Waves: Modeling, Analysis, and Numerics
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Linear Waves: From Physics to Numerics

Part I

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Contents

Generalities

Wave Phenomena

"... a wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation" — Whitham, G. B. (1974). *Linear and Nonlinear Waves.*

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$$\Delta := \underbrace{\nabla \cdot \nabla}\ = \underbrace{\text{div grad}}$$

Cartesian coordinates: $\displaystyle \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Cylindrical coordinates: $\displaystyle \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$ $\quad (x = r\cos\theta,\ y = r\sin\theta,\ z = z)$

Spherical coordinates: $\displaystyle \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$

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d'Alembert, J.-R.. (1747). treatise on *Vibrating Strings*.

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Let $\hat{u}(k,\omega)$ be the Fourier transform of $u(x,t)$:

$$\hat{u}(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\, \mathrm{e}^{-i(kx-\omega t)}\, \mathrm{d}x\, \mathrm{d}t,$$

(extended by 0 for $t < 0$)

then

$$u(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}(k,\omega)\, \mathrm{e}^{i(kx-\omega t)}\, \mathrm{d}k\, \mathrm{d}\omega.$$

Plugging into the wave equation gives the **dispersion relation**:

$$\omega^2 = c^2 k^2.$$

(in general, $\omega = \omega(k)$, for other linear wave equations)

Thus, the wave equation solution is supported at $\omega = \pm c|k|$, and it can be expressed as

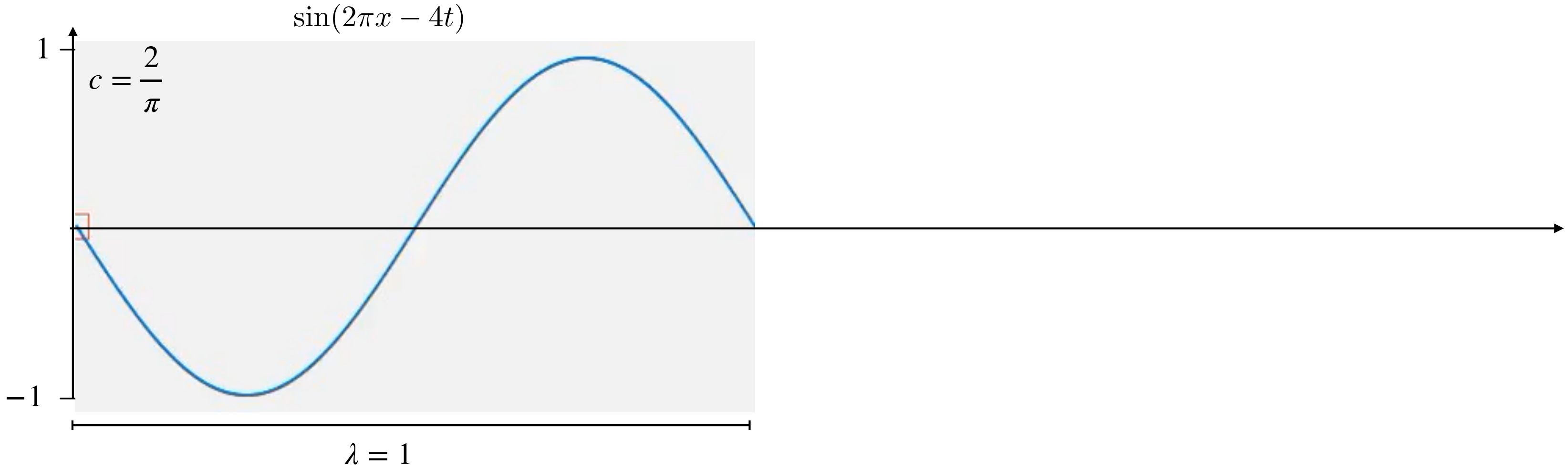
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[A(k)\, \mathrm{e}^{i(kx-c|k|t)} + B(k)\, \mathrm{e}^{i(kx+c|k|t)}\right] \mathrm{d}k,$$

where $A(k)$ and $B(k)$ are determined by the initial conditions.

Simple Linear Waves

Linear wave solutions can often be expressed as superpositions of simpler waves of the form:

$$u(x,t) = A(k)\sin(kx - \omega(k)t + \phi), \quad x, t \in \mathbb{R},$$



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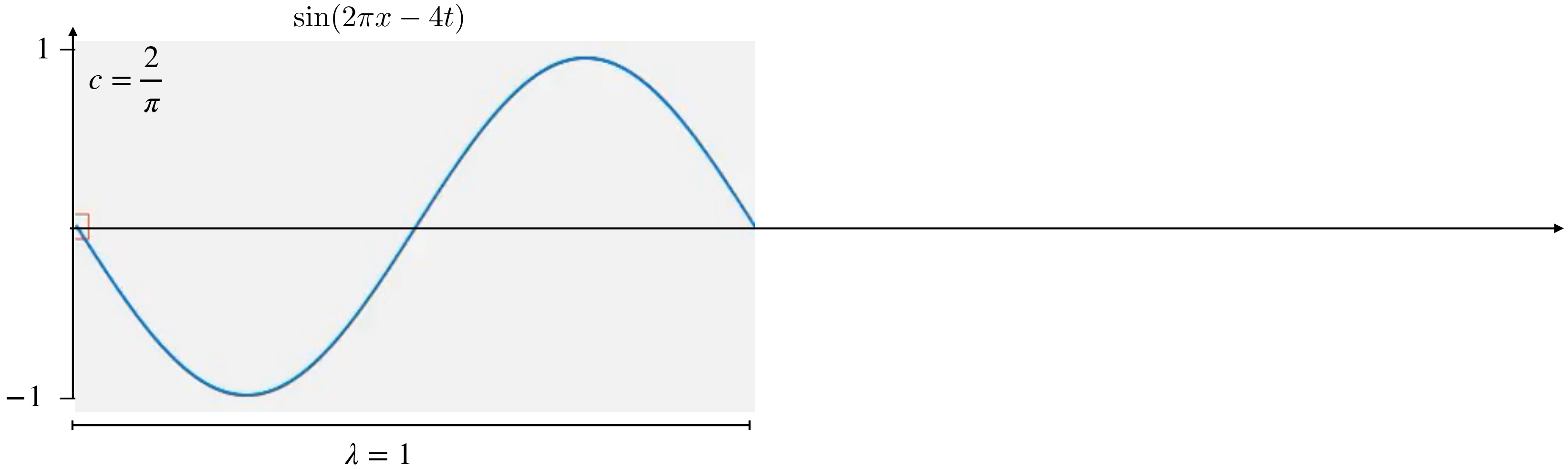

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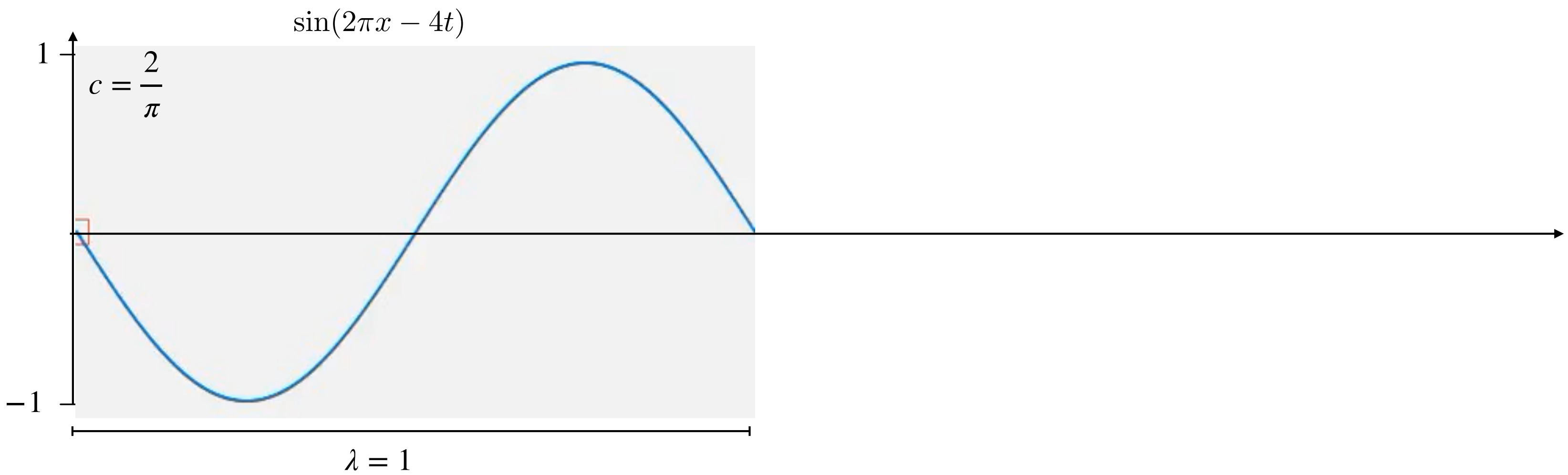

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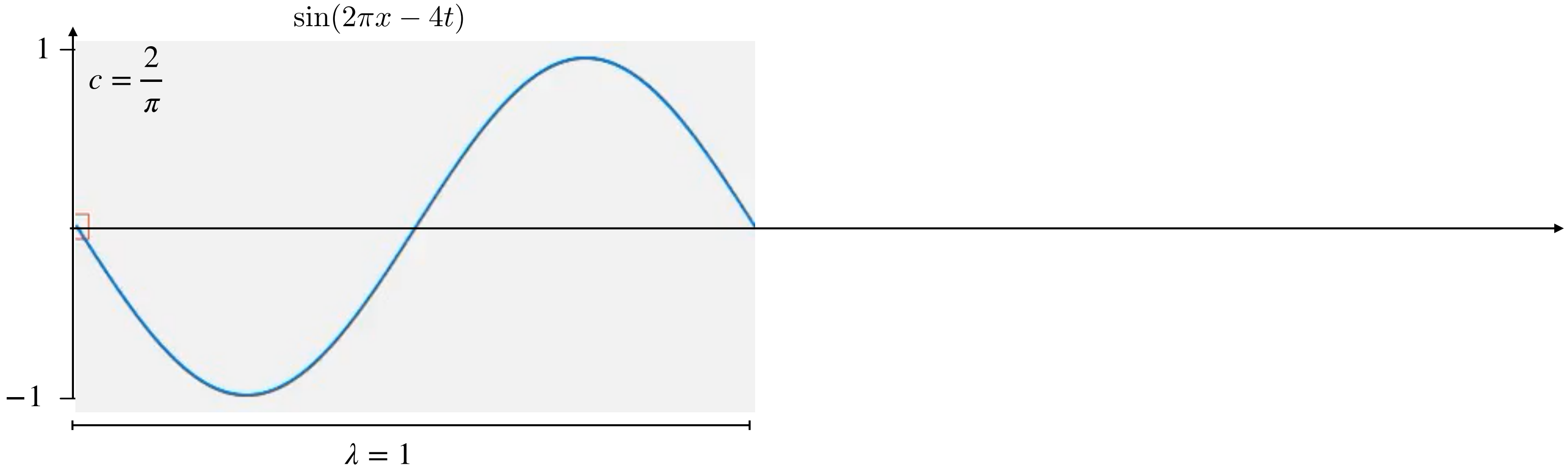

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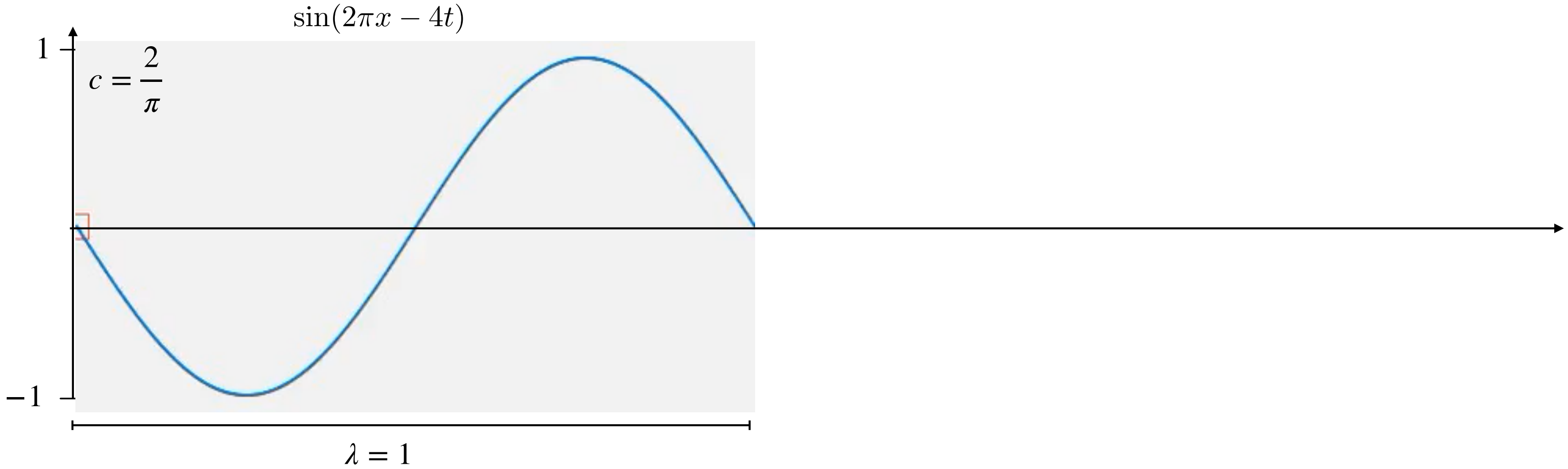
$\lambda = \dfrac{2\pi}{k}$: wavelength
$f = \dfrac{\omega}{2\pi}$: frequency

$v_p = \dfrac{\omega}{k}$: phase velocity
$v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$: group velocity



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$$u(x,t) = A(k)\sin(kx - \omega(k)t + \phi), \quad x, t \in \mathbb{R},$$

kx controls the spatial oscillation

$\omega(k)t$ controls the temporal oscillation

x : spatial coordinate
t : temporal variable
A : amplitude

k : wavenumber
ω : angular frequency
ϕ : phase

$\lambda = \dfrac{2\pi}{k}$: wavelength
$f = \dfrac{\omega}{2\pi}$: frequency

$v_p = \dfrac{\omega}{k}$: phase velocity
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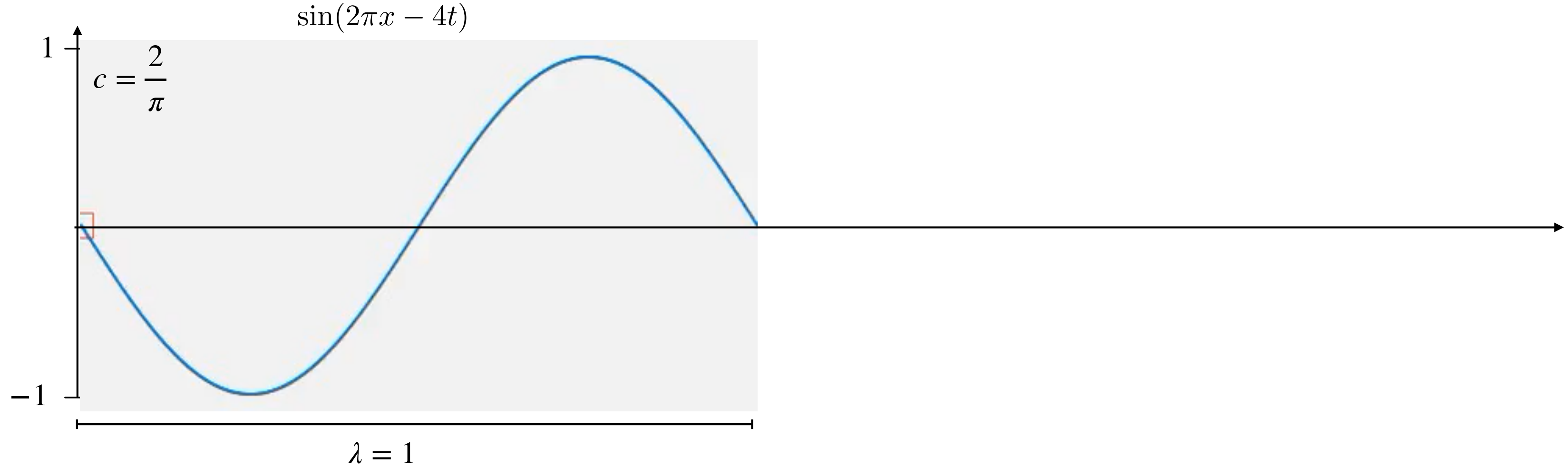

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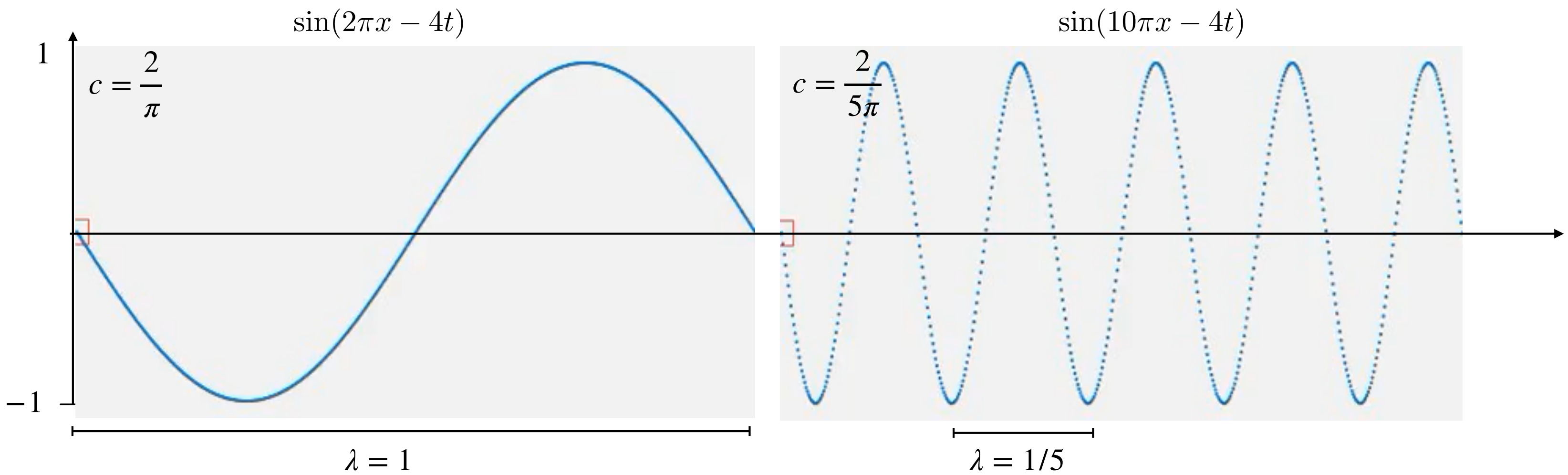

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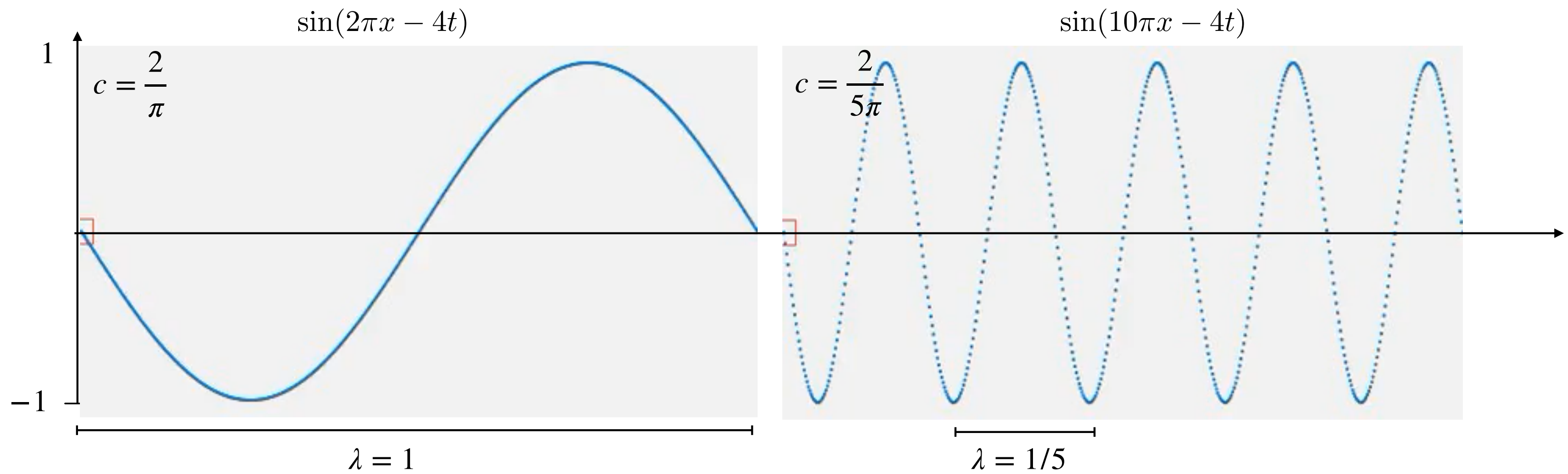

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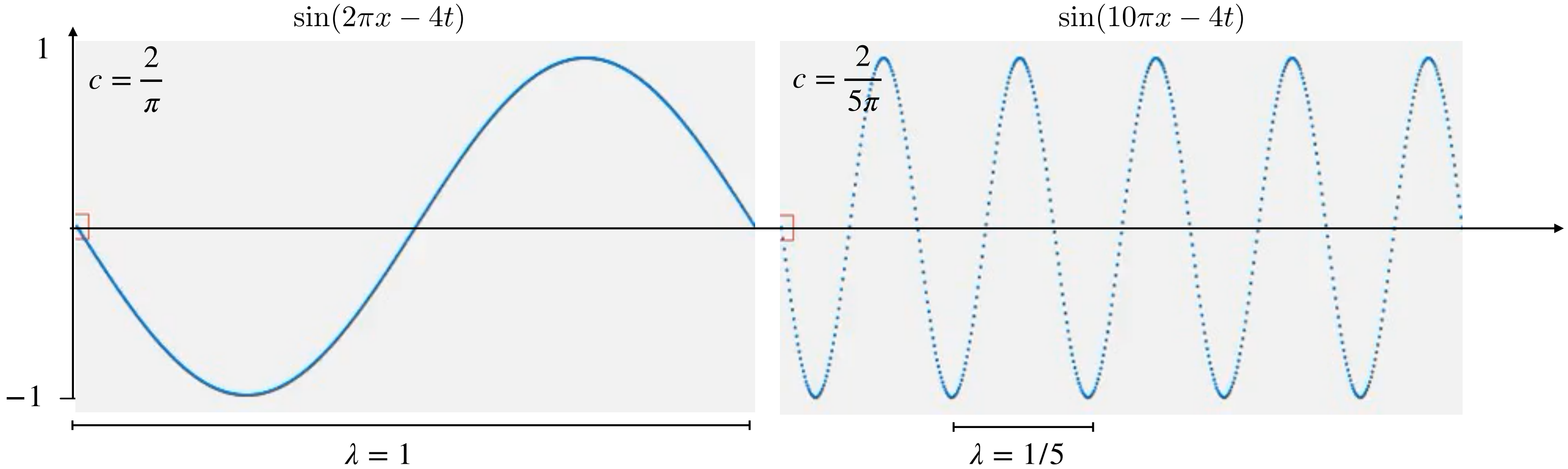

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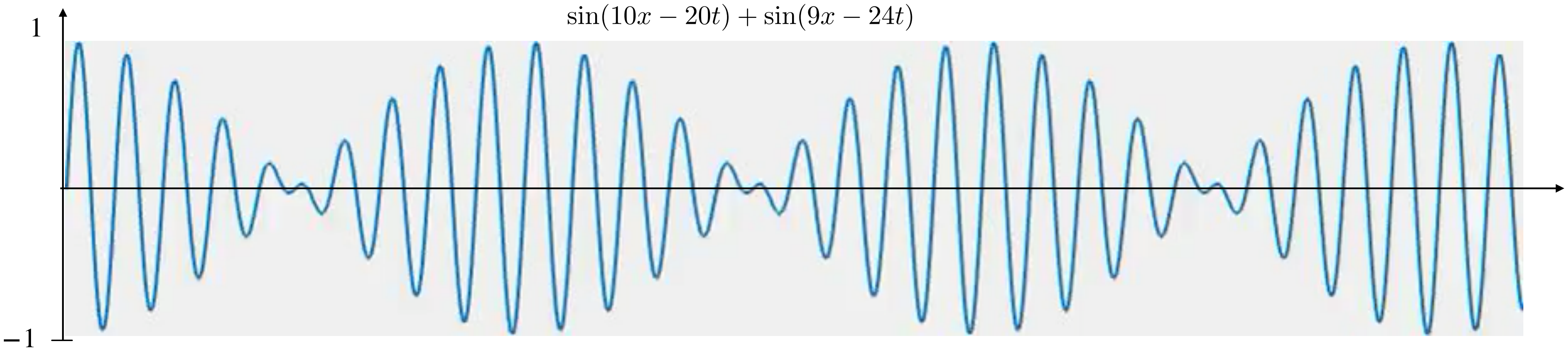

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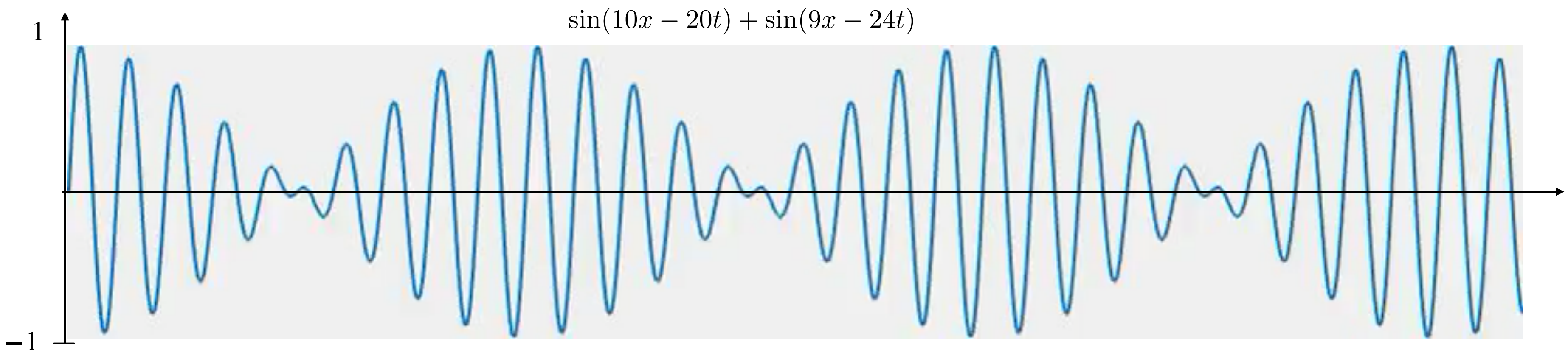

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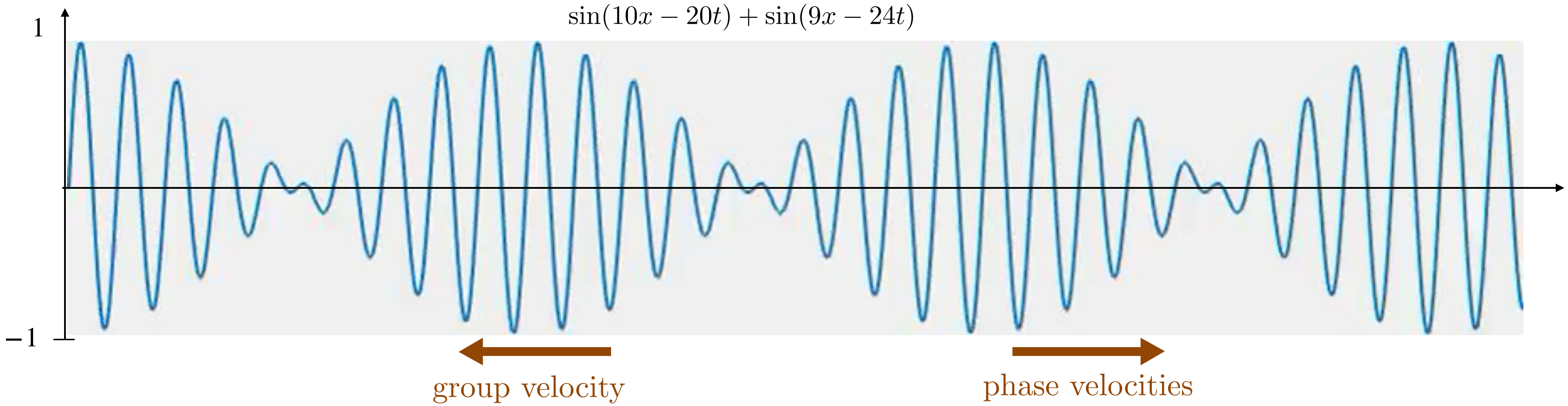
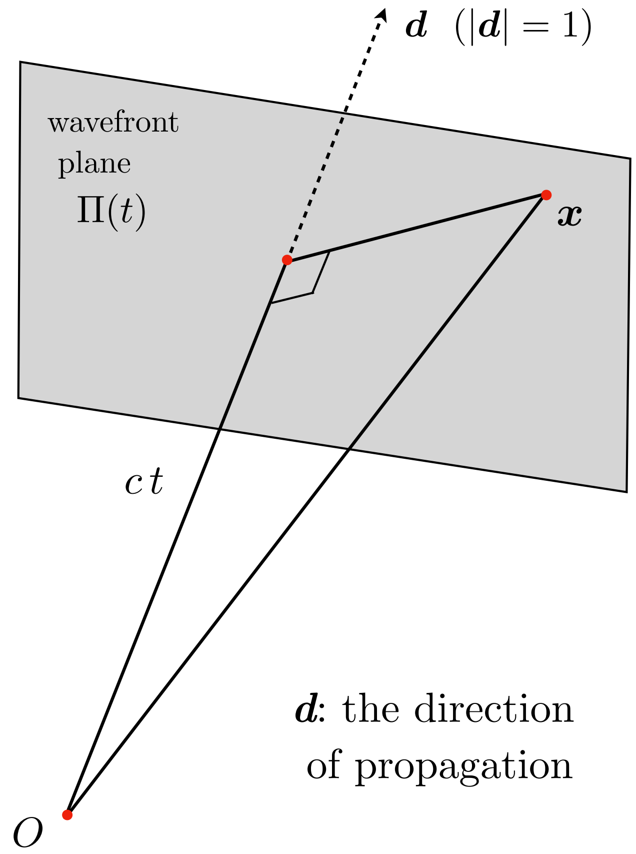


Simple Linear Waves

- Planewaves: $u(\boldsymbol{x}, t) = f(\boldsymbol{x} \cdot \boldsymbol{d} - ct)$

  Constant value at every point $\boldsymbol{x}$ on the plane $\Pi(t)$



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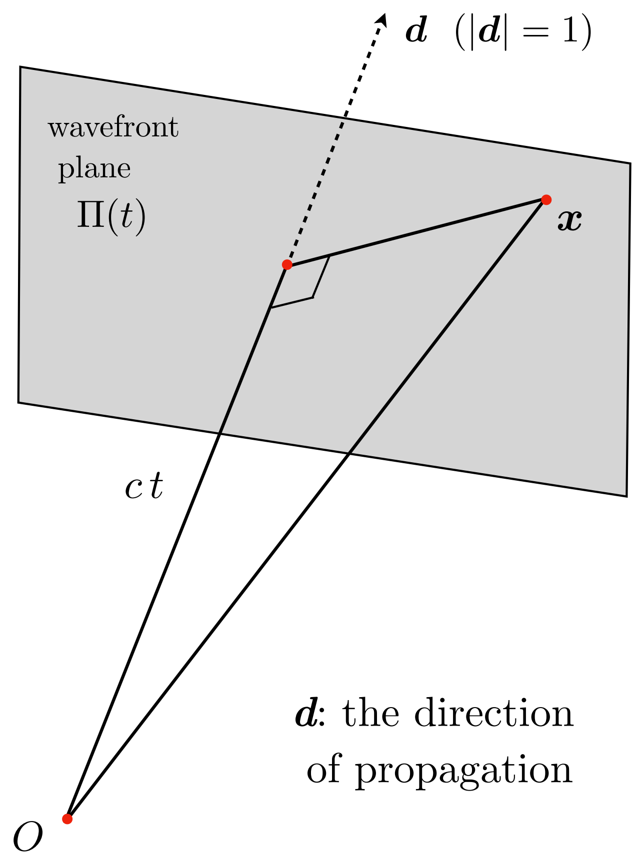
Planewaves of the form:

$$u(\boldsymbol{x}, t) = \mathrm{e}^{ik\boldsymbol{x}\cdot\boldsymbol{d} - i\omega t}$$

are solutions of the (homogeneous) wave equation



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  The wave equation admits solutions of the form: $u(\boldsymbol{x}, t) = \mathrm{e}^{-i\omega t}\, v(\boldsymbol{x})$

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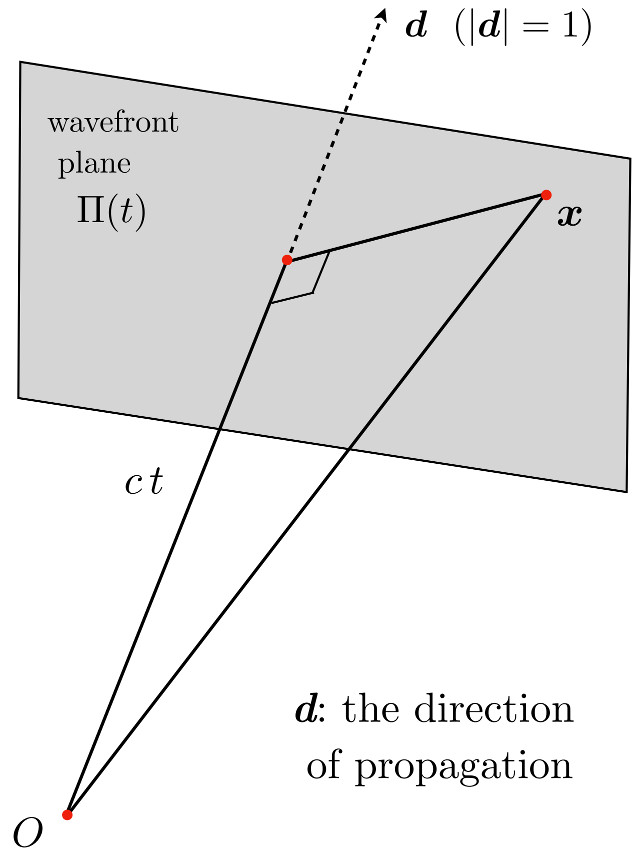
  Angular spectrum representation



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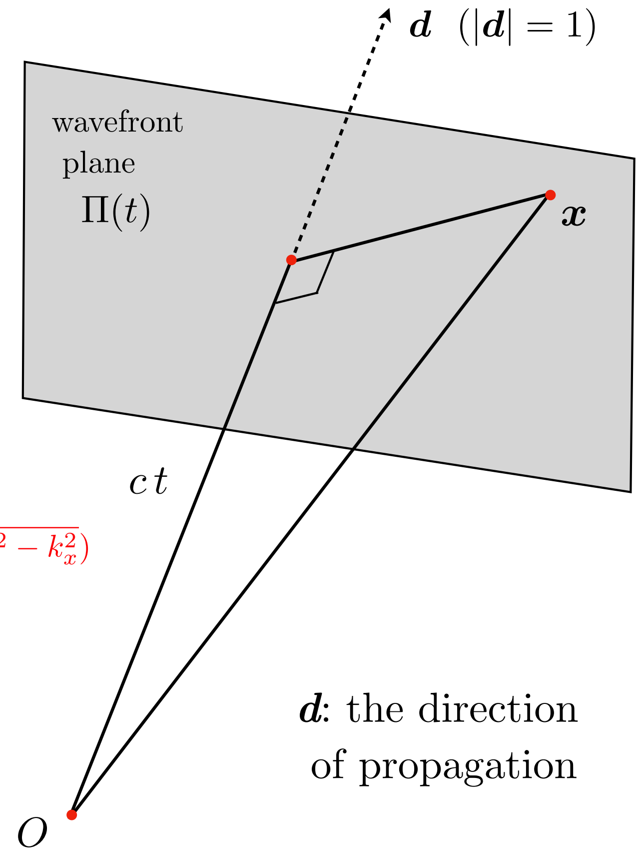

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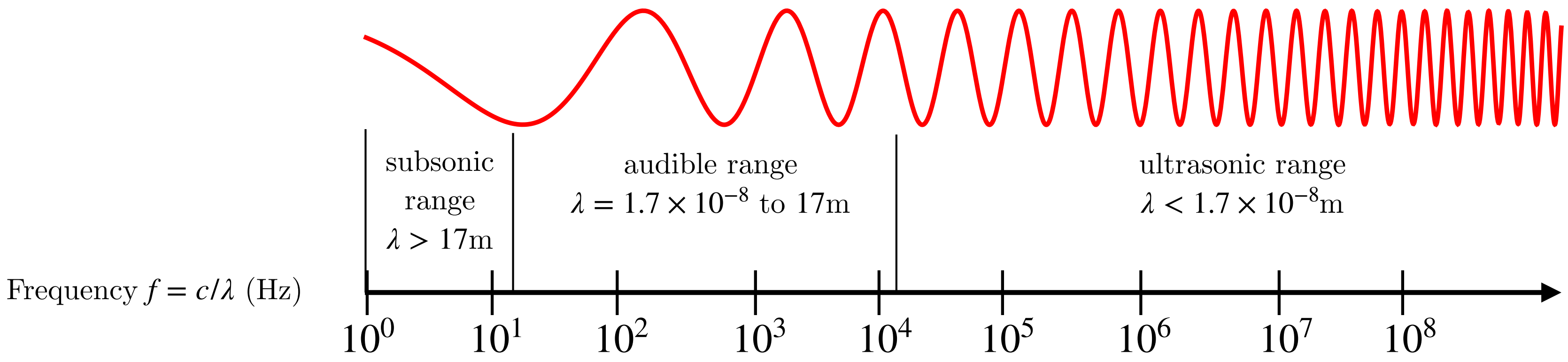
Acoustic Waves: Applications



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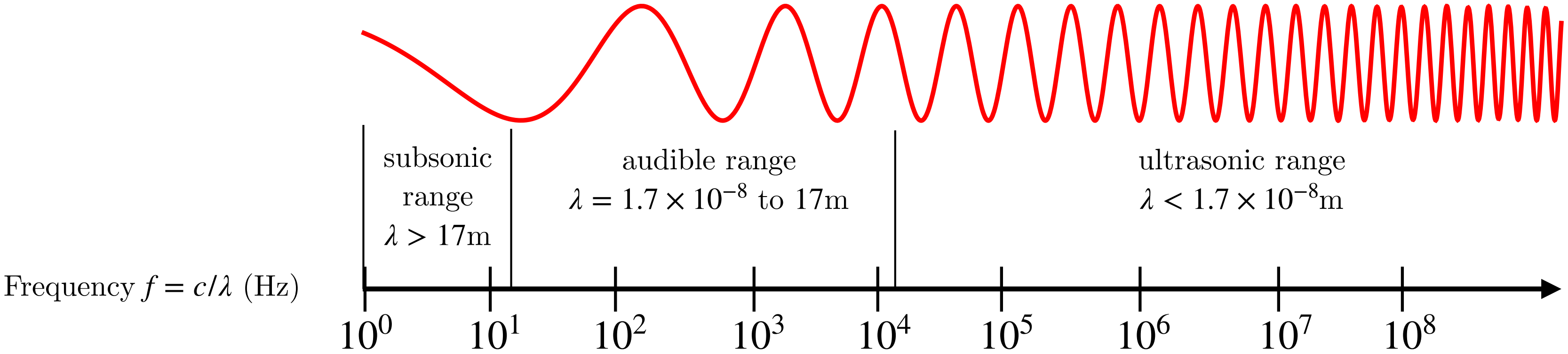
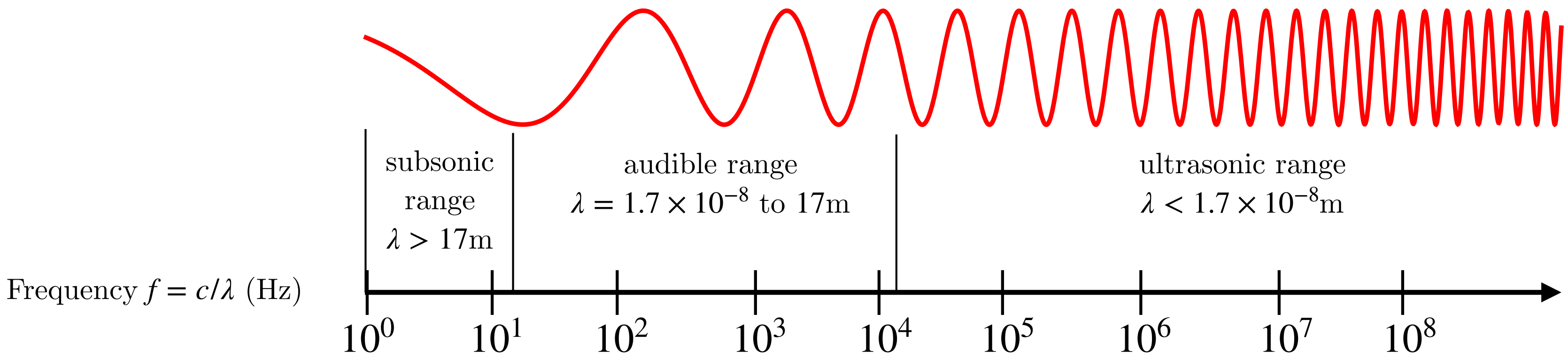


www.wikipedia.com

Infrasound monitoring station
$\lambda = 10$ to 10^3m

Acoustic Waves: Applications



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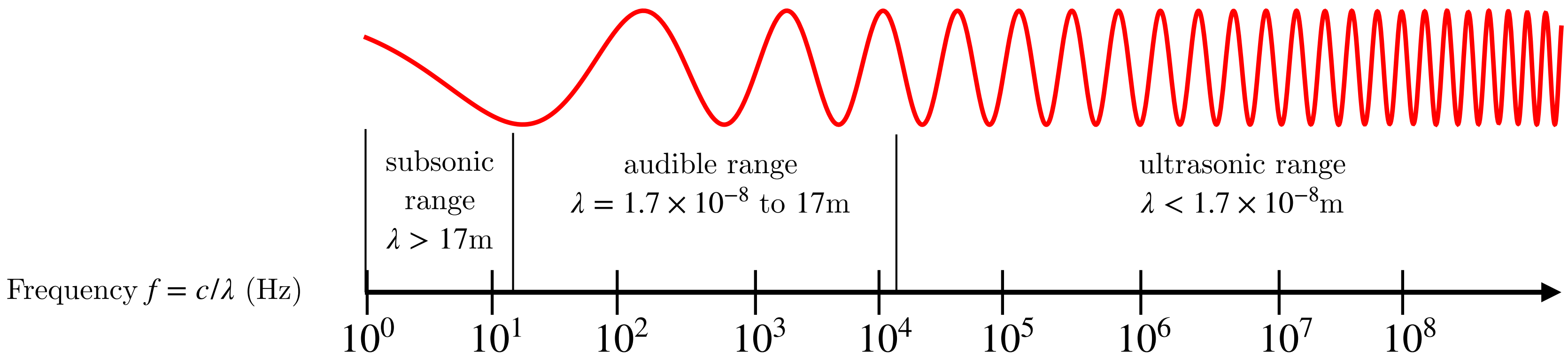
Acoustic Waves: Applications



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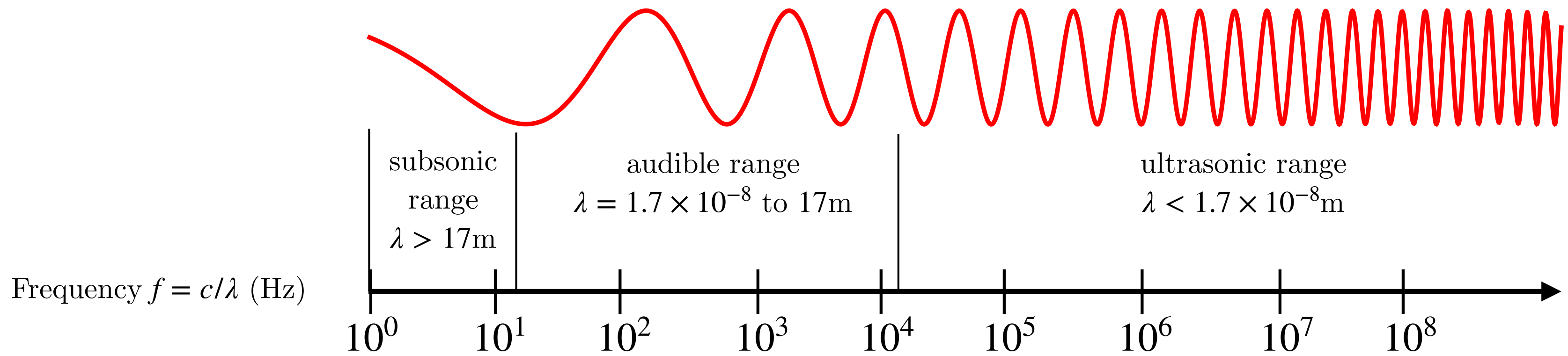
Room acoustics

Acoustic Waves: Applications



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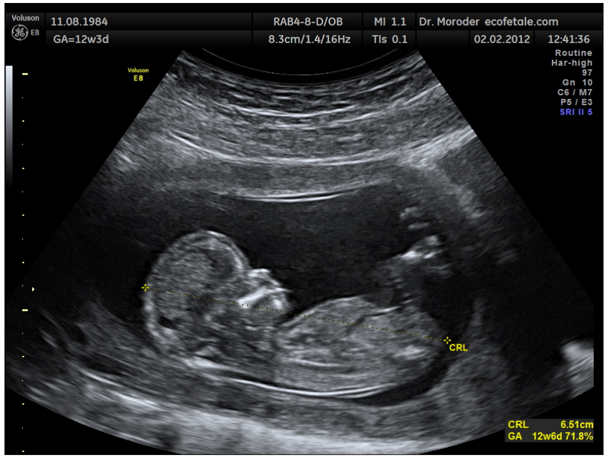
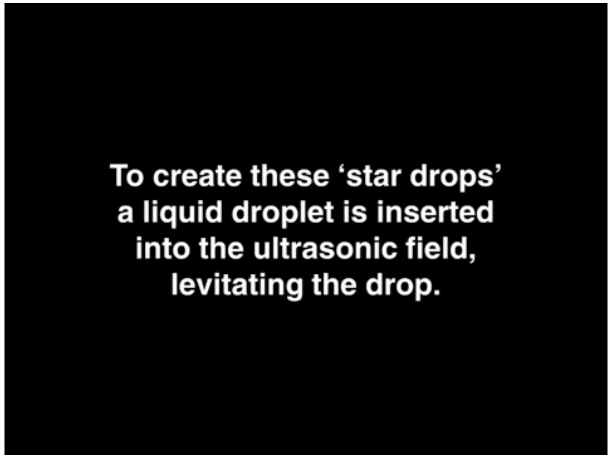


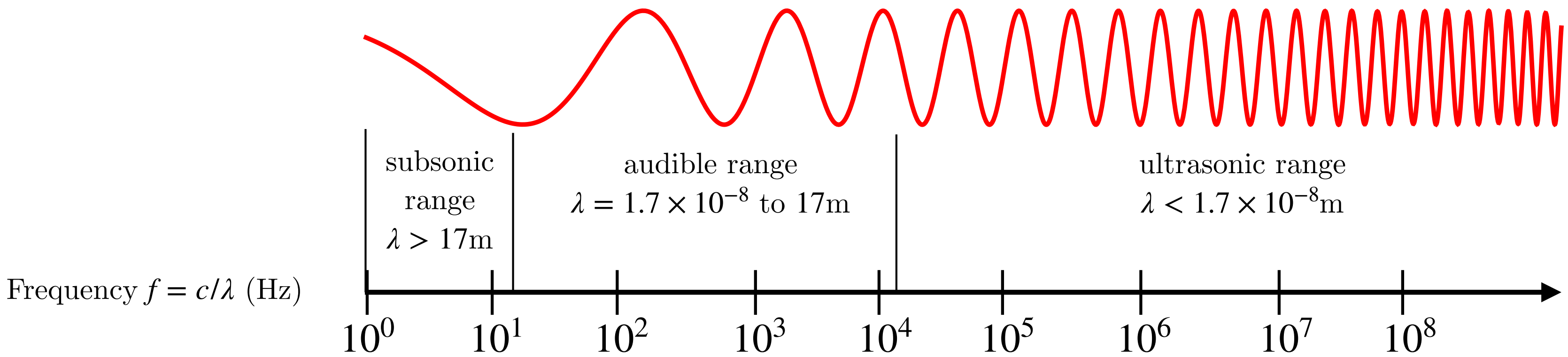
Ultrasound imaging



Droplet levitation

Sonoluminescence

Acoustic Waves: Applications



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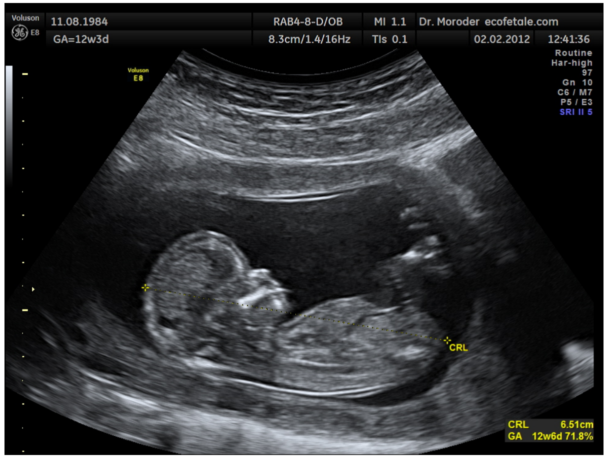

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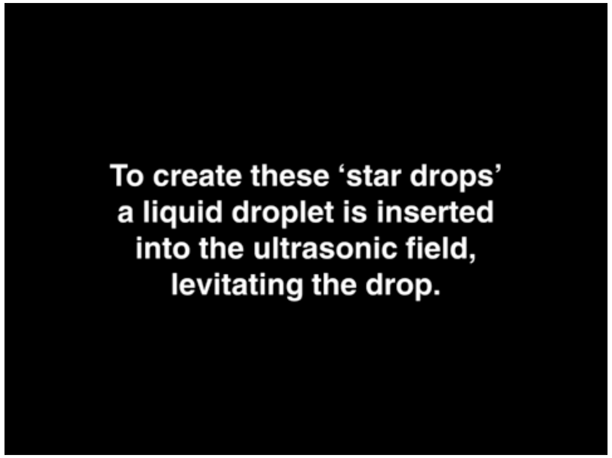

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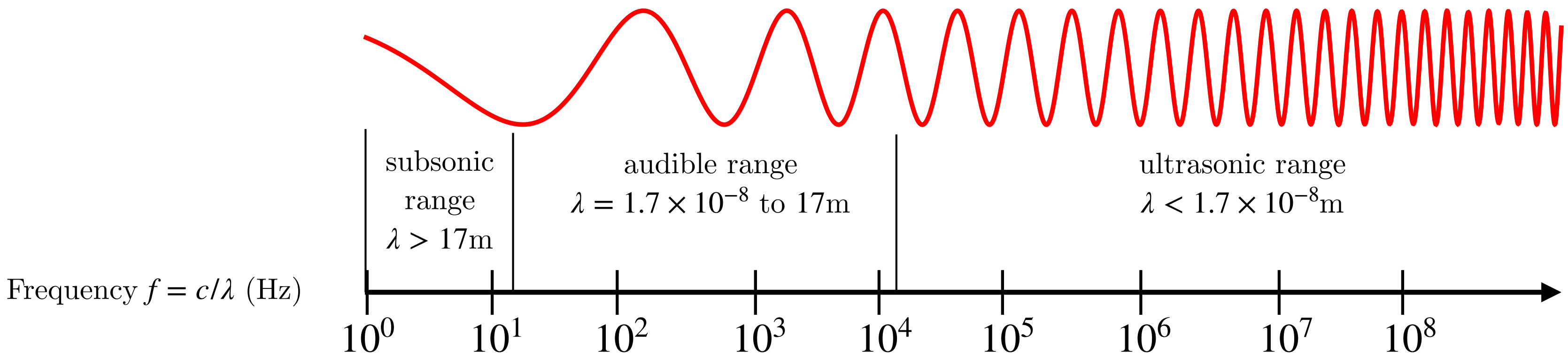

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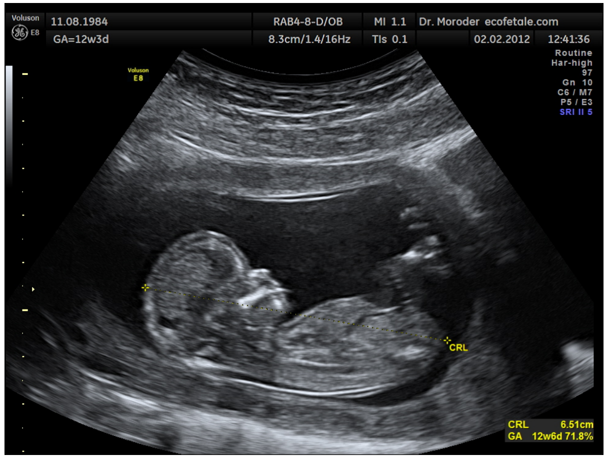

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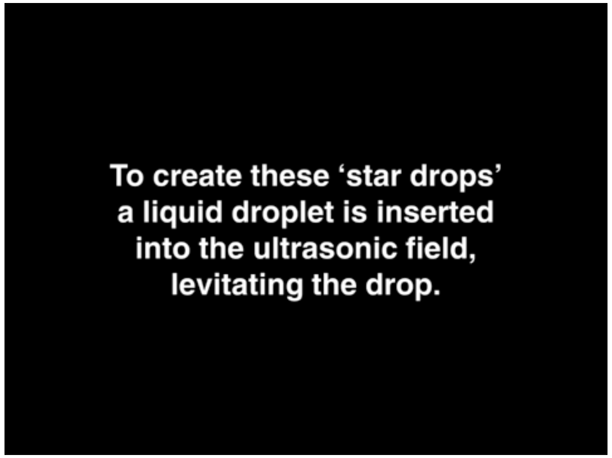

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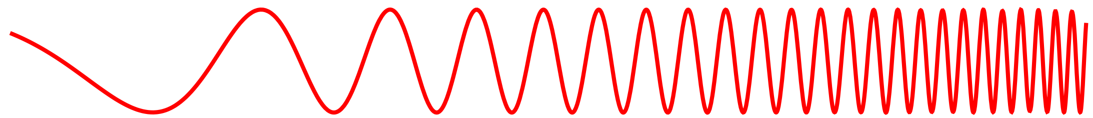
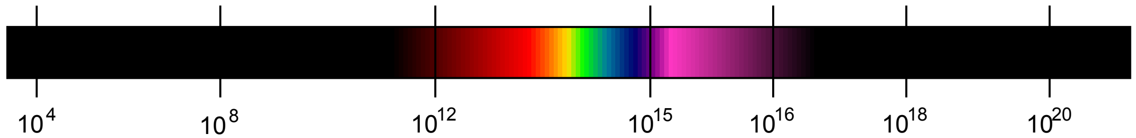
Electromagnetic Waves: Applications

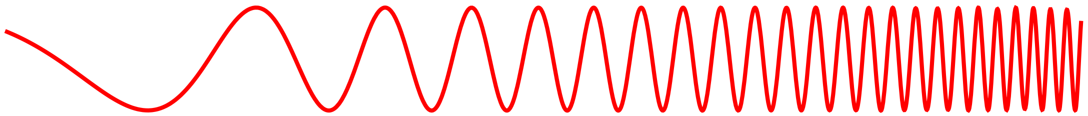


Type of radiation	Radio waves	Microwaves	Infrared	Visible	Ultraviolet	X-rays	Gamma rays
Wavelength $\lambda = c/f$ (m)	10^{3}	10^{-2}	10^{-5}	5×10^{-7}	10^{-8}	10^{-10}	10^{-12}

Frequency $f = c/\lambda$ (Hz)



Electromagnetic Waves: Applications



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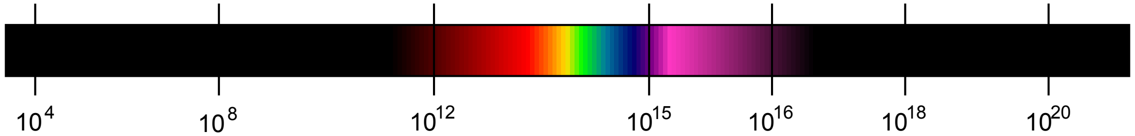
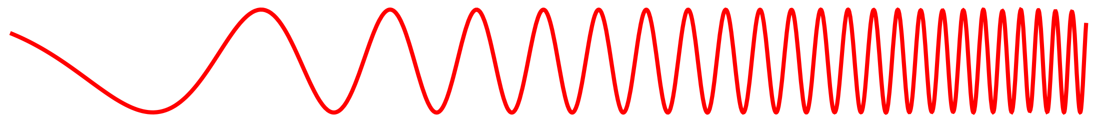


www.wikipedia.com

Telecommunications
$\lambda = 1$ to 10^3m

Electromagnetic Waves: Applications



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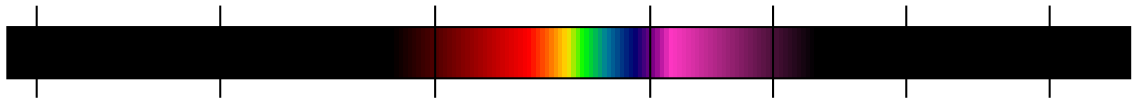

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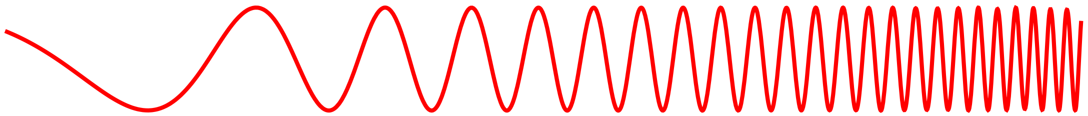
Atacama Large Millimeter Array
$\lambda = 3 \times 10^{-4}$ to 3.6×10^{-3} m

Electromagnetic Waves: Applications



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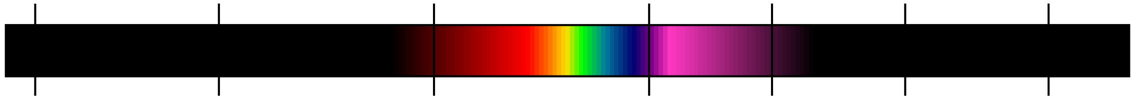

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Thermography
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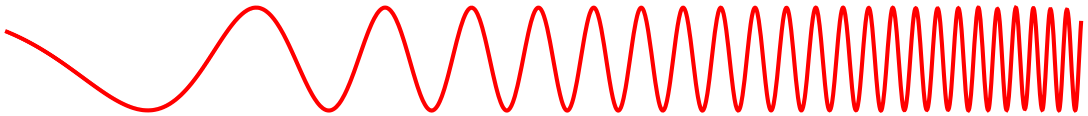
Electromagnetic Waves: Applications



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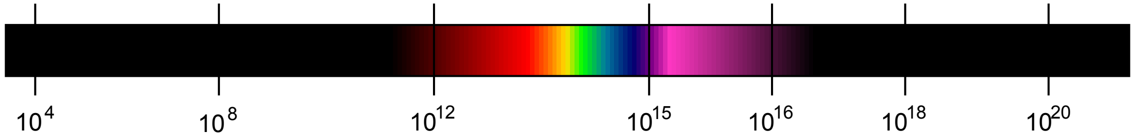
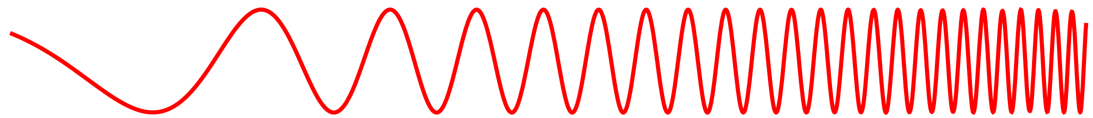


www.aminerals.cl

Solar power plants
$\lambda = 4 \times 10^{-7}$ to 7×10^{-7}m

Electromagnetic Waves: Applications



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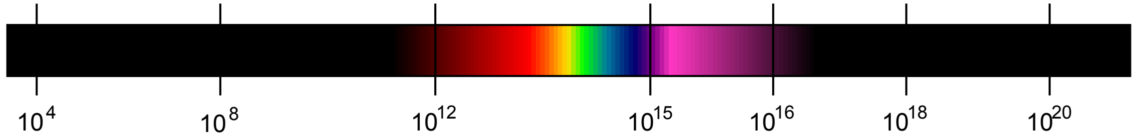

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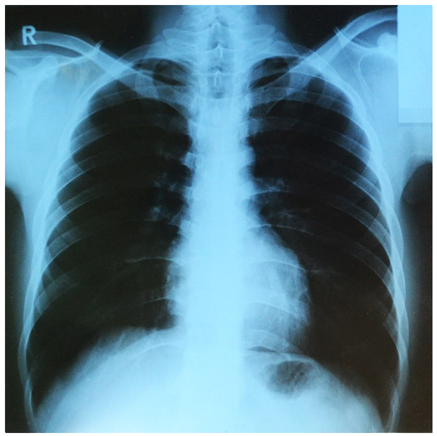
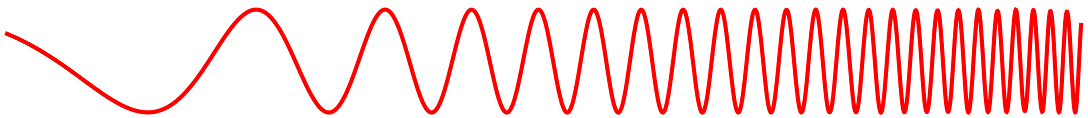
www.wikipedia.com



X-ray imagining
$\lambda = 10^{-11}$ to 10^{-8}m

Electromagnetic Waves: Applications



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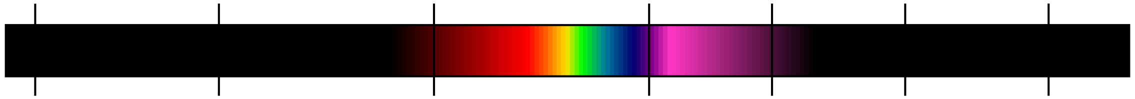

www.aminerals.cl

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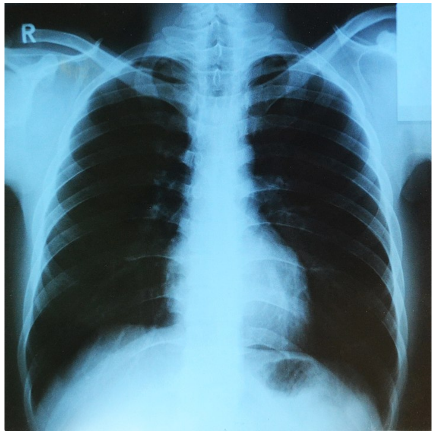
www.wikipedia.com



X-ray imagining
$\lambda = 10^{-11}$ to 10^{-8}m

www.siemens-healthineers.com

Positron Emission Tomography (PET)
$\lambda = 2.5 \cdot 10^{-12}$m

Acoustic Waves

Acoustic Waves

Velocity field: $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$

Pressure: $p = p(\boldsymbol{x}, t)$

Density: $\rho = \rho(\boldsymbol{x}, t)$

Inviscid fluid: $\mu = 0$ (viscosity)

Acoustic Waves

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Density: $\rho = \rho(\boldsymbol{x}, t)$

Pressure: $p = p(\boldsymbol{x}, t)$

Inviscid fluid: $\mu = 0$ (viscosity)

- Continuity equation (mass conservation): $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Acoustic Waves

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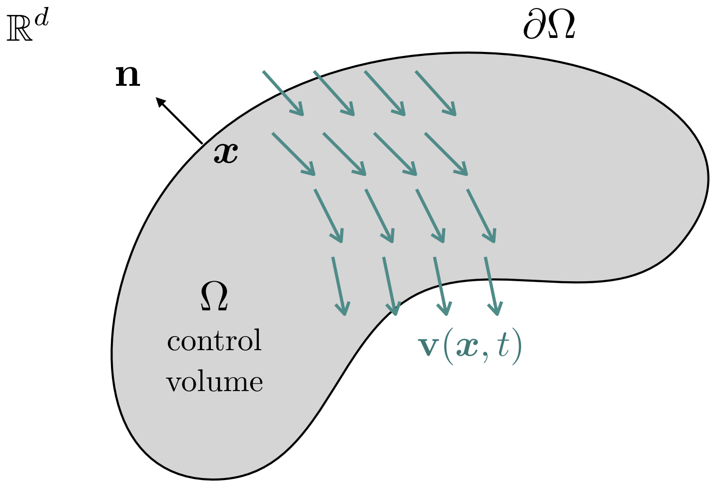

total enclosed mass: $m(t) = \int_\Omega \rho(\boldsymbol{x}, t)\, d\boldsymbol{x}$

$$\dot{m}(t) = \int_\Omega \frac{\partial \rho(\boldsymbol{x}, t)}{\partial t}\, d\boldsymbol{x} = -\underbrace{\int_{\partial\Omega} \rho(\boldsymbol{x}, t)\, \mathbf{v}(\boldsymbol{x}, t) \cdot \mathbf{n}(\boldsymbol{x})\, ds}_{\text{net mass flux}}$$

$$\overset{\text{Green's theorem}}{=} -\int_\Omega \nabla \cdot (\rho(\boldsymbol{x}, t)\mathbf{v}(\boldsymbol{x}, t))\, d\boldsymbol{x}$$

The control volume is arbitrary $\Longrightarrow \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Acoustic Waves

Velocity field: $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$

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We consider **small perturbations $\mathbf{v}$, p, and ρ to a steady background state $\mathbf{v}^{(0)}$, $p^{(0)}$, and $\rho^{(0)}$**. For simplicity, the background medium is assumed to be homogeneous, at rest, and in equilibrium, i.e., $\rho^{(0)}$ **is constant, $\mathbf{v}^{(0)} = \mathbf{0}$, and** $\nabla p^{(0)} = 0$.

formal asymptotic expansion
$$\begin{cases} \rho = \rho^{(0)} + \varepsilon \rho^{(1)} + O(\varepsilon^2) \\ p = p^{(0)} + \varepsilon p^{(1)} + O(\varepsilon^2) \\ \mathbf{v} = \mathbf{v}^{(0)} + \varepsilon \mathbf{v}^{(1)} + O(\varepsilon^2) \end{cases} \qquad (|\varepsilon| \ll 1)$$

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linear waves

Acoustic Waves: Linearization

- Continuity equation: $$\frac{\partial}{\partial t}\left(\rho^{(0)}+\varepsilon\rho^{(1)}\right)+\nabla\cdot\left\{\left(\rho^{(0)}+\varepsilon\rho^{(1)}\right)\left(\mathbf{v}^{(0)}+\varepsilon\mathbf{v}^{(1)}\right)\right\}=\mathbf{0}$$

Acoustic Waves: Linearization

- Continuity equation: $$\frac{\partial}{\partial t}\left(\rho^{(0)} + \varepsilon\rho^{(1)}\right) + \nabla \cdot \left\{\left(\rho^{(0)} + \varepsilon\rho^{(1)}\right)\left(\mathbf{v}^{(0)} + \varepsilon\mathbf{v}^{(1)}\right)\right\} = \mathbf{0}$$

at $O(1): \quad \partial_t \rho^{(0)} + \nabla \cdot \left(\rho^{(0)}\mathbf{v}^{(0)}\right) = 0$

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(background
is at rest)

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$$\partial_t\rho^{(0)} = 0 \Longrightarrow \rho^{(0)} = \rho^{(0)}(\boldsymbol{x})$$

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$$\partial_t\rho^{(0)}=0\Longrightarrow\rho^{(0)}=\rho^{(0)}(\boldsymbol{x})$$

(background is homogeneous)

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(Taylor expansion around $\rho=\rho^{(0)}$)

$$=p^{(0)}+\varepsilon p^{(1)}+O\left(\varepsilon^2\right)$$

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matching at $O(\epsilon)$: $\quad p^{(1)}=P'(\rho^{(0)})\rho^{(1)}$

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we define: $c:=\sqrt{P'(\rho_0)}$

Linear Acoustic Waves

Summarizing, at $O(\varepsilon)$ we have

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Wave equation: $$\frac{\partial^2 p^{(1)}}{\partial t^2}(\boldsymbol{x}, t) - c^2 \Delta p^{(1)}(\boldsymbol{x}, t) = F(\boldsymbol{x}, t)$$

external source

Speed of sound: $$c := \sqrt{P'(\rho_0)}$$

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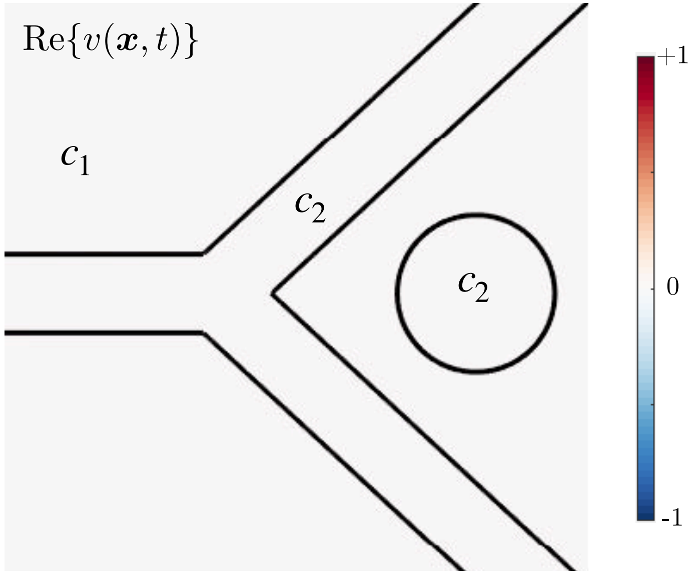
(the velocity field $\mathbf{v}^{(1)}$ and the mass density $\rho^{(1)}$ also satisfy the wave equation)

Time-Harmonic Scalar Waves

Forced wave equation: $$\frac{\partial^2 v}{\partial t^2} - c^2 \Delta v = F(\boldsymbol{x}, t), \quad v(\boldsymbol{x}, 0) = v_t(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d$$

time-harmonic source



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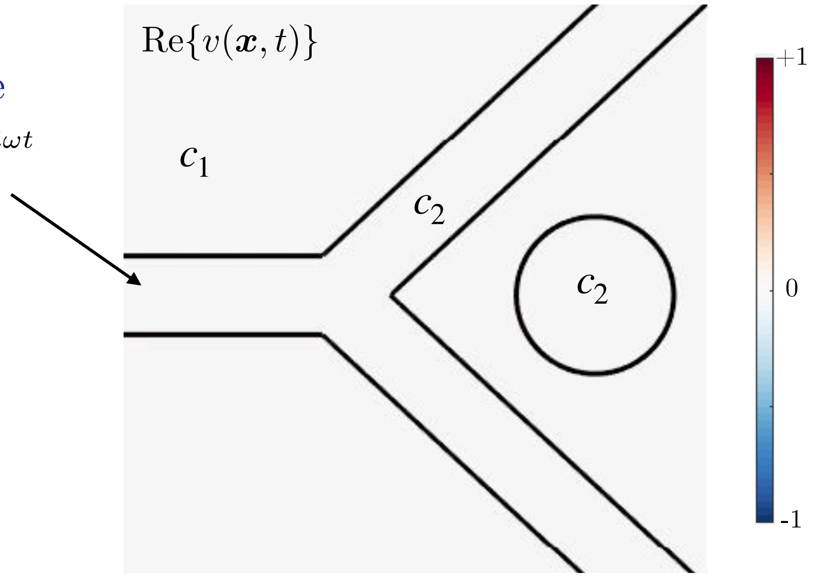
time-harmonic source

$$F(\boldsymbol{x}, t) = \delta_0(\boldsymbol{x})\, \mathrm{e}^{-i\omega t}$$



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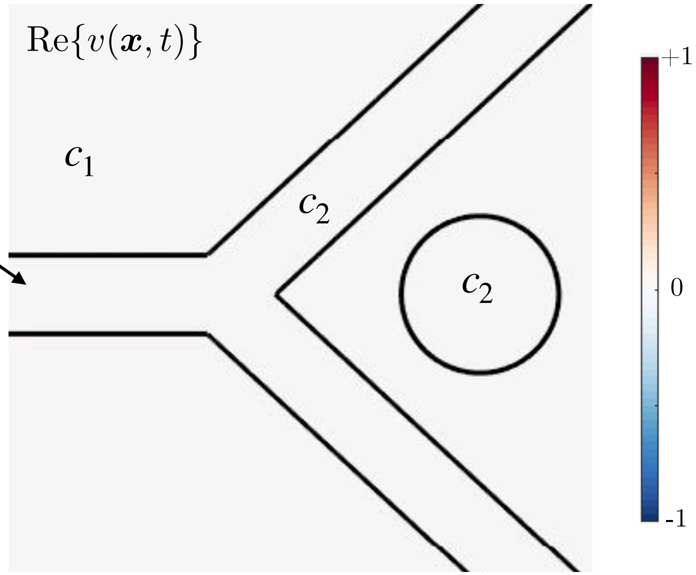

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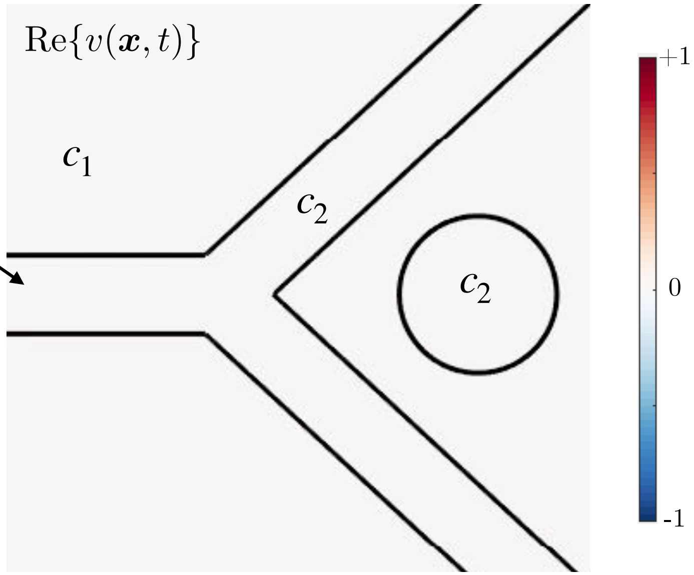

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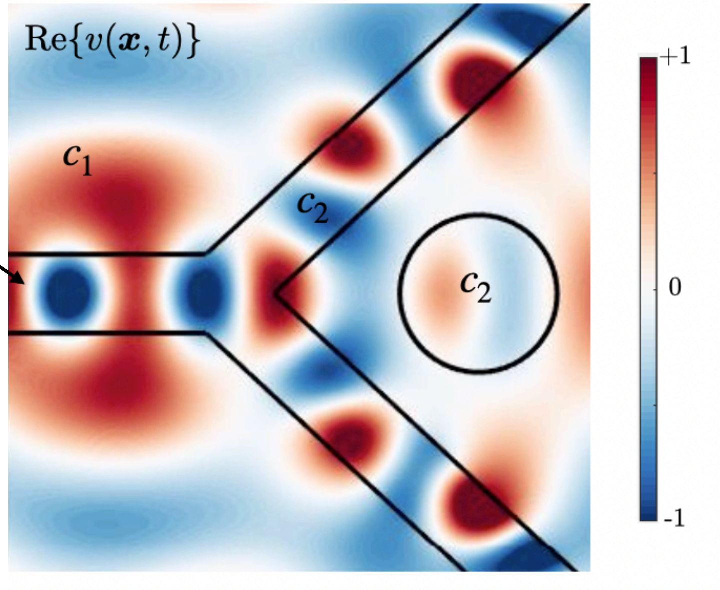
Time-Harmonic Scalar Waves

Forced wave equation: $v(\boldsymbol{x}, t) = u(\boldsymbol{x})\,\mathrm{e}^{-i\omega t} + o(1) \quad \text{as } t \to \infty$

time-harmonic source

$F(\boldsymbol{x}, t) = \delta_0(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}$



$\operatorname{Re}\{u(\boldsymbol{x})\}$

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Time-Harmonic Scalar Waves: The Helmholtz Equation

Consider the wave equation with a harmonic source term:

$$\frac{\partial^2 v}{\partial t^2} - c^2 \Delta v = f(\boldsymbol{x}) \mathrm{e}^{-i\omega t}, \quad v(\boldsymbol{x}, 0) = v_t(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d$$

(angular frequency: $\omega > 0$)
($c =$ constant)

Time-Harmonic Scalar Waves: The Helmholtz Equation

Consider the wave equation with a harmonic source term:

$$\frac{\partial^2 v}{\partial t^2} - c^2 \Delta v = f(\boldsymbol{x}) e^{-i\omega t}, \quad v(\boldsymbol{x}, 0) = v_t(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d$$

(angular frequency: $\omega > 0$)
($c =$ constant)

It can be shown that (under certain assumptions on f, e.g., $f \in C_0^\infty(\mathbb{R}^d)$)

$$v(\boldsymbol{x}, t) = u(\boldsymbol{x})\, e^{-i\omega t} + o(1) \quad \text{as } t \to \infty$$

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where u satisfies the **Helmholtz equation:** $\Delta u + k^2 u = f \quad \text{en} \quad \mathbb{R}^d$

where $k = \dfrac{\omega}{c}$ is the **wavenumber** and ω is the angular frequency

The **Sommerfeld radiation condition** is needed: $\displaystyle \lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial |\boldsymbol{x}|} - iku \right) = 0$

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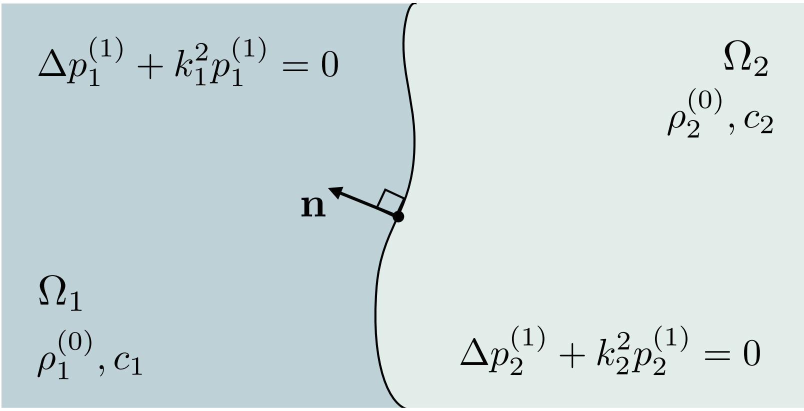
This result is known as the **limiting amplitude principle**

D. M. Eidus, *The principle of limiting amplitude*, Russian Mathematical Surveys, 24(2):97–167, 1969.

A.G. Ramm, *Scattering by Obstacles*. Vol. 21. Springer Science & Business Media, 1986.

Acoustic Transmission/Boundary Conditions

- Sound transmission between two fluids:



Two conditions:

1. Dynamic condition: $p_1^{(1)}\mathbf{n} = p_2^{(1)}\mathbf{n}$
   (balance of normal stresses)

2. Kinematic condition: $\mathbf{v}_1^{(1)} \cdot \mathbf{n} = \mathbf{v}_2^{(1)} \cdot \mathbf{n}$
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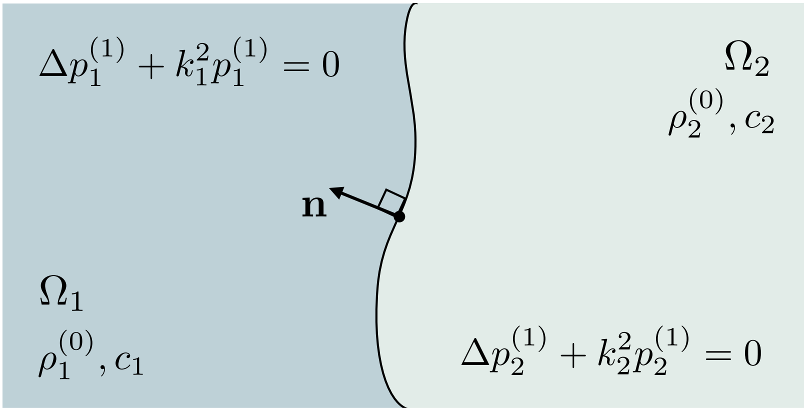

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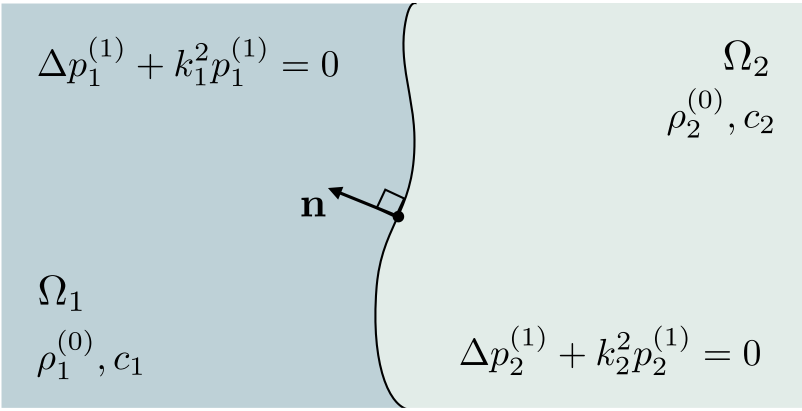

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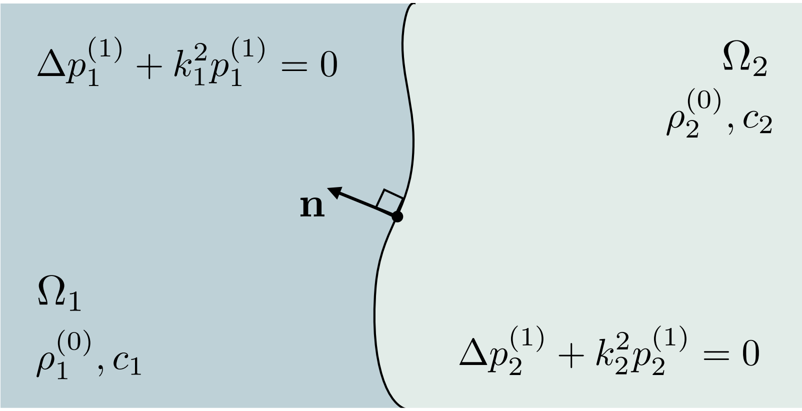

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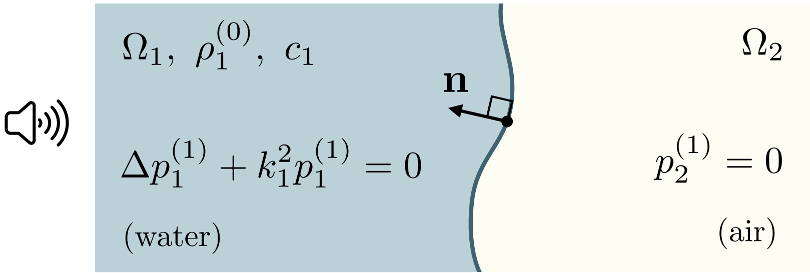

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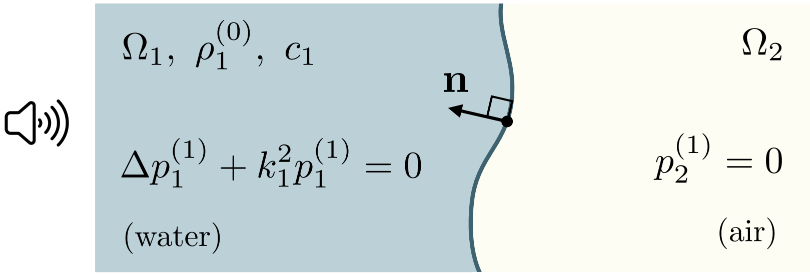

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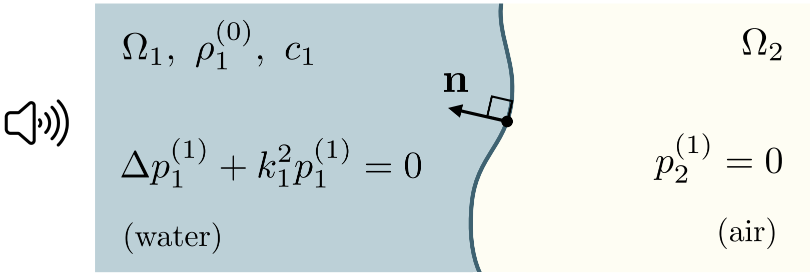

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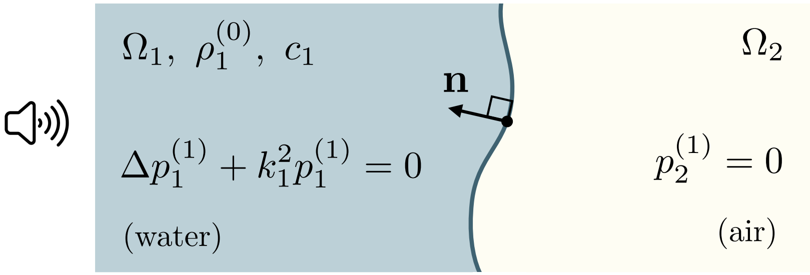

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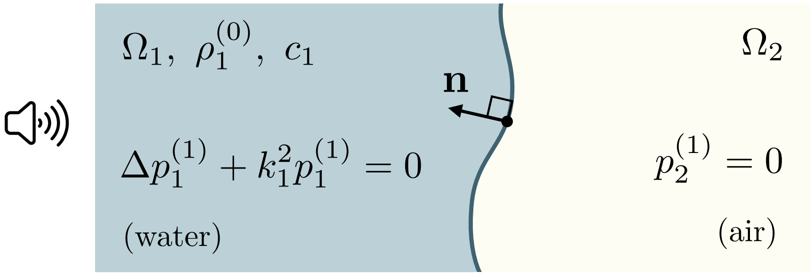

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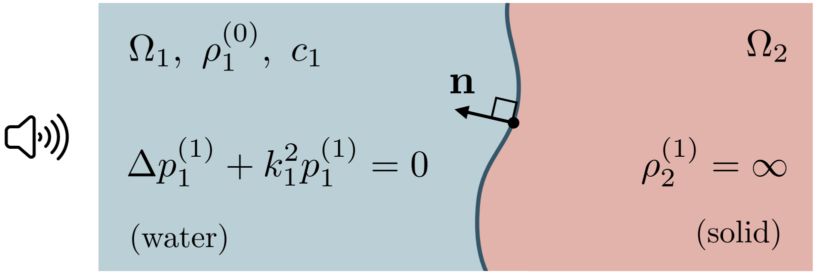

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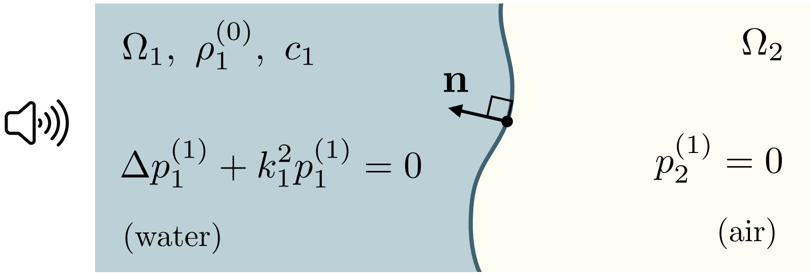

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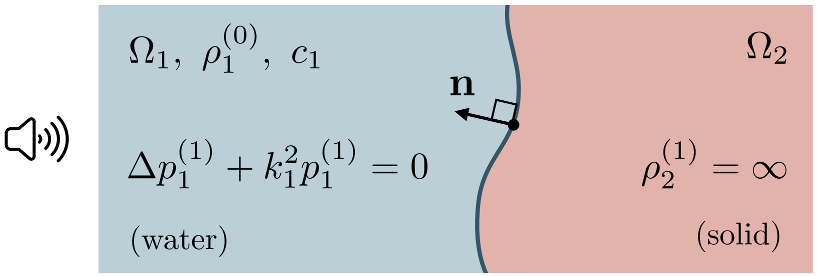

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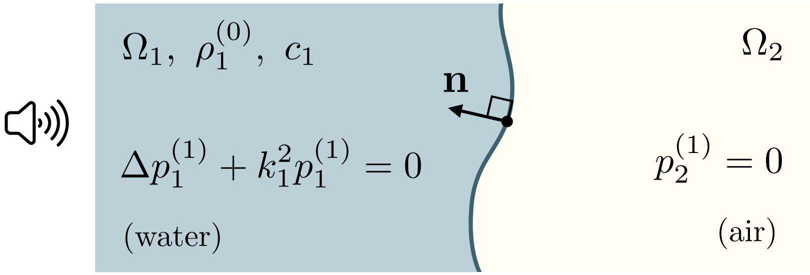

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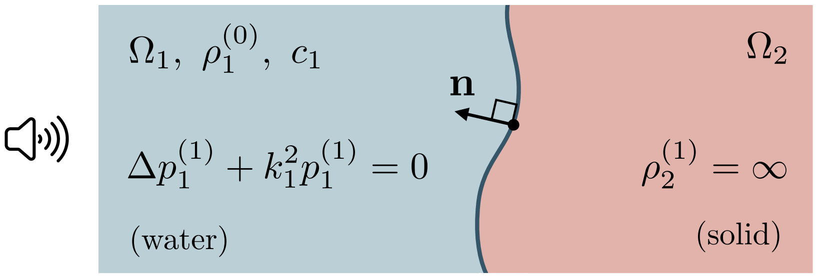

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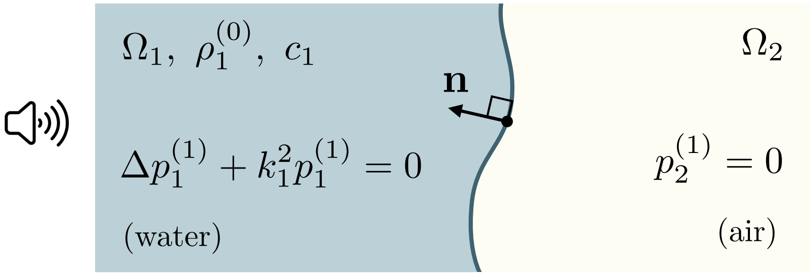

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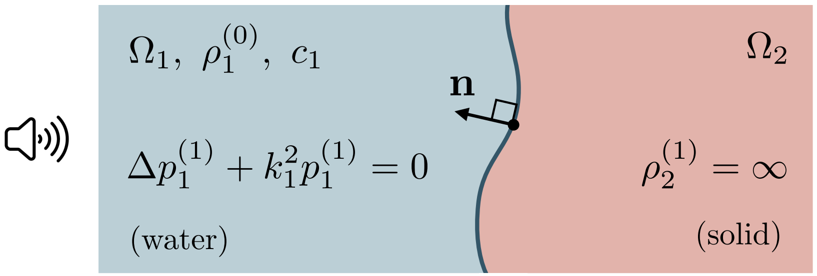

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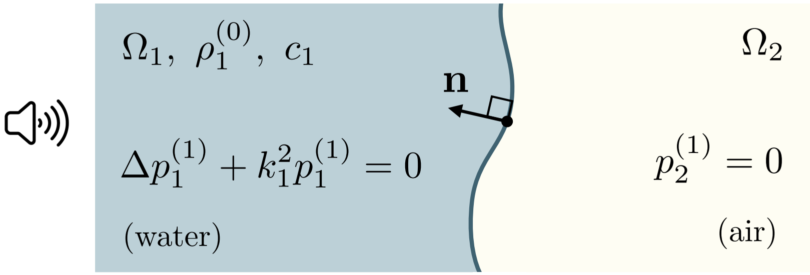

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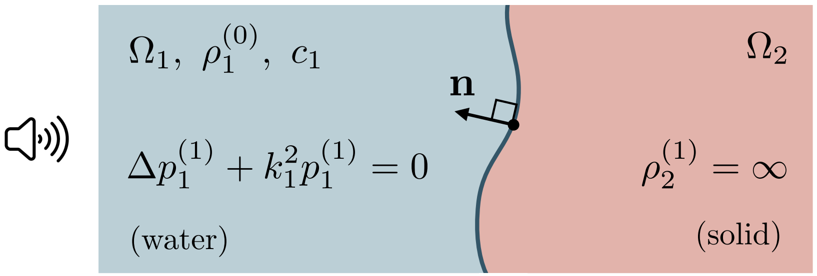

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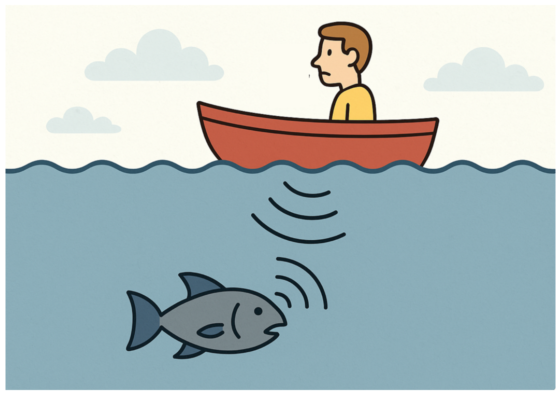

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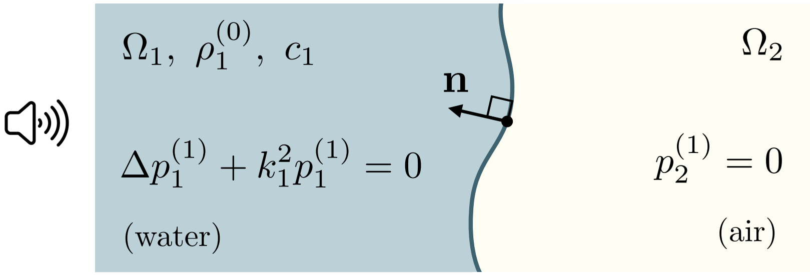

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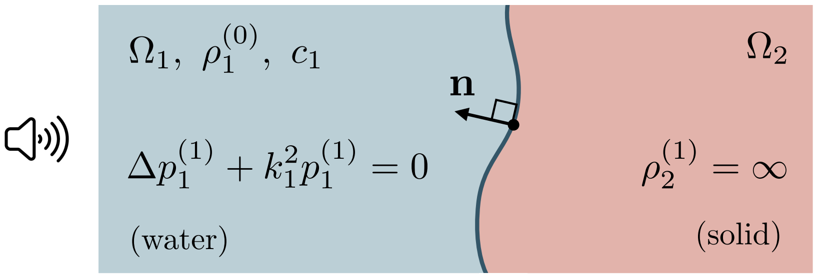

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- Impedance boundary condition: Somebody gives you the surface acoustic impedance: $Z := \dfrac{p_1^{(1)}}{\mathbf{v}_1^{(1)} \cdot \mathbf{n}}$.

Substitute $\mathbf{v}_1^{(1)} \cdot \mathbf{n}$ in the momentum equation: $Z = \dfrac{p_1^{(1)}}{\frac{1}{i\omega\rho_1^{(0)}}\frac{\partial p_1^{(1)}}{\partial n}} = i\omega\rho_1^{(0)} \dfrac{p_1^{(1)}}{\frac{\partial p_1^{(1)}}{\partial n}}$.

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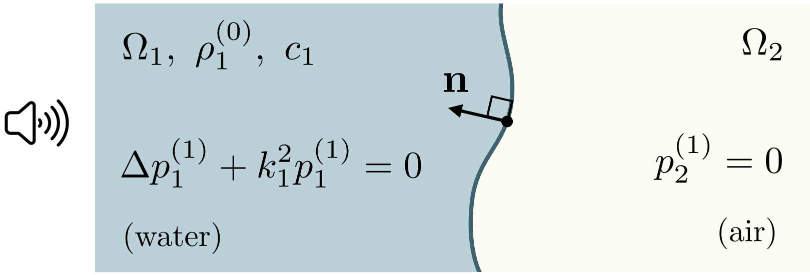

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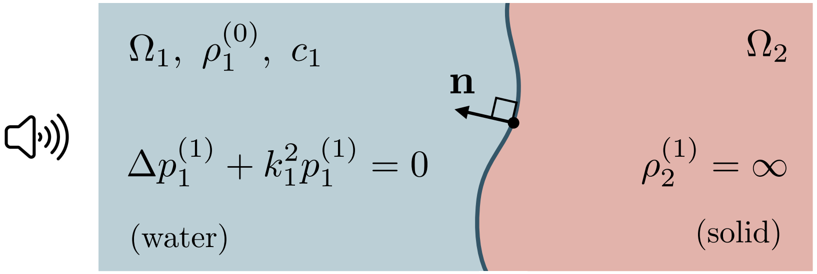

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Robin boundary condition: $\dfrac{\partial p_1^{(1)}}{\partial n} = \dfrac{i\omega\rho_1^{(0)}}{Z} p_1^{(1)} \quad \text{on} \quad \Gamma$

... it approximates how **porous** or **layered materials** absorb sound energy rather than reflecting it completely (as in sound-soft) or not at all (as in sound-hard).

Electromagnetic waves

Electromagnetic Fields

Electric field: $\mathcal{E}(\boldsymbol{x}, t)$

Electric current density: $\mathcal{J}(\boldsymbol{x}, t)$

Magnetic field: $\mathcal{H}(\boldsymbol{x}, t)$

Magnetic induction: $\mathcal{B}(\boldsymbol{x}, t)$

Electric displacement field: $\mathcal{D}(\boldsymbol{x}, t)$

Charge density: $\varrho(\boldsymbol{x}, t)$

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Charge conservation: $$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho \,\mathrm{d}V = -\int_S \mathcal{J} \cdot \mathrm{d}\mathbf{S} \quad \underset{\text{(Green's theorem)}}{\Longrightarrow} \quad \nabla \cdot \mathcal{J} = -\frac{\partial \varrho}{\partial t}$$

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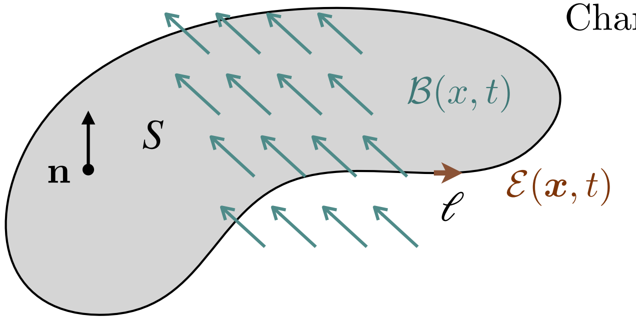

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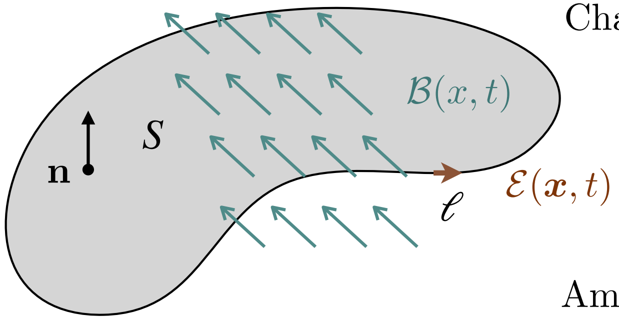

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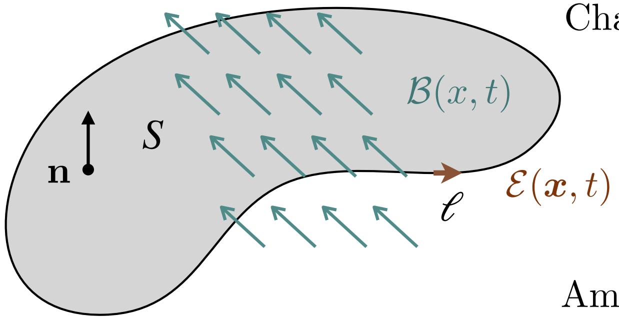
- Fundamental laws:



Charge conservation: $$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho \, \mathrm{d}V = -\int_S \mathcal{J}\cdot \mathrm{d}\mathbf{S} \quad \Longrightarrow \quad \nabla\cdot\mathcal{J} = -\frac{\partial\varrho}{\partial t}$$
(Green's theorem)

Faraday's law: $$\oint_\ell \mathcal{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathcal{B}\cdot \mathrm{d}\mathbf{S} \quad \Longrightarrow \quad \nabla\times\mathcal{E} = -\frac{\partial\mathcal{B}}{\partial t}$$
(Stokes' theorem)

Ampère's law: $$\oint_\ell \mathcal{H}\cdot \mathrm{d}\boldsymbol{\ell} = \int_S \left(\mathcal{J} + \frac{\partial\mathcal{D}}{\partial t}\right)\cdot \mathrm{d}\mathbf{S} \quad \Longrightarrow \quad \nabla\times\mathcal{H} = \mathcal{J} + \frac{\partial\mathcal{D}}{\partial t}$$
(Stokes' theorem)

- Constitutive relations (linear isotropic medium)

$\mathcal{D} = \epsilon\mathcal{E}$ Electric permittivity (dielectric constant): ϵ

$\mathcal{J} = \sigma\mathcal{E}$ Conductivity: σ

$\mathcal{B} = \mu\mathcal{H}$ Magnetic permeability: μ

Electromagnetic Waves

James Clerk Maxwell
1831-1879

Maxwell's equations
$$\begin{cases} \nabla \times \mathcal{E} = -\mu \dfrac{\partial \mathcal{H}}{\partial t} \\[2ex] \nabla \times \mathcal{H} = \epsilon \dfrac{\partial \mathcal{E}}{\partial t} + \sigma \mathcal{E} + \mathcal{J}_0 \end{cases}$$

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source

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source

Solving for the magnetic field, we obtain

$$\nabla \times \nabla \times \mathcal{H} + \mu\epsilon \frac{\partial^2 \mathcal{H}}{\partial t^2} + \mu\sigma \frac{\partial \mathcal{H}}{\partial t} = \nabla \times \mathcal{J}_0.$$

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Then, using the identity $\Delta \mathcal{H} = \nabla(\nabla \cdot \mathcal{H}) - \nabla \times \nabla \times \mathcal{H}$ and noting that $\nabla \cdot \mathcal{H} = 0$, we obtain:

$$\boxed{\frac{\partial^2 \mathcal{H}}{\partial t^2} + \frac{\sigma}{\epsilon}\frac{\partial \mathcal{H}}{\partial t} - \frac{1}{\mu\epsilon}\Delta \mathcal{H} = \frac{1}{\mu\epsilon}\nabla \times \mathcal{J}_0} \quad \left(\frac{\partial^2 \mathcal{H}}{\partial t^2} - \frac{1}{\mu\epsilon}\Delta \mathcal{H} = \frac{1}{\mu\epsilon}\nabla \times \mathcal{J}_0 \text{ when } \sigma = 0 \right)$$

where $c = 1/\sqrt{\mu\epsilon}$ is the **speed of light**.

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wave equation!

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James Clerk Maxwell
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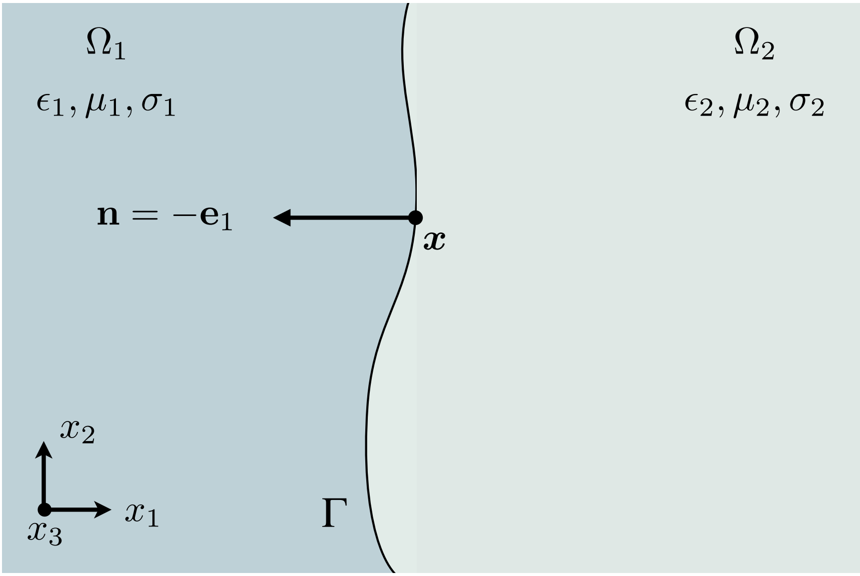
Similarly: $$\frac{\partial^2 \mathcal{E}}{\partial t^2} + \frac{\sigma}{\epsilon} \frac{\partial \mathcal{E}}{\partial t} - \frac{1}{\mu\epsilon} \Delta \mathcal{E} = -\frac{1}{\epsilon} \frac{\partial \mathcal{J}_0}{\partial t}$$

Electromagnetic Transmission/Boundary Conditions



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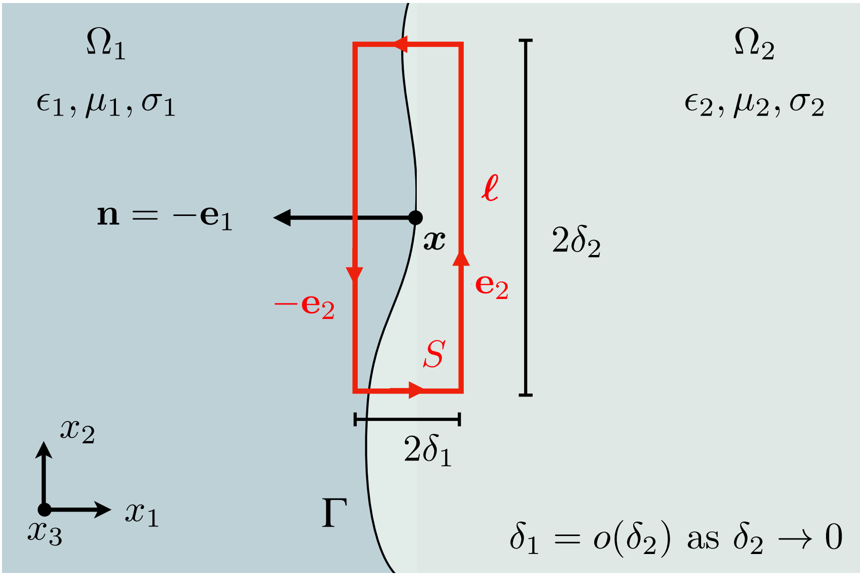
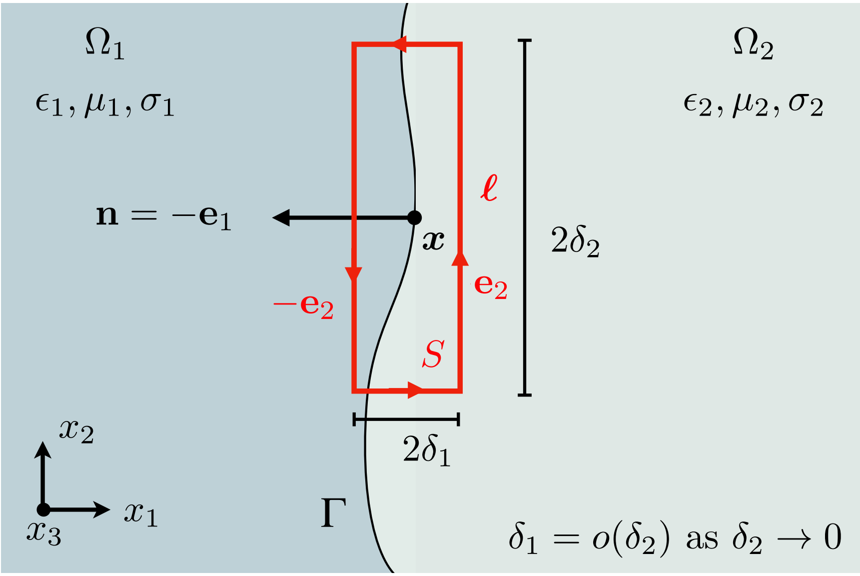


Electromagnetic Transmission/Boundary Conditions



Faraday's law:

$$\oint_{\ell} \mathcal{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathcal{B} \cdot \ \mathrm{d}\mathbf{S} = -\underbrace{\int_S \frac{\partial \mathcal{B}}{\partial t} \cdot \ \mathrm{d}\mathbf{S}}_{O(\delta_1 \delta_2)}$$

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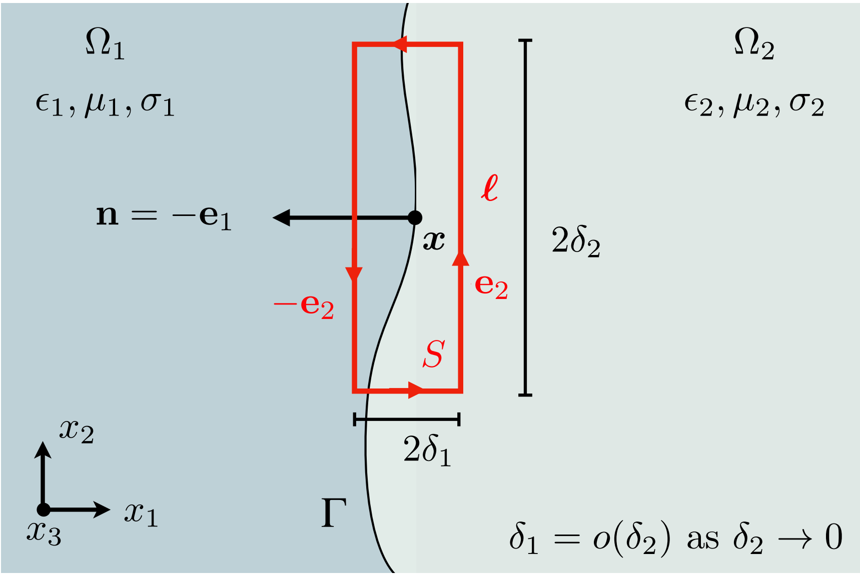

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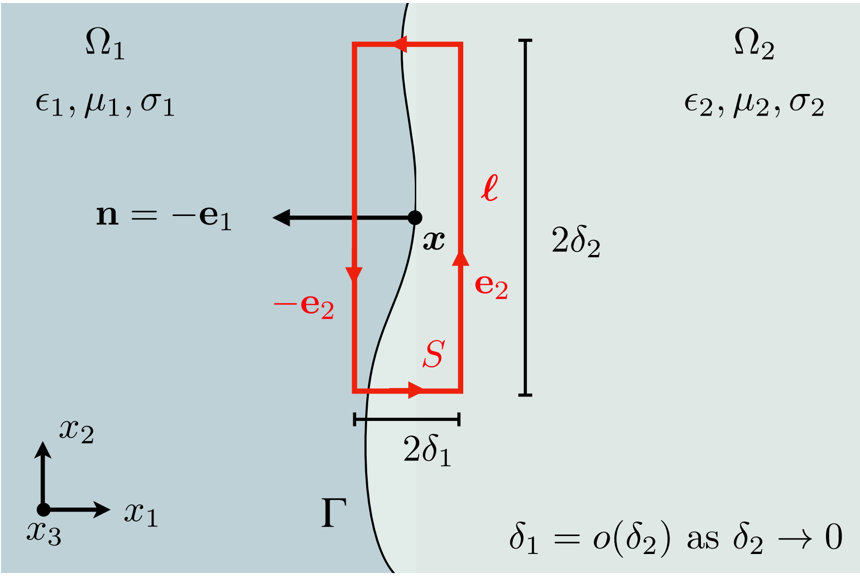
Electromagnetic Transmission/Boundary Conditions



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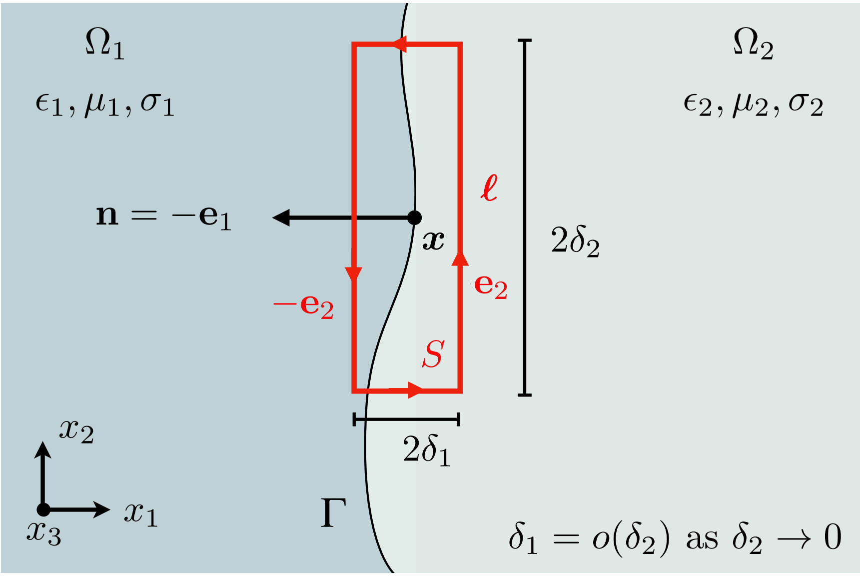
taking the limit $\delta_2 \to 0$, we arrive at (using the FTC)

$$\left(\mathcal{E}_2(\boldsymbol{x}) - \mathcal{E}_1(\boldsymbol{x})\right) \cdot \mathbf{e}_2 = 0$$

Electromagnetic Transmission/Boundary Conditions



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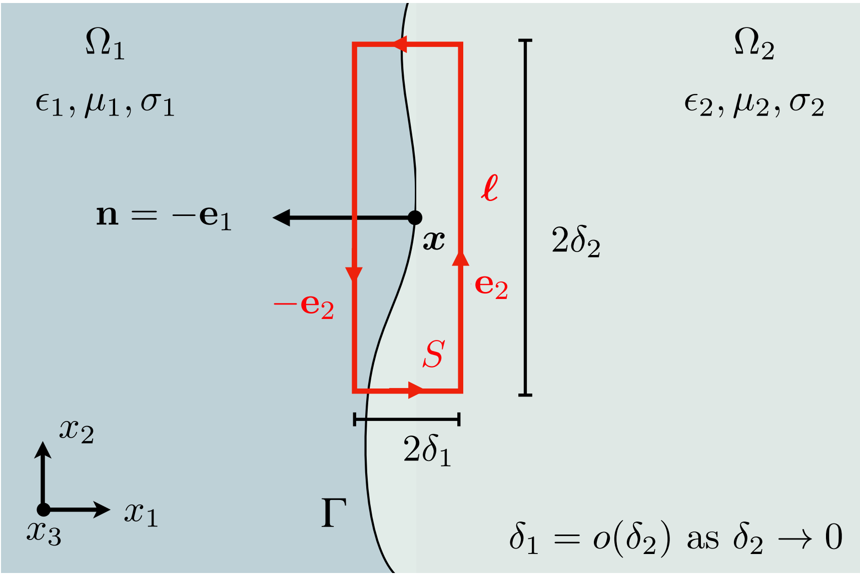
Electromagnetic Transmission/Boundary Conditions



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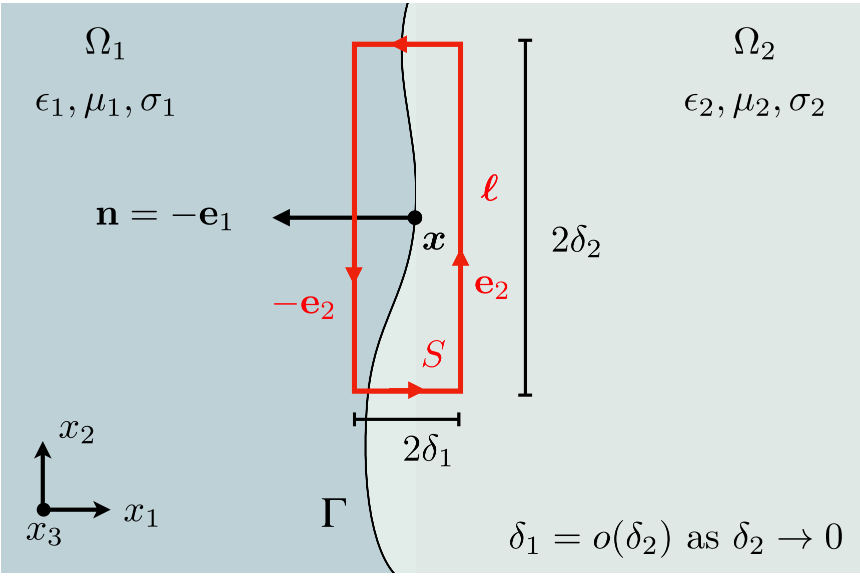

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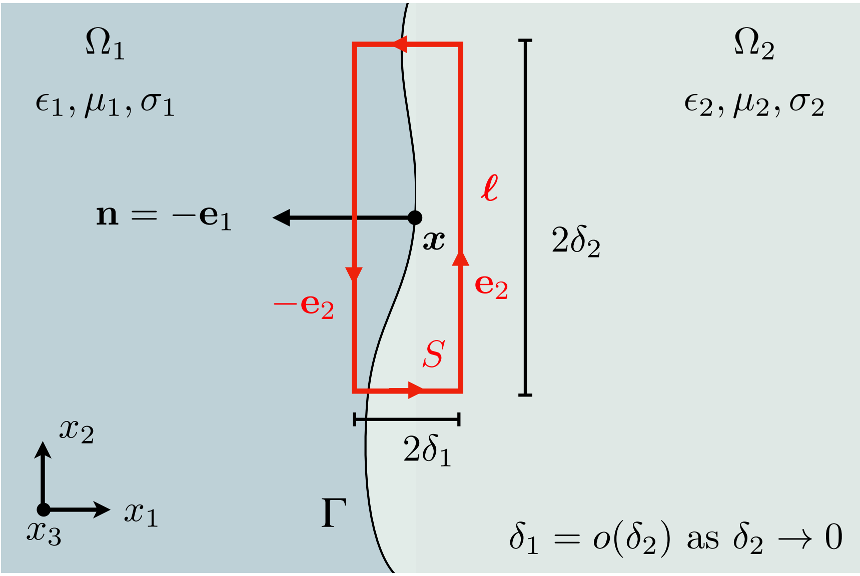
In general: $\mathbf{n} \times (\mathcal{E}_2 - \mathcal{E}_1) = \mathbf{0}$ and $\mathbf{n} \times (\mathcal{H}_2 - \mathcal{H}_1) = \mathbf{0}$ on Γ, if σ_1 and σ_2 are finite

Electromagnetic Transmission/Boundary Conditions



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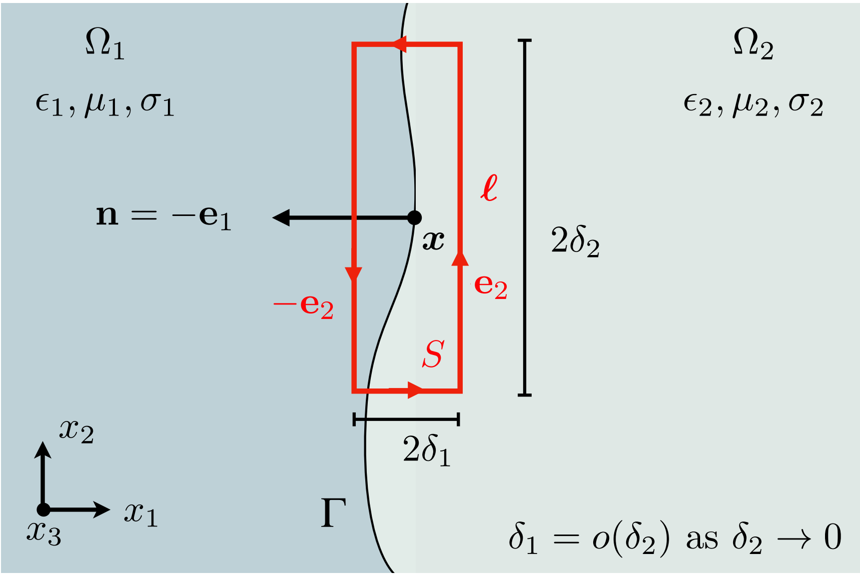
from Ampère's law

Electromagnetic Transmission/Boundary Conditions



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from Ampère's law

Perfect conductor $(\sigma_2 = \infty)$: $\boxed{\mathbf{n} \times \mathcal{E}_1 = \mathbf{0} \quad \text{and} \quad \mathbf{n} \cdot \mathcal{H}_1 = 0}$

Time-Harmonic Electromagnetic Waves

For time-harmonic fields: $\mathcal{E}(\boldsymbol{x},t) = \mathrm{Re}\left\{\mathbf{E}(\boldsymbol{x})\,\mathrm{e}^{-i\omega t}\right\}, \quad \mathcal{H}(\boldsymbol{x},t) = \mathrm{Re}\left\{\mathbf{H}(\boldsymbol{x})e^{-i\omega t}\right\},$ Maxwell's equations reduce to:

$$\text{time-harmonic Maxwell's equations} \quad \begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = \mathbf{0} \\ \\ \nabla \times \mathbf{H} + i\omega\left(\epsilon + i\dfrac{\sigma}{\omega}\right)\mathbf{E} = \mathbf{0} \end{cases}$$

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$$\Delta\mathbf{E} + k^2\mathbf{E} = \mathbf{0} \quad \text{and} \quad \Delta\mathbf{H} + k^2\mathbf{H} = \mathbf{0}, \quad \text{where} \quad k := \omega\sqrt{\mu\epsilon}$$

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Maxwell => Helmholtz,
no equivalence holds unless
$\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$

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time-harmonic Maxwell's equations $\quad \begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = \mathbf{0} \\ \nabla \times \mathbf{H} + i\omega\left(\epsilon + i\dfrac{\sigma}{\omega}\right)\mathbf{E} = \mathbf{0} \end{cases}$

Since $\nabla \times \nabla \times \mathbf{E} = -\Delta\mathbf{E} + \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{=0}) = -\Delta\mathbf{E}$ and $\nabla \times \nabla \times \mathbf{H} = -\Delta\mathbf{H}$, it follows that

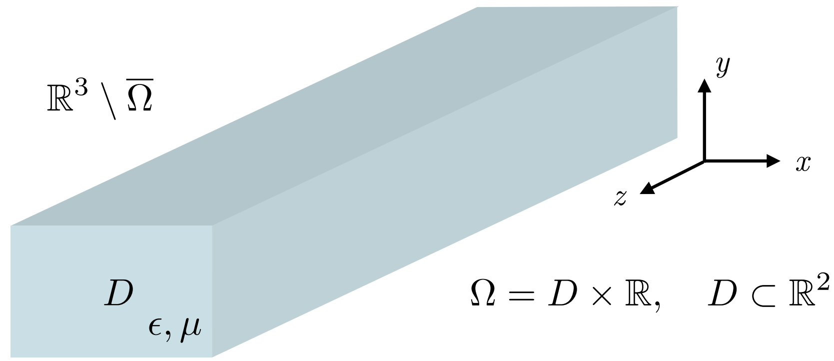
$$\Delta\mathbf{E} + k^2\mathbf{E} = \mathbf{0} \quad \text{and} \quad \Delta\mathbf{H} + k^2\mathbf{H} = \mathbf{0}, \quad \text{where} \quad k := \omega\sqrt{\mu\epsilon}$$

Maxwell => Helmholtz,
no equivalence holds unless
$\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$

As in the case of the Helmholtz equation, when solving problems in free space $\mathbb{R}^d$, $d = 2, 3$, without dissipation ($\sigma = 0$), it is necessary to impose radiation conditions:

Silver–Müller radiation conditions $\quad \begin{cases} \lim\limits_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left|\sqrt{\epsilon}\mathbf{E} - \sqrt{\mu}\mathbf{H} \times \dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right| = 0 \\ \lim\limits_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left|\sqrt{\mu}\mathbf{H} + \sqrt{\epsilon}\mathbf{E} \times \dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\right| = 0 \end{cases}$

TE/TM Mode Decomposition of Maxwell's Equations



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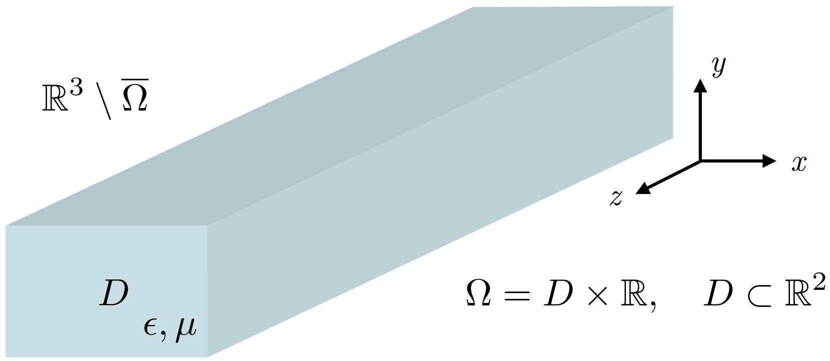

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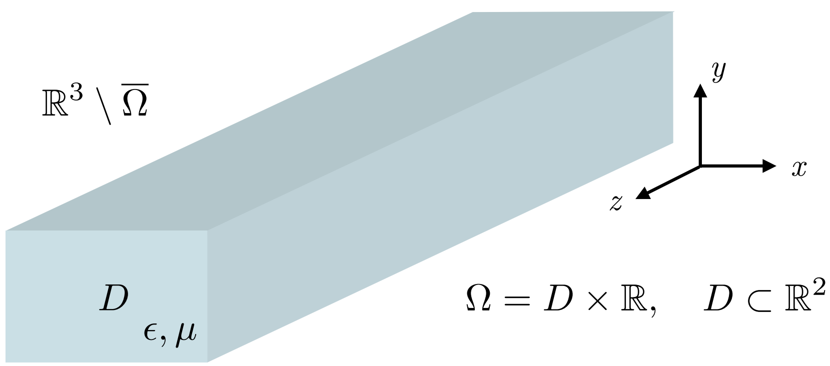

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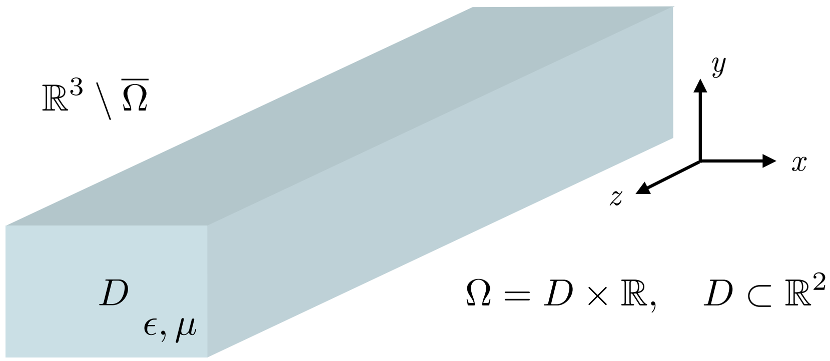

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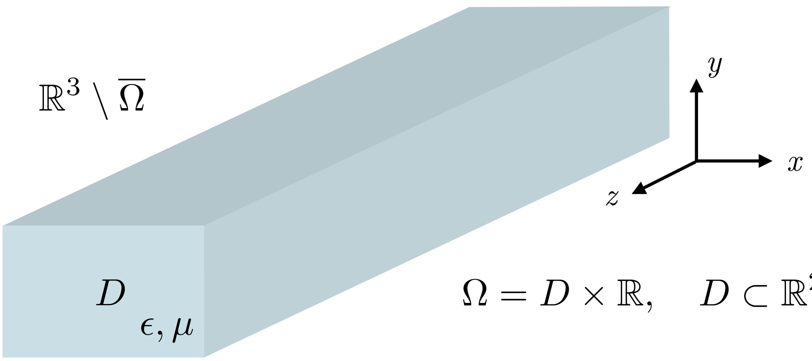

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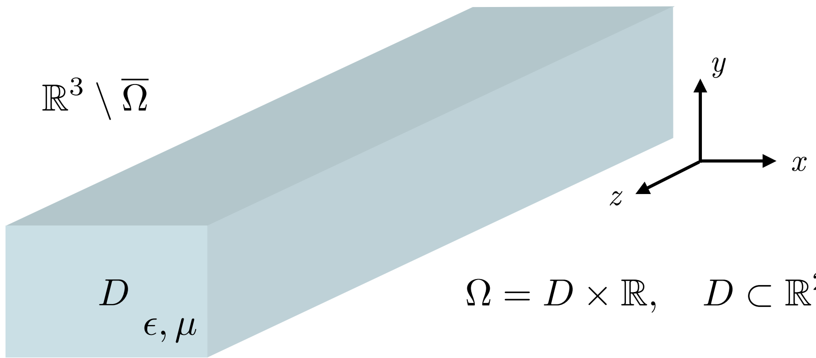

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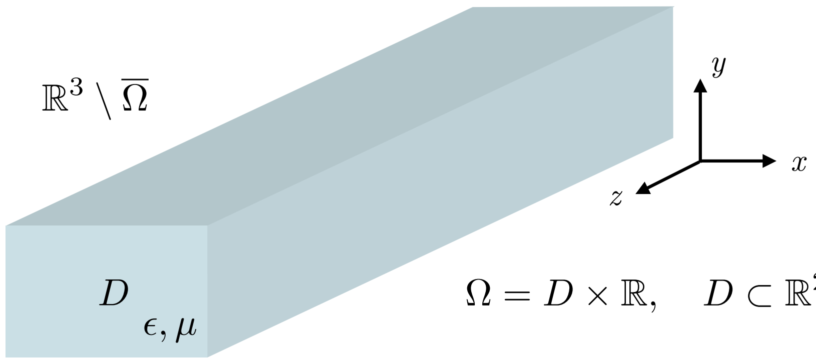

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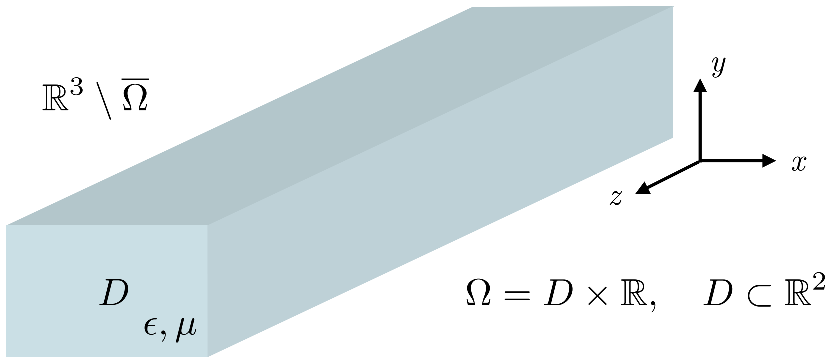

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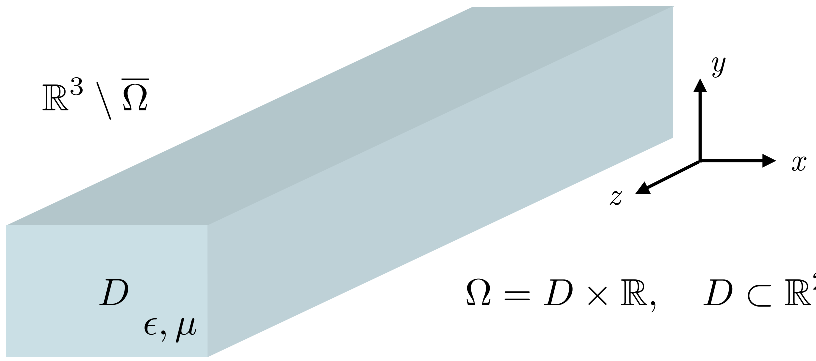

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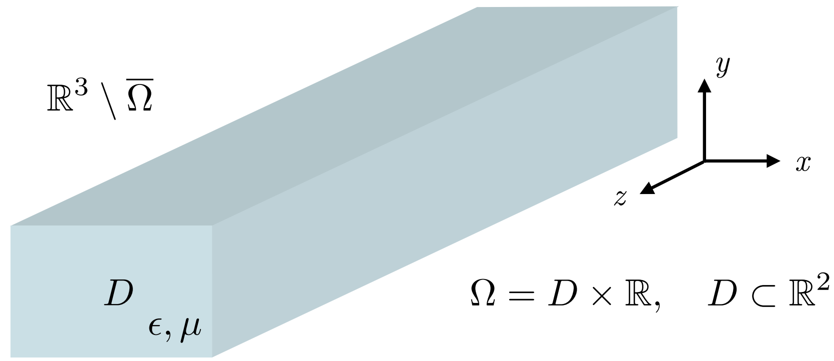

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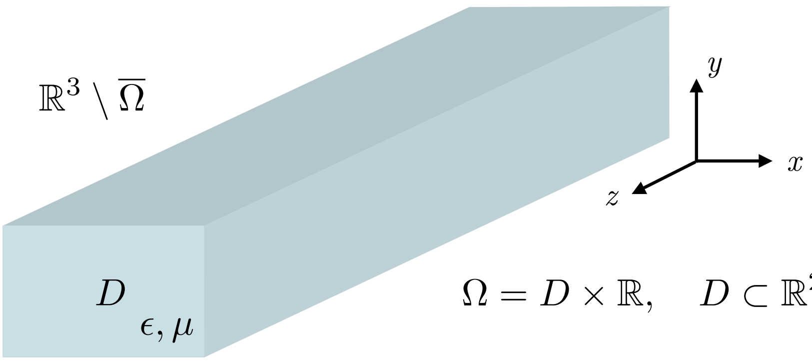

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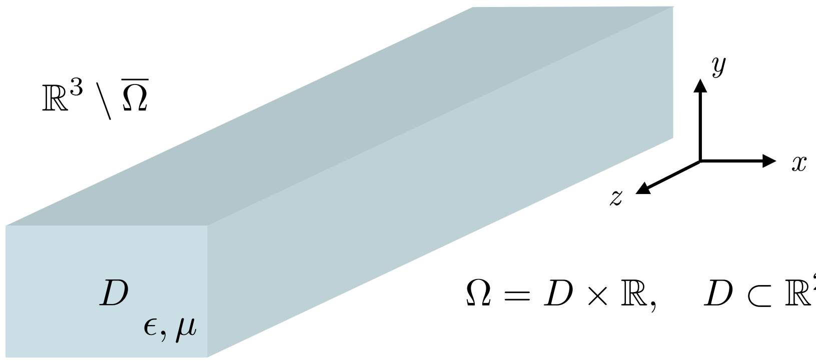

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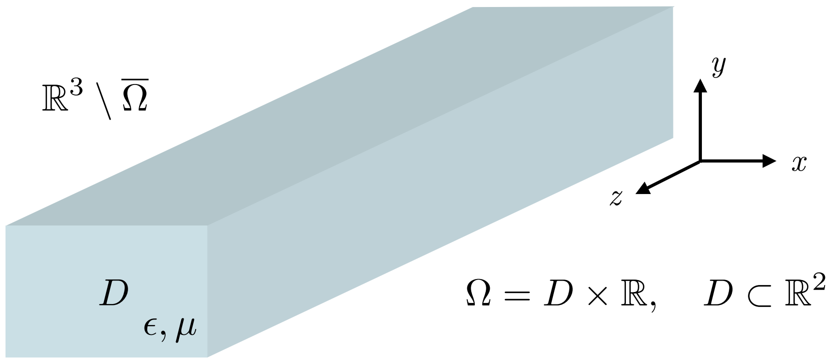

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$$\nabla \times \mathbf{H}_{\mathrm{TE}} + i\omega\epsilon\mathbf{E}_{\mathrm{TE}} = \mathbf{0} \implies \begin{cases} \dfrac{\partial H_z}{\partial y} - i\beta \boxed{H_y} + i\omega\epsilon \boxed{E_x} & = 0 \\ i\beta \boxed{H_x} - \dfrac{\partial H_z}{\partial x} + i\omega\epsilon \boxed{E_y} & = 0 \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} + i\omega\epsilon \cancel{E_z} & = 0 \end{cases} \quad \nabla \times \mathbf{E}_{\mathrm{TE}} - i\omega\mu\mathbf{H}_{\mathrm{TE}} = \mathbf{0} \implies \begin{cases} \cancel{\dfrac{\partial E_z}{\partial y}} - i\beta \boxed{E_y} - i\omega\mu \boxed{H_x} & = 0 \\ i\beta \boxed{E_x} - \cancel{\dfrac{\partial E_z}{\partial x}} - i\omega\mu \boxed{H_y} & = 0 \\ \boxed{\dfrac{\partial E_y}{\partial x}} - \boxed{\dfrac{\partial E_x}{\partial y}} - i\omega\mu H_z & = 0 \end{cases}$$

Transverse Electric (TE) mode:

solve: $\Delta H_z + (k^2 - \beta^2) H_z = 0$, and then

$$E_x = -\frac{1}{i\omega\epsilon}\left(1 - \frac{\beta^2}{\omega^2\epsilon\mu}\right)^{-1} \frac{\partial H_z}{\partial y}, \qquad H_y = \frac{\beta}{\omega\mu} E_x$$

$$E_y = \frac{1}{i\omega\epsilon}\left(1 - \frac{\beta^2}{\omega^2\epsilon\mu}\right)^{-1} \frac{\partial H_z}{\partial x}, \qquad H_x = -\frac{\beta}{\omega\mu} E_y$$

TE/TM Mode Decomposition of Maxwell's Equations

We look for solutions to Maxwell's equations of the form:

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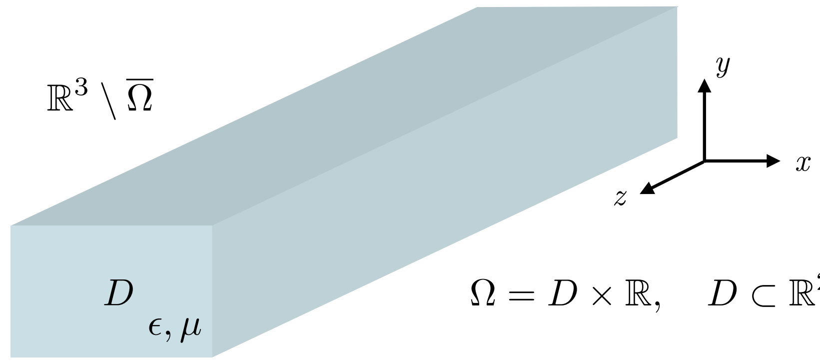

$\Omega = D \times \mathbb{R}, \quad D \subset \mathbb{R}^2$

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Helmholtz equation
in 2D

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solve: $\Delta H_z + (k^2 - \beta^2)H_z = 0$, and then

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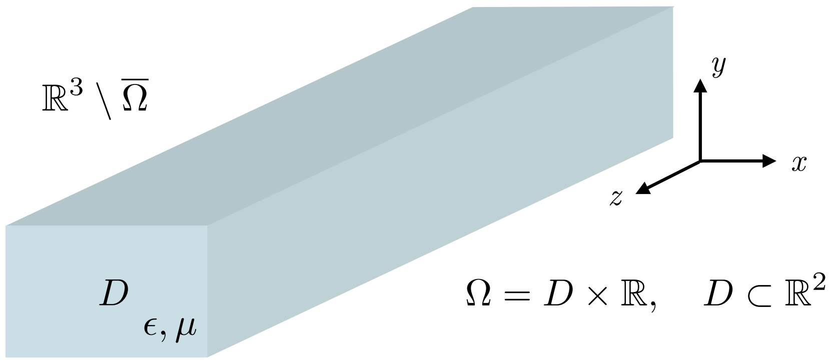

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$$\nabla \times \mathbf{H}_{\mathrm{TM}} + i\omega\epsilon\mathbf{E}_{\mathrm{TM}} = \mathbf{0} \implies \begin{cases} \cancel{\dfrac{\partial H_z}{\partial y}} - i\beta H_y + i\omega\epsilon E_x &= 0 \\ i\beta H_x - \cancel{\dfrac{\partial H_z}{\partial x}} + i\omega\epsilon E_y &= 0 \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} + i\omega\epsilon E_z &= 0 \end{cases} \quad \nabla \times \mathbf{E}_{\mathrm{TM}} - i\omega\mu\mathbf{H}_{\mathrm{TM}} = \mathbf{0} \implies \begin{cases} \dfrac{\partial E_z}{\partial y} - i\beta E_y - i\omega\mu H_x &= 0 \\ i\beta E_x - \dfrac{\partial E_z}{\partial x} - i\omega\mu H_y &= 0 \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} - i\omega\mu \cancel{H_z} &= 0 \end{cases}$$

Transverse Magnetic (TM) mode:

solve: $\Delta E_z + (k^2 - \beta^2)E_z = 0$, and then

$$H_x = \frac{1}{i\omega\epsilon}\left(1 - \frac{\beta^2}{\omega^2\epsilon\mu}\right)^{-1}\frac{\partial E_z}{\partial y}, \qquad E_y = -\frac{\beta}{\omega\mu}H_x$$

$$H_y = -\frac{1}{i\omega\epsilon}\left(1 - \frac{\beta^2}{\omega^2\epsilon\mu}\right)^{-1}\frac{\partial E_z}{\partial x}, \qquad E_x = \frac{\beta}{\omega\mu}H_y$$

TE/TM Mode Decomposition of Maxwell's Equations

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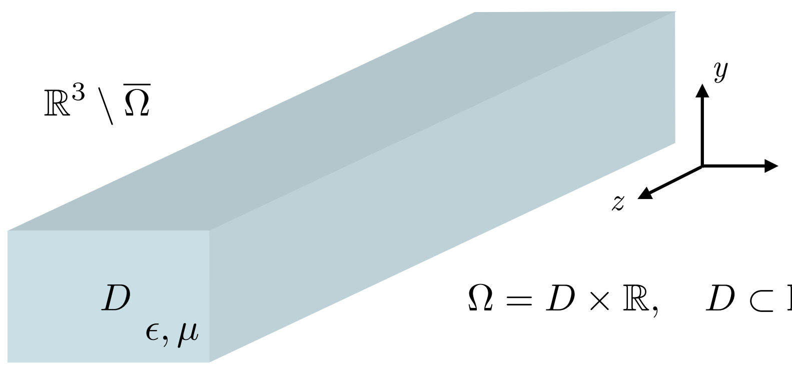

$\Omega = D \times \mathbb{R}, \quad D \subset \mathbb{R}^2$

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$$\nabla \times \mathbf{H}_{\mathrm{TM}} + i\omega\epsilon\mathbf{E}_{\mathrm{TM}} = \mathbf{0} \implies \begin{cases} \cancel{\dfrac{\partial H_z}{\partial y}} - i\beta H_y + i\omega\epsilon E_x & = 0 \\ i\beta H_x - \cancel{\dfrac{\partial H_z}{\partial x}} + i\omega\epsilon E_y & = 0 \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} + i\omega\epsilon E_z & = 0 \end{cases}$$

$$\nabla \times \mathbf{E}_{\mathrm{TM}} - i\omega\mu\mathbf{H}_{\mathrm{TM}} = \mathbf{0} \implies \begin{cases} \dfrac{\partial E_z}{\partial y} - i\beta E_y - i\omega\mu H_x & = 0 \\ i\beta E_x - \dfrac{\partial E_z}{\partial x} - i\omega\mu H_y & = 0 \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} - i\omega\mu \cancel{H_z} & = 0 \end{cases}$$

Helmholtz equation
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Transverse Magnetic (TM) mode:

solve: $\Delta E_z + (k^2 - \beta^2)E_z = 0$, and then

$$H_x = \frac{1}{i\omega\epsilon}\left(1 - \frac{\beta^2}{\omega^2\epsilon\mu}\right)^{-1}\frac{\partial E_z}{\partial y}, \qquad E_y = -\frac{\beta}{\omega\mu}H_x$$

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TE/TM Mode Decomposition of Maxwell's Equations

Consider the transmission problem:

$$\nabla \times \mathbf{H}_j + i\omega\epsilon_j \mathbf{E}_j = \mathbf{0} \quad \text{in } \Omega_j,$$
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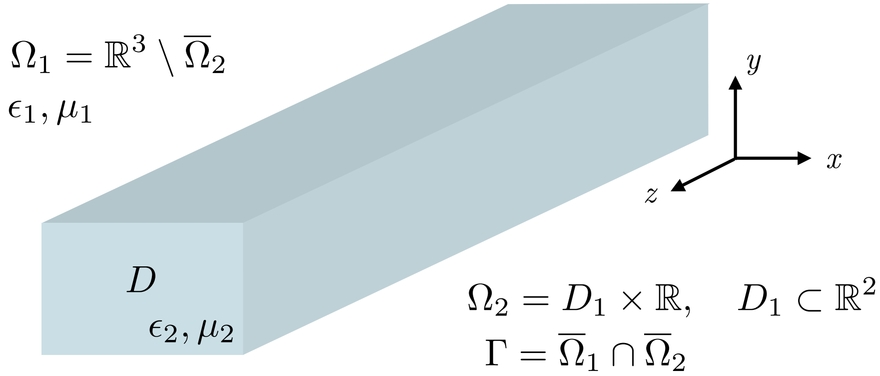

TE/TM Mode Decomposition of Maxwell's Equations

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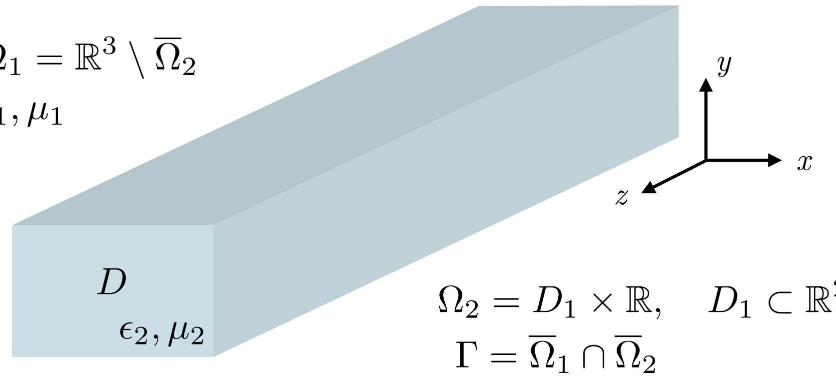
$\Omega_1 = \mathbb{R}^3 \setminus \overline{\Omega}_2$

ϵ_1, μ_1



$\Omega_2 = D_1 \times \mathbb{R}, \quad D_1 \subset \mathbb{R}^2$

$\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$

The problem reduces to a problem in $\mathbb{R}^2$ for the (scalar) Helmholtz equation.

$$\Delta H_{j,z} + (k_j^2 - \beta^2) H_{j,z} = 0 \quad \text{in } D_j,$$
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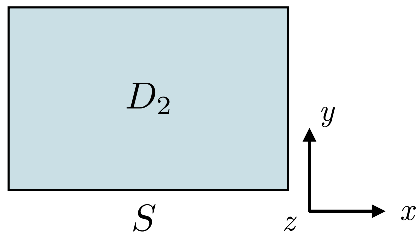

TE/TM Mode Decomposition of Maxwell's Equations

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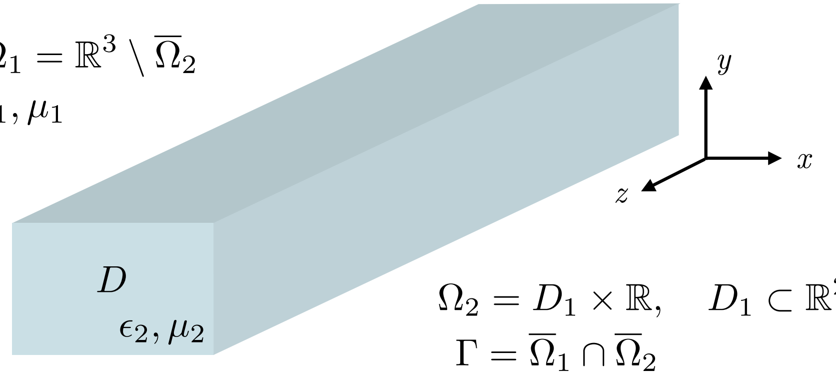

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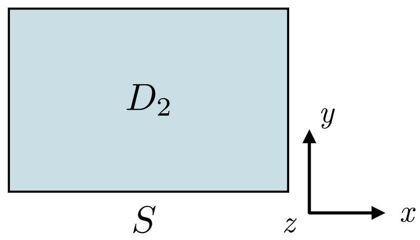

with transmission conditions across $S = \overline{D}_1 \cap \overline{D}_2$:

$$E_{2z} = E_{1z}, \quad a_2^E \frac{\partial H_{2z}}{\partial n} + b_2 \frac{\partial E_{2z}}{\partial \tau} = a_1^E \frac{\partial H_{1z}}{\partial n} + b_1 \frac{\partial E_{1z}}{\partial \tau}, \quad b_j = \frac{\beta}{\omega\left(k_j^2 - \beta^2\right)},$$
$$H_{2z} = H_{1z}, \quad a_2^H \frac{\partial E_{2z}}{\partial n} - b_2 \frac{\partial H_{2z}}{\partial \tau} = a_1^H \frac{\partial E_{1z}}{\partial n} - b_1 \frac{\partial H_{1z}}{\partial \tau}, \quad a_j^E = \frac{\mu_j}{k_j^2 - \beta^2}, \quad a_j^H = \frac{\varepsilon_j}{k_j^2 - \beta^2}$$

TE/TM Mode Decomposition of Maxwell's Equations

Consider the transmission problem:

$$\begin{aligned} \nabla \times \mathbf{H}_j + i\omega\epsilon_j \mathbf{E}_j &= \mathbf{0} \quad \text{in } \Omega_j, \\ \nabla \times \mathbf{E}_j - i\omega\mu_j \mathbf{H}_j &= \mathbf{0} \quad \text{in } \Omega_j, \qquad j = 1,2 \end{aligned}$$

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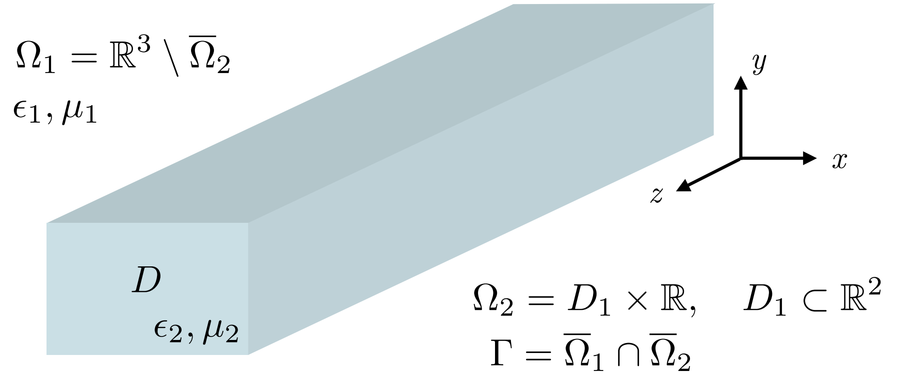

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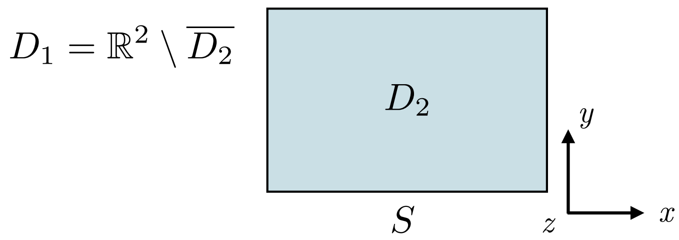

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tangential derivatives

TE/TM Mode Decomposition of Maxwell's Equations

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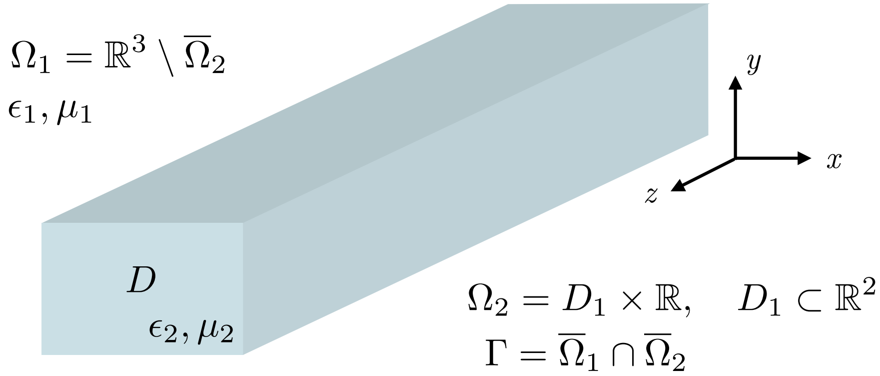

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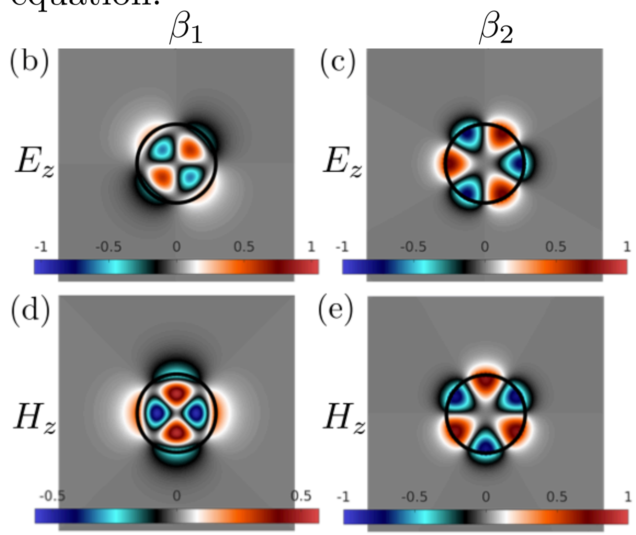
with transmission conditions across $S = \overline{D}_1 \cap \overline{D}_2$:

When $\epsilon_1 < \epsilon_2$, it can be shown that there exists a **finite number** of real values $\beta_\ell \in \mathbb{R}$ for which nontrivial solutions to the homogeneous problem exist.

guided modes →



TE/TM Mode Decomposition of Maxwell's Equations

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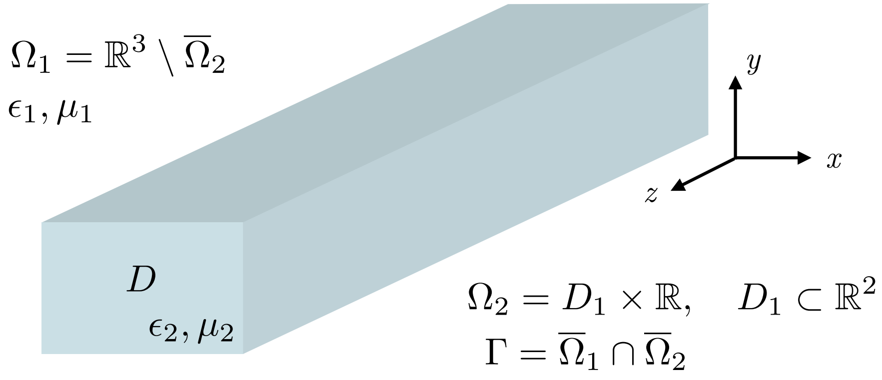

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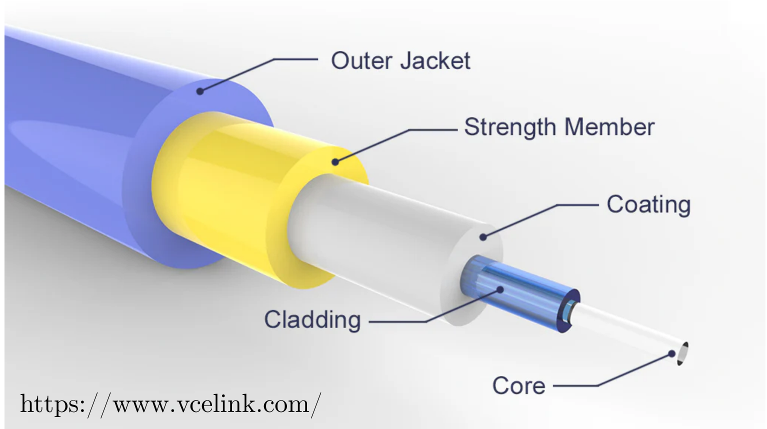
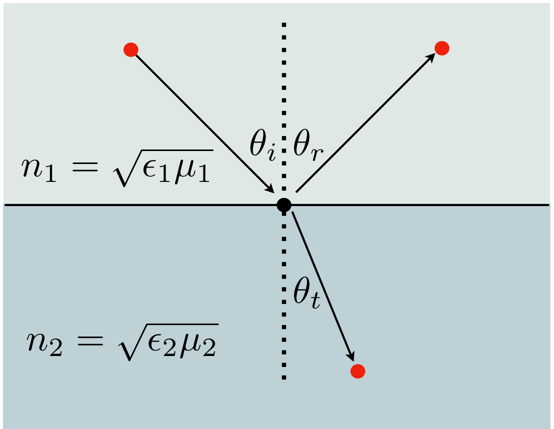
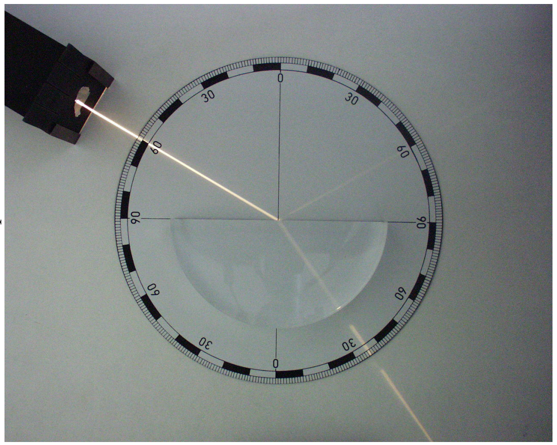
applications in optical fiber:



Snell-Descartes Laws of Reflection and Refraction





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- **Law of Reflection:**

$$\sin\theta_i = \sin\theta_r$$

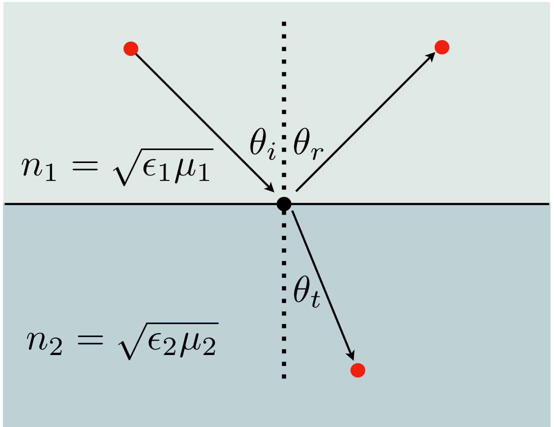
- **Law of Refraction (Snell's Law):**

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

where:

- θ_i: angle of incidence
- θ_r: angle of reflection
- θ_t: angle of transmission (refraction)
- n_1, n_2: refractive indices of the media

Snell-Descartes Laws of Reflection and Refraction



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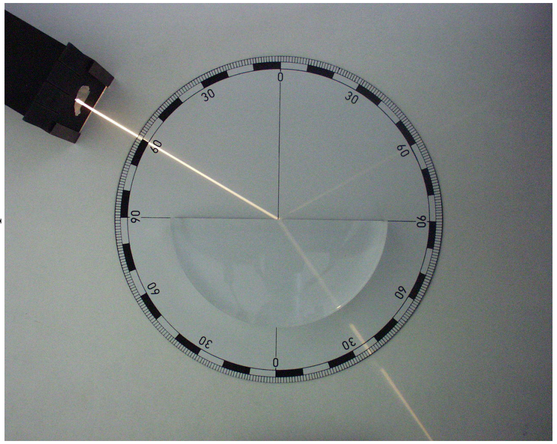

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www.wikipedia.com

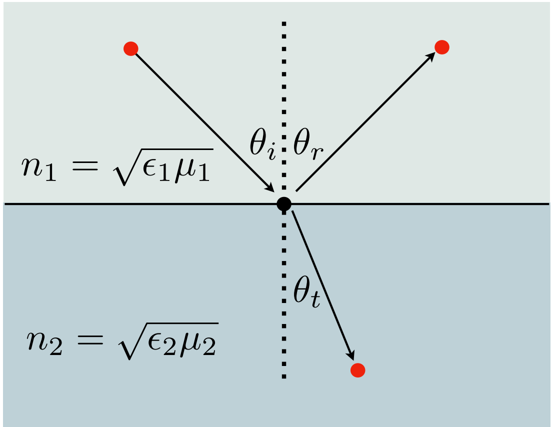
René Descartes
1596-1650

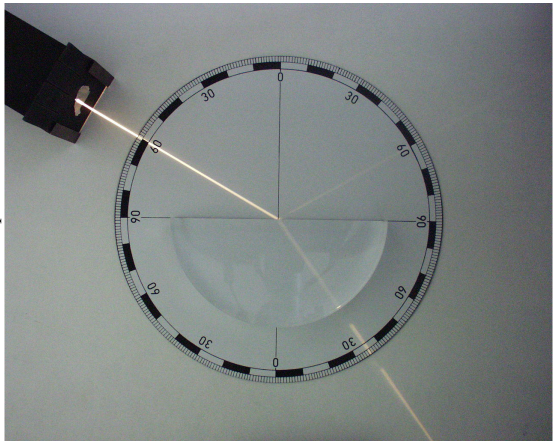
www.wikipedia.com

Willebrord Snell van Royen
1580-1626

~ 1621

Snell-Descartes Laws of Reflection and Refraction





www.wikipedia.com

- **Law of Reflection:**

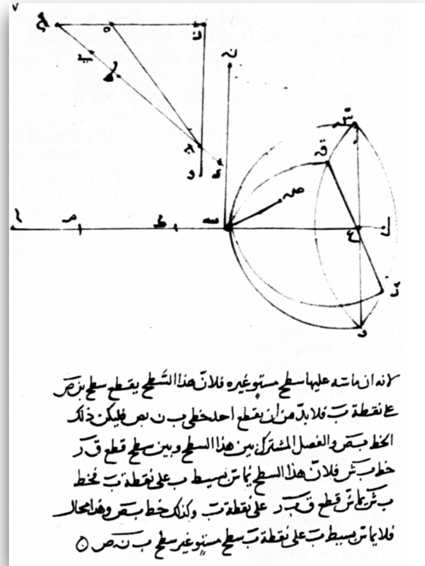
$$\sin\theta_i = \sin\theta_r$$

- **Law of Refraction (Snell's Law):**

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

where:

- θ_i: angle of incidence
- θ_r: angle of reflection
- θ_t: angle of transmission (refraction)
- n_1, n_2: refractive indices of the media



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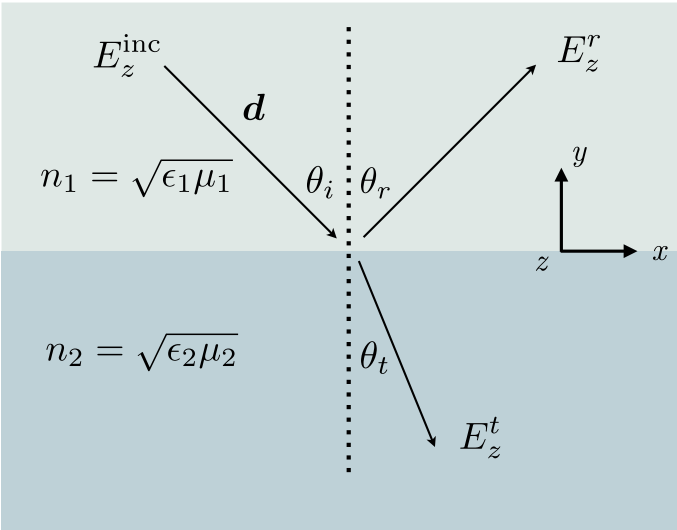

Planewave: $E_z^{\mathrm{inc}}(x, y) = \mathrm{e}^{ik_1 \boldsymbol{x}\cdot\boldsymbol{d}} = \mathrm{e}^{ik_1(x\sin\theta_i - y\cos\theta_i)}$

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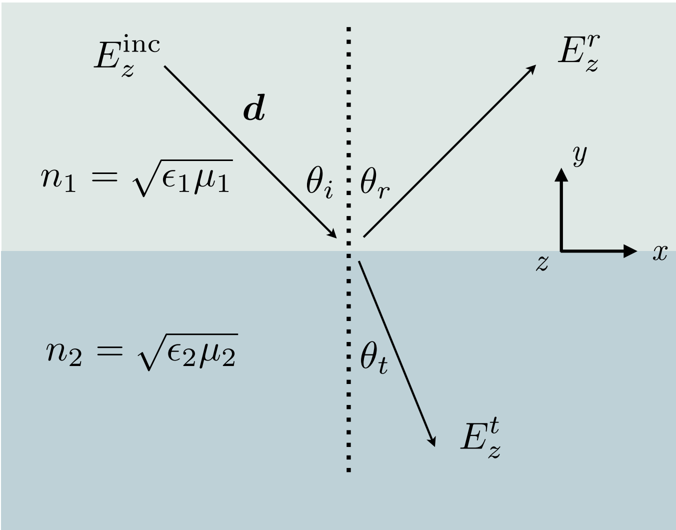

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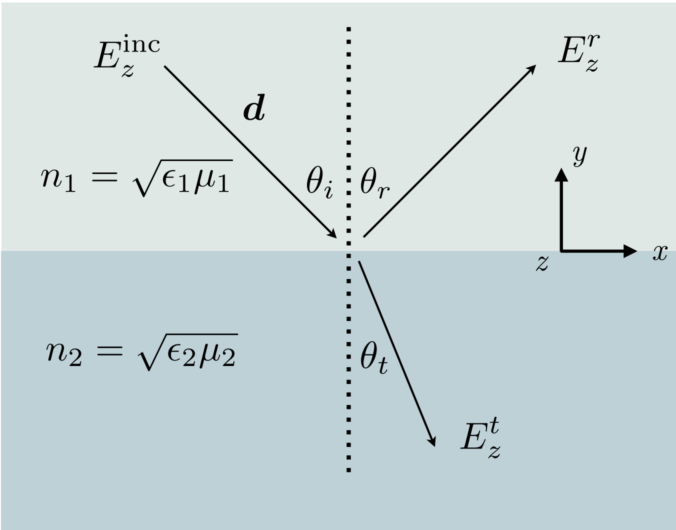
Total field: $$E_z = \begin{cases} E_z^{\mathrm{inc}} + E_z^r, & y > 0 \\ E_z^t, & y < 0 \end{cases}$$

Snell-Descartes Laws of Reflection and Refraction



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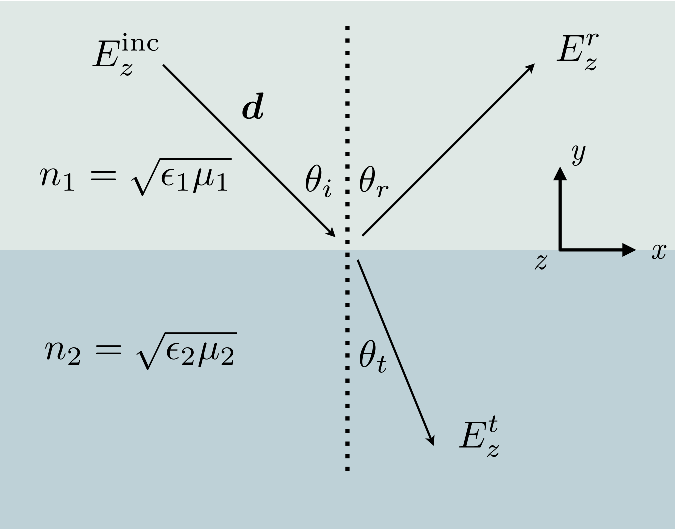
Maxwell's equations:
$$\Delta E_z^r + k_1^2 E_z^r = 0 \text{ in } \{y > 0\}$$
$$\Delta E_z^t + k_2^2 E_z^t = 0 \text{ in } \{y < 0\}$$
$(\beta = 0)$

Snell-Descartes Laws of Reflection and Refraction



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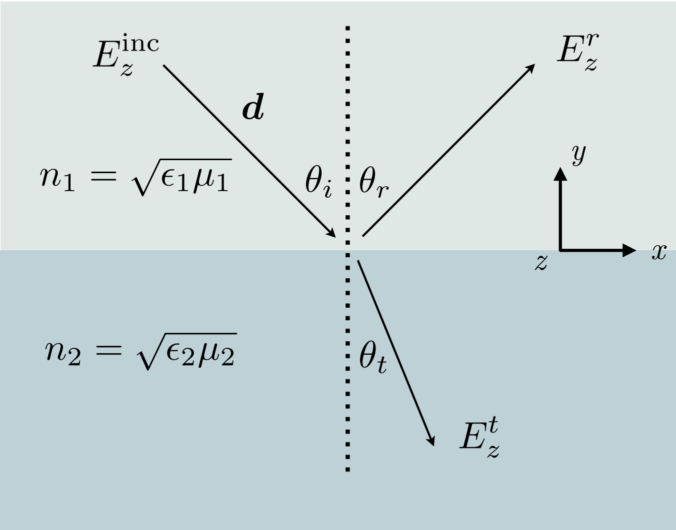
$(\beta = 0)$

Transmission conditions:
$$E_z^r + E_z^{\text{inc}} = E_z^t \quad \text{at} \quad \{y = 0\}$$
$(\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0})$

Snell-Descartes Laws of Reflection and Refraction



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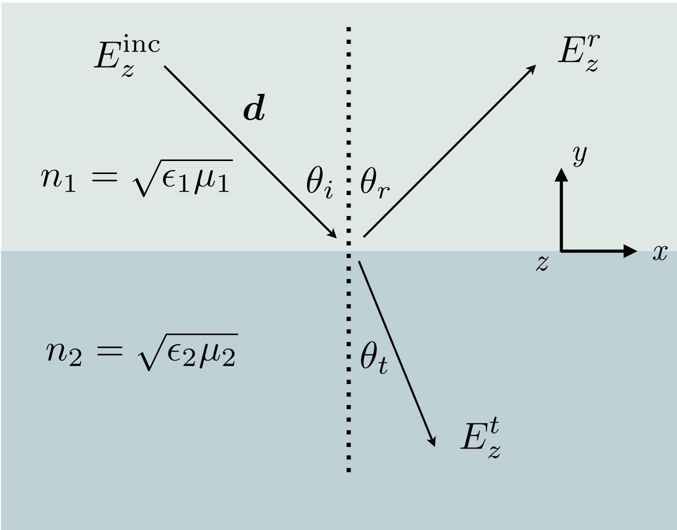
Transmission conditions: $E_z^r + E_z^{\text{inc}} = E_z^t \text{ at } \{y = 0\}$

$(\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0})$

By separation of variables:

$$E_z^r(x,y) = Re^{i(xk_{1x}+yk_{1y})}, k_{1x}^2 + k_{1y}^2 = k_1^2 \implies k_1\sin\theta_r = k_{1x} \text{ and } k_1\cos\theta_r = k_{1y}$$
$$E_z^t(x,y) = Te^{i(xk_{2x}-yk_{2y})}, k_{2x}^2 + k_{2y}^2 = k_2^2 \implies k_2\sin\theta_t = k_{2x} \text{ and } k_2\cos\theta_t = k_{2y}$$

Snell-Descartes Laws of Reflection and Refraction



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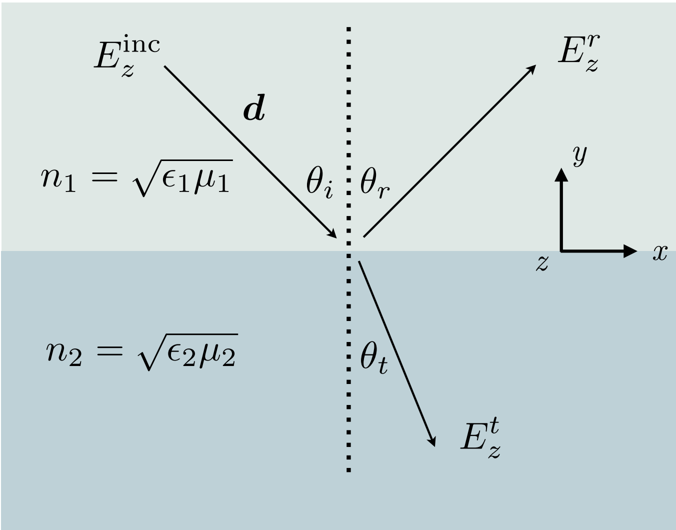
$$E_z^r(x,y) = Re^{i(xk_{1x}+yk_{1y})}, k_{1x}^2 + k_{1y}^2 = k_1^2 \Longrightarrow k_1 \sin\theta_r = k_{1x} \text{ and } k_1\cos\theta_r = k_{1y}$$
$$E_z^t(x,y) = Te^{i(xk_{2x}-yk_{2y})}, k_{2x}^2 + k_{2y}^2 = k_2^2 \Longrightarrow k_2 \sin\theta_t = k_{2x} \text{ and } k_2\cos\theta_t = k_{2y}$$

From the transmission condition:
$$e^{ik_1 x\sin\theta_i} + Re^{ixk_{1x}} = Te^{ixk_{2x}} \Longrightarrow k_1 \sin\theta_i = k_{1x} = k_{2y}$$

Snell-Descartes Laws of Reflection and Refraction



Planewave: $E_z^{\mathrm{inc}}(x,y) = \mathrm{e}^{ik_1\boldsymbol{x}\cdot\boldsymbol{d}} = \mathrm{e}^{ik_1(x\sin\theta_i - y\cos\theta_i)}$

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$(\mathbf{n}\times(\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0})$

By separation of variables:

$$E_z^r(x,y) = Re^{i(xk_{1x}+yk_{1y})}, k_{1x}^2 + k_{1y}^2 = k_1^2 \Longrightarrow k_1\sin\theta_r = k_{1x} \text{ and } k_1\cos\theta_r = k_{1y}$$

$$E_z^t(x,y) = Te^{i(xk_{2x}-yk_{2y})}, k_{2x}^2 + k_{2y}^2 = k_2^2 \Longrightarrow k_2\sin\theta_t = k_{2x} \text{ and } k_2\cos\theta_t = k_{2y}$$

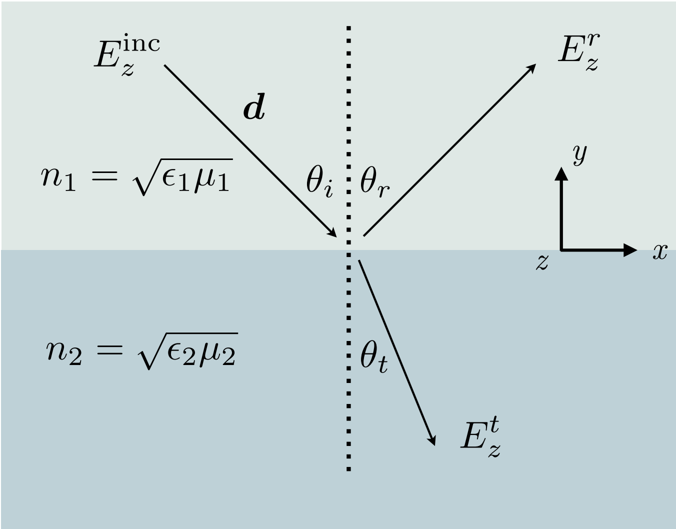
From the transmission condition:

$$e^{ik_1 x\sin\theta_i} + Re^{ixk_{1x}} = Te^{ixk_{2x}} \Longrightarrow k_1\sin\theta_i = k_{1x} = k_{2y}$$

Therefore:

$$k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t$$

Snell-Descartes Laws of Reflection and Refraction



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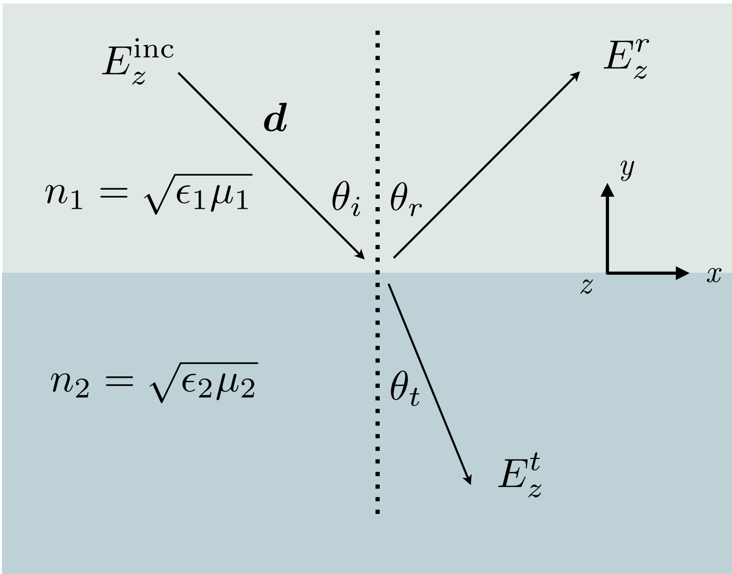
By separation of variables:
$$E_z^r(x,y) = R e^{i(xk_{1x} + yk_{1y})}, k_{1x}^2 + k_{1y}^2 = k_1^2 \implies k_1 \sin\theta_r = k_{1x} \text{ and } k_1 \cos\theta_r = k_{1y}$$
$$E_z^t(x,y) = T e^{i(xk_{2x} - yk_{2y})}, k_{2x}^2 + k_{2y}^2 = k_2^2 \implies k_2 \sin\theta_t = k_{2x} \text{ and } k_2 \cos\theta_t = k_{2y}$$

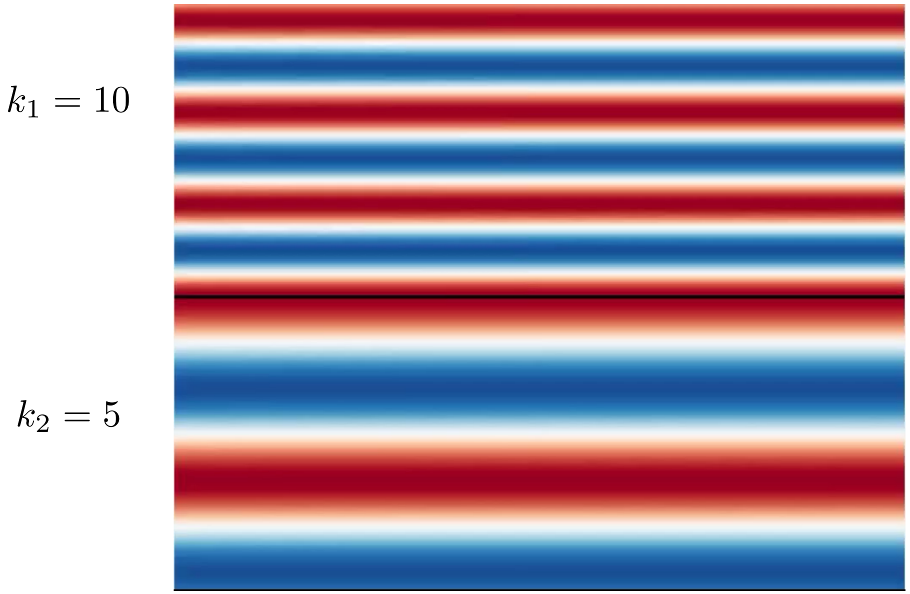
From the transmission condition:
$$e^{ik_1 x \sin\theta_i} + R e^{ixk_{1x}} = T e^{ixk_{2x}} \implies k_1 \sin\theta_i = k_{1x} = k_{2y}$$

Therefore: $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$ **Snell-Descartes law!**

Total Internal Reflection

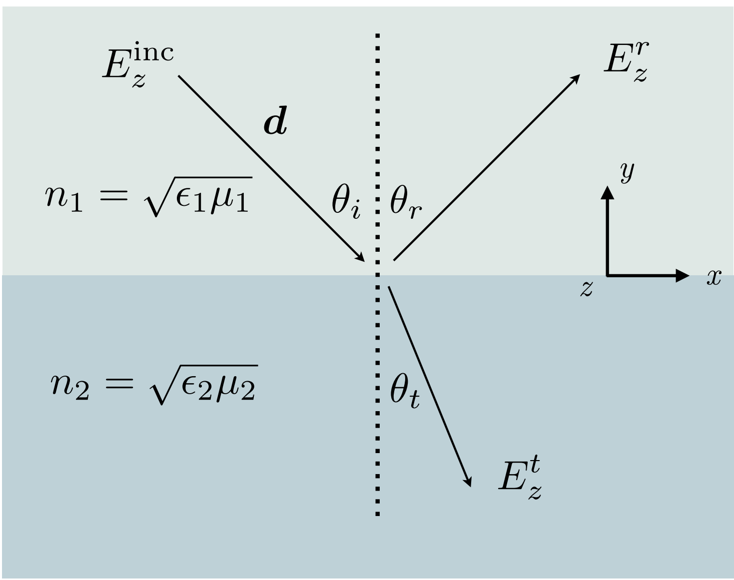
Snell-Descartes Law: $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$

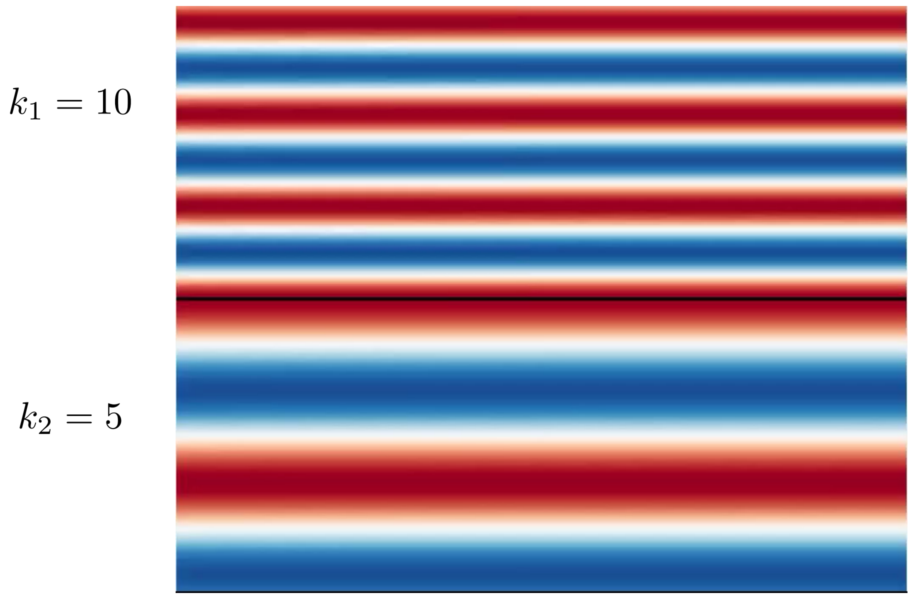




Total Internal Reflection

Snell-Descartes Law: $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$ What happens if $\dfrac{k_1}{k_2}\sin\theta_i = \sin\theta_t > 1$?





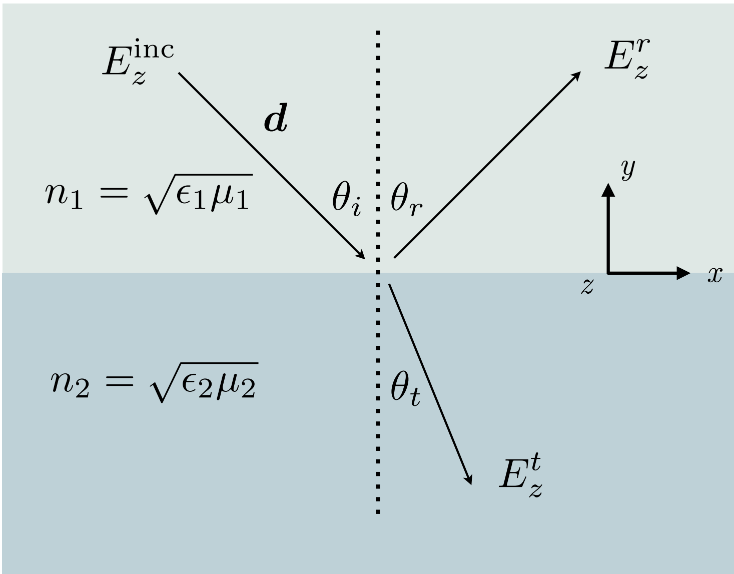
Total Internal Reflection

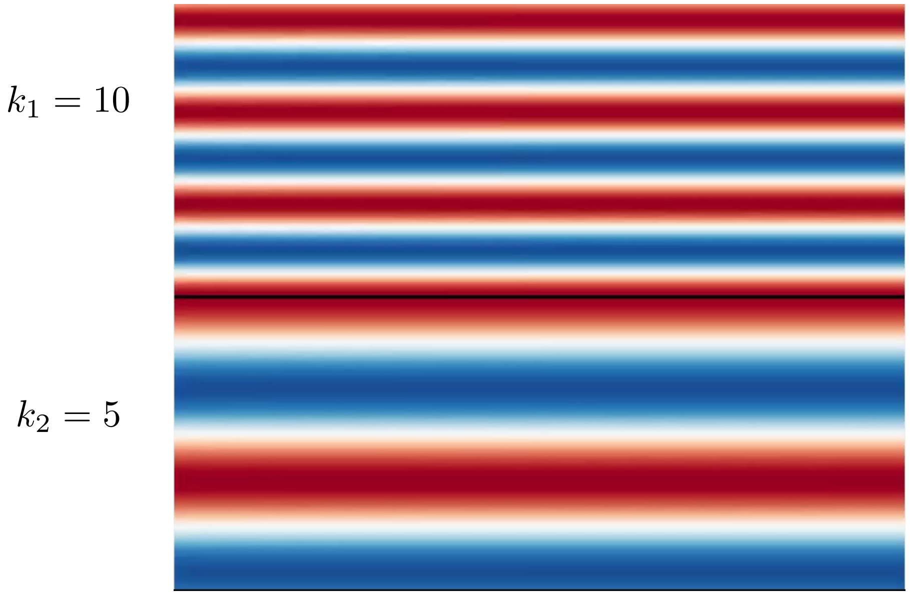
Snell-Descartes Law: $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$

What happens if $\dfrac{k_1}{k_2}\sin\theta_i = \sin\theta_t > 1$?

$$k_{2y} = k_2\sqrt{1-\sin^2\theta_t} = i\left(\frac{k_1^2\sin^2\theta_i}{k_2^2} - 1\right)^{1/2}$$

$E_z^t(x,y) = T\,\mathrm{e}^{i(xk_{2x} - yk_{2y})}$ decays exponentially fast as $y \to -\infty$ (total reflection).





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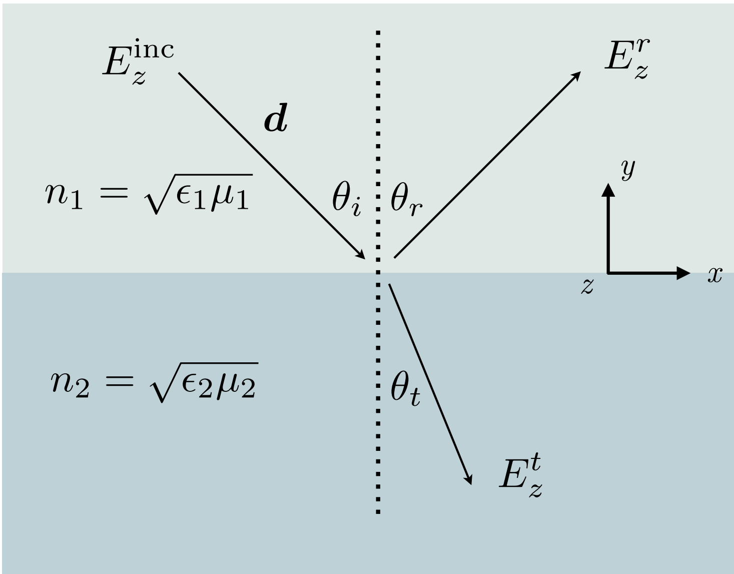
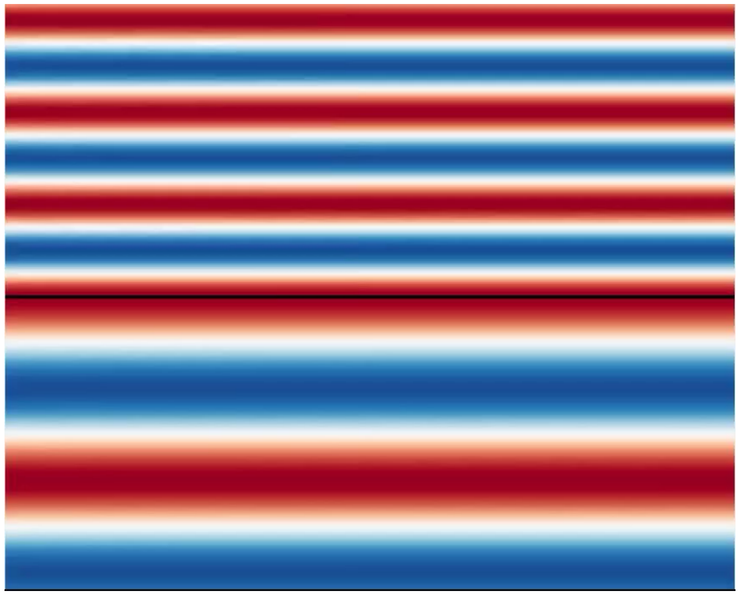


$k_1 = 10$



$k_2 = 5$

Total Internal Reflection

Snell-Descartes Law: $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$

What happens if $\dfrac{k_1}{k_2}\sin\theta_i = \sin\theta_t > 1$?

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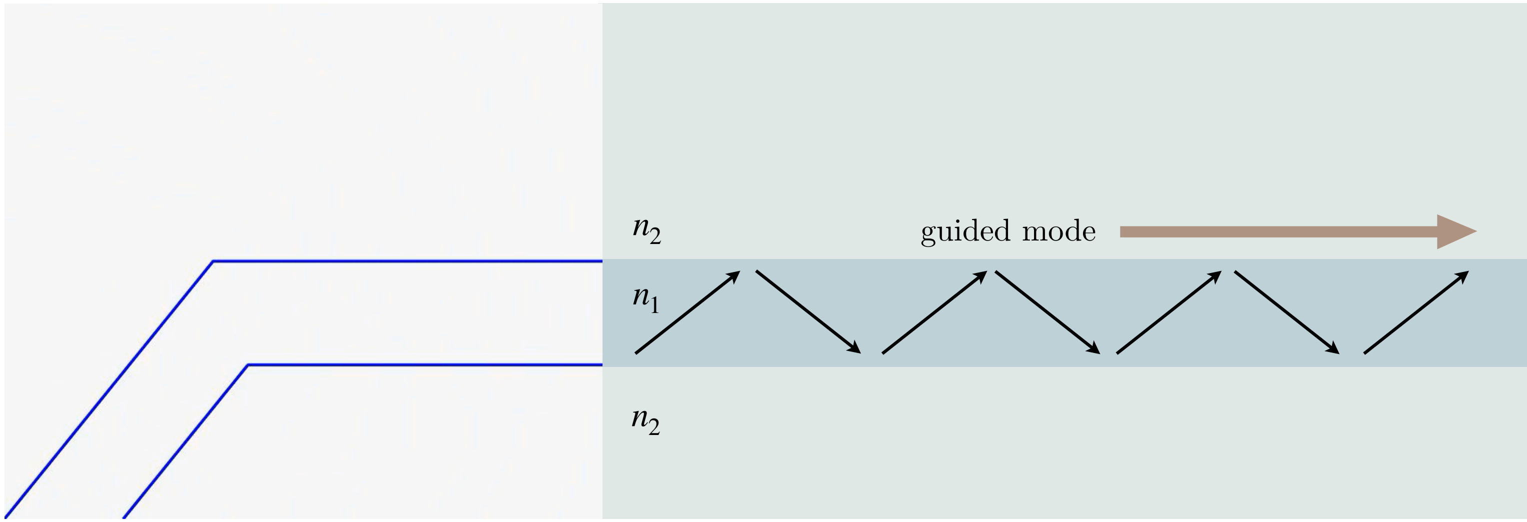

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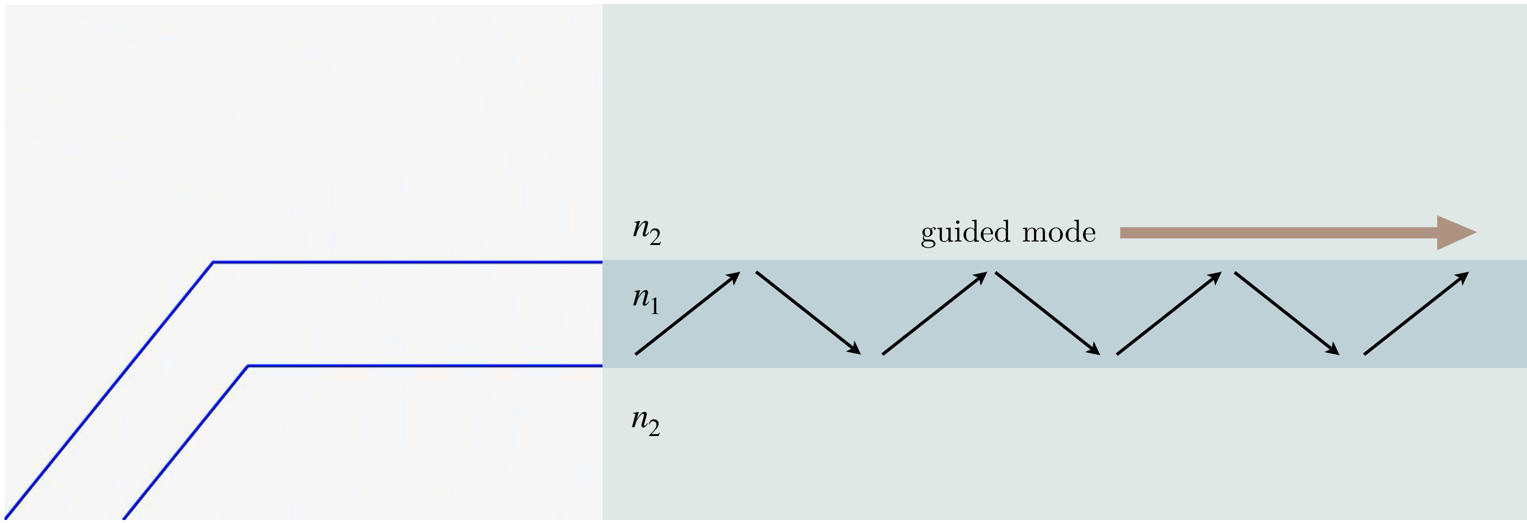
$E_z^t(x,y) = Te^{i(xk_{2x}-yk_{2y})}$ decays exponentially fast as $y \to -\infty$ (total reflection).

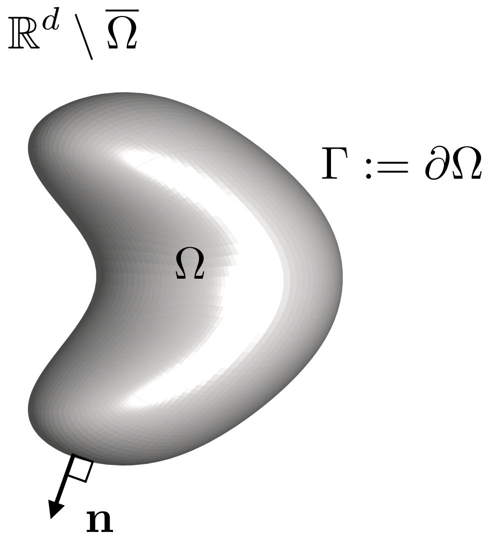


Problems of Scattering

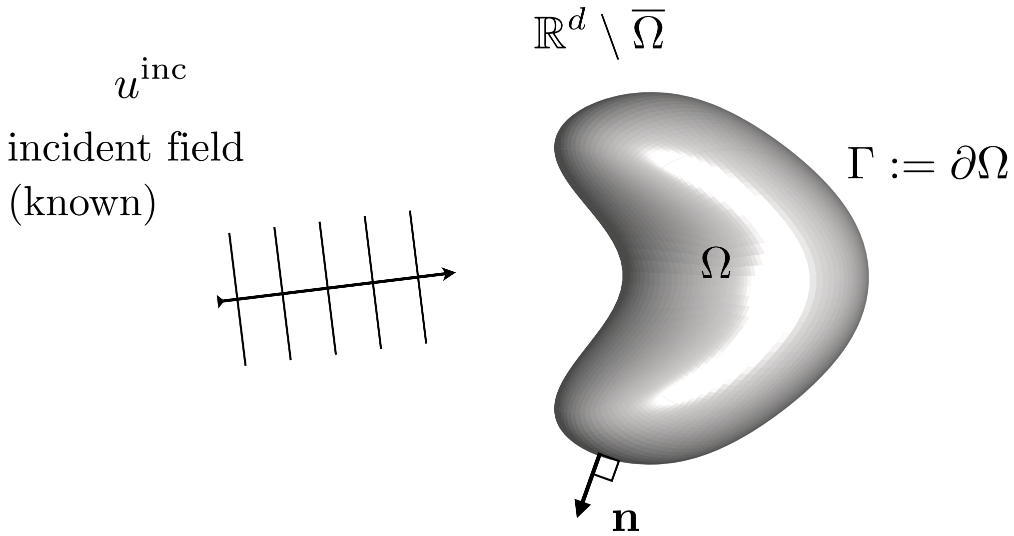
Problems of Scattering

(mainly, what the rest of this course is about)

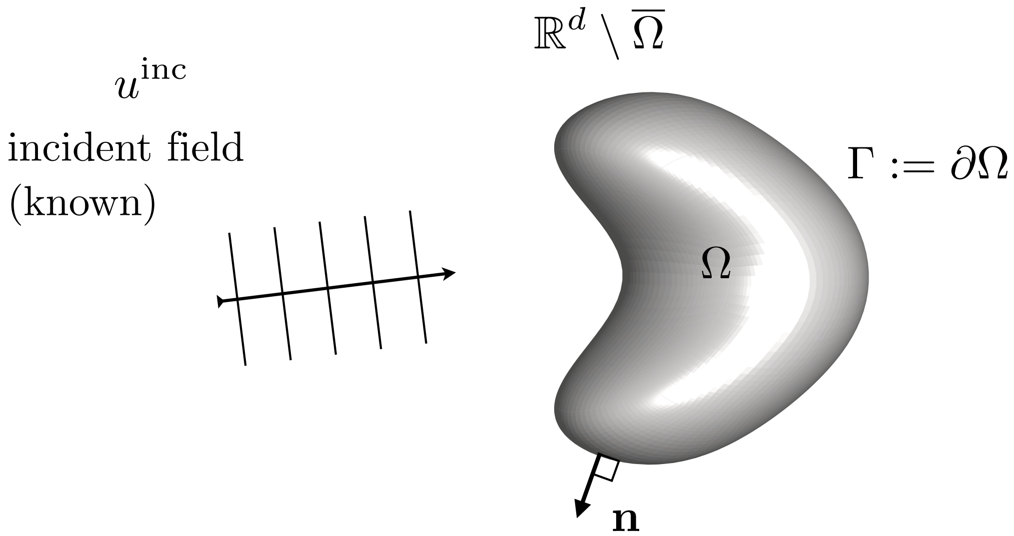
Scattering by Bounded Obstacles



Scattering by Bounded Obstacles



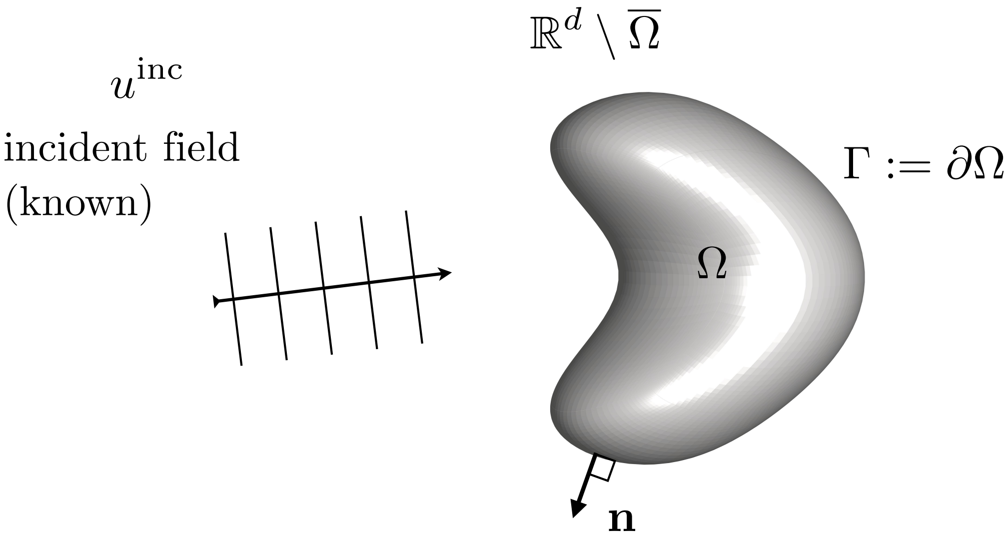
Scattering by Bounded Obstacles



Helmholtz equation:

$$\Delta u^{\mathrm{inc}} + k^2 u^{\mathrm{inc}} = 0 \quad \text{in} \quad D \supset \Gamma$$

Scattering by Bounded Obstacles



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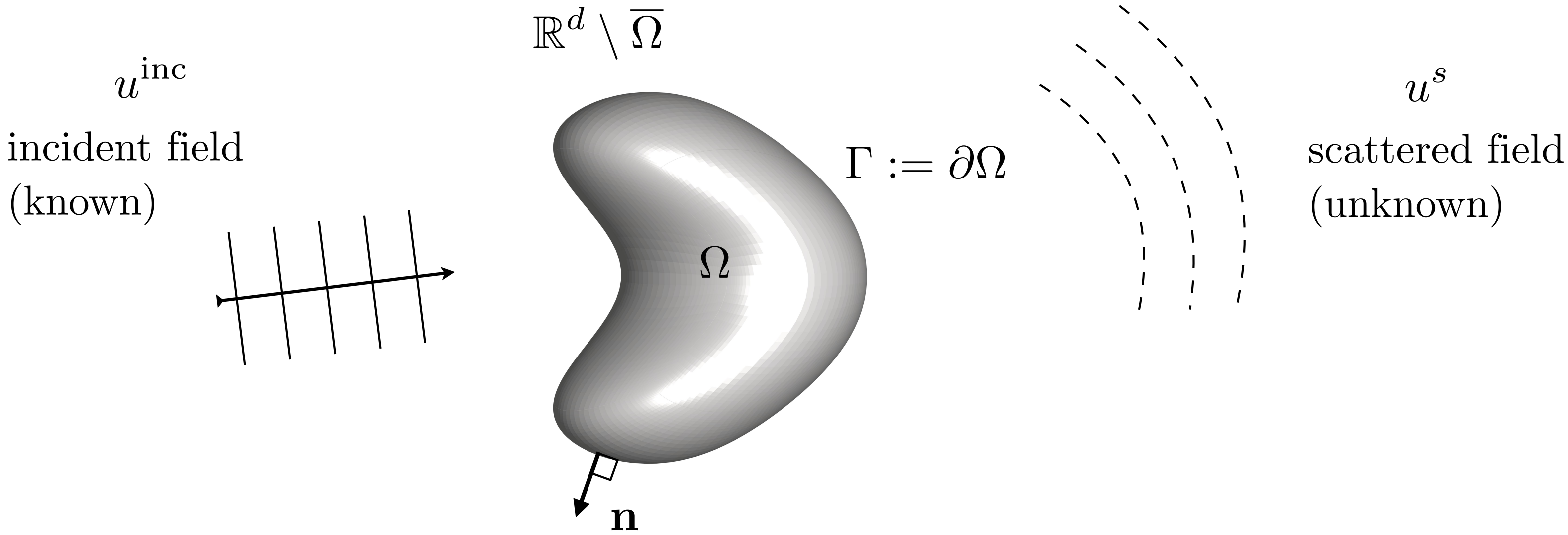
Time convention: $e^{-i\omega t}$; angular frequency: $\omega > 0$;
wavenumber: $k = \omega\sqrt{\epsilon\mu} > 0$

Scattering by Bounded Obstacles



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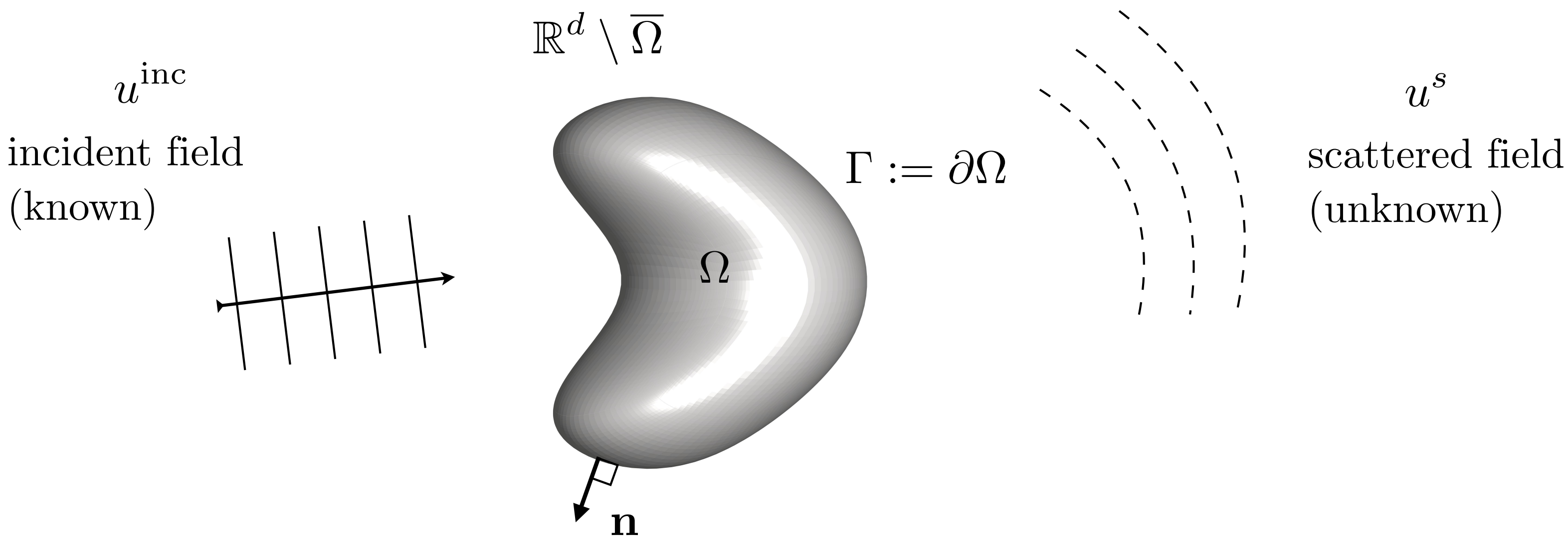

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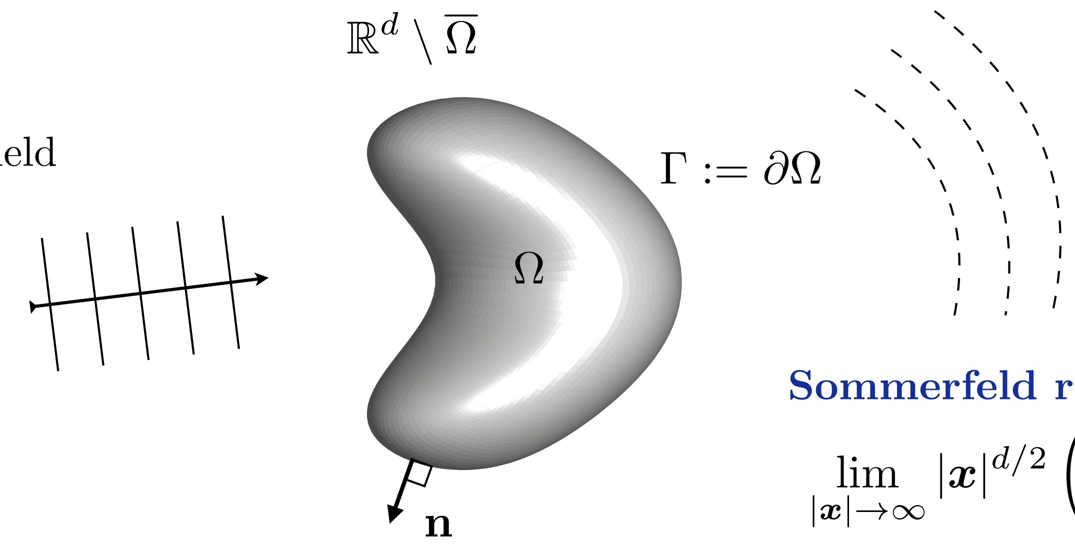
Scattering by Bounded Obstacles

u^{inc}
incident field
(known)



u^s
scattered field
(unknown)

Sommerfeld radiation condition:

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}|^{d/2} \left(\frac{\partial u^s}{\partial |\boldsymbol{x}|} - iku^s \right) = 0$$

Helmholtz equation:

$$\Delta u^{\text{inc}} + k^2 u^{\text{inc}} = 0 \quad \text{in} \quad D \supset \Gamma$$

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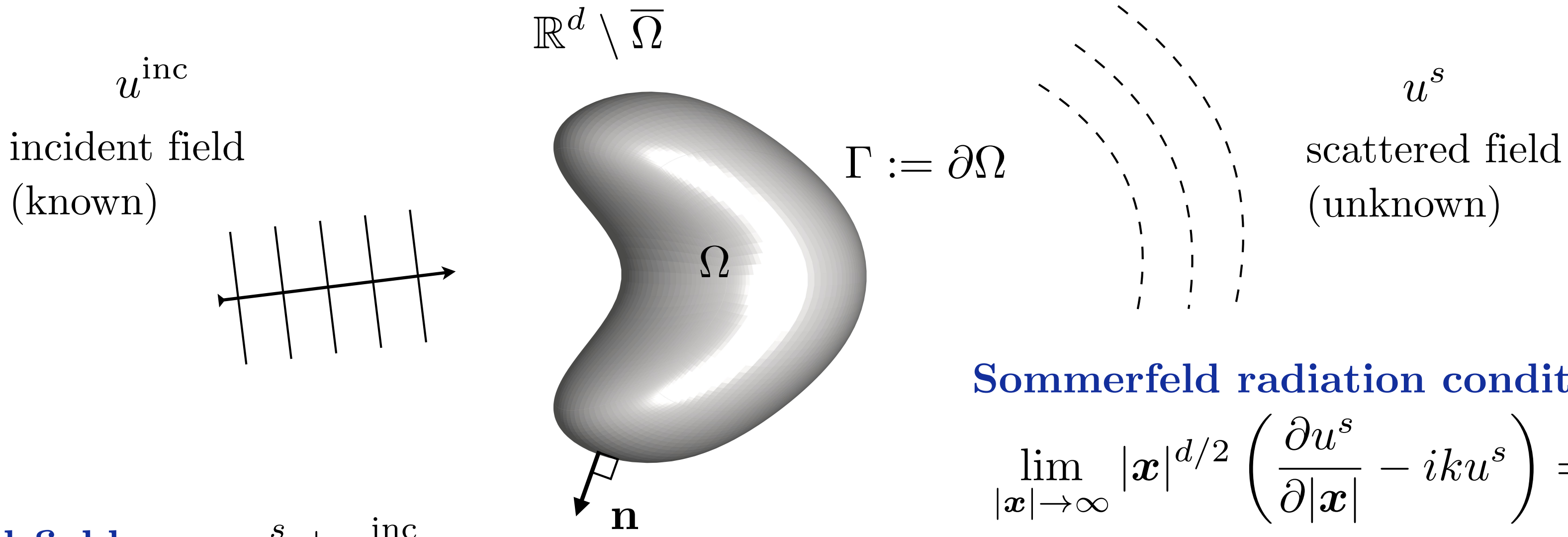
Scattering by Bounded Obstacles



Total field: $u = u^s + u^{\text{inc}}$

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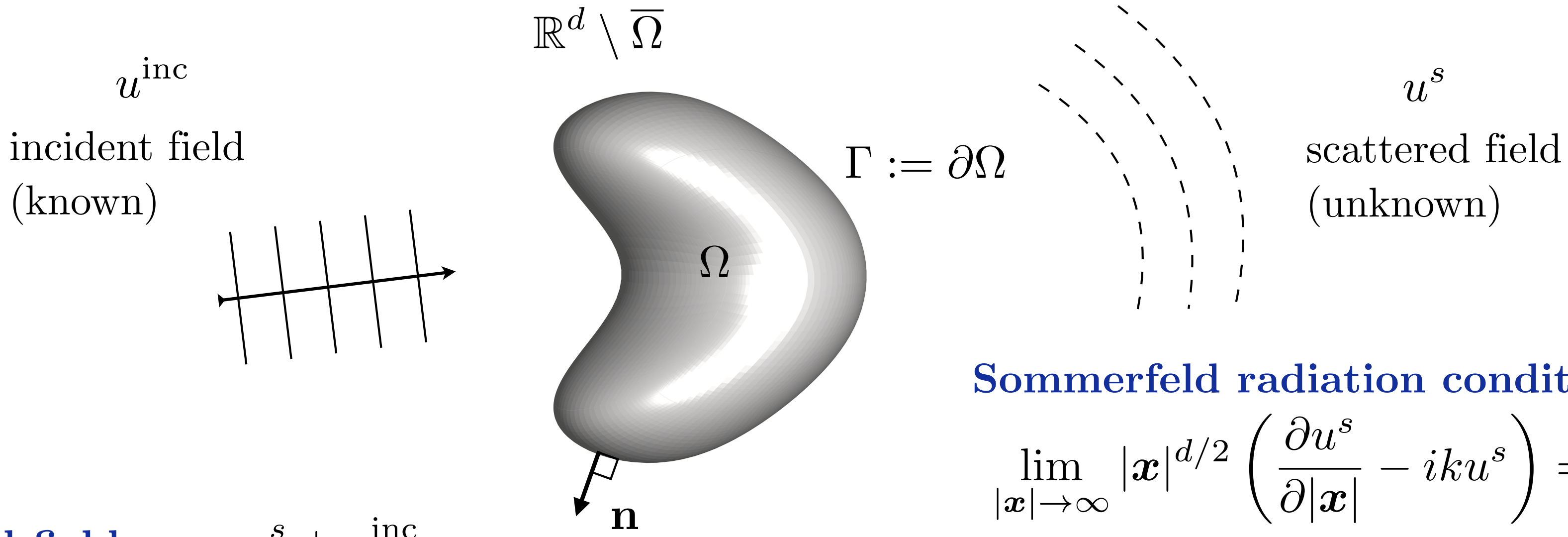
Scattering by Bounded Obstacles



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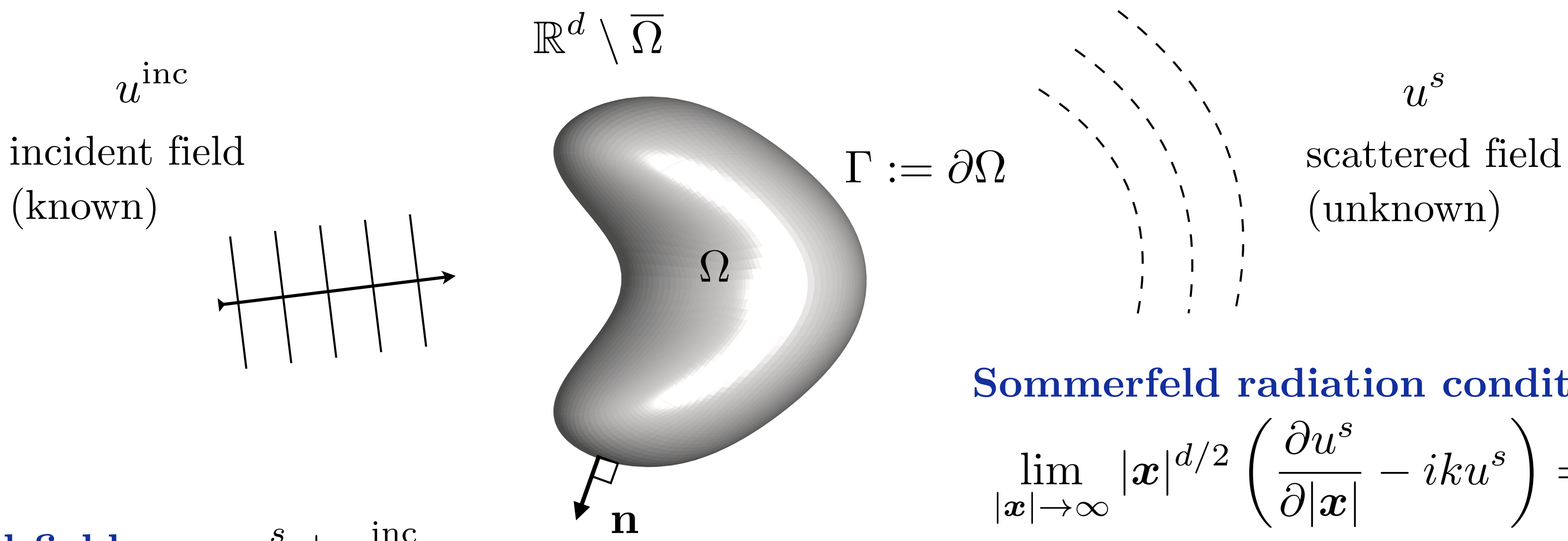
$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}$$

Sound-soft boundary condition:

$$u = 0 \quad \text{on} \quad \Gamma$$

Time convention: $\mathrm{e}^{-i\omega t}$; angular frequency: $\omega > 0$;
wavenumber: $k = \omega\sqrt{\epsilon\mu} > 0$

Scattering by Bounded Obstacles



Total field: $u = u^s + u^{\text{inc}}$

Sound-soft boundary condition:

$$u = 0 \quad \text{on} \quad \Gamma$$

Boundary condition

$$u^s = -u^{\text{inc}} \quad \text{on} \quad \Gamma$$

Sommerfeld radiation condition:

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}|^{d/2} \left(\frac{\partial u^s}{\partial |\boldsymbol{x}|} - iku^s \right) = 0$$

Helmholtz equation:

$$\Delta u^{\text{inc}} + k^2 u^{\text{inc}} = 0 \quad \text{in} \quad D \supset \Gamma$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}$$

Time convention: $\mathrm{e}^{-i\omega t}$; angular frequency: $\omega > 0$;
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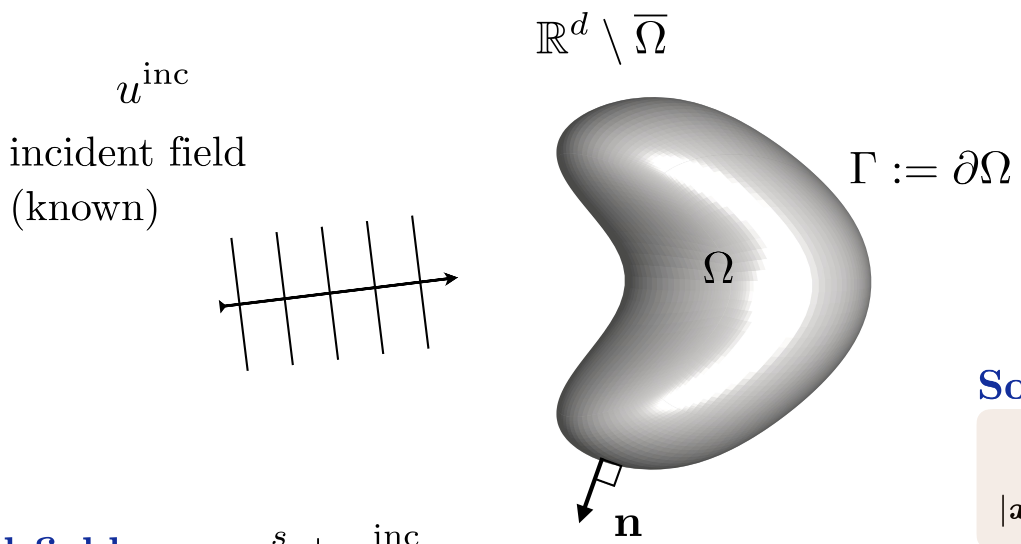
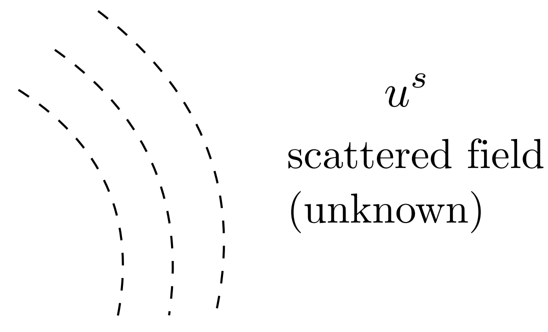
Scattering by Bounded Obstacles





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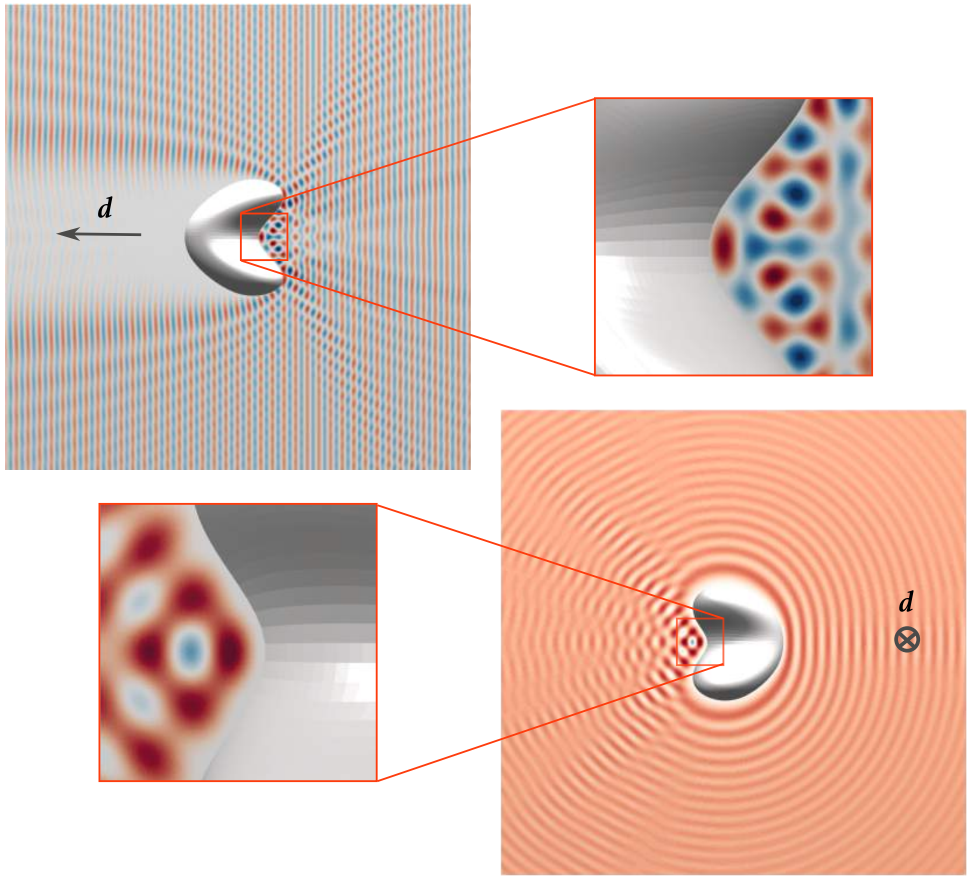
Scattering by Bounded Obstacles

$$u^{\text{inc}}(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot\boldsymbol{x}}$$

incident planewave



$$u = u^s + u^{\text{inc}}$$

total field satisfies a
boundary condition

($u = 0$ on Γ in this case)

P.-A., C., Turc, C., & Faria, L. (2019). Planewave density interpolation methods for 3D Helmholtz boundary integral equations. *SIAM Journal on Scientific Computing*, *41*(4), A2088-A2116

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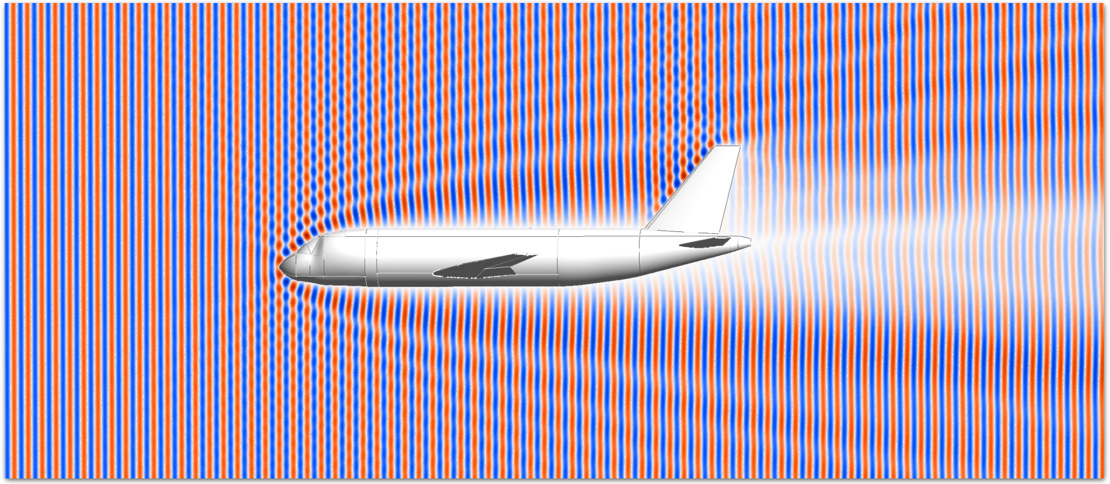
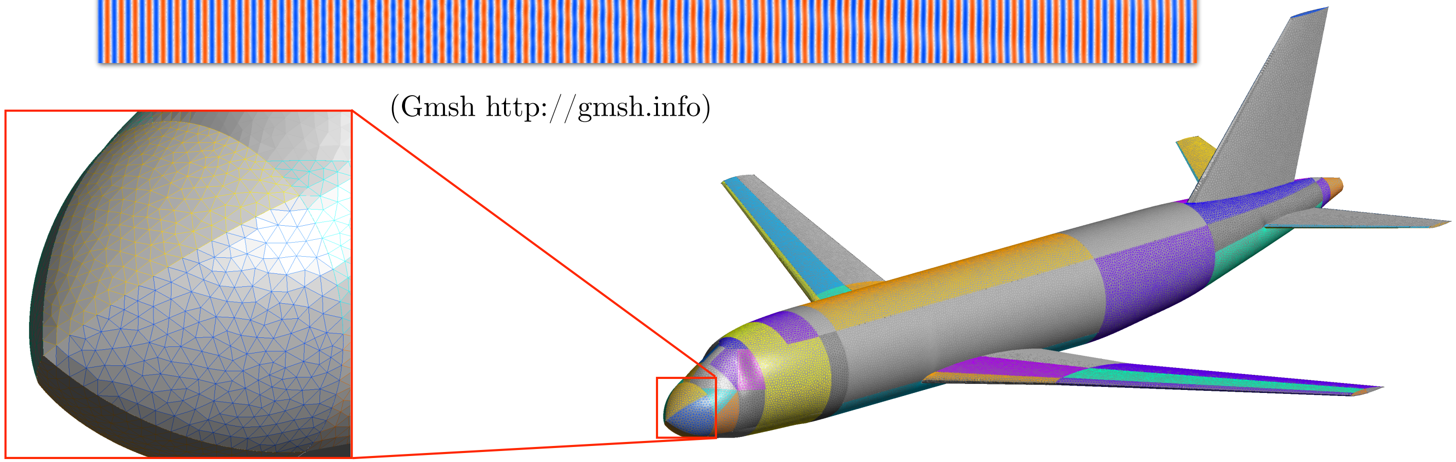


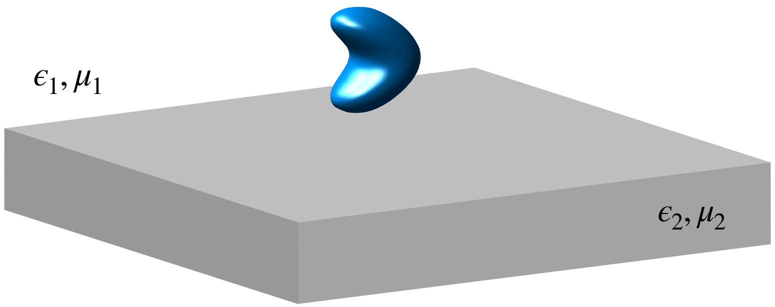
(Gmsh http://gmsh.info)

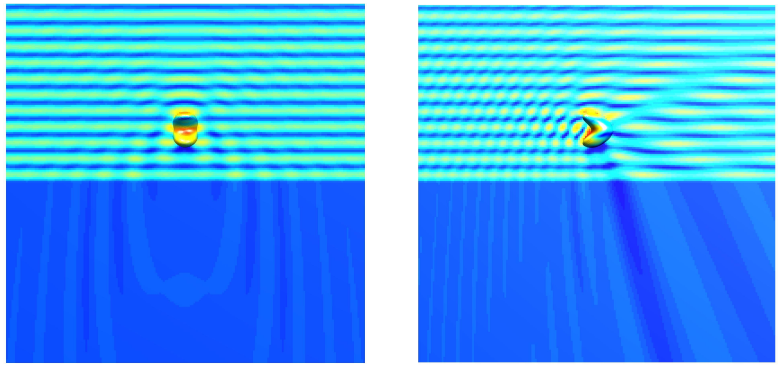


Faria, L., **P.-A., C.**, & Bonnet (2020). General-purpose kernel regularization of boundary integral equations via density interpolation. In preparation.

Scattering by Unbounded Obstacles

- Layered media scattering





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Scattering by Unbounded Obstacles

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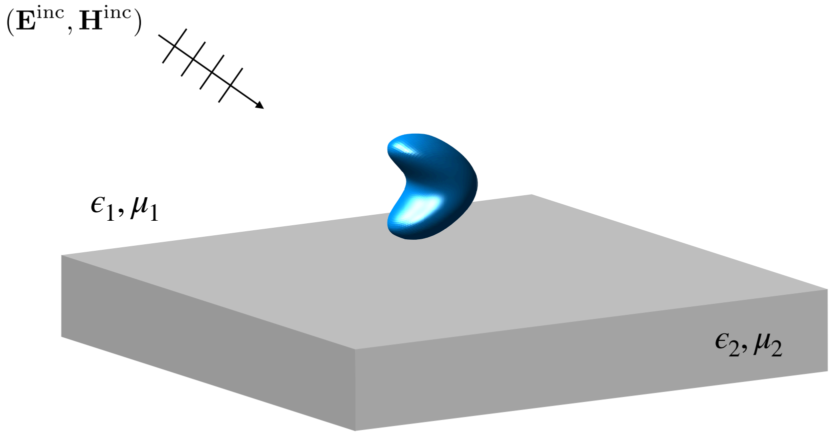

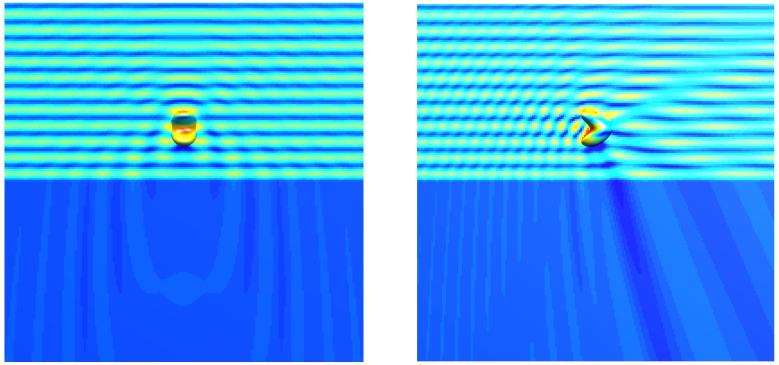

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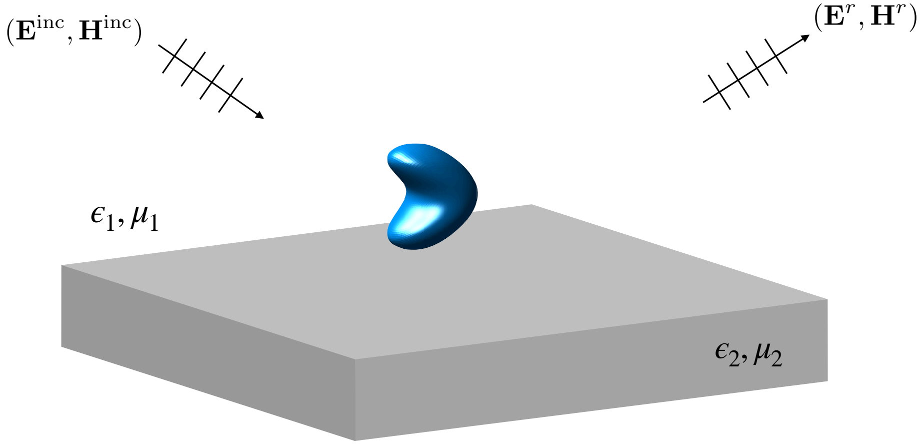

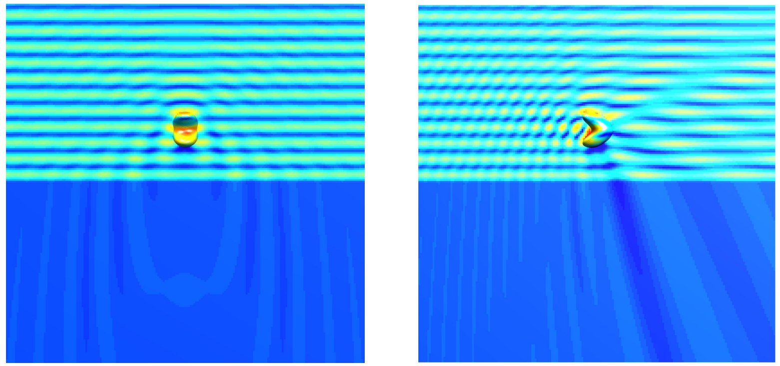

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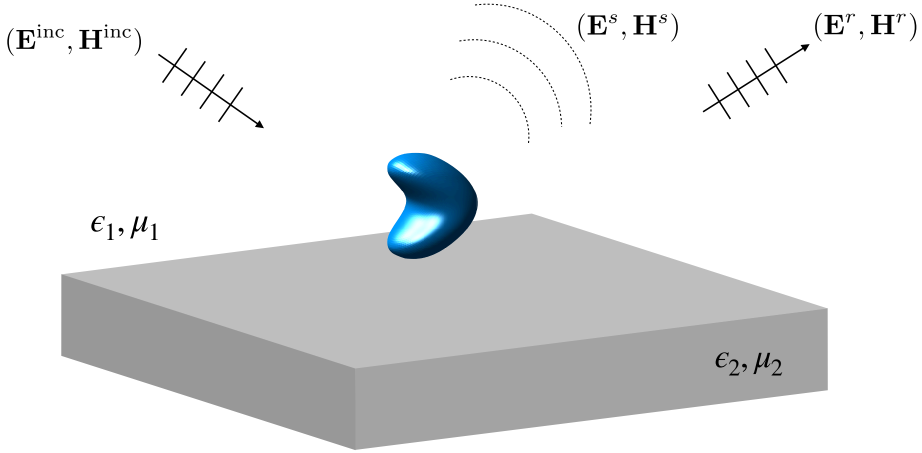

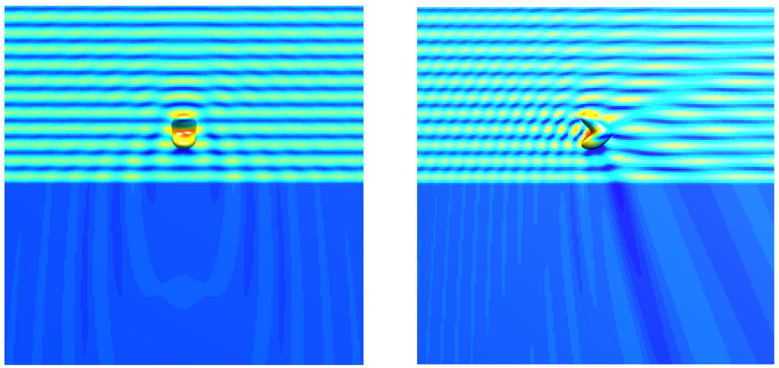

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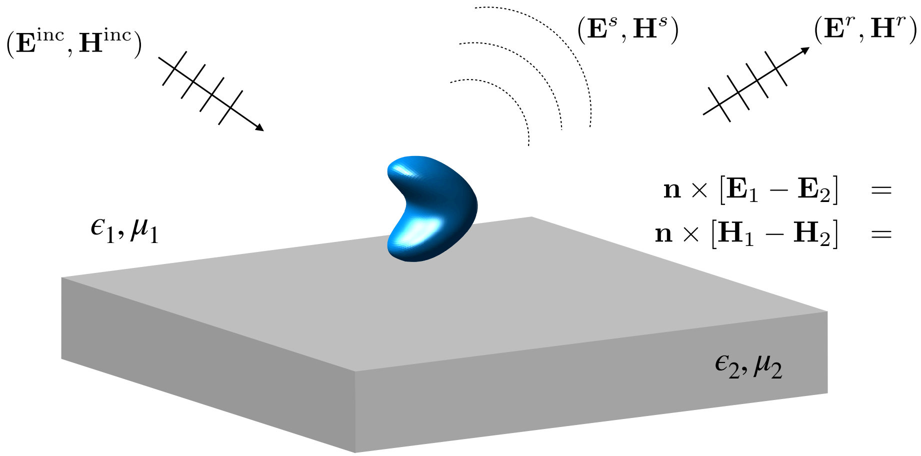

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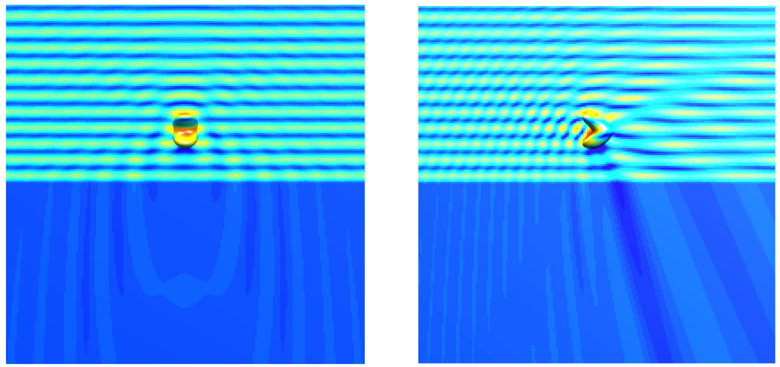

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Scattering by Unbounded Obstacles

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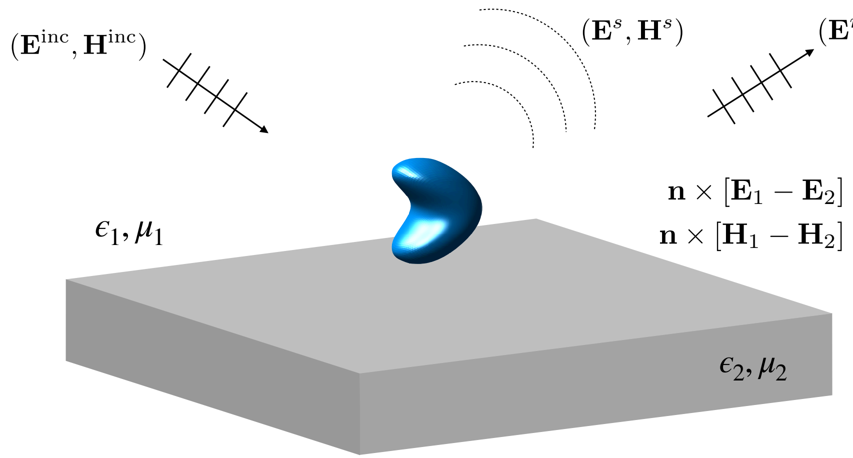

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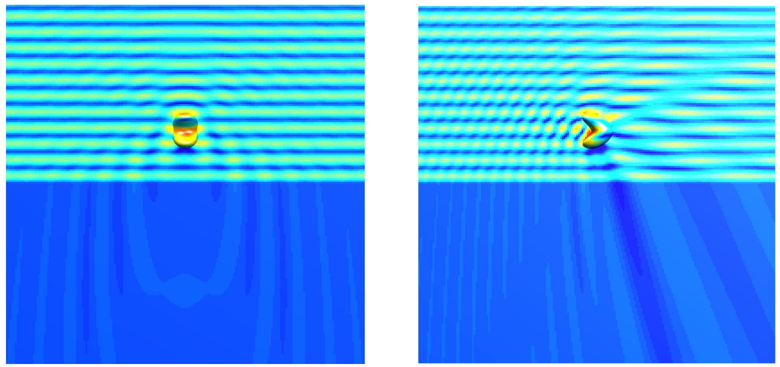

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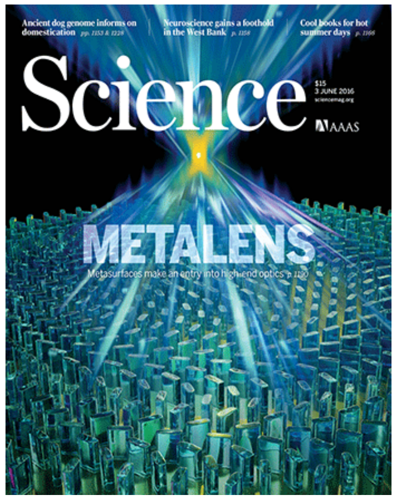
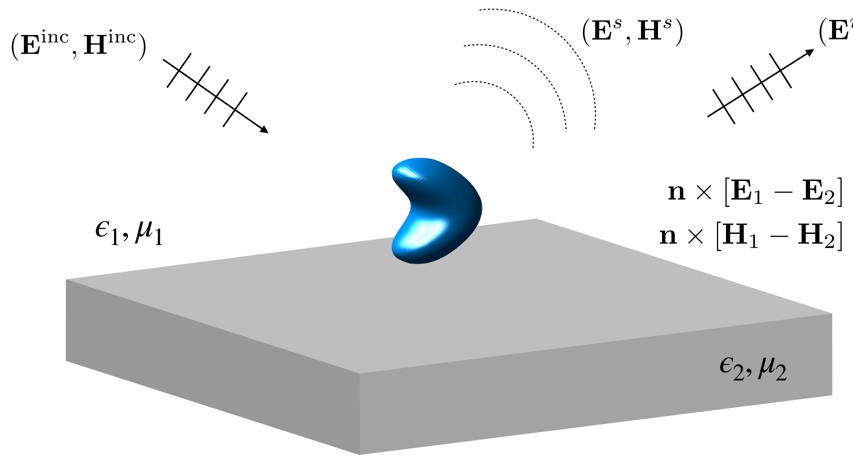


Scattering by Unbounded Obstacles

- Layered media scattering



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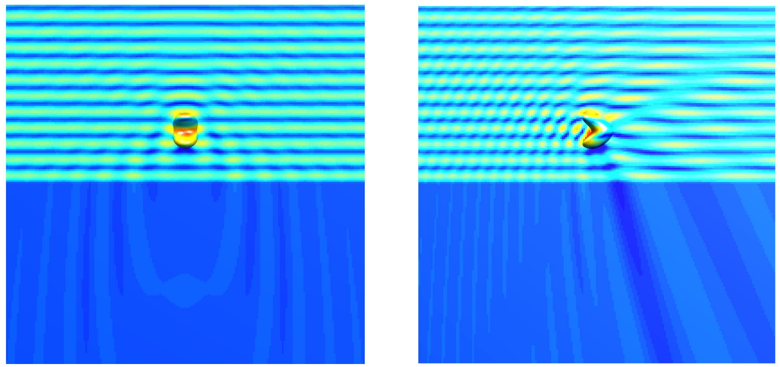

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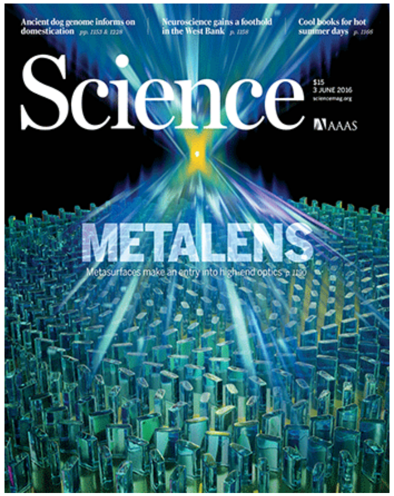

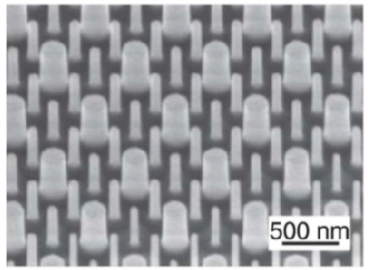

Arbabi, E., et al. "Multiwavelength metasurfaces through spatial multiplexing." *Scientific Reports* 6 (**2016**): 32803.

$(1\text{nm} = 1 \times 10^{-9}\text{m})$

Scattering by Unbounded Obstacles

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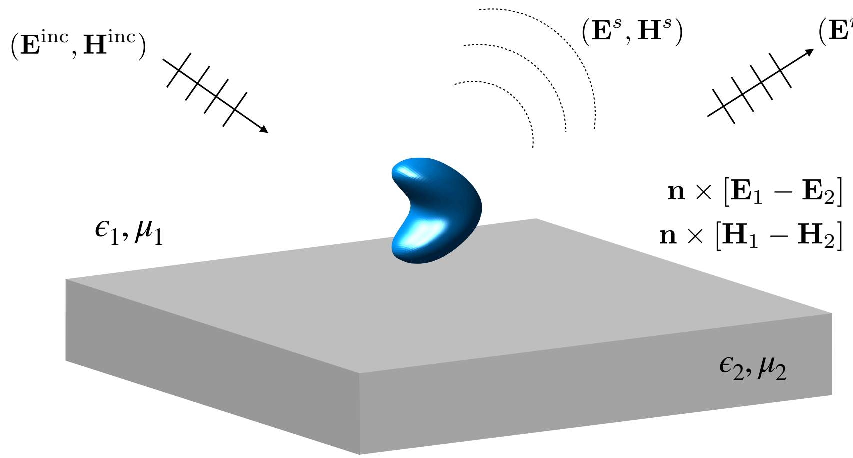

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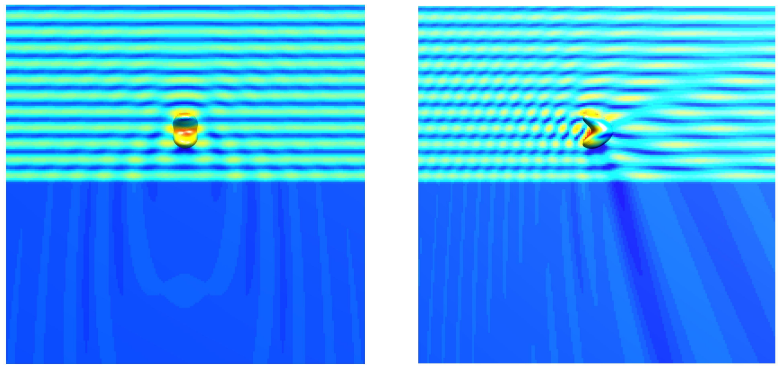

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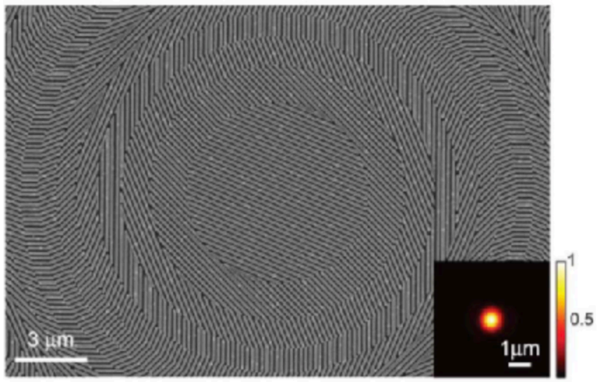

$(1\mu\text{m} = 1 \times 10^{-6}\text{m})$

Lin, D., et al. "Dielectric gradient metasurface optical elements." *Science* 345.6194 (**2014**): 298-302.

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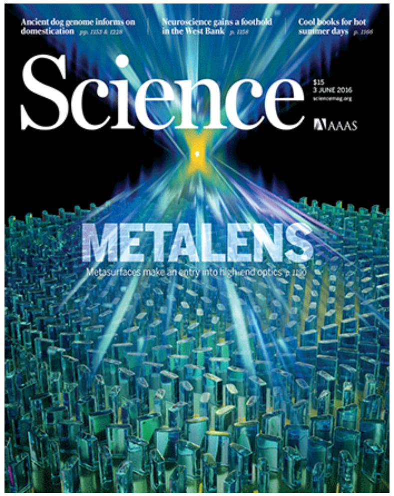

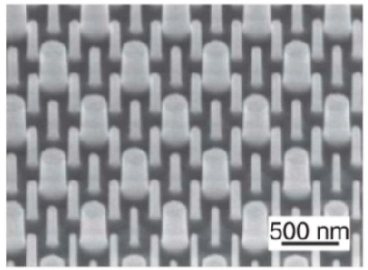

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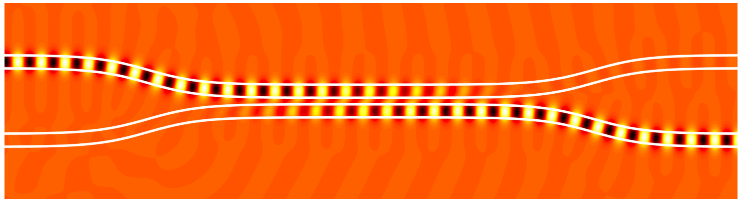
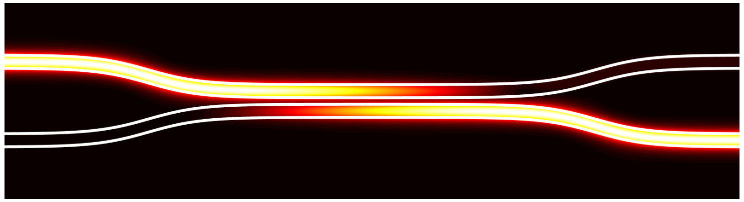
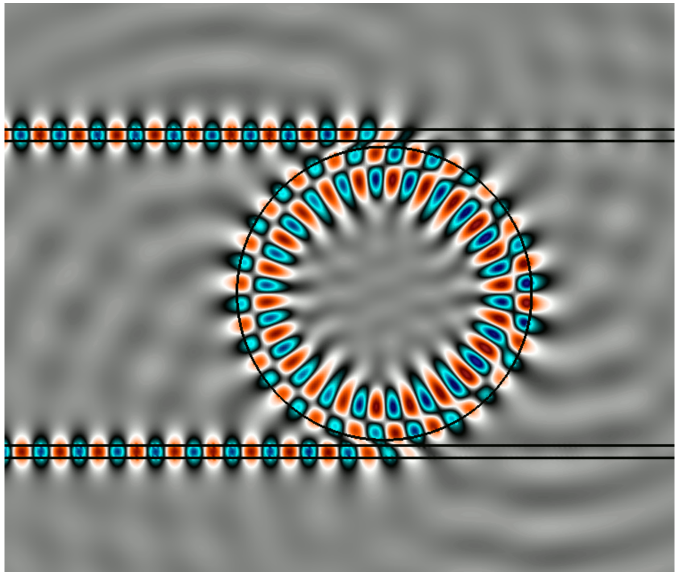
Scattering by Unbounded Obstacles

- Waveguides







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Scattering by Unbounded Obstacles

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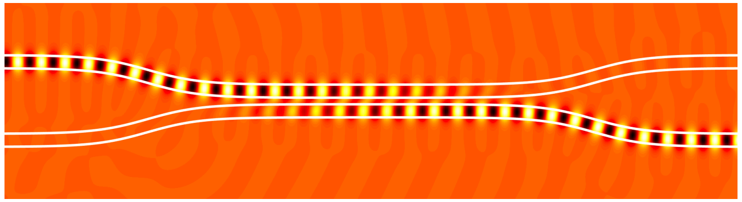

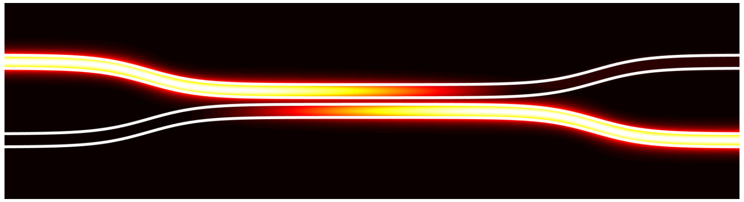

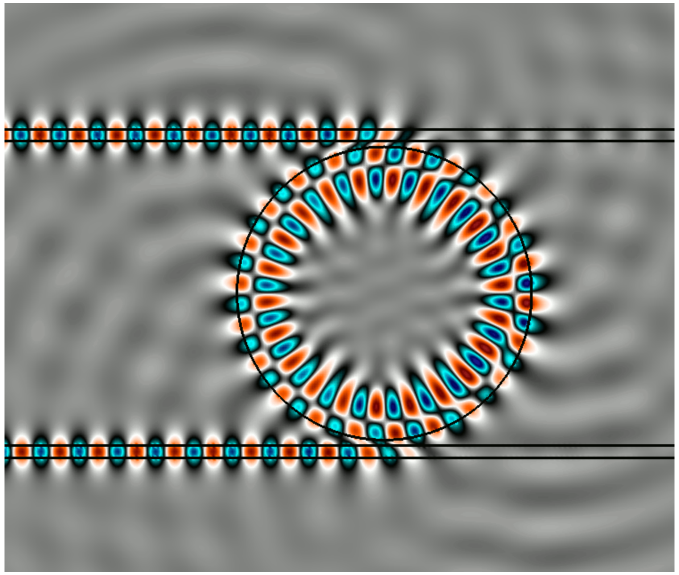

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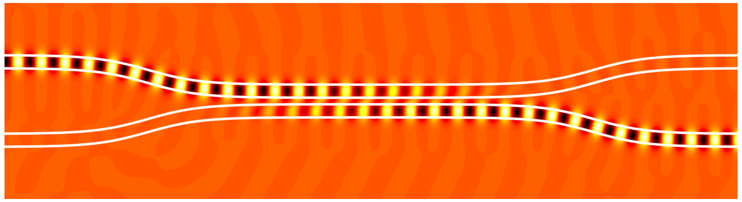

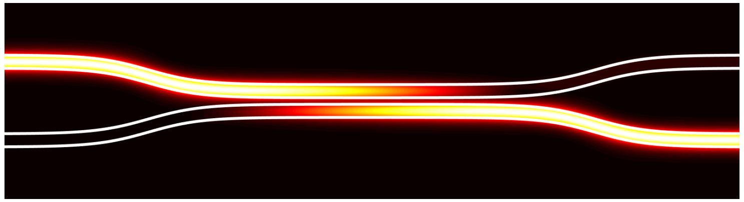

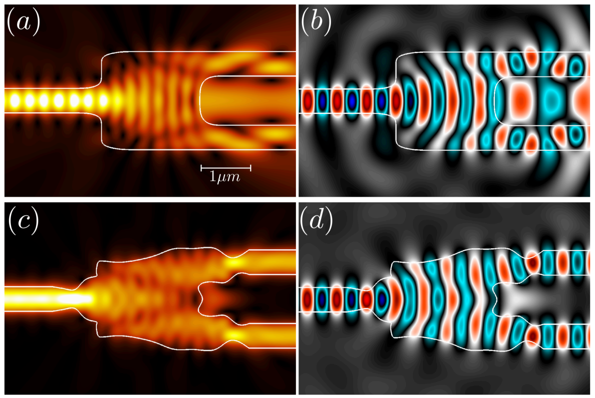

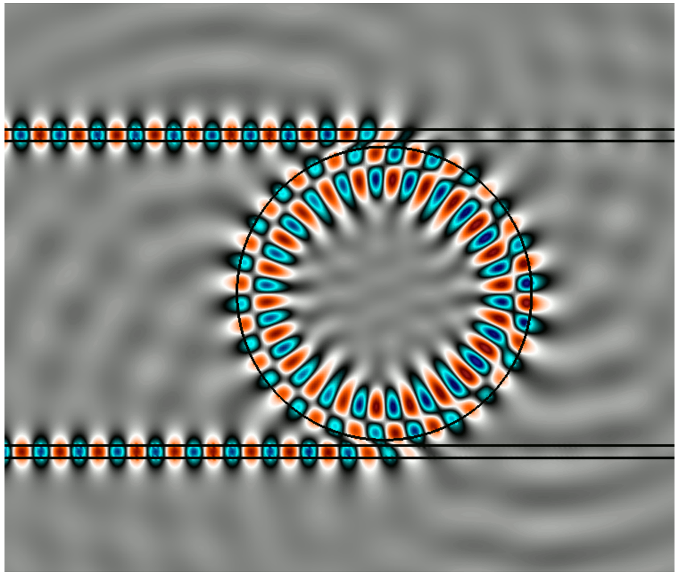

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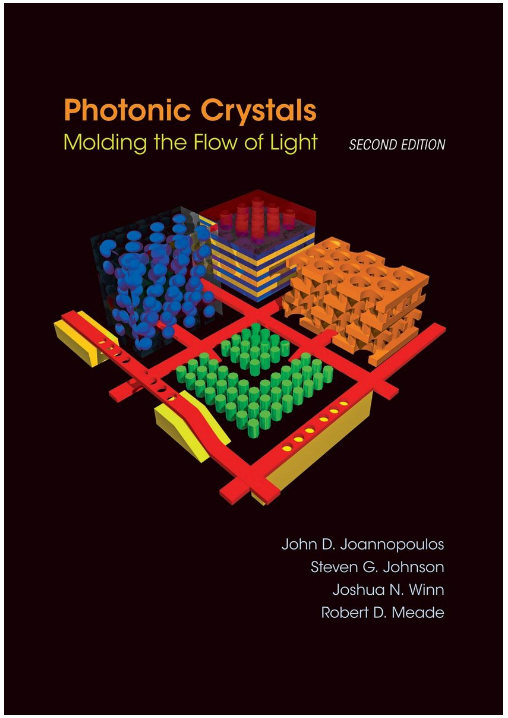
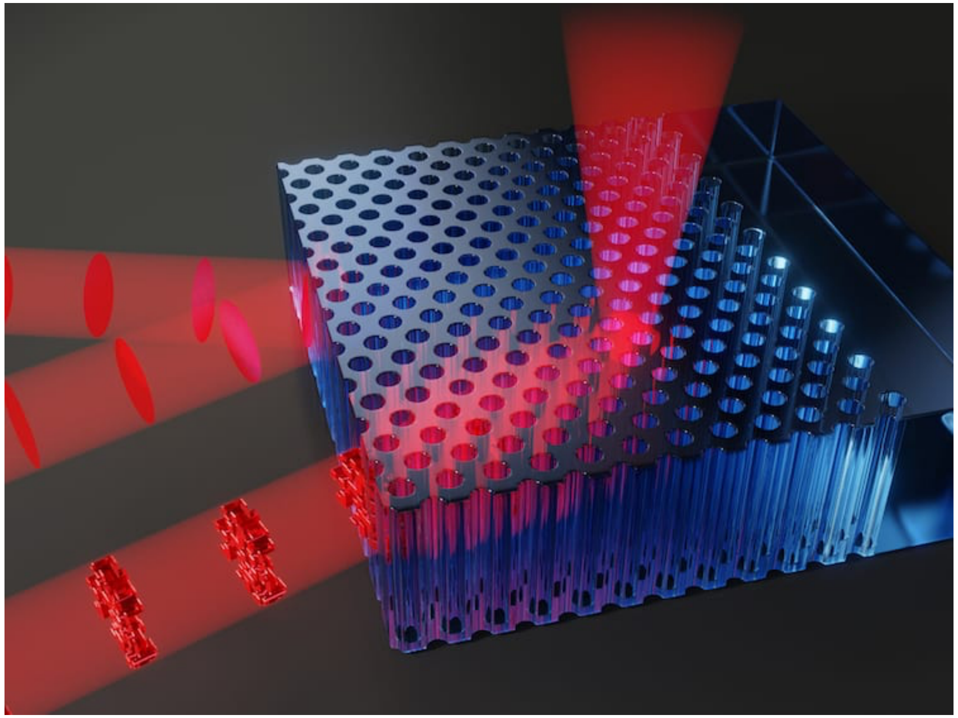
Sideris, C., Garza, E., & Bruno, O. P. (2019). Ultrafast Simulation and Optimization of Nanophotonic Devices with Integral Equation Methods. *ACS Photonics*, *6*(12), 3233-3240.

Scattering by Unbounded Obstacles

- Periodic media



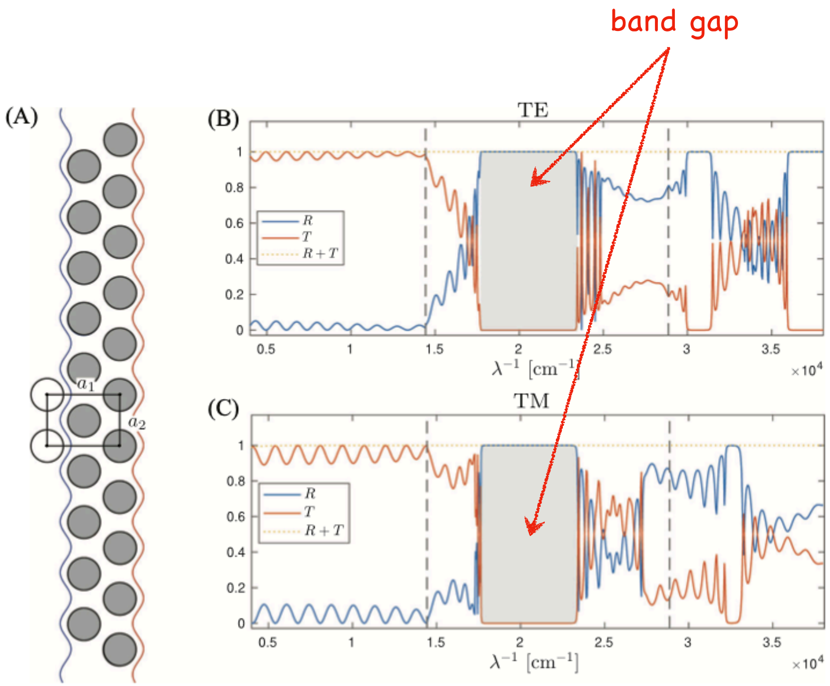


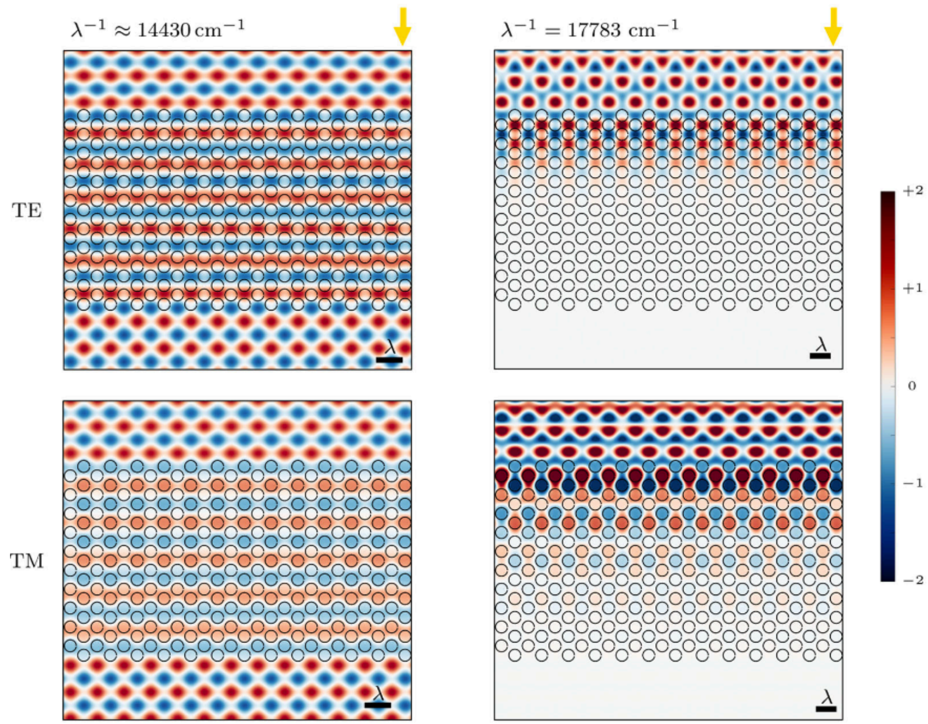
https://physicsworld.com/

Photonic crystals allow the manipulation of light propagation using periodic structures that **create allowed and forbidden frequency bands**, enabling precise control of optical behavior

Scattering by Unbounded Obstacles

- Periodic media





Strauszer-Caussade, T., Faria, L. M., Fernandez-Lado, A., & **P.-A., C.** (2023). Windowed Green Function method for wave scattering by periodic arrays of 2D obstacles. Studies in Applied Mathematics, 150(1), 277-315.

References

- Balanis, C. A. (2012). *Advanced Engineering Electromagnetics*. Wiley.
- Born, M., & Wolf, E. (1999). *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*. Cambridge University Press.
- Filippi, P. (1999). Acoustics: Basic Physics, Theory, and Methods. Academic Press.
- Lax, P. D., & Phillips, R. S. (1967). *Scattering Theory*. Academic Press.
- Pike, E. R., & Sabatier, P. C. (Eds.). (2002). *Scattering: Scattering and Inverse Scattering in Pure and Applied Science* (Vols. 1–2). Academic Press.
- Whitham, G. B. (1974). *Linear and Nonlinear Waves*. Wiley-Interscience.