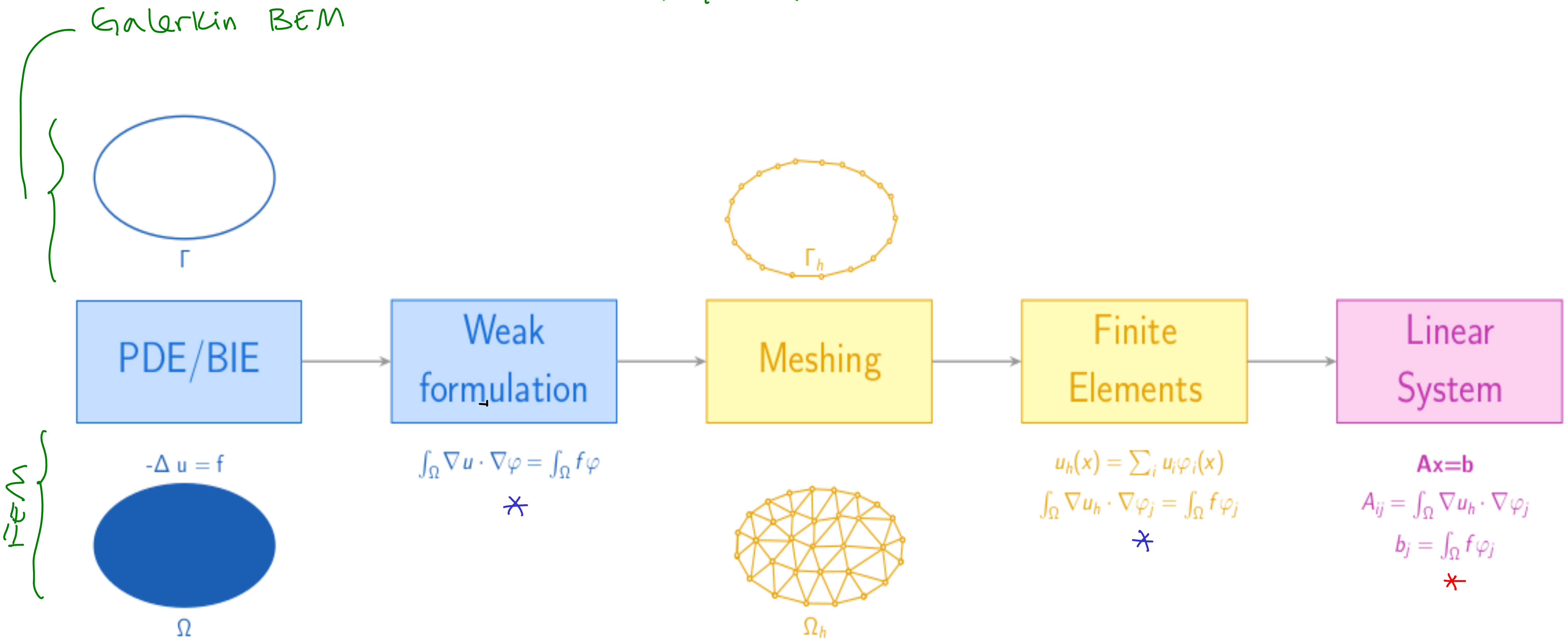
DISCLAIMER: This document contains the "whiteboard notes" taken during the "Introduction to FEM for Helmholtz" at the waves summer school in August 2025.

Therefore, they are not complete and contain some simplifications for the sake of time, etc. Moreover, the document has not been proof-read and may therefore have some minor mistakes/typos that are typical during lectures

In the coming days I will add the corresponding references and sources for the pictures that are not my own.

Figure 1



Galerkin BEM
Γ
Γ_h
PDE/BIE
Weak formulation
Meshing
Finite Elements
Linear System
FEM
$-\Delta u = f$
$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi$
$u_h(x) = \sum_i u_i \varphi_i(x)$
$\int_\Omega \nabla u_h \cdot \nabla \varphi_j = \int_\Omega f \varphi_j$
Ax=b
$A_{ij} = \int_\Omega \nabla u_h \cdot \nabla \varphi_j$
$b_j = \int_\Omega f \varphi_j$
Ω
Ω_h

INTRO TO FEM FOR HELMHOLTZ

Three lectures:

① FEM in a nutshell

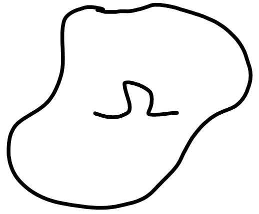
② Helmholtz problems: variational formulation & discretization

③ Numerical challenges

LECTURE 1: FEM IN A NUTSHELL

Let $\Omega \subset \mathbb{R}^n$ be a "regular enough" domain
Let us consider the BVP

$$\begin{cases} Lu = f & \text{in } \Omega \\ + \text{ boundary conditions} & \text{on } \partial\Omega \end{cases}$$



$L \hat{=}$ operator corresponding to the PDE

$L = -\Delta$, $L = -\Delta - k^2 \, Id$,
$L = \partial_t - \Delta$, etc.

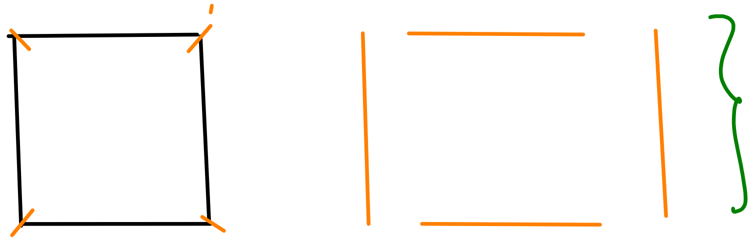
[Figure 1]

Let us illustrate this with the Laplacian.
First consider Homogeneous Dirichlet problem

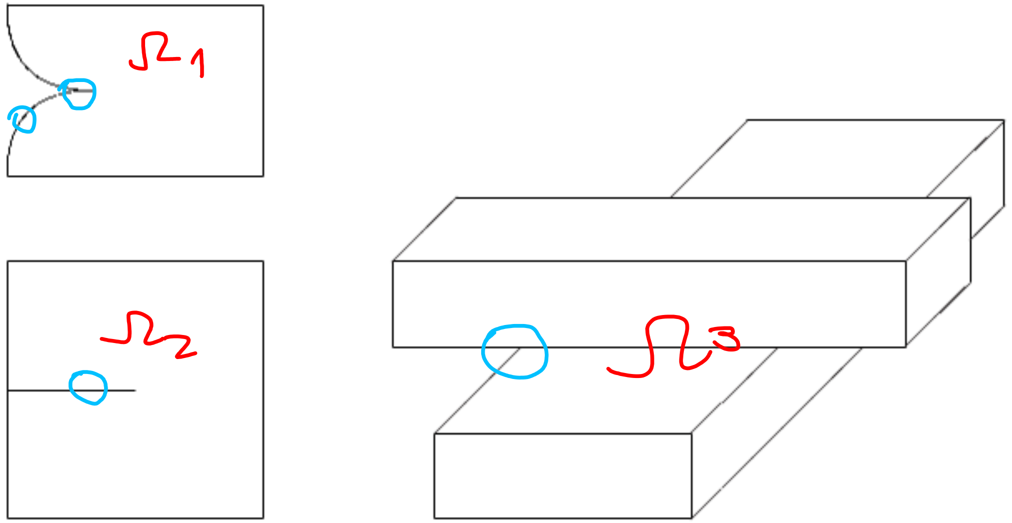
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

1.1 What are our assumptions on Ω?

We assume Ω to be Lipschitz

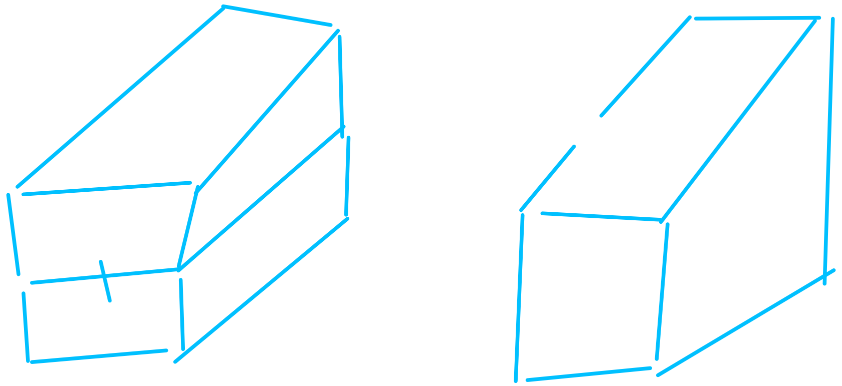


Lipschitz because we can split ∂Ω into pieces that can be parametrized by Lipschitz functions :)



not Lipschitz! :(

Discussion



1.2 Weak formulation

1.2.1. Main ingredients

▷ integration by parts

(1D) $\int_a^b u(x)\, v'(x)\, dx = uv\Big|_{x=a}^{b} - \int_a^b u'(x)\, v(x)\, dx$ (E1)

$\forall u, v \in C^1[a,b]$

(2D-3D)

$$\int_\Omega \operatorname{div} \underline{w}(x)\, v(x)\, d\Omega = \int_{\partial\Omega} (\underline{w}(x)\cdot \underline{n}(x))\, v(\underline{x})\, d\sigma - \int_\Omega \underline{w}(x)\cdot \nabla v(x)\, d\Omega \quad (E2)$$

$v \in C^1(\bar\Omega)$

$\underline{w} \in [C^1(\bar\Omega)]^n$

1.2.2. Weak formulation for the Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \qquad (E3.1) \\ u = 0 & \text{on } \partial\Omega \qquad (E3.2) \end{cases}$$

i) Let $v \in H_0^1(\Omega) = \overline{C_0^\infty}^{\|\cdot\|_{H^1}} \overset{\text{“ ”}}{=} \{ v \in H^1(\Omega) \text{ st } v|_{\partial\Omega} = 0 \}$

then

$$\int_\Omega -\Delta u\, v\, d\Omega = \int_\Omega f\, v\, d\Omega \qquad \forall v \in H_0^1(\Omega)$$

ii) integration by parts $\left(-\Delta u = -\operatorname{div}(\nabla u) \right)$

$$\int_\Omega \nabla u(x)\cdot \nabla v(x)\, d\Omega - \cancelto{0}{\int_{\partial\Omega} (\nabla u \cdot \underline{n})\, v\, d\sigma}$$ as $v \in H_0^1(\Omega)$.

$$= \int_\Omega f(x)\, v(x)\, d\Omega \qquad \forall v \in H_0^1(\Omega)$$

1st step $\quad -\Delta u = f$

Last step

Find $u \in H_0^1(\Omega)$ st

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \qquad \forall v \in H_0^1(\Omega).$$

Q where does f live?

What do we need for $\int_\Omega f v \, d\Omega$ to be well-defined.

1) f to be integrable.

2) $\int_\Omega f v \, d\Omega < \infty$

if we use L^2 duality, Cauchy-Schwarz gives you

$$\int_\Omega f v \, d\Omega \leq \|v\|_{H_0^1(\Omega)} \|f\|_{\underbrace{[H_0^1(\Omega)]'}_{\text{dual space with respect to } L^2(\Omega) \ (\text{aka } H^{-1}(\Omega))}}$$

So now $f = \delta_0 \in H^{-1}(\Omega)$ is allowed.

Let us rewrite the weak formulation we got as follows:

Given $f \in [H_0^1(\Omega)]'$, find $u \in H_0^1(\Omega)$ st

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, d\Omega}_{=: a(u,v)} = \underbrace{\int_\Omega f v \, d\Omega}_{=: \ell(v)} \qquad \forall v \in H_0^1(\Omega).$$

1.3. Existence and uniqueness

1.3.1. The simplest case

▷ Let X Hilbert space, $a \in \underbrace{\mathcal{L}(X \times X, \mathbb{R})}_{\text{is a continuous bilinear form}}$, $\ell \in \mathcal{L}(X, \mathbb{R})$ [ℓ is a bounded linear form].

the general variational problem can be written as:

$$\boxed{\begin{array}{l} \text{Find } u \in X \text{ st} \\ a(u,v) = \ell(v) \qquad \forall v \in X \qquad (P1) \end{array}}$$

Thm: Lax-Milgram

If $\exists\ \alpha_a, C_a > 0$ st

$$\alpha_a \|u\|_X^2 \leq |a(u,u)| \qquad \text{(coercivity)}$$
$$|a(u,u)| \leq C_a \|u\|_X^2 \qquad \text{(continuity)}$$

then $\exists!\ u \in X$ that solves (P1)

1.3.2 Galerkin method

Let $X_N \subset X$ st $\dim(X_N) < \infty$ [finite dimensional]

$$\boxed{\begin{array}{l}\text{Find } u_N \in X_N \text{ st} \\ \qquad a(u_N, v_N) = \ell(v_N) \quad \forall v_N \in X_N\end{array}} \quad (P1N)$$

▷ **Cea's lemma**

If $\exists\, \alpha_a, C_a > 0$ st $\alpha_a \|u\|_X^2 \leq |a(u,u)|$
$|a(u,u)| \leq C_a \|u\|_X^2$

then $\exists!\, \tilde{u} \in X$ that solves (P1),
$\exists!\, \tilde{u}_N \in X_N$ that solves (P1N),
and they satisfy

$$\|\tilde{u} - \tilde{u}_N\|_X \leq C_{qo} \min_{w_N \in X_N} \|\tilde{u} - w_N\|_X \qquad (E4)$$

with $C_{qo} = C_a / \alpha_a$.

Best approximation

▷ **Approximation property** The discrete space $X_N \subset X$ fulfills the approximation property if

$$\lim_{N \to \infty} \inf_{v_N \in X_N} \|v - v_N\| = 0 \quad \forall v \in X.$$

* Cea's lemma + approx. property ⇒ convergence.
i.e. $\lim_{N\to\infty} \tilde{u}_N \to \tilde{u}$

1.4 Discretization → Finite elements

1.4.1 Concrete example

Recall the weak formulation for the Dirichlet Poisson problem (E3)

Given $f \in [H_0^1(\Omega)]'$, find $u \in H_0^1(\Omega)$ st

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, d\Omega}_{a(u,v)} = \underbrace{\int_\Omega f v \, d\Omega}_{\ell(v)} \qquad \forall v \in H_0^1(\Omega). \quad (E5)$$

Let $X_N \subset H_0^1(\Omega)$ st $\dim(X_N) = N$, $X_N = \text{span}\{\varphi_i\}_{i=1}^N$
Then we can write any $u_N \in X_N$ as
$$u_N = \sum_{i=1}^N \beta_i \varphi_i$$

Then, the discretization of (ET) via the Galerkin method is
$$a(u_N, v_N) = \ell(v_N) \quad \forall v_N \in X_N$$
$$\Leftrightarrow \quad a(u_N, \varphi_j) = \ell(\varphi_j) \quad j = 1, \dots, N$$

$\Leftrightarrow$ (Plug $u_N = \sum_{i=1}^N \beta_i \varphi_i$)
$$\sum_{i=1}^N \beta_i \, a(\varphi_i, \varphi_j) = \ell(\varphi_j) \quad j = 1, \dots, N$$

$$\Leftrightarrow \quad S \underline{\beta} = \underline{f}$$

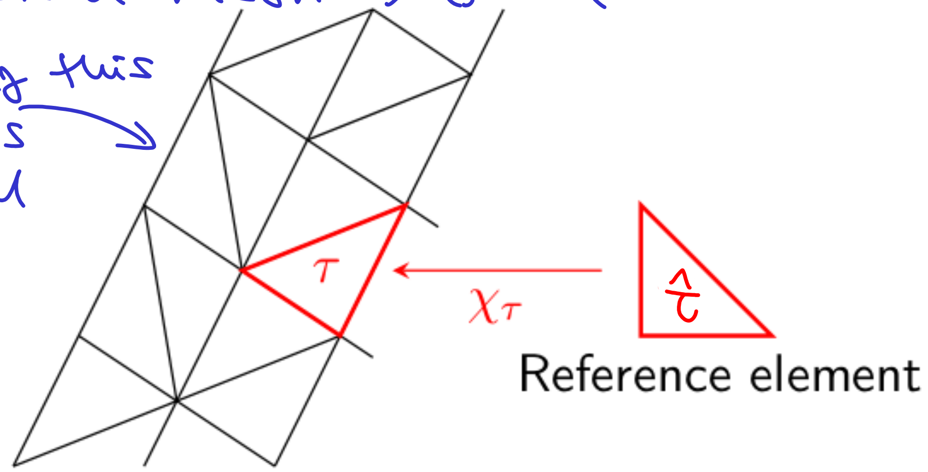
where
$$S[j,i] = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega, \quad f[j] = \int_\Omega f \varphi_j \, d\Omega$$
$\beta \triangleq$ coefficient vector for u_N.

1.4.2. Piecewise polynomial spaces X_N

i) Create a mesh $\mathcal{M}$ of Ω

ii)

Say this is $\mathcal{M}$



Then, we construct our mesh $\mathcal{M}$ as a collection of elements τ that are the image of a fixed reference element under a diffeomorphism $\chi_\tau : \hat{\tau} \to \tau$.

Let $p \in \mathbb{N}_0$ and
$\mathbb{P}_p = \{$polynomials of degree $\leq p$ on $\hat{\tau}\}$

$X_N \triangleq$ Piecewise polynomial space st for each $\tau \in \mathcal{M}$ and $w_N \in X_N$ we have
$$w_N|_\tau = \chi_\tau \circ P \quad \text{for some } P \in \mathbb{P}_p.$$

(i.e. that any element of X_N is the mapping of an element of $\mathbb{P}_p$).

You may also want to impose that all $w_N \in X_N$ are globally continuous for $X_N \subset X$.

Remark:

As $N \to \infty$, we require

- $h_{\mathcal{M}} := \max_{\tau \in \mathcal{M}} \operatorname{diam}(\tau) \to 0$ (h-FEM)
- $p \to \infty$ (p-FEM)
- $h_{\mathcal{M}} \to 0$ and $p \to \infty$ (hp-FEM)

▷ Convergence rates: For a simplex $\tau \in \mathbb{R}^d$, we define

$$h_\tau = \operatorname{diam}(\tau), \quad \left(h_{\mathcal{M}} := \max_{\tau \in \mathcal{M}} h_\tau\right)$$

shape regularity measure $\rho_\tau = h_\tau^d / |\tau|$

shape regularity of mesh $\mathcal{M}$

$$\rho_{\mathcal{M}} := \max_{\tau \in \mathcal{M}} \rho_\tau$$

Theorem [Best approx. estimates for Lagrangian FEM]

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. be a bounded polygonal / polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices.

Then, $\forall r \in \mathbb{N}$ $\exists C > 0$ depending only r and $\rho_{\mathcal{M}}$ st

$$\inf_{v_N \in X_N} \|u - v_N\|_{H^1(\Omega)} \le C \left(\frac{h_{\mathcal{M}}}{p}\right)^{\min(p+1, r)-1} \|u\|_{H^r(\Omega)}$$

* This assumes $\mathcal{M}$ is a quasi-uniform mesh.

$p \triangleq$ polynomial degree.
$r \triangleq$ "extra" regularity of u.

If $r = 4$ and $p = 3$, then $\left(\frac{h_{\mathcal{M}}}{3}\right)^3$
But $r = 1$ and $p = 3$, then $\left(\frac{h_{\mathcal{M}}}{3}\right)^0$