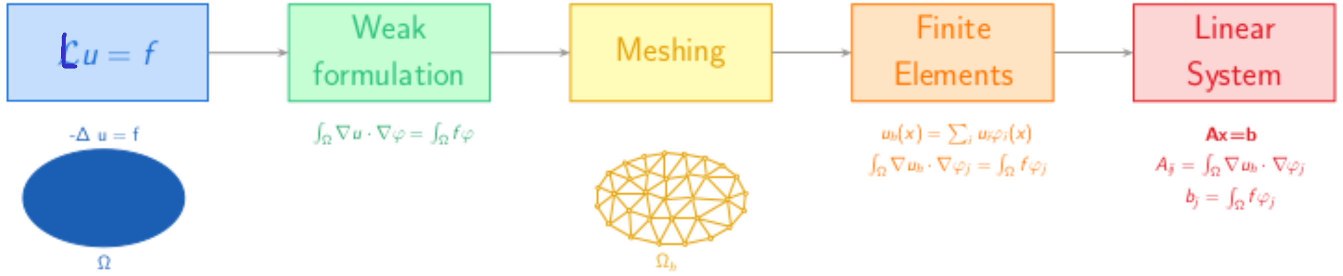
DISCLAIMER: This document contains the "whiteboard notes" taken during the "Introduction to FEM for Helmholtz" at the waves summer school in August 2025.

Therefore, they are not complete and contain some simplifications for the sake of time, etc. Moreover, the document has not been proof-read and may therefore have some minor mistakes/typos that are typical during lectures

In the coming days I will add the corresponding references and sources for the pictures that are not my own.

LECTURE 2: HELMHOLTZ MAKE THINGS A BIT COMPLEX

Brief Recap:



In L1 we discussed:

- Assumptions on Ω
- Main ingredient variational formulations
- Sobolev spaces
- Existence and uniqueness

} weak formulation

- Galerkin method

  Finite elements discretization

  Piecewise polynomial spaces X_N

  Convergence rate

} Finite elements

Recap: In the variational problem for

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we actually had that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega$$

and

$$a(u,u) = \int_\Omega |\nabla u|^2 d\Omega = |u|_{H^1(\Omega)}$$

Recall that $|u|_{H^1(\Omega)}$ is a norm in $H^1_0(\Omega)$.

Helmholtz problems: Let $\Omega \subset \mathbb{R}^d$ Lipschitz and bounded.

▷ Exterior BVP

$$(\text{BVP ext}) \begin{cases} -\Delta u - k^2 u = f & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u = g & \text{on } \partial\Omega \\ + \text{Sommerfield radiation cond.} \end{cases}$$

▷ Interior BVP

$$(\text{BVP int}) \begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

our main focus today

2.1 Variational formulation (for BVP int)

$$-\int_\Omega \Delta u\, v\, d\Omega - k^2 \int_\Omega u v\, d\Omega = \int_\Omega f v\, d\Omega \quad \forall v \in H_0^1(\Omega)$$

$$\int_\Omega \nabla u \cdot \nabla v\, d\Omega - k^2 \int_\Omega u v\, d\Omega = \int_\Omega f v\, d\Omega \quad \forall v \in H_0^1(\Omega)$$

Let us define our bilinear form to be

$$a(u,v) := \underbrace{\int_\Omega \nabla u \cdot \nabla v\, d\Omega}_{a_0(u,v)} - \underbrace{k^2 \int_\Omega u v\, d\Omega}_{c(u,v)}$$

$$a(u,u) = a_0(u,u) - c(u,u)$$

This can take any sign, so we cannot use Lax-Milgram.

2.2 Existence and uniqueness (what changes)

Let us consider something less restrictive than coercivity.

Gårding's inequality. Let X be Hilbert space, $A : X \to X'$ satisfies Gårding's inequality if there exists a compact operator $C : X \to X'$ such that

$$\langle (A+C)v, v \rangle_\Omega \geq \alpha_A \|v\|_X^2 \quad \forall v \in X$$

where $\langle \, , \rangle_\Omega$ denotes the duality pairing.

$\Leftrightarrow$ The bilinear form a satisfies Gårding's inequality if there exists a bilinear form $c(u,v) = \int_\Omega C(u)\, v\, d\Omega$ where C is a compact operator $C : X \to X'$ and such that

$$a(u,u) + c(u,u) \geq \alpha_A \|u\|_X^2 \quad \forall u \in X$$

Theorem [Fredholm alternative]
Let $K : X \to X$ be a compact op. Either
the homogeneous equation
$$(\alpha I - K)u = 0 \qquad \alpha \neq 0$$
has a non-trivial solution $u \in X$; or

the inhomogeneous equation
$$(\alpha I - K)u = f \qquad \alpha \neq 0$$
has.

What do we know?

i) $a(u,v) = a_0(u,v) + c(u,v)$

a_0 is invertible

$$c(u,v) = k^2 \int_\Omega u\, v\, d\Omega$$
$$c(u,u) = k^2 \|u\|^2_{L^2(\Omega)}$$

By Sobolev embeddings, we know that this gives us a compact perturbation

$$a(u,v) = \underbrace{a_0(u,v)}_{\text{coercive}} - \underbrace{c(u,v)}_{\text{compact}}$$

$$a(u,u) + c(u,u) = a_0(u,u) \geq |u|^2_{H^1(\Omega)}$$

Gårding's inequality ✓

$$a_0(u,v) = \langle A_0 u, v \rangle$$

then
$$a(u,v) = \langle A_0 u, v \rangle - \langle C u, v \rangle$$
so
$$a(A_0^{-1} u, v) = \langle u, v \rangle - k^2 \langle A_0^{-1} u, v \rangle$$
$$= \langle \underbrace{(Id - A_0^{-1} k^2)}_{Id - K} u, v \rangle$$

we can use Fredholm alternative

Corollary Let $A: X \to X'$ be a bounded linear op. satisfying Gårding's inequality.
If A is also injective, then $\exists!$ solution $u \in X$ of the operator equation $Au = f$

Now, let us use this for Helmholtz.
When can we guarantee injectivity?

Proposition:
If $k^2 = \lambda_i$, $\lambda_i \hat{=}$ eigenvalue of the Laplacian with Dirichlet bc. (interior problem)

$$\begin{cases} -\Delta u_i = \lambda_i u_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

then, there exist non-trivial solution of the Homogeneous Dirichlet BVP for Helmholtz.

If k^2 is not an eigenvalue of the Dirichlet eigenvalue problem for the Laplacian, then we have that $A := -\Delta u - k^2 Id$ is injective ($A: H_0^1(\Omega) \to H^{-1}(\Omega)$).

2.3 Discretization

Let us again consider piecewise polynomial spaces X_N.

Ⓠ When do we have the approximation property?

By Whittaker-Shannon-Nyquist criterion we expect

$$\dim(X_N) \sim k^d \qquad (d \hat{=} \text{dimension of } \Omega \subset \mathbb{R}^d).$$

necessary and sufficient to maintain accuracy as $k \to \infty$.

Conclusion after playing with the code
Larger k needs smaller h.

2.3.1 Convergence

Thm ("Cea's Lemma v2") let $a \in \mathcal{L}(X \times X, \mathbb{R})$ satisfy the Gårding inequality and let the discrete stability condition

$$\alpha_A \|w_N\|_X \leq \sup_{v_N \in X_N \setminus \{0\}} \frac{a(w_N, v_N)}{\|v_N\|_X} \qquad (E1)$$

be satisfied for all $w_N \in X_N$.
Then there exists a unique solution $u_N \in X_N$ of

$$a(u_N, v_N) = a_0(u_N, v_N) - c(u_N, v_N) = \ell(v_N)$$

$$\forall v_N \in X_N.$$

This unique solution satisfies

$$\|u_N\|_X \leq \frac{1}{\alpha_A} \|f\|_{X'}$$

and the error estimate

$$\|u - u_N\|_X \leq \left(1 + \frac{C_A}{\alpha_A}\right) \inf_{v_N \in X_N} \|u - v_N\|_X$$

Thm let $a(u,v) = \langle Au, v \rangle$ be such that it satisfies Gårding's inequality and A is injective.

let $X_N \subset X$ be a dense sequence of spaces.
Then, there exists an index $N_0 \in \mathbb{N}_0$ st the discrete inf-sup condition (E1) is satisfied for $N \geq N_0$.