Exercises

Exercise 1 (Bias-variance decomposition)

Prove the bias-variance decomposition: for $Q, Q_L \in L^2(\Omega, \mathcal{H})$, with $\mathcal{H}$ a separable Hilbert space,

$$\mathbb{E}\left[\|E_M(Q_L(\vartheta)) - \mathbb{E}\left[Q(\vartheta)\right]\|_{\mathcal{H}}^2\right] = \|\mathbb{E}\left[Q_L(\vartheta) - Q(\vartheta)\right]\|_{\mathcal{H}}^2 + \frac{1}{M}\mathrm{Var}\left[Q(\vartheta)\right]$$

Exercise 2 (Variance reduction with the Helmholtz equation)
Consider the Helmholtz problem

$$\begin{aligned} -\Delta u - \kappa^2 u &= 0, \quad \text{in } D := (-1,1)^2, \\ u &= g, \quad \text{on } \partial D, \end{aligned}$$

with $g(x,y) = \sin\left(\frac{\kappa x}{\sqrt{2}}\right)\sin\left(\frac{\kappa y}{\sqrt{2}}\right)$, for $(x,y) \in \partial D$. Consider a sequence of nested h-finite element discretizations with mesh sizes $(h_l)_{l\geq 0}$.

(i) Assume now that $\kappa = \kappa_0(1+\varepsilon)$, with $\varepsilon \sim \mathcal{U}([-0.25, 0.25])$. For the quantity of interest the solution u itself as a function in $H^1(D)$, which exponents α and β do you expect in the assumptions for the multilevel Monte Carlo complexity theorem?

(ii) Test now this numerically, plotting Monte Carlo sample estimates for $\|\mathbb{E}[u_l - u]\|_{H^1(D)}$ and $\mathrm{Var}[u_l - u_{l-1}]$ in dependence of l.

Hint: To test the bias, you can use the exact solution $u(x,y) = \sin\left(\frac{\kappa x}{\sqrt{2}}\right)\sin\left(\frac{\kappa y}{\sqrt{2}}\right)$.