Mathematical Models in Nonlinear Acoustics

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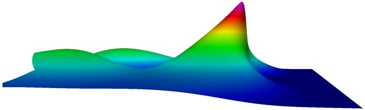


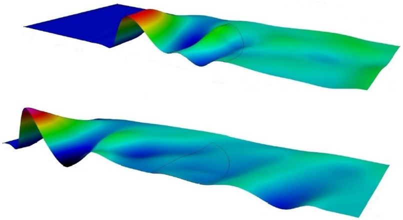
Nijmegen, August 2025

Organization

1. 9–10:30 Lecture
2. 11:00–12:30 Lecture + Problem session

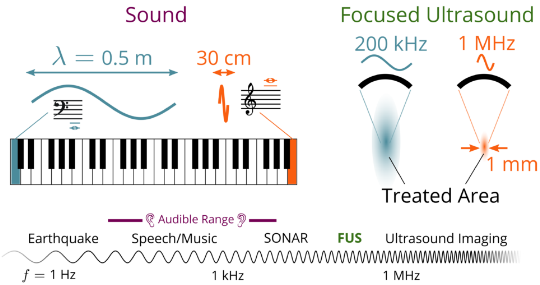
Acoustics

- Sound = Oscillatory disturbance that moves away from a source



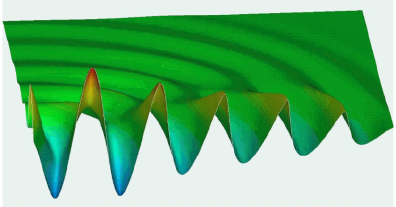
- Sound is a mechanical wave
  - it requires a medium to propagate through

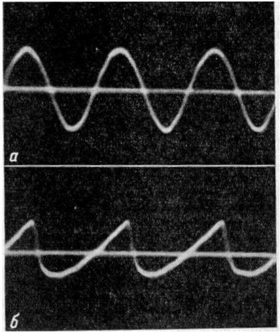
Acoustics



[Schoen Jr. & Arvanitis, https://acoustics.org/]

Nonlinear acoustics

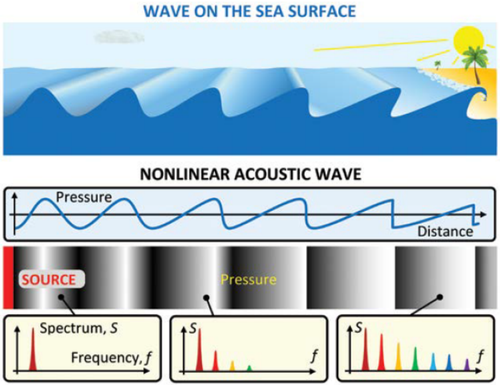




[Naugol'nykh, Romanenko, 1958]

High amplitude-to-wavelength ratio $\rightsquigarrow$ Nonlinear behavior

Nonlinear acoustic vs. water waves

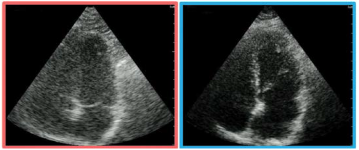


[Sapozhnikov et al., 2019]

Ultrasound applications

Diagnostic

Non-invasive imaging for real-time visualization of organs and tissues.



Linear vs. Nonlinear imaging

[Sapozhnikov et al., 2019]

Ultrasound applications

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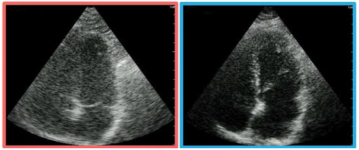
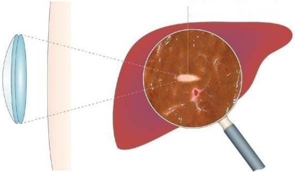


Linear vs. Nonlinear imaging

[Sapozhnikov et al., 2019]

Therapeutic

Includes cancer therapy, targeted drug delivery, and gene therapy.



Cancer therapy with HIFU

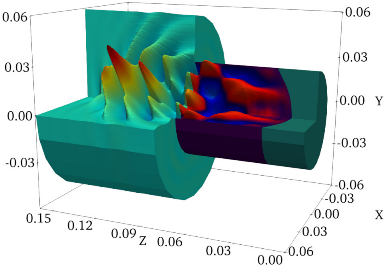
[Kennedy, 2005]

Ultrasound applications

Donders Institute for Brain, Cognition, and Behaviour, Radboud University

Modeling

- Modeling as a tool in advancing ultrasound applications



Simulation of an elasto-acoustic system

[Muhr, Nikolić, & Wohlmuth, 2022]

Modeling considerations

The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

if often not valid in practice.

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The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

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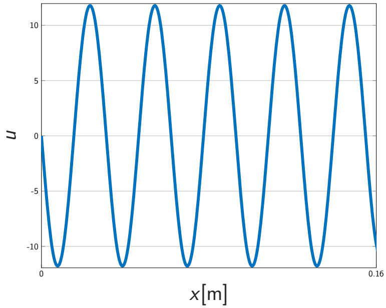
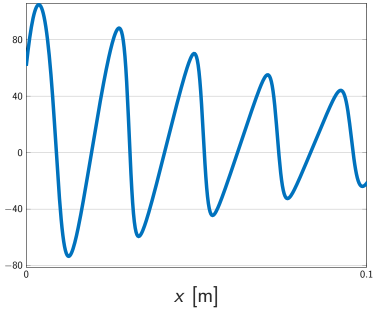
1. At high frequencies/intensities, nonlinear effects are prominent

Linear acoustics



Nonlinear acoustics



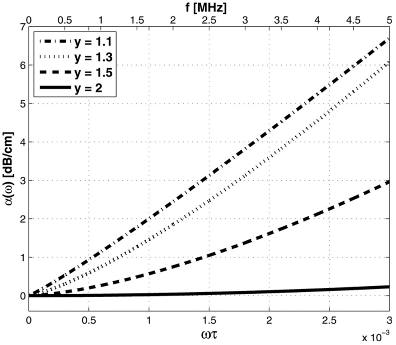
Modeling considerations

The lossless linear wave equation

$$u_{tt} - c^2 \Delta u = 0$$

if often not valid in practice.

2 In biological media, observed attenuation obeys a power law



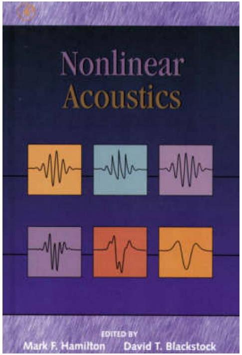
$$\text{attenuation} \sim |\omega|^{y-1}$$

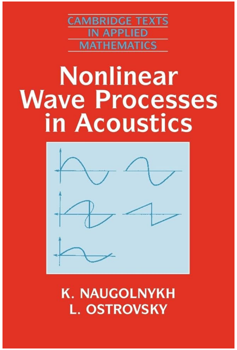
source: [Prieur & Holm, 2011]

Next

1. Models of nonlinear acoustic waves in fluids
2. Energy methods in the analysis
3. Time-fractional attenuation in biological media

Modeling literature





papers: [Westervelt, 1963], [Blackstock, 1963], [Kuznetsov, 1973], …

Acoustic field

- Sound wave perturbs background properties

$$\begin{aligned} \text{density:} \qquad & \rho = \rho_0 + \rho', \\ \text{pressure:} \qquad & p = p_0 + p', \\ \text{velocity:} \qquad & \boldsymbol{v} = \boldsymbol{v}' \end{aligned}$$

- Assumptions: $\boldsymbol{v}_0 = \mathbf{0}$, p_0, $\rho_0 = \text{const.}$

Governing system of equations in viscous fluids

1. Conservation of mass

$$\boxed{\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0}$$

2. Momentum equation

$$\boxed{\rho \boldsymbol{v}_t + \frac{\rho}{2} \nabla (\boldsymbol{v} \cdot \boldsymbol{v}) + \nabla p' = \left(\frac{4}{3}\mu + \mu_b \right) \Delta \boldsymbol{v}}$$

μ ... shear viscosity, μ_b ... bulk viscosity

Governing system of equations

3. Expanded pressure-density relation $p = p(\rho)$

$$\boxed{p - p_0 \approx c^2(\rho - \rho_0) + \frac{c^2}{\rho_0}\frac{B}{2A}(\rho - \rho_0)^2}$$

c ... speed of sound in the fluid, $A = \rho_0\left(\frac{\partial p}{\partial \varrho}\right) \approx \varrho_0 c^2$, $B = \rho_0^2\left(\frac{\partial^2 p}{\partial \varrho^2}\right)$

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- Nonlinearity parameter $\frac{B}{A}$ can be used for tissue characterization

$\rightsquigarrow$ *Acoustic nonlinearity parameter tomography*

Approximating the governing system

1. Only first- and second-order terms are retained

   - first-order terms $\sim$ linear with respect to alternating quantities
   - second-order terms $\sim$ quadratic with respect to the alternating quantities and dissipative terms

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   - first-order terms $\sim$ linear with respect to alternating quantities
   - second-order terms $\sim$ quadratic with respect to the alternating quantities and dissipative terms

For example, $\left(\frac{4}{3}\mu + \mu_b\right)\Delta \boldsymbol{v}$ is a second-order term

Approximating the governing system

2. Substitution corollary: any factor in a second-order term may be replaced by its first-order approximation

First-order approximations

$$\text{linear mass equation:} \quad \nabla \cdot \boldsymbol{v} = -\frac{1}{\rho_0}\rho'_t = -\frac{1}{\rho_0}\frac{p'_t}{c^2},$$

$$\text{linear momentum equation:} \quad \rho_0 \boldsymbol{v}'_t + \nabla p' = 0,$$

$$\text{linear state equation:} \quad \rho' = \frac{p'}{c^2}$$

Approximating the governing system

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For example, $\rho' \boldsymbol{v}_t \approx \dfrac{p'}{c^2}\left(-\dfrac{1}{\rho_0}\nabla p'\right) = -\dfrac{1}{2}\dfrac{1}{\rho_0 c^2}\nabla p'^2$

What is the resulting acoustic wave equation?

Simplifying the conservation of momentum

$$\boxed{\rho \boldsymbol{v}_t + \frac{\rho}{2} \nabla (\boldsymbol{v} \cdot \boldsymbol{v}) + \nabla p' = \left(\frac{4}{3}\mu + \mu_b \right) \Delta \boldsymbol{v}}$$

Or, equivalently,

$$\boxed{(\rho_0 + \rho')\boldsymbol{v}_t + \frac{\rho_0 + \rho'}{2} \nabla (\boldsymbol{v} \cdot \boldsymbol{v}) + \nabla p' = \left(\frac{4}{3}\mu + \mu_b \right) \Delta \boldsymbol{v}}$$

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We use

$$\frac{\rho_0 + \rho'}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v}) \approx \frac{\rho_0}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v})$$

because $\frac{\rho'}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v})$ is of third order

Simplifying the momentum equation

$$\boxed{(\rho_0 + \rho')\boldsymbol{v}_t + \frac{\rho_0}{2}\nabla\left(\boldsymbol{v}\cdot\boldsymbol{v}\right) + \nabla p' = \left(\frac{4}{3}\mu + \mu_b\right)\Delta\boldsymbol{v}}$$

Rewrite it as

Simplifying the momentum equation

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Rewrite it as

$$\boxed{\rho_0\boldsymbol{v}_t + \nabla p' = \left(\frac{4}{3}\mu + \mu_b\right)\Delta\boldsymbol{v} - \rho'\boldsymbol{v}_t - \frac{\rho_0}{2}\nabla\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)}$$

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Via the substitution corollary:

$$-\rho'\boldsymbol{v}_t \approx -\frac{p'}{c^2}\left(-\frac{1}{\rho_0}\nabla p'\right) = \frac{1}{2}\frac{1}{\rho_0 c^2}\nabla(p')^2$$

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$$\left(\frac{4}{3}\mu + \mu_b\right)\Delta\mathbf{v} \approx -\frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\nabla p'_t$$

Simplifying the conservation of momentum

Modified Euler equation

$$\rho_0 \boldsymbol{v}_t + \nabla p' = \frac{1}{2}\frac{1}{\rho c^2}\nabla (p')^2 - \frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\nabla p'_t - \frac{\rho_0}{2}\nabla\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)$$

Applying the divergence operator yields

Simplifying the conservation of momentum

Modified Euler equation

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Applying the divergence operator yields

$$\rho_0 \nabla\cdot\boldsymbol{v}_t + \Delta p' = \frac{1}{2}\frac{1}{\rho_0 c^2}\Delta (p')^2 - \frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\Delta p'_t - \frac{\rho_0}{2}\Delta\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)$$

Simplifying the mass conservation

$$\boxed{\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0}$$

We can rewrite it as

$$\boxed{\rho'_t + \rho_0 \nabla \cdot \boldsymbol{v} = -\rho' \nabla \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \rho'}$$

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$$-\boldsymbol{v} \cdot \nabla \rho' \approx -\boldsymbol{v} \cdot \nabla \frac{p'}{c^2} = \frac{\rho_0}{c^2} \boldsymbol{v} \cdot \boldsymbol{v}_t = \frac{1}{2} \frac{\rho_0}{c^2} (\boldsymbol{v}^2)_t$$

Simplifying the mass conservation

$$\boxed{\rho'_t + \rho_0 \nabla \cdot \boldsymbol{v} = \frac{1}{2}\frac{\rho_0}{c^4}(p'^2)_t + \frac{1}{2}\frac{\rho_0}{c^2}(\boldsymbol{v}^2)_t}$$

We can express ρ'_t using the (differentiated) state equation and $(\rho')^2 \approx \frac{1}{c^4}(p')^2$

$$\boxed{\frac{1}{c^2}p'_t - \frac{1}{\rho_0 c^4}\frac{B}{2A}(p'^2)_t + \rho_0 \nabla \cdot \boldsymbol{v} = \frac{1}{2}\frac{1}{\rho_0 c^4}(p'^2)_t + \frac{1}{2}\frac{\rho_0}{c^2}(\boldsymbol{v}^2)_t}$$

Combining the two simplified equations

$$\boxed{\frac{1}{c^2}p'_{tt} - \frac{1}{\rho_0 c^4}\frac{B}{2A}(p'^2)_{tt} + \rho_0 \nabla\cdot\boldsymbol{v}_t = \frac{1}{2}\frac{1}{\rho_0 c^4}(p'^2)_{tt} + \frac{1}{2}\frac{\rho_0}{c^2}(\boldsymbol{v}^2)_{tt}}$$

$$-$$

$$\boxed{\rho_0 \nabla\cdot\boldsymbol{v}_t + \Delta p' = \frac{1}{2}\frac{1}{\rho_0 c^2}\Delta(p')^2 - \frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\Delta p'_t - \frac{\rho_0}{2}\Delta\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)}$$

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Using $(p'^2)_{tt} \approx c^2 \Delta p'^2$ yields

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Using $(\boldsymbol{v}\cdot\boldsymbol{v})_{tt} \approx c^2\Delta(\boldsymbol{v}\cdot\boldsymbol{v})$ yields

$$\boxed{\frac{1}{c^2}p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\Delta p'_t = \frac{1}{\rho_0 c^4}\frac{B}{2A}((p')^2)_{tt} + \frac{\rho_0}{c^2}(\boldsymbol{v}^2)_{tt}}$$

The resulting pressure-velocity equation

$$\frac{1}{c^2} p'_{tt} - \Delta p' - \frac{1}{\rho_0 c^2} \left(\frac{4}{3}\mu + \mu_b \right) \Delta p'_t = \frac{1}{\rho_0 c^4} \frac{B}{2A} ((p')^2)_{tt} + \frac{\rho_0}{c^2} (\boldsymbol{v}^2)_{tt}$$

To express the equation in terms of one unknown, we can work with

the **acoustic velocity potential ψ**:

$$\boldsymbol{v}' = -\nabla \psi$$
$$p' = \rho_0 \psi_t$$

Kuznetsov equation

$$\boxed{\frac{1}{c^2}\psi_{ttt} - \Delta\psi_t - \frac{1}{\rho_0 c^2}\left(\frac{4}{3}\mu + \mu_b\right)\Delta\psi_{tt} = \frac{1}{c^4}\frac{B}{2A}(\psi_t^2)_{tt} + \frac{1}{c^2}(\nabla\psi\cdot\nabla\psi)_{tt}}$$

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Integrating in time yields

Kuznetsov equation

$$\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + \nabla\psi\cdot\nabla\psi\right)_t$$

Here $b = \frac{1}{\rho_0}\left(\frac{4}{3}\mu + \mu_b\right)$ is called the *sound diffusivity*.

Further simplification

Kuznetsov equation

$$\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + \nabla\psi\cdot\nabla\psi\right)_t$$

If $c^2|\nabla\psi|^2 \approx \psi_t^2$ holds (this will be the case sufficiently far from the source), then

$$\boxed{\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \frac{1}{c^2}\left(1 + \frac{B}{A}\right)(\psi_t^2)_t}$$

Westervelt equation

$$\boxed{\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \frac{1}{c^2}\left(1 + \frac{B}{A}\right)(\psi_t^2)_t}$$

Time-differentiating and using $\rho_0\psi_t = p'$ yields

Westervelt equation

$$p'_{tt} - c^2\Delta p' - b\Delta p'_t = \frac{1}{\rho_0 c^2}\left(1 + \frac{B}{2A}\right)(p'^2)_{tt}$$

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Peter Westervelt (1919–2015)

P. Westervelt, *Parametric acoustic array*, J. Acoust. Soc. Am., 1963. $(b = 0)$

Other wave equations

Kuznetsov equation

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$$\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + \nabla\psi\cdot\nabla\psi\right)_t$$

Using $(\psi_t^2)_t \approx 2\psi_t\, c^2\Delta\psi$ leads to

Blackstock equation

$$\psi_{tt} - \left(c^2 + \frac{B}{A}\psi_t\right)\Delta\psi - b\Delta\psi_t = (\nabla\psi\cdot\nabla\psi)_t$$

Other wave equations

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Effective speed is $\sqrt{c^2 + \frac{B}{A}\psi_t}$

Summary so far: Different models

Kuznetsov equation

$$\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + \nabla\psi\cdot\nabla\psi\right)_t$$

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$$\psi_{tt} - \left(c^2 + \frac{B}{A}\psi_t\right)\Delta\psi - b\Delta\psi_t = (\nabla\psi\cdot\nabla\psi)_t$$

Westervelt equation

$$p'_{tt} - c^2\Delta p' - b\Delta p'_t = \frac{1}{\rho_0 c^2}\left(1 + \frac{B}{2A}\right)((p')^2)_{tt}$$

Energy methods in the analysis

Westervelt equation

- Consider the initial boundary-value problem:

$$\boxed{\begin{aligned} u_{tt} - c^2 \Delta u - b\Delta u_t &= k(u^2)_{tt} && \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \partial\Omega \times (0,T) \\ (u, u_t) &= (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}}$$

u ... acoustic pressure $(= p')$

$\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ ... "nice" bounded domain

$T > 0$ ... final time, $\quad k = \dfrac{1}{\rho_0 c^2}\left(1 + \frac{B}{2A}\right)$

(other BCs are also of interest but more technical to analyze)

Literature on the analysis of nonlinear acoustic waves

- **Well-posedness & qualitative behavior**

[Hughes, Kato, Marsden 1977] ($b = 0$)

[Kawashima, Shibata 1992]

[Mizohata, Ukai 1993] (Kuznetsov eq.)

[Kaltenbacher, Lasiecka 2009, 2011] (Westervelt eq., Kuznetsov eq.)

[Meyer, Wilke 2011] (max L^p regularity approach)

[Dörfler, Gerner, Schnaubelt 2016] ($b = 0$)

[Kaltenbacher, Rundell 2021] (Westervelt eq. with fractional damping)

[Kaltenbacher, Meliani, Nikolić 2023] (acoustic eq. with fractional damping)

. . .

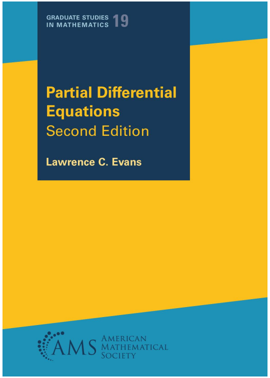
Well-posedness analysis in a linear setting

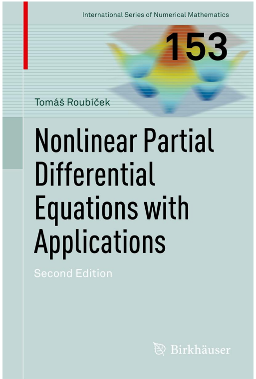
- Consider first the linear initial boundary-value problem:

$$
\boxed{
\begin{aligned}
&u_{tt} - c^2\Delta u - b\Delta u_t = 0 && \text{in } \Omega \times (0, T) \\
&u = 0 && \text{on } \partial\Omega \times (0, T) \\
&(u, u_t) = (u_0, u_1) && \text{in } \Omega \times \{0\}
\end{aligned}
}
$$

(we have set $k = 0$)

Literature on the Faedo–Galerkin method





named after Boris Galerkin (1871–1945) and Alessandro Faedo (1913–2001)

Steps of the Faedo–Galerkin method

1. Discretize the problem in space
    - Semi-discrete problem is a system of n ODEs
    - We can exploit existence theory for ODEs

Steps of the Faedo–Galerkin method

2. Uniformly bound the energy of the approximate solution u^n

- Canonical testing function is u_t^n
- Canonical energy functional is

$$\boxed{E(u^n) = \frac{1}{2}\|u_t^n(t)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u^n(t)\|^2_{L^2(\Omega)}}$$

- Here we also have dissipation

$$\boxed{D(u^n) = b\int_0^t \|\nabla u_t^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s}$$

and we will bound $E(u^n) + D(u^n)$

Steps of the Faedo–Galerkin method

3. Extract a (weakly) convergent subsequence $\{u^{n_l}\}_{l\geq 1}$

   - Check if the limit u is a solution of the original problem
   - Check if this is the only solution

$\square$

Notation: Bochner spaces

- $L^p(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \to X$, such that

$$\boxed{\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{1/p} < \infty, \quad 1 \leq p < \infty}$$

- $L^\infty(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \to X$, such that

$$\boxed{\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X < \infty}$$

- $W^{1,p}(0, T; X)$ denotes the set of Bochner measurable functions $u : (0, T) \to X$, such that

$$\boxed{u \in L^p(0, T; X), \;\; u_t \in L^p(0, T; X), \quad 1 \leq p \leq \infty}$$

Weak form of the problem

- Multiply the equation
$$u_{tt} - c^2 \Delta u - b \Delta u_t = 0$$
with $\phi \in L^2(0, T; H_0^1(\Omega))$ and integrate over space and time

Weak form of the problem

- Multiply the equation
$$u_{tt} - c^2 \Delta u - b \Delta u_t = 0$$
with $\phi \in L^2(0, T; H_0^1(\Omega))$ and integrate over space and time

- Weak form: find $u \in \mathcal{U}$, such that

$$\boxed{\int_0^T \left\{ \langle u_{tt}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + c^2 (\nabla u, \nabla \phi)_{L^2(\Omega)} + b(\nabla u_t, \nabla \phi)_{L^2(\Omega)} \right\} \mathrm{d}t = 0}$$

for all test functions $\phi \in L^2(0, T; H_0^1(\Omega))$ with $(u, u_t)|_{t=0} = (u_0, u_1)$

Step 1: Discretization in space

- $\{w_i\}_{i=1}^{\infty}$ eigenfunctions for the Dirichlet–Laplacian

$$\begin{cases} -\Delta w_i = \lambda_i w_i \\ w_i\big|_{\partial\Omega} = 0 \end{cases}$$

  - *an orthogonal basis of* $H_0^1(\Omega)$
  - *an orthonormal basis of* $L^2(\Omega)$

so that

$$(w_i, w_j)_{L^2(\Omega)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad (\nabla w_i, \nabla w_j)_{L^2(\Omega)} = 0, \ i \neq j$$

Step 1: Discretization in space

- Fix $n \in \mathbb{N}$ and let $V_n = \text{span}\{w_1, \dots, w_n\}$

$$\boxed{u^n(t) = \sum_{k=1}^{n} \xi_i^n(t) w_i(x) \in V_n}$$

- The coefficients $\xi_i^n : [0, T] \to \mathbb{R}$ are unknown

Step 1: Discretization in space

- Approximate solution $u^n(t) = \sum_{i=1}^{n} \xi_i^n(t) w_i(x)$ satisfies

$$\boxed{(u_{tt}^n, \phi^n)_{L^2(\Omega)} + c^2(\nabla u^n, \nabla \phi^n)_{L^2(\Omega)} + b(\nabla u_t^n, \phi^n)_{L^2(\Omega)} = 0}$$

for all $\phi^n \in V_n$, a.e. in time, and we set

$$\xi_i^n(0) = (u_0, w_i)_{L^2(\Omega)}, \qquad \xi_{i,t}^n(0) = (u_1, w_i)_{L^2(\Omega)}$$

Step 1: Discretization in space

- For each $n \in \mathbb{N}$, the semi-discrete problem reduces to

$$\boxed{M\boldsymbol{\xi}^n_{tt} + K\boldsymbol{\xi}^n + C\boldsymbol{\xi}^n_t = 0 \quad \text{in} \quad [0, T]}$$

supplemented by initial conditions, where $\boldsymbol{\xi}^n = [\xi^n_1 \;\; \ldots \;\; \xi^n_n]^T$

- Existence theorems for ODEs yield a unique $\xi^n_i \in C^2[0, T]$ for each $i \in [1, n]$ and thus a unique

$$u^n \in C^2([0, T]; V_n)$$

Step 2: Uniform bounds in n

Testing with $\phi^n = u_t^n \in V_n$ leads to

$$\int_0^t \left\{(u_{tt}^n, u_t^n)_{L^2(\Omega)} + c^2(\nabla u^n, \nabla u_t^n)_{L^2(\Omega)} + b(\nabla u_t^n, \nabla u_t^n)_{L^2(\Omega)}\right\} \, \mathrm{d}s = 0$$

Step 2: Uniform bounds in n

Testing with $\phi^n = u_t^n \in V_n$ leads to

$$\int_0^t \left\{(u_{tt}^n, u_t^n)_{L^2(\Omega)} + c^2(\nabla u^n, \nabla u_t^n)_{L^2(\Omega)} + b(\nabla u_t^n, \nabla u_t^n)_{L^2(\Omega)}\right\} \,\mathrm{d}s = 0$$

- Integration by parts in time yields

$$\int_0^t (u_{tt}^n, u_t^n)_{L^2(\Omega)} \,\mathrm{d}t = \frac{1}{2}\|u_t^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t$$

$$c^2 \int_0^t (\nabla u^n, \nabla u_t^n)_{L^2(\Omega)} \,\mathrm{d}t = \frac{c^2}{2}\|\nabla u^n(s)\|_{L^2(\Omega)}^2 \Big|_0^t$$

Step 2: Uniform bounds in n

Energy identity

$$\frac{1}{2}\|u_t^n(s)\|^2_{L^2(\Omega)}\Big|_0^t + \frac{c^2}{2}\|\nabla u^n(s)\|^2_{L^2(\Omega)}\Big|_0^t + b\int_0^t \|\nabla u_t^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s = 0$$

Step 2: Uniform bounds in n

Energy identity

$$\frac{1}{2}\|u_t^n(s)\|_{L^2(\Omega)}^2\Big|_0^t + \frac{c^2}{2}\|\nabla u^n(s)\|_{L^2(\Omega)}^2\Big|_0^t + b\int_0^t \|\nabla u_t^n(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s = 0$$

- Equivalently,

$$\begin{aligned}\frac{1}{2}\|u_t^n(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u^n(t)\|_{L^2(\Omega)}^2 + b\int_0^t \|\nabla u_t^n(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \\ = \frac{1}{2}\|u_t^n(0)\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u^n(0)\|_{L^2(\Omega)}^2\end{aligned}$$

Step 2: Uniform bounds in n

- The derived identity implies that

$$E[u^n](t) + D[u^n](t) = E[u^n](0), \quad t \in (0, T)$$

and thus we have a uniform bound with respect to n:

$$\boxed{\frac{1}{2}\|u_t^n(t)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u^n(t)\|^2_{L^2(\Omega)} + b\int_0^t \|\nabla u_t^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq C, \quad t \in (0, T)}$$

- We can additionally estimate $\|u_{tt}^n\|_{L^2(0,T;H^{-1}(\Omega))}$ via the semi-discrete PDE

Step 3: Passing to the limit

- The Faedo–Galerkin method is combined with a compactness argument

> *Any bounded set in a reflexive Banach space is weakly sequentially compact:*
>
> any sequence in a bounded set has a weakly converging subsequence.

- Weak convergence in a Banach space B:

> **Def.** Let u^n, $u \in B$. If for any $f \in B'$ (dual space)
>
> $$f(u^n) \to f(u)$$
>
> as $n \to \infty$, then we say that
>
> $$u^n \rightharpoonup u \quad \text{weakly in } B.$$

Step 3: Passing to the limit

- Thus $\{u^n\}_{n\geq 1}$ has a convergent subsequence

$$u^n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega))$$

and

$$u_t^n \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;H_0^1(\Omega))$$

and

$$u_{tt}^n \rightharpoonup u_{tt} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega))$$

Step 3: Passing to the limit

Passing to the limit shows that u satisfies

$$\int_0^T \left\{ \langle u_{tt}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + c^2 (\nabla u, \nabla \phi)_{L^2(\Omega)} + b(\nabla u_t, \nabla \phi)_{L^2(\Omega)} \right\} \mathrm{d}t = 0$$

for all $\phi \in L^2(0, T; H_0^1(\Omega))$

Step 3: Passing to the limit

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for all $\phi \in L^2(0, T; H_0^1(\Omega))$

It remains to prove that

- $(u, u_t)|_{t=0} = (u_0, u_1)$
- u satisfies the same energy bound
- the constructed solution u is the only solution

$\square$

What changes in the Faedo–Galerkin method for nonlinear acoustic equations?

Westervelt equation

- Consider the initial boundary-value problem:

$$\boxed{\begin{aligned} & u_{tt} - c^2\Delta u - b\Delta u_t = k(u^2)_{tt} && \text{in } \Omega \times (0,T) \\ & u = 0 && \text{on } \partial\Omega \times (0,T) \\ & (u, u_t) = (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}}$$

for simplicity, let $k = \text{const.} > 0$

(the proof can be easily adapted to $k \in L^\infty(\Omega)$)

Westervelt equation

- Rewrite the equation:

$$\boxed{\begin{aligned} &((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0 && \text{in } \Omega \times (0,T) \\ &u = 0 && \text{on } \partial\Omega \times (0,T) \\ &(u, u_t) = (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}}$$

Approach in the analysis

Consider a linearization

$$((1-2k u^*)u_t)_t - c^2\Delta u - b\,\Delta u_t = 0$$

Approach in the analysis

Consider a linearization

$$((1-2ku^*)u_t)_t - c^2\Delta u - b\,\Delta u_t = 0$$

Define the mapping

$$\mathcal{T} : u^* \mapsto u$$

with

- u^* in a ball $\mathbb{B} \subset \mathcal{U}$ (solution space)
- u solves the linearized problem

The **fixed point** $u^* = u$ solves the nonlinear problem.

Things to keep in mind

Quasilinear evolution

$$\boxed{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0}$$

Needed to avoid degeneracy:

$$1-2ku > 0 \quad \text{in } \Omega \times (0,T)$$

Things to keep in mind

Quasilinear evolution

$$\boxed{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0}$$

Needed to avoid degeneracy:

$$u < \frac{1}{2k} \quad \text{in } \Omega \times (0, T)$$

Things to keep in mind

Quasilinear evolution

$$\boxed{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0}$$

Needed to avoid degeneracy:

$$\|u(t)\|_{L^\infty(\Omega)} < \frac{1}{2k}, \quad t \in (0,T)$$

Things to keep in mind

Quasilinear evolution

$$\boxed{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0}$$

Approach via an embedding:

$$\|u(t)\|_{L^\infty(\Omega)} \leq C\|u(t)\|_{H^2(\Omega)} < \frac{1}{2k}, \quad t \in (0,T)$$

Things to keep in mind

Quasilinear evolution

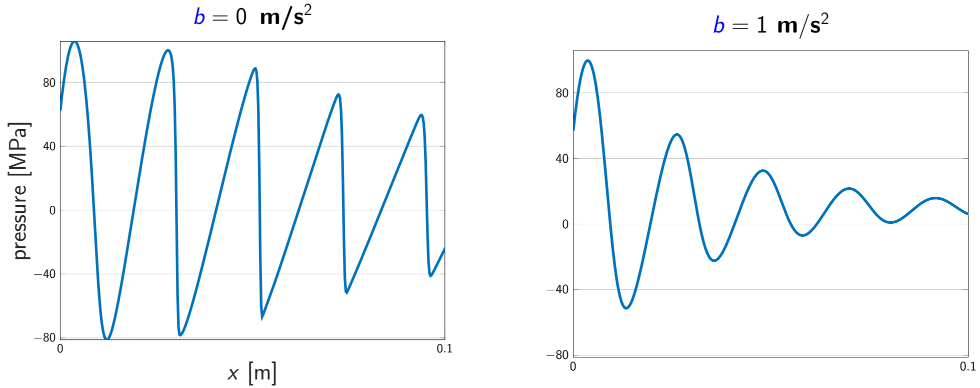
$$\boxed{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0}$$

Approach via an embedding combined with smallness of data:

$$\|u(t)\|_{H^2(\Omega)} \leq C(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}) < \frac{1}{2k}, \quad t \in (0, T)$$

Things to keep in mind: Strong damping

$$((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0$$



Notion of the solution

To avoid degeneracy, we need $u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$

$\rightsquigarrow$ we require that u satisfies

$$\boxed{\begin{gathered}\int_0^T \int_\Omega \left\{((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t\right\} \phi \,\mathrm{d}x\mathrm{d}t = 0 \\ \text{for all } \phi \in L^2(0,T; L^2(\Omega)) \text{ with } (u,u_t)|_{t=0} = (u_0,u_1)\end{gathered}}$$

The fixed-point argument

Apply Banach's fixed-point theorem on

$$\boxed{\mathcal{T} : u^* \mapsto u}$$

mapping u^* from the ball

$$\mathbb{B} = \{u^* \in \mathcal{U} : \|u^*\|_{\mathcal{U}} \leq R,\ (u^*, u^*_t)|_{t=0} = (u_0, u_1)\}$$

to the solution u of

$$\boxed{((1 - 2ku^*)u_t)_t - c^2\Delta u - b\,\Delta u_t = 0}$$

We need that $\mathcal{U} \hookrightarrow L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$.

The fixed-point argument

The mapping $\mathcal{T} : u^* \mapsto u$ is well-defined if the linear problem is well-posed:

$$\boxed{((1-2ku^*)u_t)_t - c^2\Delta u - b\,\Delta u_t = 0}$$

- We can rely on the Galerkin approach to prove it
- We need a bound on $\|u(t)\|_{H^2(\Omega)}$

The fixed-point argument

The mapping $\mathcal{T} : u^* \mapsto u$ is well-defined if the linear problem is well-posed:

$$\boxed{((1-2ku^*)u_t)_t - c^2\Delta u - b\,\Delta u_t = 0}$$

- We can rely on the Galerkin approach to prove it
- We need a bound on $\|u(t)\|_{H^2(\Omega)}$
- Is the linearized problem non-degenerate? Yes, for small R:

$$\|u^*\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\|u^*\|_{\mathcal{U}} \leq R$$

Galerkin method for the linearization

We set up the semi-discretization as before and consider

$$\int_\Omega \left\{((1-2ku^*)u_t^n)_t - c^2\Delta u^n - b\Delta u_t^n\right\} \phi^n \, \mathrm{d}x = 0$$

for all $\phi^n \in V_n$ a.e. in time, with $(u^n, u_t^n)|_{t=0} = (u_0^n, u_1^n)$

Galerkin method for the linearization

We set up the semi-discretization as before and consider

$$\boxed{\begin{gathered}\int_\Omega \left\{((1-2ku^*)u_t^n)_t - c^2\Delta u^n - b\Delta u_t^n\right\}\phi^n\,\mathrm{d}x = 0\\ \text{for all } \phi^n \in V_n \text{ a.e. in time, with } (u^n, u_t^n)|_{t=0} = (u_0^n, u_1^n)\end{gathered}}$$

Testing strategy: $\phi^n = -\Delta u^n \in V_n$

Galerkin method for the linearization

We set up the semi-discretization as before and consider

$$\boxed{\begin{aligned}&\int_\Omega \left\{((1-2ku^*)u_t^n)_t - c^2\Delta u^n - b\Delta u_t^n\right\}\phi^n \,\mathrm{d}x = 0\\ &\text{for all } \phi^n \in V_n \text{ a.e. in time, with } (u^n, u_t^n)|_{t=0} = (u_0^n, u_1^n)\end{aligned}}$$

Testing strategy: $\phi^n = -\Delta u^n \in V_n$

$$b\int_0^t (\Delta u_t^n, \Delta u^n)_{L^2(\Omega)}\,\mathrm{d}s = \frac{b}{2}\|\Delta u^n(s)\|^2_{L^2(\Omega)}\Big|_0^t$$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = -\Delta u^n$ leads to

$$\boxed{\begin{aligned} &\frac{b}{2}\|\Delta u^n(s)\|^2_{L^2(\Omega)}\Big|_0^t + c^2\int_0^t \|\Delta u^n\|^2_{L^2(\Omega)}\,\mathrm{d}s \\ &\leq \int_0^t \|(1-2ku^*)u^n_{tt} - 2ku^*_t u^n_t\|_{L^2(\Omega)}\|\Delta u^n\|_{L^2(\Omega)}\,\mathrm{d}s \end{aligned}}$$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = -\Delta u^n$ leads to

$$\boxed{\begin{aligned}&\frac{b}{2}\|\Delta u^n(s)\|^2_{L^2(\Omega)}\Big|_0^t + c^2\int_0^t \|\Delta u^n\|^2_{L^2(\Omega)}\,\mathrm{d}s\\ \leq &\int_0^t \|(1-2ku^*)u^n_{tt} - 2ku^*_t u^n_t\|_{L^2(\Omega)}\|\Delta u^n\|_{L^2(\Omega)}\,\mathrm{d}s\end{aligned}}$$

Additional testing needed: $\phi^n = u^n_{tt} \in V_n$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = u_{tt}^n$ and using

$$
\begin{aligned}
\int_0^t ((1-2ku^*)u_{tt}^n, u_{tt}^n)_{L^2(\Omega)}\,\mathrm{d}s &= \int_0^t \|\sqrt{1-2ku^*}\,u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\geq \gamma \int_0^t \|u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}
$$

Faedo–Galerkin method for the linearization

Testing with $\phi^n = u_{tt}^n$ and using

$$\int_0^t ((1-2ku^*)u_{tt}^n, u_{tt}^n)_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t \|\sqrt{1-2ku^*}\,u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s$$
$$\geq \gamma \int_0^t \|u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s$$

leads to

$$\gamma \int_0^t \|u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{b}{2}\|\nabla u_t^n(s)\|^2_{L^2(\Omega)}\Big|_0^t$$
$$\leq \frac{c^4}{2\varepsilon}\int_0^t \|\Delta u^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{1}{\varepsilon}k^2 \int_0^t \|u_t^*\|^2_{L^3(\Omega)}\|u_t^n\|^2_{L^6(\Omega)}\,\mathrm{d}s$$
$$+\,\varepsilon \int_0^t \|u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s$$

for any $\varepsilon > 0$

Faedo–Galerkin method for the linearization

Adding the two estimates leads to

$$\boxed{\begin{aligned}&\frac{\gamma}{4}\int_0^t \|u_{tt}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{b}{2}\|\nabla u_t^n(t)\|^2_{L^2(\Omega)} + \frac{b}{2}\|\Delta u^n(t)\|^2_{L^2(\Omega)}\\ &\leq C(u^*, T)\left(\|u_1\|^2_{H^1(\Omega)} + \|u_0\|^2_{H^2(\Omega)}\right)\end{aligned}}$$

- The n-uniform bound is the key step in the Galerkin procedure
- Solution space

$$\begin{aligned}\mathcal{U} = \{u \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) : u_t \in L^\infty(0,T;H_0^1(\Omega))\\ u_{tt} \in L^2(0,T;L^2(\Omega))\}\end{aligned}$$

Faedo–Galerkin method for the linearization

Adding the two estimates leads to

$$\boxed{\begin{aligned}&\frac{\gamma}{4}\int_0^t \|u^n_{tt}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s+\frac{b}{2}\|\nabla u^n_t(t)\|^2_{L^2(\Omega)}+\frac{b}{2}\|\Delta u^n(t)\|^2_{L^2(\Omega)}\\ &\leq C(u^*,T)\left(\|u_1\|^2_{H^1(\Omega)}+\|u_0\|^2_{H^2(\Omega)}\right)\end{aligned}}$$

- The n-uniform bound is the key step in the Galerkin procedure
- Solution space

$$\begin{aligned}\mathcal{U}=\{u\in L^\infty(0,T;H^1_0(\Omega)\cap H^2(\Omega)):u_t\in L^\infty(0,T;H^1_0(\Omega))\\ u_{tt}\in L^2(0,T;L^2(\Omega))\}\end{aligned}$$

- We can bootstrap regularity to $u_t\in L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega))$

The fixed-point argument

To apply the Banach fixed-point theorem, we need to prove that

- $\mathcal{T}(B) \subset B$
- $\mathcal{T}$ is strictly contractive

We can guarantee this by decreasing the radius of the ball

$$\boxed{\mathbb{B} = \{u^* \in \mathcal{U} : \|u^*\|_{\mathcal{U}} \leq R, \ (u^*, u_t^*)|_{t=0} = (u_0, u_1)\}}$$

which reduces the size of initial data

$\square$

Local well-posedness

Theorem. Let $b > 0$ and $(u_0, u_1) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega)$. There exists data size $r = r(T) > 0$ such that if

$$\|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)} \leq r^2,$$

then there is a unique solution $u \in \mathcal{U}$. The solution satisfies the estimate

$$\begin{aligned} &\|u_{tt}\|^2_{L^2(L^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^2(\Omega))} + \|u_t\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(H^2(\Omega))} \\ &\leq C(T)\left(\|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}\right). \end{aligned}$$

details: [Kaltenbacher & Lasiecka, 2009], try in the problem session

On the Kuznetsov equation

$$\boxed{(1 - 2ku_t)u_{tt} - c^2\Delta u - b\Delta u_t = (\nabla u \cdot \nabla u)_t}$$

- Here the non-degeneracy condition is

$$1 - 2ku_t \geq \gamma > 0$$

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> Smoother data are needed to ensure non-degeneracy and treat the gradient terms.

details: [Mizohata, Ukai, 1993]

On the Blackstock equation

$$\boxed{u_{tt} - c^2(1 + 2ku_t)\Delta u - b\Delta u_t = (\nabla u \cdot \nabla u)_t}$$

- Here $1 + 2ku_t$ is allowed to degenerate
- A bound on $\|u_t\|_{L^2(L^\infty(\Omega))}$ is still needed to handle the nonlinear terms
- Regularity-wise it is in between Westervelt and Kuznetsov equation

details: [Fritz, Nikolić, Wohlmuth, 2019], [Nikolić, Said-Houari, , 2023]

An open question

Well-posedness in heterogeneous media:

$$\boxed{((1-2ku)u_t)_t - \operatorname{div}(c^2(x)\nabla u) - \operatorname{div}(b(x)\nabla u_t) = 0}$$

where $c,\, b \in L^\infty(\Omega)$ with $c \geq c_0 > 0$, $b \geq b_0 > 0$ a.e.

Issue: Insufficient global spacial regularity because $u(t) \notin H^2(\Omega)$

Attenuation in biological media

Attenuation in biological media

- In biological media, attenuation follows a power law in terms of frequency

$$\text{attenuation} \sim \omega^{\alpha}, \quad \alpha \in (0,1)$$

How to model this in the time-domain acoustic equation? We know that

Attenuation in biological media

- In biological media, attenuation follows a power law in terms of frequency

$$\text{attenuation} \sim \omega^{\alpha}, \quad \alpha \in (0,1)$$

How to model this in the time-domain acoustic equation? We know that

$$u_t \xrightarrow{\mathcal{F}} (\imath\omega)\hat{u}$$

$$\partial_t^n u \xrightarrow{\mathcal{F}} (\imath\omega)^n \hat{u}$$

$$\partial_t^{\alpha} u \xrightarrow{\mathcal{F}} (\imath\omega)^{\alpha} \hat{u} \quad (\text{assuming } u(0) = 0)$$

Westervelt eq. with time-fractional attenuation

$$\boxed{u_{tt} - \Delta u - b\Delta\partial_t^\alpha u = \frac{\beta_a}{\rho_0 c^2}((u)^2)_{tt}, \quad \alpha \in (0,1)}$$

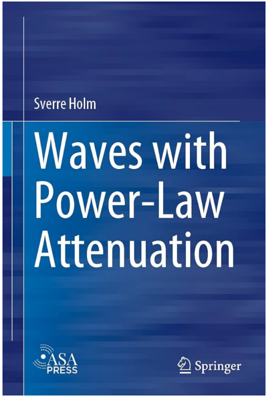
where

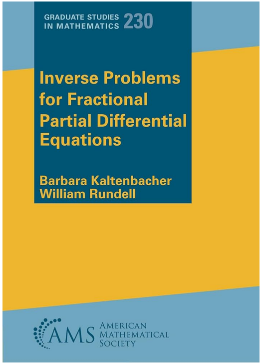
$$b = \frac{1}{\rho_0}\tau^{\alpha-1}\left(\frac{4}{3}\mu + \mu_b\right) \qquad \beta_a = 1 + \frac{B}{2A}$$

$\tau > 0$ ... relaxation parameter

modeling details: [Prieur & Holm, 2011]

Literature on fractional modeling





A glimpse into **fractional calculus**:

How to define $\partial_t^\alpha(\cdot)$?

Leibniz (1646–1716)

l'Hôpital (1661–1704)

Letter, September 30th 1695

Can the meaning of derivatives with integer order be generalized to derivatives with non-integer order?

Motivation behind fract. integration

- **Cauchy formula** for repeated integration

$$\mathrm{I}^n u(t) = \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{n-1}} u(\sigma_n)\,\mathrm{d}\sigma_n \dots \mathrm{d}\sigma_1$$

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$$= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)\, \mathrm{d}s$$

Fractional integration

- The Riemann–Liouville integral

$$\boxed{\mathrm{I}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, \mathrm{d}s, \qquad \alpha > 0}$$

- Gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, \mathrm{d}t, \quad \Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N}$$

$$\Gamma(z+1) = z\Gamma(z)$$

Fractional integration

$$\mathrm{I}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s, \qquad \alpha > 0$$

- Well-defined a.e. for $u \in L^1(0,T)$
- Can be seen using Young's convolution inequality

$$\|\mathfrak{K} * u\|_{L^1(0,T)} \leq \|\mathfrak{K}\|_{L^1(0,T)} \|u\|_{L^1(0,T)}$$

Example: Fractional integral of a constant function

$$\mathsf{I}^{\alpha} 1 =$$

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$$= \frac{1}{\alpha\Gamma(\alpha)}t^{\alpha}$$

$$= \frac{1}{\Gamma(\alpha+1)}t^{\alpha}$$

Fractional derivatives

- Notion of a fractional derivative is less straightforward

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- Idea: Differentiation as inverse of integration

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or

$$\mathrm{D}_t^\alpha u = \mathrm{D}_t \mathrm{I}^{1-\alpha} u$$

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Not the same! unless $u(0) = 0$

Djrbashian–Caputo derivative

Mkhitar Djrbashian (1918–1994)

Michele Caputo (1927–)

$$\boxed{\partial_t^{\alpha} u = \mathrm{I}^{1-\alpha} u_t, \qquad \alpha \in (0,1)}$$

[Djrbashian, Nauka, (1966)], [Caputo, Geophys. J. Int., (1967)]

Fractional derivative

- Written out

$$\boxed{\partial_t^{\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_t(s)\,\mathrm{d}s, \quad \alpha \in (0,1)}$$

- Well-defined for $u \in W^{1,1}(0, T)$

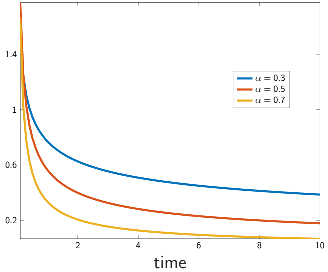
Fractional derivative

- Can be seen as a temporal convolution

$$\boxed{\partial_t^\alpha u = \mathfrak{K} * u_t}$$

using the memory kernel

$$\mathfrak{K}(s) = \frac{1}{\Gamma(1-\alpha)} s^{-\alpha}$$



$$\lim_{s \to 0^+} \mathfrak{K}(s) = +\infty$$

$$\mathfrak{K} \in L^1(0, T)$$

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$$\partial_t^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \cdot 0 \, \mathrm{d}s$$

$$= 0$$

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$$= 0$$

Side note: The Riemann–Liouville notion of the fractional derivative does not give zero

$$\mathrm{D}_t^\alpha 1 = \mathrm{D}_t \left(\mathrm{I}^{1-\alpha} 1\right) = \mathrm{D}_t \left(\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \right) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$$

Good and not so good properties

- We can impose initial conditions in the usual way

$$\boxed{\partial_t^\alpha u = f, \qquad u(0) = u_0}$$

$\rightsquigarrow$ Useful when studying fractional ODEs/PDEs

Good and not so good properties

- We can impose initial conditions in the usual way

$$\boxed{\partial_t^\alpha u = f, \qquad u(0) = u_0}$$

$\rightsquigarrow$ Useful when studying fractional ODEs/PDEs

- We know that

$$\partial_t^\alpha \partial_t u = \partial_t^{1+\alpha} u$$

but

$$\partial_t \partial_t^\alpha u = \partial_t^{\alpha+1} u + \mathfrak{K}(t) u_t(0)$$

Good and not so good properties

- No commutative or semigroup properties

$$\partial_t^\alpha \partial_t^\beta u \neq \partial_t^\beta \partial_t^\alpha u$$

$$\partial_t^\alpha \partial_t^\beta u \neq \partial_t^{\alpha+\beta} u$$

- No Leibniz rule

$$\partial_t^\alpha (uv) \neq \partial_t^\alpha u \cdot v + u \cdot \partial_t^\alpha v$$

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$\rightsquigarrow$ $\boxed{\text{No integration by parts formula}}$

Instead of integration by parts: Useful inequalities

$$\boxed{\begin{aligned} \int_0^t \int_\Omega \partial_t^\alpha y(s)\, \partial_t y(s)\,\mathrm{d}x\mathrm{d}s &\geq C_\alpha \int_0^t \|\partial_t^\alpha y(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\ \int_0^t \int_\Omega \partial_t^\alpha y(s)\, y(s)\,\mathrm{d}x\mathrm{d}s &\geq -C_\alpha \|y(0)\|^2_{L^2(\Omega)} \end{aligned}}$$

Instead of integration by parts: Useful inequalities

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⚠ We do not have a similar lower bound for

$$\int_0^t \int_\Omega \partial_t^\alpha y(s)\, y_{tt}(s)\, \mathrm{d}x\mathrm{d}s$$

What changes in the PDE analysis?

- Consider the initial boundary-value problem:

$$\boxed{\begin{aligned} & u_{tt} - c^2\Delta u - b\Delta\partial_t^\alpha u = k(u^2)_{tt} && \text{in } \Omega \times (0, T) \\ & u = 0 && \text{on } \partial\Omega \times (0, T) \\ & (u, u_t) = (u_0, u_1) && \text{in } \Omega \times \{0\} \end{aligned}}$$

∂_t^α ... Caputo partial derivative of order $\alpha \in (0, 1)$

$\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ ... smooth bounded domain

$T > 0$ ... final time

Approach in the well-posedness analysis

As before, consider a linearization

$$((1-2k{\color{blue}u^*})u_t)_t - c^2\Delta t - b\Delta\partial_t^\alpha u = 0$$

Define the mapping

$$\mathcal{T} : {\color{blue}u^*} \mapsto u,$$

with

- ${\color{blue}u^*}$ in a ball $\mathbb{B} \subset \mathcal{U}$ (solution space)
- u solves the linearized problem

The **fixed point** ${\color{blue}u^*} = u$ solves the nonlinear problem.

Differences in the Galerkin approach

- The semi-discrete problem is now a system of integro-differential equations
- Previous testing strategy does not work
  - If we test with u_{tt}^n, we cannot integrate by parts in time to bound from below

$$b \int_0^t (\nabla \partial_t^\alpha u^n, \nabla u_{tt}^n)_{L^2(\Omega)} \, \mathrm{d}s$$

Differences in energy estimates

- We need to use test functions of higher order

$$\boxed{\phi^n = \Delta^2 u_t^n}$$

- After testing and integrating over Ω,

$$(\mathfrak{a}(u^*)u_{tt}^n - c^2\Delta u^n - b\Delta\partial_t^\alpha u^n - 2ku_t^* u_t^n,\ \Delta^2 u_t^n)_{L^2(\Omega)} = 0$$

where

$$\mathfrak{a}(u^*) = 1 - 2ku^*$$

Energy estimates

- After testing with $-\Delta u_t^n$, we can use the lower bound

$$\boxed{\begin{aligned} &-b\int_0^t (\Delta\partial_t^\alpha u^n, \Delta^2 u_t^n)_{L^2(\Omega)}\,\mathrm{d}s \\ &= b\int_0^t (\nabla\Delta\partial_t^\alpha u^n, \nabla\Delta u_t^n)_{L^2(\Omega)}\,\mathrm{d}s \\ &\geq b\,C_\alpha \int_0^t \|\nabla\Delta\partial_t^\alpha u^n\|^2_{L^2(\Omega)}\,\mathrm{d}s \end{aligned}}$$

Energy estimates

- In this manner, we arrive at

$$\frac{1}{2}\|\sqrt{\mathfrak{a}}\Delta u_t^n\|_{L^2}^2\Big|_0^t + \frac{c^2}{2}\|\nabla\Delta u^n(s)\|_{L^2}^2\Big|_0^t + C_\alpha\int_0^t \|(\nabla\Delta\partial_t^\alpha u^n)(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s$$

$$\leq \frac{1}{2}\int_0^t (\mathfrak{a}_t\Delta u_t^n, \Delta u_t^n)_{L^2(\Omega)}\,\mathrm{d}s + \int_0^t (u_{tt}^n\,\Delta\mathfrak{a} + 2\nabla u_{tt}^n\cdot\nabla\mathfrak{a}, \Delta u_t^n)_{L^2(\Omega)}\,\mathrm{d}s$$

- Additional estimate of $\|u_{tt}^n\|_{H^1(\Omega)}$ follows by using the semi-discrete PDE

Energy estimates

- Energy functionals are higher-order in space

$$\boxed{E[u](t) = \|u_t(t)\|^2_{H^2(\Omega)} + \|u(t)\|^2_{H^3(\Omega)}}$$

- Solution space now is

$$\begin{aligned}\mathcal{U} = \{u : u \in L^\infty(0,T; H^3_\diamondsuit(\Omega) \cap W^{1,\infty}(0,T; H^1_0(\Omega) \cap H^2(\Omega)),\\ \partial_t^\alpha u \in L^2(0,T; H^3_\diamondsuit(\Omega))\}\end{aligned}$$

where

$$H^3_\diamondsuit(\Omega) = \{u \in H^3(\Omega) : \ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}$$

Local well-posedness

Theorem. Let the assumptions on $\mathfrak{K}$ hold and let

$$(u_0, u_1) \in H^3_\diamondsuit(\Omega) \times \left(H^1_0(\Omega) \cap H^2(\Omega)\right).$$

There exists data size $r = r(T) > 0$, such that if

$$\|u_0\|^2_{H^3(\Omega)} + \|u_1\|^2_{H^2(\Omega)} \leq r^2,$$

then there is a unique solution in $\mathcal{U}$, which satisfies

$$\|u\|^2_{\mathcal{U}} \leq C(T)(\|u_0\|^2_{H^3(\Omega)} + \|u_1\|^2_{H^2(\Omega)}).$$

details: [Kaltenbacher, Meliani, & Nikolić, 2023]

An open question

Analysis is uniform in b (we are not exploiting damping)

- Can we exploit damping to reduce the regularity assumptions on data?

On numerical analysis

On numerical analysis

A finite element semi-discretization of the strongly damped equation:

Find $u_h : [0, T] \to V_h$, such that

$$\begin{cases} (((1-2ku_h)u_{h,t})_t, \phi_h)_{L^2} + (c^2 \nabla u_h + b\nabla u_{h,t}, \nabla \phi_h)_{L^2} = 0 \\ \text{for all } \phi_h \in V_h \text{ a.e. in time, with} \\ (u_h, u_{h,t})|_{t=0} = (u_{0h}, u_{1h})|_{t=0} \end{cases}$$

Delicate point: In general, discrete solutions are not smooth enough

- Inverse inequalities or discrete embeddings have to be employed

$$\|u_h(t)\|_{L^\infty(\Omega)} \leq C h^{-d/2} \|u_h(t)\|_{L^2(\Omega)} \quad \text{or} \quad \|u_h(t)\|_{L^\infty(\Omega)} \leq C \|\Delta_h u_h(t)\|_{L^2(\Omega)}$$

On numerical analysis

- Still many open questions in this area
- Some recent advances:
  - **FEM** semi-discretization

    [Nikolić, Wohlmuth SIMA, 2019], [Hochbruck, Maier, IMAJNA, 2022], . . .
  - **Space-time** finite element methods

    [Gómez & Nikolić, IMAJNA, 2025]
  - **Multiharmonic approaches** $\rightsquigarrow$ connection to Helmholtz problems

    [Kaltenbacher & Rainer, ESAIM: M2AN, 2025]
  - Efficient methods for **time-fractional** models

    [Baker, Banjai, & Ptashnyk, Math. Comput., 2022]

Summary: Nonlinear acoustic models

- Nonlinear acoustic modeling by retaining second-order terms

Westervelt equation

$$((1-2ku)u_t)_t - c^2\Delta u - b\Delta u_t = 0$$

- Analysis via energy arguments $\rightsquigarrow$ small-data well-posedness
- Time-fractional attenuation invokes higher smoothness arguments

Next $\Rightarrow$ Problem sheet with Westervelt equation