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WAVES.NL Summer School | Nijmegen
Assignment 1 | 25.8.2025
Due Date: -

Exercise 1. (Polynomial velocity expansions)
We expand the velocity profile $u(t,x,\zeta)$ in Legendre polynomials as follows:

$$u(t,x,\zeta) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x) \cdot \phi_i(\zeta),$$

where the first three Legendre polynomials are given by:

$$\phi_1(\zeta) = 1 - 2\zeta, \qquad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2, \qquad \phi_3(\zeta) = 1 - 12\zeta + 30\zeta^2 - 20\zeta^3,$$

with normalization $\phi_i(0) = 1$ and orthogonality on $[0,1]$

$$\int_0^1 \phi_i(\zeta)\phi_j(\zeta)d\zeta = \frac{\delta_{ij}}{2i+1}.$$

Compute the values of the variables $u_m, \alpha_1, \alpha_2, \alpha_3$ for the following velocity profiles:

a) Constant profile: $u(t,x,\zeta) = 0.25$.

b) Linear profile: $u(t,x,\zeta) = 0.5\zeta$.

c) Quadratic profile: $u(t,x,\zeta) = 1.5\zeta(1-\zeta)$.

Exercise 2. (SWME friction term)
The transformed equation includes a friction term $-\frac{1}{\rho}\partial_\zeta \tilde{\sigma}_{xz}$.
Compute the final term in the moment equations obtained by projection with test function $\psi_j = \phi_j$ using

- Expansion $u(t,x,\zeta) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x) \cdot \phi_i(\zeta)$.
- Orthogonal Legendre basis with normalization $\phi_i(\zeta)|_{\zeta=0} = 1$.
- Newtonian friction law in the bulk: $\zeta \in [0,1]$: $\frac{1}{\rho}\tilde{\sigma}_{xz} = \frac{\nu}{h} \cdot \partial_\zeta u(\zeta)$ and:
  a) No slip boundary condition at the top: $\zeta = 1 \Rightarrow \tilde{\sigma}_{xz}(1) = 0$.
  b) Slip boundary condition at the bottom: $\zeta = 0 \Rightarrow \frac{1}{\rho}\tilde{\sigma}_{xz}(0) = \frac{\nu}{\lambda} \cdot u(0)$ with slip length λ and viscosity coefficient ν

Exercise 3. (Non-orthonormal, linearized Grad model)
Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t,x,c) + c\partial_x f(t,x,c) = 0, \tag{1}$$

with $c \in \mathbb{R}$ and the following ansatz:

$$f(t,x,c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^{M} f_i(t,x) \cdot He_i(c)$$

with orthogonal, but non-orthonormal Hermite basis polynomials $He_i(c)$:

$$\int_{\mathbb{R}} He_i(c) \cdot He_j(c) \cdot w(c) dc = j!\delta_{i,j},$$

$$cHe_i(c) = He_{i+1}(c) + i \cdot He_{i-1}(c),$$

for weight function $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$.

Exercise 4. (kinetic equation)
Derive a moment model for the following equation:

$$\frac{\partial}{\partial t} f(t,x,c) + c\partial_x f(t,x,c) = 0,$$

with $c \in \mathbb{R}$ and the following ansatz:

$$f(t,x,c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}} \sum_{i=0}^{M} \alpha_i(t,x) \cdot He_i(c),$$

for orthonormal Hermite polynomials $He_i(c)$ following the recursions:

$$cHe_i(c) = \sqrt{i+1}He_{i+1}(c) + \sqrt{i}He_{i-1}(c),$$

$$\int He_i(c)He_j(c) \cdot w(c) dc = \delta_{ij},$$

for weight function $w(c) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$.

Exercise 5. (Uncertainty quantification for the hot shower problem)
The hot shower model is given by the delay differential equation:

$$\dot{x}(t) = -(K + w) \cdot x(t - \tau),$$

with

x : target temperature difference

w : uniformly distributed uncertainty $w \sim U(-0.1, 0.1)$

K : reaction parameter

τ : delay.

a) Rewrite the model with normalized uncertainty $w \sim U(-1, 1)$.

b) Use the polynomial chaos expansion (PCE) $x(t, w) = \sum_{i=0}^{N} x_i(t)\phi_i(w)$, with ϕ_i Legendre polynomials, orthonormal on $[-1, 1]$, to derive a stochastic Galerkin model for the evolution of the coefficients x_i in matrix-vector form.