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Free-surface waves using extended shallow water models part 2

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WAVES.NL Summer school, Nijmegen, 25 August 2025

Schedule

| Time | Monday | Tuesday | Wednesday | Thursday | Friday |
|-------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 8:50–9:00 | <i>Opening</i> | | | | |
| 9:00–10:30 | L3 | L5 | L2 | L4 | L6 |
| 10:30–11:00 | <i>Coffee break</i> | <i>Coffee break</i> | <i>Coffee break</i> | <i>Coffee break</i> | <i>Coffee break</i> |
| 11:00–12:30 | L1 | L1 | L2 | L4 | L6 |
| 12:30–13:30 | <i>Lunch</i> | <i>Lunch</i> | <i>Lunch</i> | <i>Lunch</i> | <i>Lunch</i> |
| 13:30–15:00 | L3 | L5 | L3 | L5 | |
| 15:00–15:30 | <i>Coffee break</i> | <i>Coffee break</i> | | | |
| 15:30–17:00 | Poster session | L1 | | | |
| 17:45–19:00 | | | <i>Social event</i> | | |

L1: Mon 11-12:30

- overview
- motivation
- derivation

L2: Tue 11-12:30

- analysis

L3: Tue 15:30-17

- selected papers
- outlook

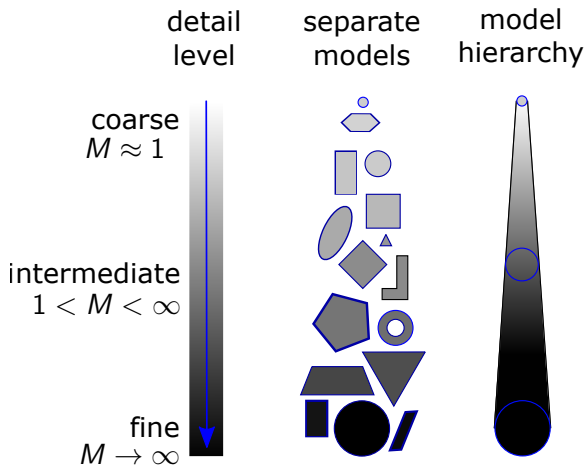
Content of this talk

1 Repetition

2 Analysis

1 Repetition

Hierarchical mathematical modeling



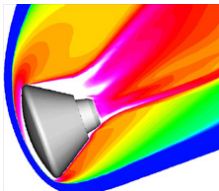
Hierarchical moment models

Advantages

1. general derivation
 2. structure preserving
 3. accurate results
- ⇒ adaptive simulations

Motivation: Rarefied gases and shallow flows

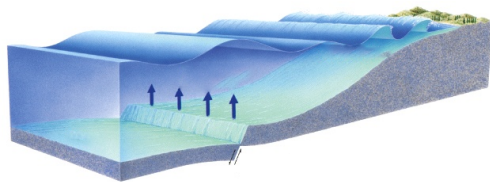
a) rarefied gases



Scale is the *Knudsen number*

$$Kn = \frac{\text{mean free path length}}{\text{reference length}} = \frac{l}{L}$$

b) shallow flows



Scale is the *shallowness*

$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

b)

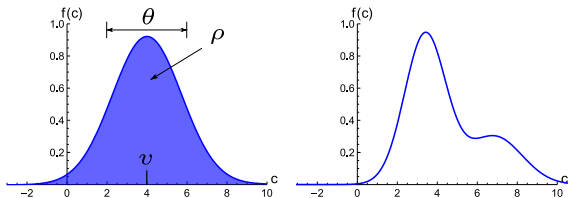


Model equation: Rarefied gases and shallow flows

a) rarefied gases

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

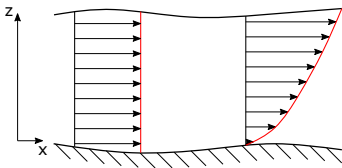


Euler equations \rightarrow ?

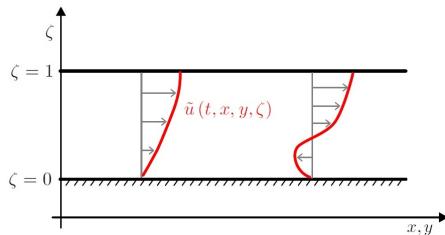
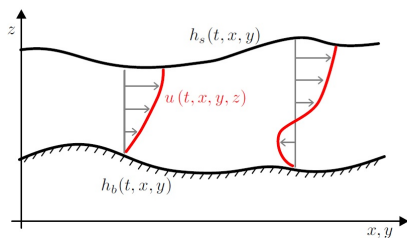
b) shallow flows

Incompressible Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$



SWE \rightarrow ?

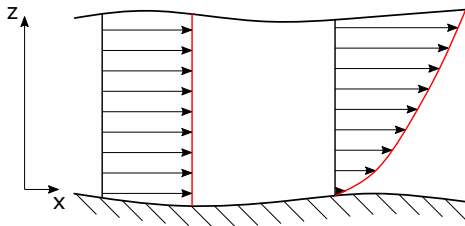


$$z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h}$$

$$z \in [h_b(t, x), h_s(t, x)] \quad \Rightarrow \quad \zeta \in [0, 1]$$

Represent variations over depth with polynomials

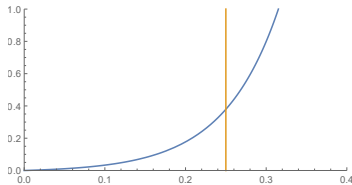
$$u(t, x, z) = \underbrace{u_m(t, x)}_{\text{mean of } u} + \sum_{i=1}^M \alpha_i(t, x) \underbrace{\phi_i\left(\frac{z - h_b}{h_s - h_b}\right)}_{\phi_i(\zeta)}$$



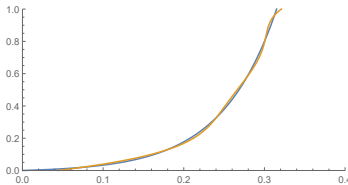
Represent variations over depth with polynomials

$$u(t, x, z) = \underbrace{u_m(t, x)}_{\text{mean of } u} + \sum_{i=1}^M \alpha_i(t, x) \underbrace{\phi_i\left(\frac{z - h_b}{h_s - h_b}\right)}_{\phi_i(\zeta)}$$

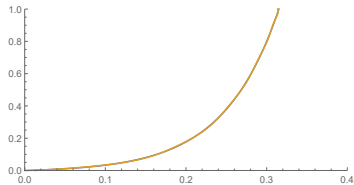
$M = 0$



$M = 5$



$M = 10$



Moment models

1. underlying model equation

$$\mathcal{D}(\mathbf{U}(t, \mathbf{x}, \mathbf{y})) = 0$$

2. expansion with ansatz

$$\mathbf{U}_{\mathbb{M}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{M}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^{\mathbf{U}}(\mathbf{y})$$

3. *moment* projection

$$\int_{\Omega} \mathcal{D}(\mathbf{U}_{\mathbb{M}}(t, \mathbf{x}, \mathbf{y})) \cdot \Psi_j^{\mathbf{U}}(\mathbf{y}) \, d\mathbf{y} \text{ for } j \in \mathbb{M}$$

Moment model

Hierarchical system of lower-dimensional PDEs for $\mathbf{U}_i(t, \mathbf{x})$

Moment models [GRAD, 1949], [KOWALSKI, TORRILHON, 2018]

1. underlying model equation: incompressible NSE

$$\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g} \quad (*)$$

2. expansion: polynomial ansatz

$$u(t, x, \zeta) = u_m(t, x) + \sum_{i=1}^M \alpha_i(t, x) \phi_i(\zeta)$$

3. *moment* projection: depth integration

$$\int_0^1 (*) \cdot \phi_j(\zeta) d\zeta, \quad j = 0, \dots, M$$

Moment model

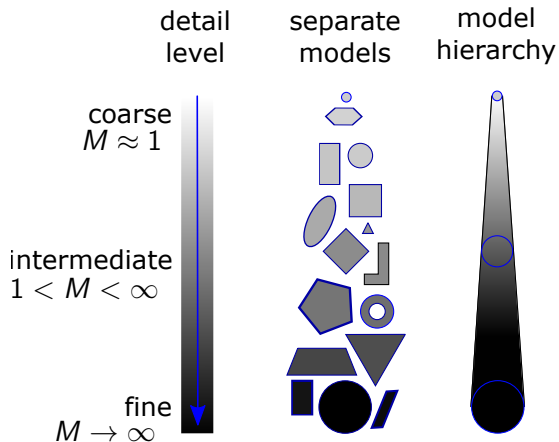
Hierarchical system of lower-dimensional PDEs for $h(t, x)$, $u_m(t, x)$, $\alpha_i(t, x)$

($M = 0$)

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} \end{pmatrix} = - \begin{pmatrix} 0 \\ gh \partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν .

General derivation of hierarchical moment models



Ansatz:

$$\mathbf{U}_{\mathbb{M}}(t, \mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbb{M}} \mathbf{U}_i(t, \mathbf{x}) \cdot \Phi_i^U(\mathbf{y})$$

Projection:

$$\int_{\Omega} \mathcal{D}(\mathbf{U}_{\mathbb{M}}(t, \mathbf{x}, \mathbf{y})) \cdot \Psi_j^U(\mathbf{y}) d\mathbf{y} \text{ for } j \in \mathbb{M}$$

Other models

- uncertainty quantification
- traffic flow

$$M = 1$$

First order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta)$, $\phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{pmatrix}$$

Shallow Water Moment Equations [KOWALSKI, TORRILHON, 2019]

$M = 2$

Second order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta) + \alpha_2 \phi_2(\zeta)$, $\phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} + \frac{1}{3} h \alpha_1^2 + \frac{1}{5} h \alpha_2^2 \\ 2hu_m \alpha_1 + \frac{4}{5} h \alpha_1 \alpha_2 \\ 2hu_m \alpha_2 + \frac{2}{3} h \alpha_1^2 + \frac{2}{7} h \alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3 \left(u_m + \alpha_1 + \alpha_2 + 4 \frac{\lambda}{h} \alpha_1 \right) \\ 5 \left(u_m + \alpha_1 + \alpha_2 + 12 \frac{\lambda}{h} \alpha_2 \right) \end{pmatrix}.$$

2 Analysis

Question: What are desirable model properties?

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- high accuracy
- low complexity
- efficiency
- adaptivity
- extendability
- analytical form

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- hyperbolicity
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(✓)

?

2.1 conservation

Conservation properties

Second order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta) + \alpha_2 \phi_2(\zeta)$, $\phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

Conservation of mass ✓

no conservation of momentum with bottom force (as expected) ✓

non-conservative form of equations

2.2 hyperbolicity

Definition (hyperbolicity)

A PDE of the form

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) + A \frac{\partial}{\partial x} \mathbf{u}(t, x) = 0,$$

for $\mathbf{u}: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is hyperbolic if A can be diagonalized with real eigenvalues.

Hyperbolicity remarks

- Hyperbolic systems can (locally) be decomposed into a system of scalar PDEs. We use the decomposition $A = V\Lambda V^{-1}$ with $\Lambda = \text{diag}(\text{EV}(A))$ and $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ with \mathbf{v}_i the eigenvectors of A . Then we use the new variables $\mathbf{w} = V^{-1}\mathbf{v}$ and $\mathbf{v} = V\mathbf{w}$ to write

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{u}(t, x) + A\frac{\partial}{\partial x}\mathbf{u}(t, x) &= 0, \\ \Rightarrow V\frac{\partial}{\partial t}\mathbf{w}(t, x) + AV\frac{\partial}{\partial x}\mathbf{w}(t, x) &= 0, \\ \Rightarrow V^{-1}V\frac{\partial}{\partial t}\mathbf{w}(t, x) + V^{-1}V\Lambda V^{-1}V\frac{\partial}{\partial x}\mathbf{w}(t, x) &= 0, \\ \Rightarrow \frac{\partial}{\partial t}\mathbf{w}(t, x) + \Lambda\frac{\partial}{\partial x}\mathbf{w}(t, x) &= 0.\end{aligned}$$

This is a system of decoupled PDEs in the variable \mathbf{w} .

- Hyperbolicity is lost if eigenvalues are complex or if there exists no full set of eigenvectors.
- Hyperbolic systems describe the propagation of information with real, bounded propagation speeds.

($M = 0$)

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} \end{pmatrix} = - \begin{pmatrix} 0 \\ gh \partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν .

($M = 0$)

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g \frac{h^2}{2} \end{pmatrix} = - \begin{pmatrix} 0 \\ gh \partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν .

Propagation speeds are

$$\lambda_{1,2} = u_m \pm \sqrt{gh}.$$

Shallow Water Moment Equations [KOWALSKI, TORRILHON, 2019]

$$M = 1$$

First order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta)$, $\phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{pmatrix}$$

Propagation speeds are

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \alpha_1^2} \text{ and } \lambda_3 = u_m.$$

Shallow Water Moment Equations [KOWALSKI, TORRILHON, 2019]

$M = 2$

Second order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta) + \alpha_2 \phi_2(\zeta)$, $\phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1) \\ 5(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2) \end{pmatrix}.$$

Propagation speeds: ?

Propagation speeds ($M = 2$)

$$\partial_t \mathbf{u}_M + \mathbf{A}(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{0}$$

variable set

$$\mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

$$\mathbf{A}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\ -2\alpha_1 u_m - \frac{4}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m + \alpha_2 & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 - 2u_m\alpha_2 - \frac{2}{7}\alpha_2^2 & 2\alpha_2 & -\frac{\alpha_1}{3} & u_m + \frac{3\alpha_2}{7} \end{pmatrix} \quad (M = 2)$$

Propagation speeds ($M = 2$)

$$\partial_t \mathbf{u}_M + \mathbf{A}(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{0}$$

variable set

$$\mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

$$\mathbf{A}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\ -2\alpha_1 u_m - \frac{4}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m + \alpha_2 & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 - 2u_m\alpha_2 - \frac{2}{7}\alpha_2^2 & 2\alpha_2 & -\frac{\alpha_1}{3} & u_m + \frac{3\alpha_2}{7} \end{pmatrix} \quad (M = 2)$$

eigenvalues can become complex \Rightarrow loss of hyperbolicity \nexists

Propagation speeds ($M = 4$) rarefied gases

$$\partial_t \mathbf{u}_M + \mathbf{A}(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{0}$$

rarefied gases

$$\mathbf{u}_M = (\rho, v, \theta, f_3, f_4, \dots, f_M)^T \in \mathbb{R}^{M+1}$$

$$\mathbf{A}_{\text{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix} \quad (M = 4)$$

eigenvalues can become imaginary \Rightarrow loss of hyperbolicity

Breakdown of hyperbolicity

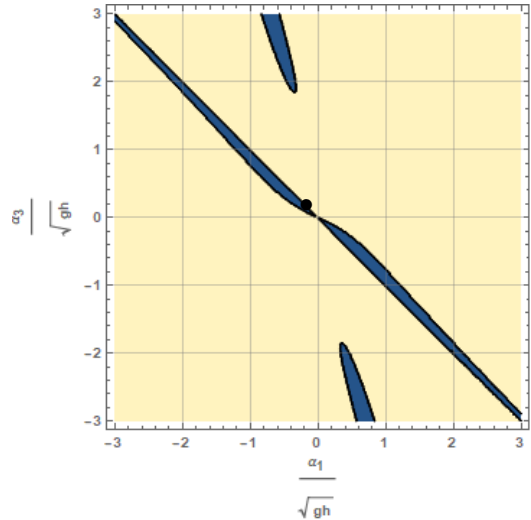
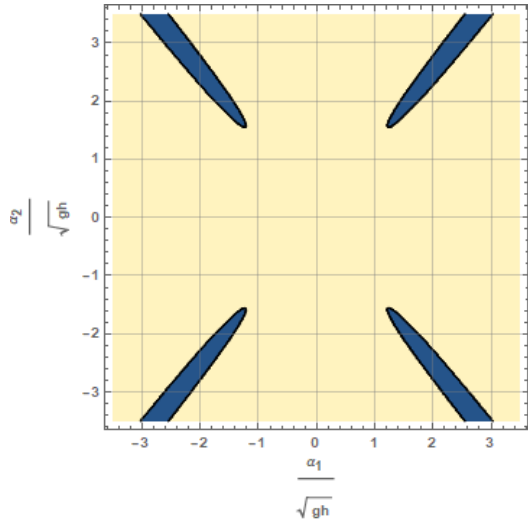


Figure: Second order (left) and third order (right, for $\alpha_2 = 0$)

Breakdown of hyperbolicity

Simulation test case

Simple transport of smooth wave

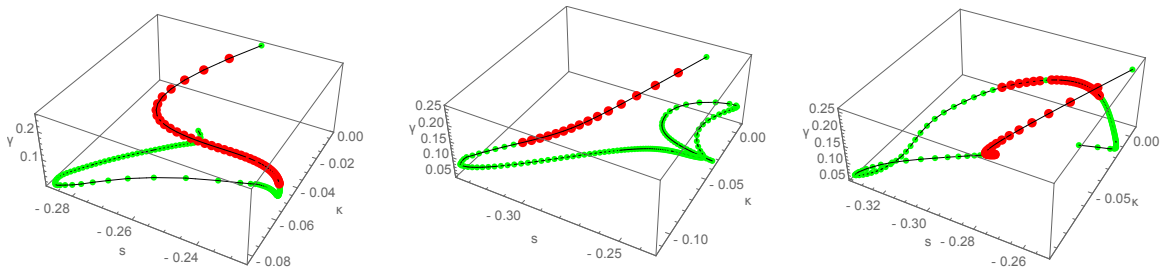
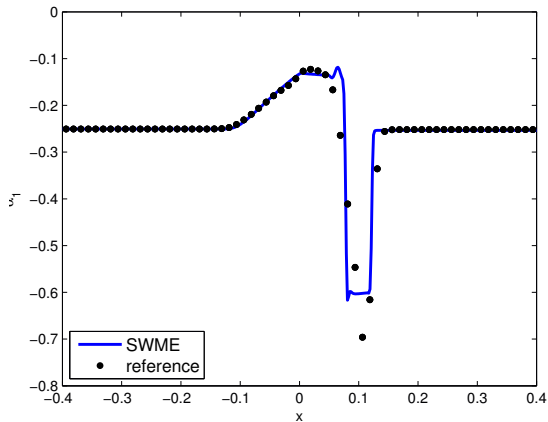


Figure: Hyperbolic breakdown (red) for $x_1 = -0.5$; $x_2 = 0$; $x_3 = 0.5$.

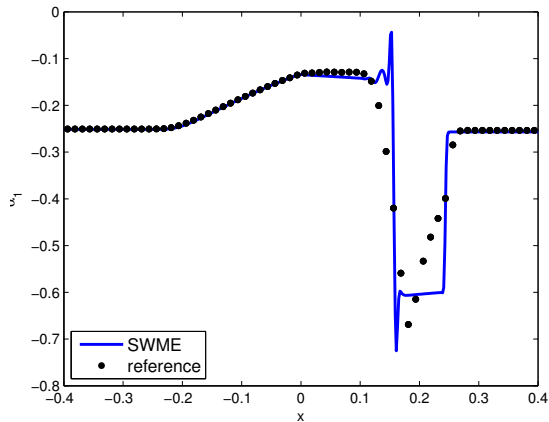
Hyperbolicity breakdown

Solution loses hyperbolicity directly after the first time step.

Instability



(a) $\alpha_1, t = 0.05$



(b) $\alpha_1, t = 0.1$

Figure: Unstable dam break simulation of SWME for $N = 3$.

Idea

- Change system matrix to obtain hyperbolicity
- Preserve structure and conservation of mass

HSWME

- Linearization around $(h, u_m, \alpha_1, 0, \dots, 0)$
- hyperbolic for all $N \in \mathbb{N}$

β -HSWME

- HSWME plus additional parameters for more physical eigenvalues
- hyperbolic for all $N \in \mathbb{N}$

$$\partial_t \mathbf{u}_M + \mathbf{A}_H(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M)$$

Variable vector

$$\mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

Example $M = 2$:

$$\mathbf{A}_H(\mathbf{u}_M) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} & 2u_m & \frac{2\alpha_1}{3} & 0 \\ -2\alpha_1 u_m & 2\alpha_1 & u_m & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 & 0 & -\frac{\alpha_1}{3} & u_m \end{pmatrix}$$

$$\partial_t \mathbf{u}_M + \mathbf{A}_H(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M)$$

Variable vector

$$\mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

general M :

$$\mathbf{A}_H(\mathbf{u}_M) = \begin{pmatrix} 1 & & & & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & & & \\ & & & \ddots & \ddots & & & \\ & & & & \frac{N-1}{2N-1}\alpha_1 & \frac{N+1}{2N+1}\alpha_1 & & \\ & & & & & u_m & & \end{pmatrix}$$

Theorem

The eigenvalues of the system matrix $\mathbf{A}_H(\mathbf{u}_M) \in \mathbb{R}^{(M+2) \times (M+2)}$ are the real numbers

$$\begin{aligned}\lambda_{1,2} &= u_m \pm \sqrt{gh + \alpha_1^2} \\ \lambda_{i+2} &= u_m + c_i \cdot \alpha_1, \quad i = 1, \dots, M\end{aligned}$$

with $c_i \in \mathbb{R}$.

The HSWME system is thus globally hyperbolic.

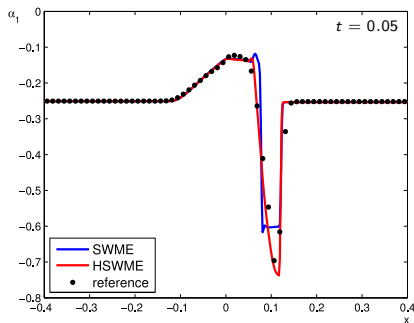
Remarks:

- Analytical form of characteristic polynomial [JK, ROMINGER, 2020]
- General hyperbolicity proof [HUANG, JK, YONG, 2022]
- Explicit characteristic polynomial JK, submitted]

$$\partial_t \mathbf{u}_M + \mathbf{A}_M \partial_x \mathbf{u}_M = \mathbf{0}, \quad \mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

Loss of hyperbolicity

- \mathbf{A}_M has complex eigenvalues
- solution breakdown



$$\tilde{\mathbf{A}}_M = \begin{pmatrix} 1 & & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} \lambda_{1,2} &= u_m \pm \sqrt{gh + \alpha_1^2}, \\ \lambda_{2+i} &= u_m + c_i \alpha_1, \quad i = 1, \dots, M \end{aligned}$$

$$\frac{\partial U_{con}}{\partial t} + A_{con}^{SWME} \frac{\partial U_{con}}{\partial x} = 0,$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -u_m + gh - \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} & 2u_m & \frac{2}{3}\alpha_1 & \dots & \frac{2}{2N+1}\alpha_N \\ -2u_m\alpha_1 - \sum_{j,k=1}^N A_{1jk}\alpha_j\alpha_k & 2\alpha_1 & & & \\ \vdots & \vdots & & \mathcal{A} & \\ -2u_m\alpha_N - \sum_{j,k=1}^N A_{Njk}\alpha_j\alpha_k & 2\alpha_N & & & \end{pmatrix},$$

with block matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$, $\mathcal{A}_{i,l} = \sum_{j=1}^N (B_{ilj} + 2A_{ijl})\alpha_j + u_m\delta_{i,l}$

with Kronecker delta $\delta_{i,j}$ and A, B , coefficients defined in [TORRILHON, KOWALSKI, 2019].

HSWME

- Linearization around $(h, hu_m, h\alpha_1, 0, \dots, 0)$

$$\partial_t \mathbf{u}_N + \mathbf{A}(\mathbf{u}_N) \partial_x \mathbf{u}_N = \mathbf{S}(\mathbf{u}_N), \quad \mathbf{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\mathbf{A}(\mathbf{u}_N) = \begin{pmatrix} 1 & & & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & \frac{N-1}{2N-1}\alpha_1 & \frac{N+1}{2N+1}\alpha_1 & \\ & & & & & u_m & \end{pmatrix}$$

β -HSWME

- HSWME plus additional parameters for different eigenvalues

$$\partial_t \mathbf{u}_N + \mathbf{A}(\mathbf{u}_N) \partial_x \mathbf{u}_N = \mathbf{S}(\mathbf{u}_N), \quad \mathbf{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\mathbf{A}(\mathbf{u}_N) = \begin{pmatrix} 1 & & & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & & \frac{2N^2-N-1}{2N^2+N-1}\alpha_1 & u_m \end{pmatrix}$$

SWLME

- Keep first 2 equations exactly; neglect other higher order products, $\alpha_i \alpha_j \approx 0$ for $i > 1$

$$\partial_t \mathbf{u}_N + \mathbf{A}(\mathbf{u}_N) \partial_x \mathbf{u}_N = \mathbf{S}(\mathbf{u}_N), \quad \mathbf{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\mathbf{A}(\mathbf{u}_N) = \begin{pmatrix} 1 & & & & & \\ -u_m^2 + gh - \sum_{i=1}^N \frac{3\alpha_i^2}{2i+1} & 2u_m & \frac{2}{3}\alpha_1 & \dots & \dots & \frac{2}{2N+1}\alpha_N \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & & & \\ -2u_m\alpha_2 & 2\alpha_2 & & u_m & & \\ \vdots & \vdots & & & \ddots & \\ -2u_m\alpha_N & 2\alpha_N & & & & u_m \end{pmatrix}$$

Primitive regularization [JK, submitted]

New idea:

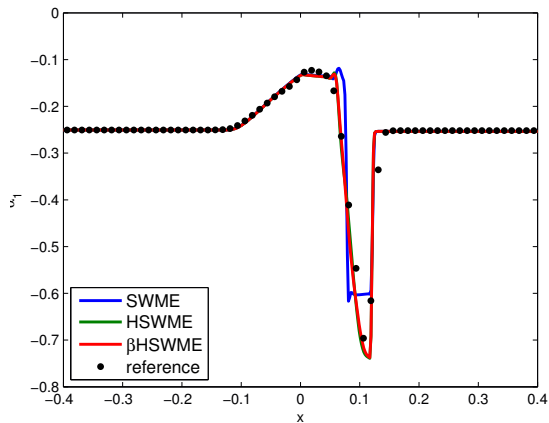
- trafo to primitive variables, linearize last eqns $(h, u_m, \alpha_1, 0, \dots, 0)$, trafo back

$$A = \begin{pmatrix} 1 & & & & & & & \\ -u_m^2 + gh - \sum_{i=1}^N \frac{\alpha_1^2}{2i+1} & 2u_m & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 & \dots & \frac{2}{2i+1}\alpha_i & \dots & \frac{2}{2N+1}\alpha_N \\ -2u_m - \frac{3}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & & & \\ -u_m\alpha_2 - \frac{4}{7}\alpha_1\alpha_3 - \frac{2}{3}\alpha_1^2 & \alpha_2 & \frac{1}{3}\alpha_1 & u_m & \ddots & & & \\ \vdots & \vdots & & \ddots & \ddots & \frac{i+1}{2i+1}\alpha_1 & & \\ -u_m\alpha_i - \frac{i-1}{2i-1}\alpha_1\alpha_{i-1} - \frac{i+1}{2i+1}\alpha_i\alpha_{i+1} & \alpha_i & & \frac{i-1}{2i-1}\alpha_1 & u_m & \ddots & & \\ \vdots & \vdots & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 & \\ -u_m\alpha_N - \frac{N-1}{2N-1}\alpha_1\alpha_N & \alpha_N & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{pmatrix}$$

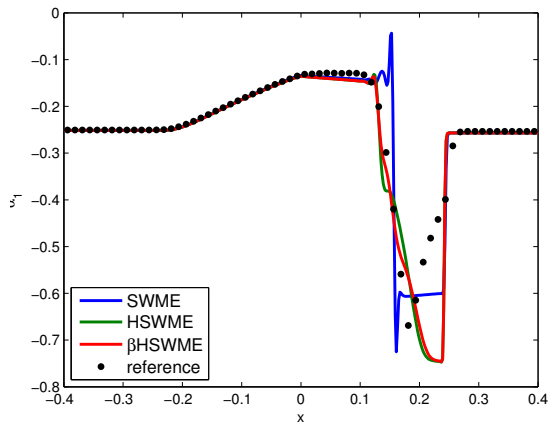
- Hyperbolic Shallow Water Moment Equations [JK, ROMINGER, 2020]
- Shallow Water Linearized Moment Equations [JK, PIMENTEL-GARCIA, 2022]
- Primitive variable regularization [JK, submitted]
- axisymmetric quasi-2D [VERBIEST, JK, 2025] and 2D [BAUERLE et al., 2025]

2.3 accuracy

Hyperbolic regularization



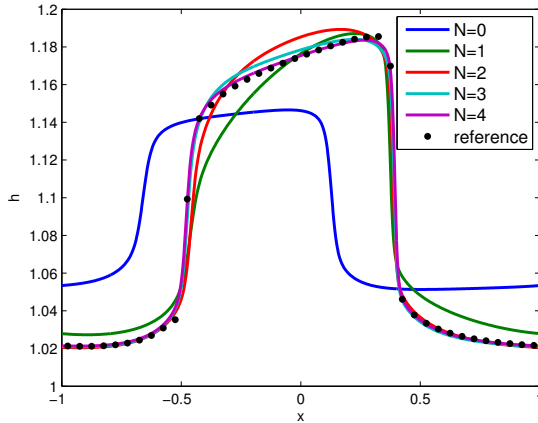
(a) $\alpha_1, t = 0.05$



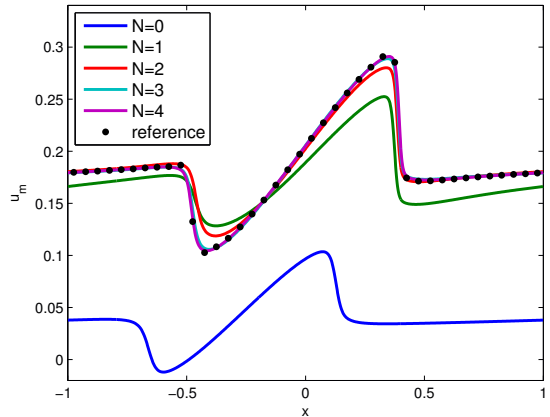
(b) $\alpha_1, t = 0.1$

Figure: Now stable dam break simulation of HSWME, β -HSWME for $N = 3$.

Smooth test case, HSWME



(a) HSWME, h



(b) HSWME, u_m

Figure: Smooth test case for HSWME for varying N .

Smooth test case, convergence

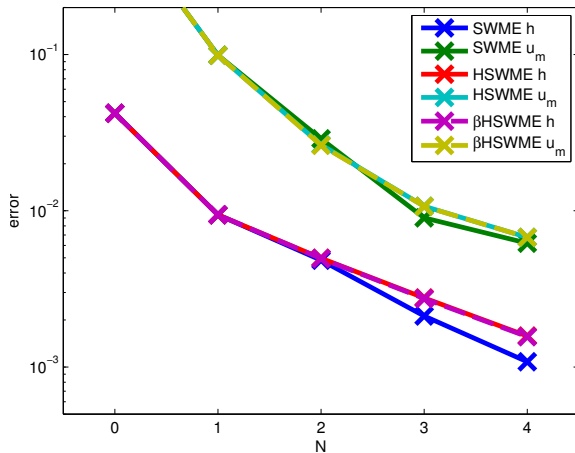
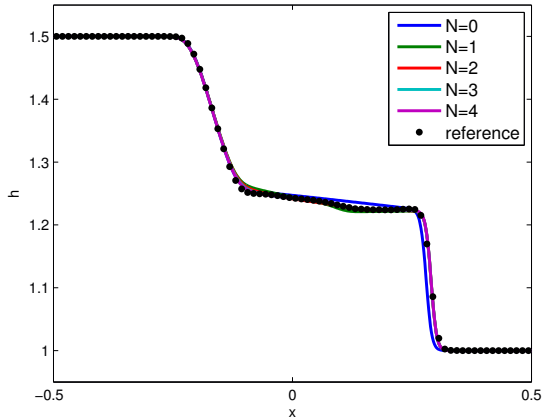
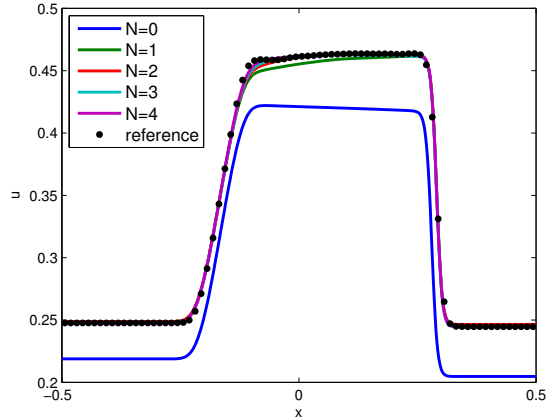


Figure: Error convergence of smooth test case.

Dam break test case, HSWME



(a) HSWME, h_m



(b) HSWME, u_m

Figure: Dam break test case for HSWME for varying N .

Dam break test case, convergence

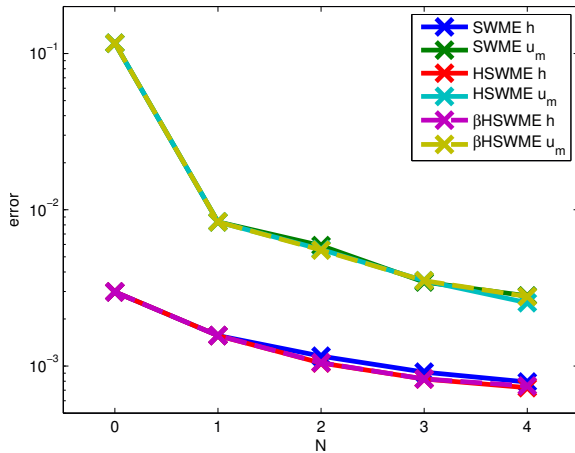


Figure: Error convergence of dam break test case.

2.4 stability

We consider the PDEs

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) + A \frac{\partial}{\partial x} \mathbf{u}(t, x) = -\frac{1}{\tau} B \mathbf{u}.$$

Definition

A PDE system is called linearly stable for a linearisation if possible wave solutions of the form $\mathbf{u}(t, x) = \mathbf{A}e^{-(\kappa x - \omega t)}$ are damped in time, i.e. $\text{Im}(\omega) < 0$.

We assume linearisation around some equilibrium or steady state and use a wave ansatz:

$$\mathbf{u}(t, x) = \mathbf{A}e^{-(\kappa x - \omega t)},$$

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) = -i\omega \mathbf{u}(t, x), \quad \frac{\partial}{\partial x} \mathbf{u}(t, x) = i\kappa \mathbf{u}(t, x)$$

Example 1: Relaxation system

We consider the relaxation system

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) = -\frac{1}{\tau} B \mathbf{u}(t, x),$$

Inserting the wave ansatz:

$$-i\omega \mathbf{u}(t, x) = -\frac{1}{\tau} B \mathbf{u}(t, x),$$

$$\left(-\frac{i}{\tau} B - \omega I \right) \mathbf{u}(t, x) = 0.$$

ω is the solution of an eigenvalue problem:

$$\omega = \text{EV} \left(-\frac{i}{\tau} B \right) = -\frac{i}{\tau} \text{EV}(B).$$

For stability, the eigenvalues of B have to meet condition $\text{Re}(\text{EV}(B)) > 0$

Example 2: Hyperbolic PDE System

We consider the hyperbolic PDE system

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) + A \frac{\partial}{\partial x} \mathbf{u}(t, x) = 0,$$

Inserting the wave ansatz:

$$-i\omega \mathbf{u}(t, x) + i\kappa A \mathbf{u}(t, x) = 0,$$

which in turn leads to the condition

$$(\kappa A - \omega I) \mathbf{u}(t, x) = 0.$$

Therefore, ω is the solution of an eigenvalue problem:

$$\omega = \text{EV}(\kappa A) = \kappa \text{EV}(B).$$

For stability, the eigenvalues of A have to all be real; otherwise complex conjugated unstable eigenvalues would exist. This leads to the condition $\text{Im}(\text{EV}(A)) = 0$, i.e., hyperbolicity.

Example 3: Hyperbolic Relaxation System

We consider the hyperbolic relaxation system

$$\frac{\partial}{\partial t} \mathbf{u}(t, x) + A \frac{\partial}{\partial x} \mathbf{u}(t, x) = -\frac{1}{\tau} B \mathbf{u}.$$

Inserting the wave ansatz:

$$-i\omega \mathbf{u}(t, x) + i\kappa \mathbf{u}(t, x) = -\frac{1}{\tau} B \mathbf{u}(t, x),$$

which in turn leads to the condition

$$\left(\kappa A - \frac{i}{\tau} B - \omega I \right) \mathbf{u}(t, x) = 0.$$

Therefore, ω is the solution of an eigenvalue problem:

$$\omega = \text{EV} \left(\kappa A - \frac{i}{\tau} B \right).$$

stability is not clear a priori.

Equilibrium manifolds of Shallow Water Moment Equations

Model:

$$\partial_t \mathbf{u}_M + \mathbf{A}_M \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M), \quad \mathbf{u}_M \in \mathbb{R}^{M+2}$$

$$\mathbf{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1) \left(u_m + \sum_{j=1}^N \alpha_j \right) - \frac{\nu}{h} \sum_{j=1}^N C_{ij} \alpha_j$$

Newtonian fluid: slip length λ and viscosity ν

Definition (Equilibrium manifold)

Friction terms vanish in equilibrium: $\mathcal{E} = \{\mathbf{u}_M : \mathbf{S}(\mathbf{u}_M) = \mathbf{0}\}$

Water-at-rest equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1) \left(u_m + \sum_{j=1}^M \alpha_j \right) - \frac{\nu}{h} \sum_{j=1}^M C_{ij} \alpha_j$$

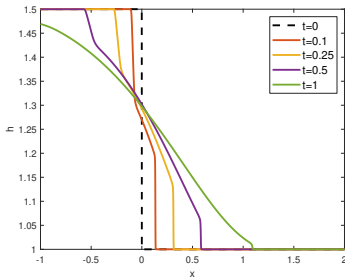
Water-at-rest is in equilibrium

$$\mathcal{E} = \{ \mathbf{u}_M : u_m = \alpha_1 = \dots = \alpha_M = 0 \}$$

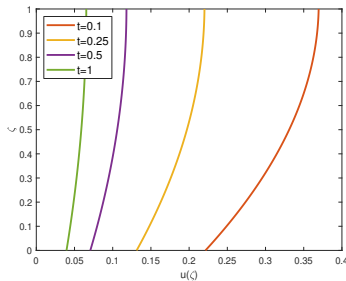
$$\Rightarrow u(t, x, z) = u_m(t, x) + \sum_{i=1}^M \alpha_i(t, x) \phi_i \left(\frac{z - h_b}{h} \right) = 0$$

Water-at-rest convergence for $\lambda = 1$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the water-at-rest equilibrium with time

Constant-velocity equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1) \left(u_m + \sum_{j=1}^M \alpha_j \right) - \frac{\nu}{h} \sum_{j=1}^M C_{ij} \alpha_j$$

If $\lambda \gg h$ (perfect slip limit)

$$S_i = -\frac{\nu}{h} \sum_{j=1}^M C_{ij} \alpha_j$$

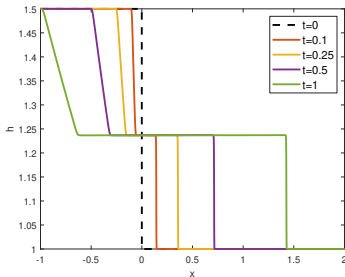
Constant-velocity is in equilibrium

$$\mathcal{E} = \{ \mathbf{u}_M : \alpha_1 = \dots = \alpha_N = 0 \}$$

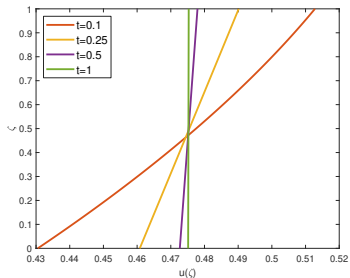
$$\Rightarrow u(t, x, z) = u_m(t, x) + \sum_{i=1}^M \alpha_i(t, x) \phi_i \left(\frac{z - h_b}{h} \right) = u_m(t, x)$$

Constant-velocity convergence for $\lambda = 10$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the constant-velocity equilibrium with time

Bottom-at-rest equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1) \left(u_m + \sum_{j=1}^M \alpha_j \right) - \frac{\nu}{h} \sum_{j=1}^M C_{ij} \alpha_j$$

If $\lambda \ll h$ (no-slip limit)

$$S_i = -\frac{\nu}{\lambda}(2i+1) \left(u_m + \sum_{j=1}^M \alpha_j \right)$$

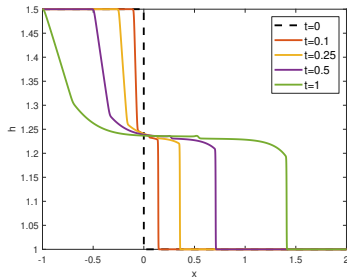
Bottom-at-rest is in equilibrium

$$\mathcal{E} = \{ \mathbf{u}_M : u_m + \sum_{j=1}^M \alpha_j = 0 \}$$

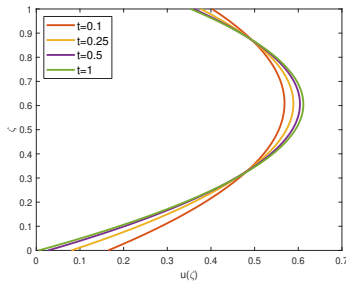
$$\Rightarrow u(t, x, h_b) = u_m(t, x) + \sum_{i=1}^M \alpha_i(t, x) \phi_i \left(\frac{h_b - h_b}{h} \right) = u_m(t, x) + \sum_{i=1}^M \alpha_i(t, x) = 0$$

Bottom-at-rest convergence for $\lambda = 10^{-3}$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the bottom-at-rest equilibrium with time

$$\partial_t \mathbf{u}_M + \mathbf{A}_M \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M)$$

Equilibrium stability

- system is stable for small perturbation around equilibrium
- relaxation back towards equilibrium
- instabilities may or may not cause numerical problems

(I): For any $U \in \mathcal{E}$, the Jacobian $S_U(U)$ can be manipulated by an invertible $n \times n$ matrix $P = P(U)$ and an invertible $r \times r$ ($0 < r \leq n$) matrix $\hat{T}(U)$ such that

$$P(U)S_U(U) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{T}(U) \end{bmatrix} P(U), \quad \forall U \in \mathcal{E}.$$

(II): There exists a positive definite symmetrizer $A_0 = A_0(U)$ of the coefficient matrix $A(U)$ such that

$$A_0(U)A(U) = A^T(U)A_0(U), \quad \forall U \in G.$$

(III): On the equilibrium manifold \mathcal{E} , the coefficient matrix and the source term are coupled as

$$A_0(U)S_U(U) + S_U^T(U)A_0(U) \preceq -P^T(U) \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} P(U), \quad \forall U \in \mathcal{E}.$$

Equilibrium stability analysis

$$\partial_t \mathbf{u}_M + \mathbf{A}_M \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M)$$

Structural stability conditions [Yong, 1999]

1. source term jacobian is invertible
2. transport term is hyperbolic
3. coupling between source and transport term

Equilibrium stability analysis of SWME [Huang et al., 2022]

1. water-at-rest is stable
2. constant-velocity is stable
3. bottom-at-rest can be *unstable*

Equilibrium stability analysis

$$\partial_t \mathbf{u}_M + \mathbf{A}_M \partial_x \mathbf{u}_M = \mathbf{S}(\mathbf{u}_M)$$

Structural stability conditions [Yong, 1999]

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3. coupling between source and transport term

Equilibrium stability analysis of SWME [Huang et al., 2022]

1. water-at-rest is stable
2. constant-velocity is stable
3. bottom-at-rest can be *unstable*

We observed no instabilities in numerical simulations

2.5 steady states

Steady states of Shallow water equations

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(flat bottom $\partial_x b = 0$ and zero friction); the steady state fulfils

$$\partial_x (hu_m) = 0, \quad \partial_x \left(hu_m^2 + \frac{1}{2}gh^2 \right) = 0.$$

Rankine-Hugoniot conditions from a given state $(h_0, h_0 u_{m,0})$ to a state (h, hu_m) :

$$\left(\frac{h}{h_0} \right) = -\frac{1}{2} + \frac{1}{2} \cdot \sqrt{1 + 8Fr^2},$$

where Fr is the Froude number for the given state defined by

$$Fr = \frac{u_{m,0}}{\sqrt{gh_0}}.$$

Steady states of Shallow water moment equations ($M = 1$)

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = Q\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \begin{pmatrix} 0 \\ gh\partial_x b \\ 0 \end{pmatrix} - \frac{\nu}{\lambda}P,$$

For flat bottom $\partial_x b = 0$ and zero friction, the steady state fulfills

$$\begin{aligned} \partial_x(hu_m) &= 0, \\ \partial_x\left(hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2\right) &= 0, \\ \partial_x(2hu_m\alpha_1) &= u_m\partial_x(h\alpha_1), \end{aligned}$$

From the first and last equation, we obtain after some modification

$$\begin{aligned} hu_m &= \text{const}, \\ u_m = 0 \quad \text{or} \quad \frac{\alpha}{h} &= \text{const}. \end{aligned}$$

Steady states of Shallow water moment equations ($M = 1$)

Rankine-Hugoniot conditions from a given state $(h_0, h_0 u_{m,0}, h_0 \alpha_0)$ to a state $(h, h u_m, h \alpha)$:

$$(h - h_0) \left[-\frac{u_{m,0}^2}{gh_0} + \frac{1}{2} \left(\left(\frac{h}{h_0} \right)^2 + \left(\frac{h}{h_0} \right) \right) + \frac{1}{3} \frac{\alpha_0^2}{gh_0} \left(\left(\frac{h}{h_0} \right)^3 + \left(\frac{h}{h_0} \right)^2 + \left(\frac{h}{h_0} \right) \right) \right] = 0.$$

dimensionless flow numbers:

$$Fr = \frac{u_{m,0}}{\sqrt{gh_0}}, \quad M\alpha = \frac{\alpha_0}{u_{m,0}},$$

use $y = \frac{h}{h_0}$:

$$h = h_0 \quad \vee \quad -Fr^2 + \frac{1}{2} (y^2 + y) + \frac{1}{3} M\alpha^2 Fr^2 (y^3 + y^2 + y) = 0.$$

third order polynomial with two parameters Fr and $M\alpha$.

Steady states of Shallow water moment equations ($M = 2$)

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix}$$

No analytical solution possible.

Linearised SWME [PIMENTEL, JK, 2022] (example: $M = 2$)

$$\mathbf{A}(\mathbf{u}_M) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\ -2\alpha_1 u_m - \frac{4}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m + \alpha_2 & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 - 2u_m\alpha_2 - \frac{2}{7}\alpha_2^2 & 2\alpha_2 & -\frac{\alpha_1}{3} & u_m + \frac{3\alpha_2}{7} \end{pmatrix}$$

SWLME idea:

In higher-order equations, assume near equilibrium: $\alpha_i = \mathcal{O}(\epsilon)$, neglect terms $\mathcal{O}(\epsilon^2)$

$$\mathbf{A}_L(\mathbf{u}_M) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\alpha_1^2}{3} - u_m^2 + gh - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & 0 \\ -2u_m\alpha_2 & 2\alpha_2 & 0 & u_m \end{pmatrix}$$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \dots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix} = Q \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} \quad (1)$$

The non-conservative term simplifies to $Q = \text{diag}(0, 0, u_m, \dots, u_m)$.

Alternatively:

$$\begin{aligned} \partial_t h + \partial_x(hu_m) &= 0, \\ \partial_t(hu_m) + \partial_x \left(hu_m^2 + h \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} + \frac{1}{2}gh^2 \right) &= -gh\partial_x b, \\ \partial_t(h\alpha_i) + \partial_x(2hu_m\alpha_i) &= u_m\partial_x(h\alpha_i), \end{aligned}$$

SWLME

$$\begin{aligned}\partial_t h + \partial_x (hu_m) &= 0, \\ \partial_t(hu_m) + \partial_x \left(hu_m^2 + h \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} + \frac{1}{2}gh^2 \right) &= -gh\partial_x b, \\ \partial_t(h\alpha_i) + \partial_x(2hu_m\alpha_i) &= u_m\partial_x(h\alpha_i),\end{aligned}$$

SWME

$$\begin{aligned}\partial_t h + \partial_x (hu_m) &= 0, \\ \partial_t(hu_m) + \partial_x \left(hu_m^2 + \sum_{i=1}^N \frac{h\alpha_i^2}{2i+1} + \frac{1}{2}gh^2 \right) &= -gh\partial_x b, \\ \partial_t(h\alpha_i) + \partial_x(2hu_m\alpha_i + \mathfrak{A}_i) &= u_m\partial_x(h\alpha_i) - \mathfrak{B}_i,\end{aligned}$$

SWME

$$\begin{aligned}\partial_t h + \partial_x (hu_m) &= 0, \\ \partial_t (hu_m) + \partial_x \left(hu_m^2 + \sum_{i=1}^N \frac{h\alpha_i^2}{2i+1} + \frac{1}{2}gh^2 \right) &= -gh\partial_x b, \\ \partial_t (h\alpha_i) + \partial_x \left(2hu_m\alpha_i + \mathfrak{A}_i \right) &= u_m\partial_x (h\alpha_i) - \mathfrak{B}_i,\end{aligned}$$

where \mathfrak{A}_i and \mathfrak{B}_i are

$$\mathfrak{A}_i = h \sum_{j,k=1}^N A_{ijk} \alpha_j \alpha_k, \quad \mathfrak{B}_i = \sum_{j,k=1}^N B_{ijk} \alpha_k \partial_x (h\alpha_j),$$

and the coefficients are

$$A_{ijk} = (2i+1) \int_0^1 \phi_i \phi_j \phi_k d\zeta, \quad B_{ijk} = (2i+1) \int_0^1 \phi'_i \left(\int_0^\zeta \phi_j d\zeta \right) \phi_k d\zeta.$$

Theorem

The eigenvalues of the system matrix $\mathbf{A}_L(\mathbf{u}_M) \in \mathbb{R}^{(M+2) \times (M+2)}$ are the real numbers

$$\begin{aligned}\lambda_{1,2} &= u_m \pm \sqrt{gh + \sum_{i=1}^M \frac{3\alpha_i^2}{2i+1}} \\ \lambda_{i+2} &= u_m, \quad i = 1, \dots, M.\end{aligned}$$

The SWLME system is thus globally hyperbolic.

$$\begin{aligned}\partial_x(hu_m) &= 0 \\ \partial_x\left(hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 + \dots + \frac{1}{2N+1}h\alpha_N^2\right) &= 0 \\ \partial_x(2hu_m\alpha_i) &= u_m\partial_x(h\alpha_i)\end{aligned}$$

$$\begin{aligned}hu_m &= \text{const}, \\ u_m = 0 \text{ or } \frac{\alpha_i}{h} &= \text{const}, \text{ for } i = 1, \dots, N.\end{aligned}$$

dimensionless flow numbers for each variable and writing $y = \frac{h}{h_0}$

$$(M\alpha)_i = \frac{\alpha_{i,0}}{u_{m,0}}, \quad \text{for } i = 1, \dots, N,$$

$$h = h_0 \vee -Fr^2 + \frac{1}{2}(y^2 + y) + \sum_{i=1}^N \frac{1}{2i+1} (M\alpha)_i^2 Fr^2 (y^3 + y^2 + y) = 0.$$

2.6 entropy

Entropy equation for Shallow Water Equations

From standard Shallow Water Equations

$$(C) : \quad \partial_t h + \partial_x (hu_m) = 0$$

$$(M1) : \quad \partial_t (hu_m) + \partial_x \left(hu_m^2 + \frac{1}{2}gh^2 \right) = -gh\partial_x b.$$

Derive energy equation

$$(E) : \quad \partial_t \left(\frac{hu_m^2}{2} + g\frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{hu_m^3}{2} + gh u_m (h + b) \right) = 0$$

where the total energy $\frac{1}{2}hu_m^2 + \frac{1}{2}gh^2 + ghb$ is the entropy and $\frac{hu_m^3}{2} + gh u_m (h + b)$ is the entropy flux.

Derivation of entropy equation for Shallow Water Equations (1)

standard Shallow Water Equations

$$(C) : \quad \partial_t h + \partial_x (hu_m) = 0$$

$$(M1) : \quad \partial_t (hu_m) + \partial_x \left(hu_m^2 + \frac{1}{2}gh^2 \right) = -gh\partial_x b.$$

modified momentum balance

$$(M2) : \quad \partial_t (hu_m) + \partial_x (hu_m^2) + gh\partial_x (h + b) = 0.$$

(M2) - $u_m \cdot (C)$ to get

$$(A) : \quad h\partial_t u_m + hu_m\partial_x u_m + gh\partial_x (h + b) = 0.$$

Derivation of entropy equation for Shallow Water Equations (2)

modified momentum balance

$$(M2) : \quad \partial_t (hu_m) + \partial_x (hu_m^2) + gh\partial_x(h + b) = 0.$$

(M2) $- u_m \cdot$ (C) to get

$$(A) : \quad h\partial_t u_m + hu_m\partial_x u_m + gh\partial_x(h + b) = 0.$$

average (S) = $\frac{1}{2}$ (A) + $\frac{1}{2}$ (M2)

$$(S) : \quad \frac{1}{2} \left(\partial_t (hu_m) + h\partial_t u_m \right) + \frac{1}{2} \left(\partial_x (hu_m^2) + hu_m\partial_x u_m \right) + gh\partial_x(h + b) = 0.$$

Derivation of entropy equation for Shallow Water Equations (3)

$$(S) : \quad \frac{1}{2} \left(\partial_t (hu_m) + h \partial_t u_m \right) + \frac{1}{2} \left(\partial_x (hu_m^2) + hu_m \partial_x u_m \right) + gh \partial_x (h + b) = 0.$$

equation for the kinetic energy (K) by multiplying u_m to (S), then product rule

$$(K) : \quad \partial_t \left(\frac{hu_m^2}{2} \right) + \partial_x \left(\frac{hu_m^3}{2} \right) + gh u_m \partial_x (h + b) = 0,$$

where the term $k = \frac{1}{2} hu_m^2$ is the kinetic energy.

Derivation of entropy equation for Shallow Water Equations (4)

standard Shallow Water Equations

$$(C) : \quad \partial_t h + \partial_x (hu_m) = 0$$

$$(M1) : \quad \partial_t (hu_m) + \partial_x \left(hu_m^2 + \frac{1}{2}gh^2 \right) = -gh\partial_x b.$$

$$(P) : \quad \partial_t \left(g\frac{h^2}{2} + ghb \right) + g(h+b)\partial_x (hu_m) = 0,$$

where $p = \frac{1}{2}gh^2 + ghb$ denotes the potential energy.

Derivation of entropy equation for Shallow Water Equations (5)

$$(P) : \quad \partial_t \left(g \frac{h^2}{2} + ghb \right) + g(h+b) \partial_x (hu_m) = 0,$$

where $p = \frac{1}{2}gh^2 + ghb$ denotes the potential energy.

$$(K) : \quad \partial_t \left(\frac{hu_m^2}{2} \right) + \partial_x \left(\frac{hu_m^3}{2} \right) + gh u_m \partial_x (h+b) = 0,$$

where $k = \frac{1}{2}hu_m^2$ is the kinetic energy.

(E) = (P) + (K) is the total energy equation

$$(E) : \quad \partial_t \left(\frac{hu_m^2}{2} + g \frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{hu_m^3}{2} + gh u_m (h+b) \right) = 0$$

where the total energy is $k + p = \frac{1}{2}hu_m^2 + \frac{1}{2}gh^2 + ghb$

Derivation of entropy equation for SWLME

SWLME

$$\partial_t h + \partial_x (hu_m) = 0,$$

$$\partial_t (hu_m) + \partial_x \left(hu_m^2 + h \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} + \frac{1}{2} gh^2 \right) = -gh \partial_x b,$$

$$\partial_t (h\alpha_i) + \partial_x (2hu_m\alpha_i) = u_m \partial_x (h\alpha_i),$$

$$\partial_t \left(\frac{hu_m^2}{2} + \frac{h}{2} \sum_{i=1}^N \frac{\alpha_i^2}{2i+1} + g \frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{hu_m^3}{2} + \frac{3hu_m}{2} \sum_{i=1}^N \frac{\alpha_i^2}{2i+1} + gh u_m (h+b) \right) = 0,$$

where we denote the total energy by

$$e = k_\alpha + p = \frac{hu_m^2}{2} + \frac{h}{2} \sum_{i=1}^N \frac{\alpha_i^2}{2i+1} + g \frac{h^2}{2} + ghb.$$

Derivation of entropy equation for SWME (1)

SWME

$$\begin{aligned}\partial_t h + \partial_x (hu_m) &= 0, \\ \partial_t (hu_m) + \partial_x \left(hu_m^2 + \sum_{i=1}^N \frac{h\alpha_i^2}{2i+1} + \frac{1}{2}gh^2 \right) &= -gh\partial_x b, \\ \partial_t (h\alpha_i) + \partial_x \left(2hu_m\alpha_i + \mathfrak{A}_i \right) &= u_m\partial_x (h\alpha_i) - \mathfrak{B}_i,\end{aligned}$$

where \mathfrak{A}_i and \mathfrak{B}_i are

$$\mathfrak{A}_i = h \sum_{j,k=1}^N A_{ijk} \alpha_j \alpha_k, \quad \mathfrak{B}_i = \sum_{j,k=1}^N B_{ijk} \alpha_k \partial_x (h\alpha_j),$$

(SWLME is SWME with $\mathfrak{A}_i = \mathfrak{B}_i = 0$.)

Derivation of entropy equation for SWME (2)

SWME

$$\partial_t h + \partial_x (hu_m) = 0,$$

$$\partial_t (hu_m) + \partial_x \left(hu_m^2 + \sum_{i=1}^N \frac{h\alpha_i^2}{2i+1} + \frac{1}{2}gh^2 \right) = -gh\partial_x b,$$

$$\partial_t (h\alpha_i) + \partial_x (2hu_m\alpha_i + \mathfrak{A}_i) = u_m\partial_x (h\alpha_i) - \mathfrak{B}_i,$$

$$\partial_t \left(\frac{hu_m^2}{2} + \frac{h}{2} \sum_{i=1}^N \frac{\alpha_i^2}{2i+1} + g\frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{hu_m^3}{2} + \frac{3hu_m}{2} \sum_{i=1}^N \frac{\alpha_i^2}{2i+1} + gh u_m (h+b) + \hat{\mathcal{Q}} \right) = 0,$$

with

$$\hat{\mathcal{Q}} = \sum_{i,j,k=1}^N \left(\tilde{A}_{ijk} + \tilde{B}_{ijk} \right) h\alpha_i\alpha_j\alpha_k, \quad \tilde{A}_{ijk} = \frac{A_{ijk}}{2i+1}, \quad \tilde{B}_{ijk} = \frac{B_{ijk}}{2i+1}$$

and the same (!) total energy

summary

1 repetition

- Shallow Water Moment Models

2 analysis

- conservation
- hyperbolicity
- stability and equilibria
- steady states
- entropy