Free-surface waves
using extended shallow water models
part 2

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WAVES.NL Summer school, Nijmegen, 26 August 2025

Schedule

Time	Monday	Tuesday	Wednesday	Thursday	Friday
8:50–9:00	*Opening*				
9:00–10:30	L3	L5	L2	L4	L6
10:30–11:00	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
11:00–12:30	L1	L1	L2	L4	L6
12:30–13:30	*Lunch*	*Lunch*	*Lunch*	*Lunch*	*Lunch*
13:30–15:00	L3	L5	L3	L5	
15:00–15:30	*Coffee break*	*Coffee break*			
15:30–17:00	Poster session	L1			
17:45–19:00			*Social event*		

Slides at: `https://github.com/scalaura/waves_summerschool`

L1: Mon 11-12:30
- overview
- motivation
- derivation

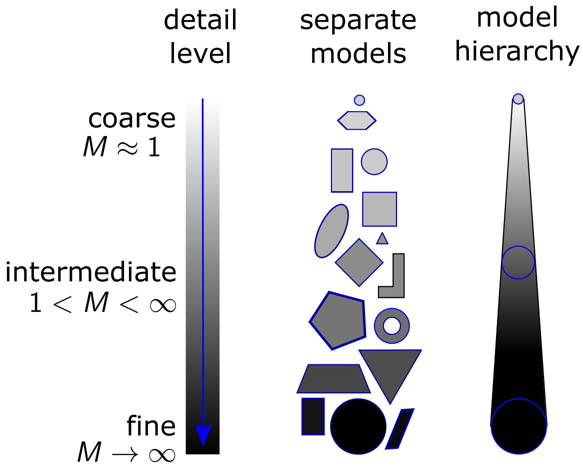
L2: Tue 11-12:30
- analysis

L3: Tue 15:30-17
- selected papers
- outlook

Content of this talk

1 Repetition

Hierarchical mathematical modeling



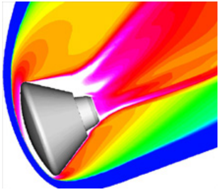
Hierarchical moment models

Advantages

1. general derivation
2. structure preserving
3. accurate results

$\Rightarrow$ adaptive simulations

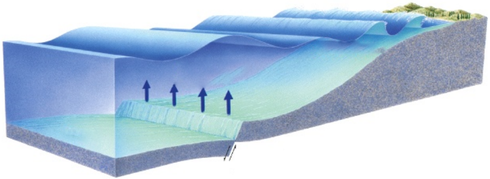
a) rarefied gases



Scale is the *Knudsen number*

$$\mathrm{Kn} = \frac{\text{mean free path length}}{\text{reference length}} = \frac{l}{L}$$

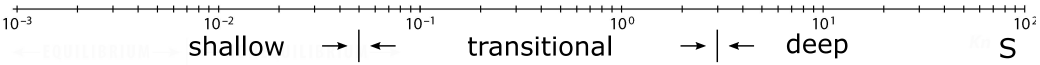
b) shallow flows



Scale is the *shallowness*

$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

b)



Model equation: Rarefied gases and shallow flows

a) rarefied gases

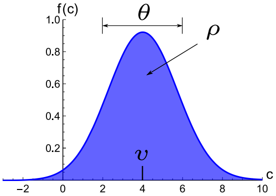
b) shallow flows

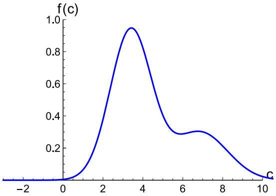
Boltzmann Transport Equation

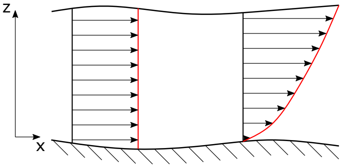
$$\frac{\partial}{\partial t} f(t, \boldsymbol{x}, \boldsymbol{c}) + c_i \frac{\partial}{\partial x_i} f(t, \boldsymbol{x}, \boldsymbol{c}) = S(f)$$

Incompressible Navier-Stokes Equations

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + g$$



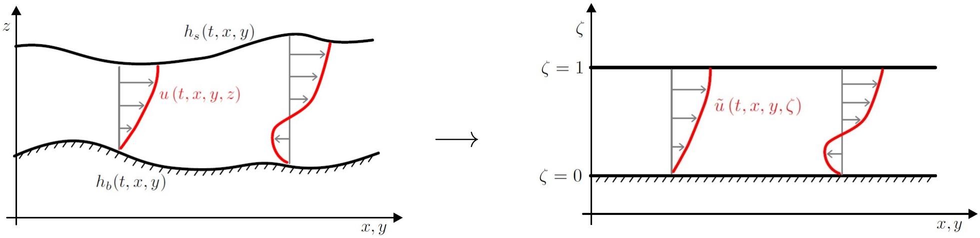




Euler equations $\rightarrow$?

SWE $\rightarrow$?

Transformation [Torrilhon, Kowalski, 2018]



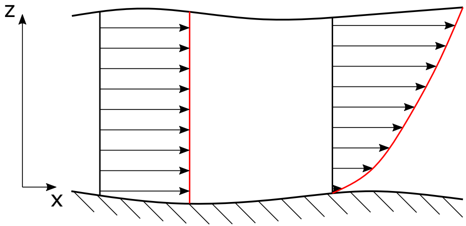
$$z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h}$$

$$z \in [h_b(t,x), h_s(t,x)] \quad \Rightarrow \quad \zeta \in [0,1]$$

Polynomial ansatz [Kowalski, Torrilhon, 2018]

Represent variations over depth with polynomials

$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} + \sum_{i=1}^{M} \alpha_i(t,x)\, \underbrace{\phi_i\left(\frac{z-h_b}{h_s-h_b}\right)}_{\phi_i(\zeta)}$$



Polynomial ansatz [Kowalski, Torrilhon, 2018]

Represent variations over depth with polynomials

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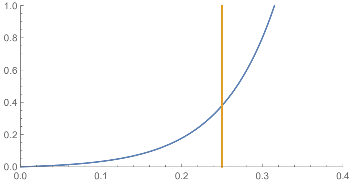
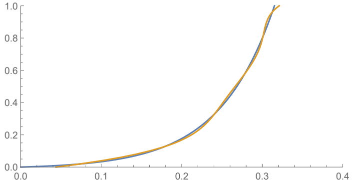
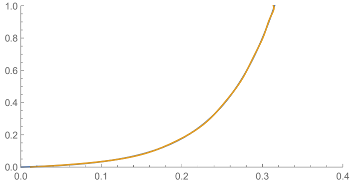
$M = 0$



$M = 5$



$M = 10$



Moment models

1. underlying model equation

$$\mathcal{D}\left(\boldsymbol{U}(t,\boldsymbol{x},\boldsymbol{y})\right) = 0$$

2. expansion with ansatz

$$\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in\mathbb{M}} \boldsymbol{U}_i(t,\boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

3. *moment* projection

$$\int_{\Omega} \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y})\right) \cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, \boldsymbol{dy} \quad \text{for } j \in \mathbb{M}$$

Moment model

Hierarchical system of lower-dimensional PDEs for $\boldsymbol{U}_i(t,\boldsymbol{x})$

Moment models [Grad, 1949], [Kowalski, Torrilhon, 2018]

1. underlying model equation: incompressible NSE

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + g \qquad (*)$$

2. expansion: polynomial ansatz

$$u(t,x,\zeta) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x)\phi_i(\zeta)$$

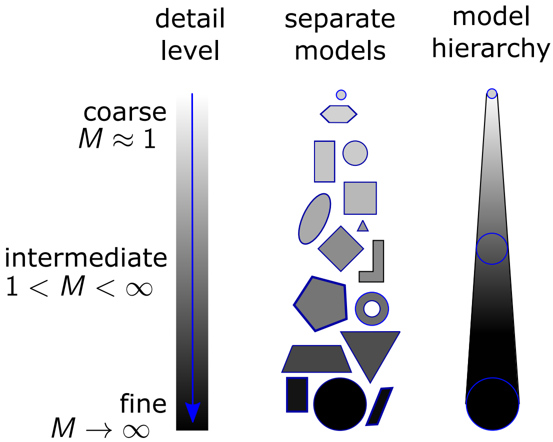
3. *moment* projection: depth integration

$$\int_0^1 (*) \cdot \phi_j(\zeta)\, d\zeta, \quad j = 0, \ldots, M$$

Moment model

Hierarchical system of lower-dimensional PDEs for $h(t,x), u_m(t,x), \alpha_i(t,x)$

General derivation of hierarchical moment models



Ansatz:

$$\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in\mathbb{M}} \boldsymbol{U}_i(t,\boldsymbol{x}) \cdot \Phi_i^{\boldsymbol{U}}(\boldsymbol{y})$$

Projection:

$$\int_\Omega \mathcal{D}\left(\boldsymbol{U}_{\mathbb{M}}(t,\boldsymbol{x},\boldsymbol{y})\right)\cdot \Psi_j^{\boldsymbol{U}}(\boldsymbol{y})\, \boldsymbol{dy} \text{ for } j \in \mathbb{M}$$

Other models

- uncertainty quantification
- traffic flow

Shallow Water Equations [Kowalski, Torrilhon, 2019]

$(M = 0)$

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} \end{pmatrix} = -\begin{pmatrix} 0 \\ gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda}\begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν.

Shallow Water Moment Equations [Kowalski, Torrilhon, 2019]

$M = 1$

First order model: $u(\zeta) = u_m + \alpha_1\phi_1(\zeta)$, $\quad \phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda}P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3\left(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1\right) \end{pmatrix}$$

$M = 2$

Second order model: $u(\zeta) = u_m + \alpha_1\phi_1(\zeta) + \alpha_2\phi_2(\zeta)$, $\quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

SWME system

$$
\begin{cases}
\partial_t h + \partial_x (h u_m) = 0, \\
\partial_t (h u_m) + \partial_x \left(h u_m^2 + h \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} \right) + gh\partial_x (b + h) = -\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^{N} \alpha_j \right), \\
\partial_t (h\alpha_i) + \partial_x \left(h \left(2u_m\alpha_i + \sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k \right) \right) = u_m \partial_x (h\alpha_i) - \sum_{j,k=1}^{N} B_{ijk}\alpha_k \partial_x (h\alpha_j) \\
\qquad -(2i+1) \left(-\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^{N} \alpha_j \right) + \frac{\nu}{h} \sum_{j=1}^{N} C_{ij}\alpha_j \right)
\end{cases}
$$

A_{ijk}, B_{ijk}, C_{ij} are constant coefficients:

$$
\frac{A_{ijk}}{2i+1} = \int_0^1 \phi_i\phi_j\phi_k d\xi, \quad \frac{B_{ijk}}{2i+1} = \int_0^1 \phi_i' \left(\int_0^\xi \phi_j d\xi \right) \phi_k d\xi, \quad \text{and} \quad C_{ij} = \int_0^1 \phi_i'\phi_j' d\xi.
$$

2 Analysis

Desirable properties

Question: What are desirable model properties?

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- high accuracy
- low complexity
- efficiency
- adaptivity
- extendability
- analytical form

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- analytical form
- conservation
- hyperbolicity
- stability
- equilibria
- steady states
- entropy

Question: What are desirable model properties?

- high accuracy
- low complexity
- efficiency
- adaptivity
- extendability
- analytical form

$(\checkmark)$

- conservation
- hyperbolicity
- stability
- equilibria
- steady states
- entropy

?

2.1 conservation

Conservation properties

Second order model: $u(\zeta) = u_m + \alpha_1\phi_1(\zeta) + \alpha_2\phi_2(\zeta)$, $\quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

Conservation of mass ✓

no conservation of momentum with bottom force (as expected) ✓

non-conservative form of equations

2.2 hyperbolicity

Hyperbolicity definition

Definition (hyperbolicity)

A PDE of the form

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) + A\frac{\partial}{\partial x}\mathbf{u}(t,x) = 0,$$

for $\mathbf{u}\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is hyperbolic if A can be diagonalized with real eigenvalues.

Hyperbolicity remarks

- Hyperbolic systems can (locally) be decomposed into a system of scalar PDEs using $A = V \Lambda V^{-1}$ with $\Lambda = \mathit{diag}(\mathrm{EV}(A))$ and $V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ with $\mathbf{v}_i$ the eigenvectors of A. New variables $\mathbf{w} = V^{-1}\mathbf{v}$ and $\mathbf{v} = V\mathbf{w}$:

$$
\begin{aligned}
& \frac{\partial}{\partial t}\mathbf{u}(t,x) + A\frac{\partial}{\partial x}\mathbf{u}(t,x) = 0, \\
\Rightarrow \quad & \frac{\partial}{\partial t}\mathbf{w}(t,x) + \Lambda\frac{\partial}{\partial x}\mathbf{w}(t,x) = 0.
\end{aligned}
$$

- Hyperbolicity is lost if eigenvalues are complex or if there exists no full set of eigenvectors.
- Hyperbolic systems describe the propagation of information with real, bounded propagation speeds.

$(M = 0)$

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} \end{pmatrix} = -\begin{pmatrix} 0 \\ gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda}\begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν.

$(M = 0)$

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} \end{pmatrix} = -\begin{pmatrix} 0 \\ gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda}\begin{pmatrix} 0 \\ u_m \end{pmatrix},$$

for slip friction law at bottom with slip length λ and viscosity ν.

Propagation speeds are

$$\lambda_{1,2} = u_m \pm \sqrt{gh}.$$

$M = 1$

First order model: $u(\zeta) = u_m + \alpha_1 \phi_1(\zeta), \quad \phi_1(\zeta) = 1 - 2\zeta$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{pmatrix}$$

Propagation speeds are

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \alpha_1^2} \text{ and } \lambda_3 = u_m.$$

$M = 2$

Second order model: $u(\zeta) = u_m + \alpha_1\phi_1(\zeta) + \alpha_2\phi_2(\zeta), \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda} P$$

with

$$P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

Propagation speeds: ?

Propagation speeds $(M = 2)$

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}(\boldsymbol{u}_M)\,\partial_x \boldsymbol{u}_M = \boldsymbol{0}$$

variable set

$$\boldsymbol{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

$$\boldsymbol{A}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\ -2\alpha_1 u_m - \frac{4}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m + \alpha_2 & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 - 2u_m\alpha_2 - \frac{2}{7}\alpha_2^2 & 2\alpha_2 & -\frac{\alpha_1}{3} & u_m + \frac{3\alpha_2}{7} \end{pmatrix} \quad (M = 2)$$

Propagation speeds $(M = 2)$

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eigenvalues can become complex $\Rightarrow$ loss of hyperbolicity $\lightning$

Propagation speeds ($M=4$) rarefied gases

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}(\boldsymbol{u}_M)\,\partial_x \boldsymbol{u}_M = \boldsymbol{0}$$

rarefied gases

$$\boldsymbol{u}_M = (\rho, v, \theta, f_3, f_4, \ldots, f_M)^T \in \mathbb{R}^{M+1}$$

$$\boldsymbol{A}_{\mathrm{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix} \quad (M=4)$$

eigenvalues can become imaginary $\Rightarrow$ loss of hyperbolicity

Breakdown of hyperbolicity

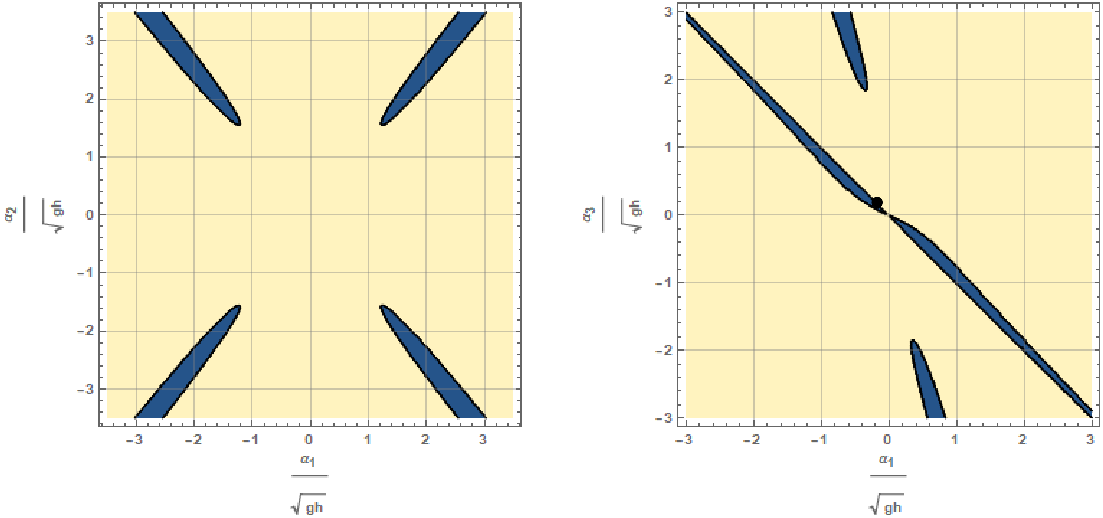


Figure: Second order (left) and third order (right, for $\alpha_2 = 0$)

Breakdown of hyperbolicity

Simulation test case

Simple transport of smooth wave

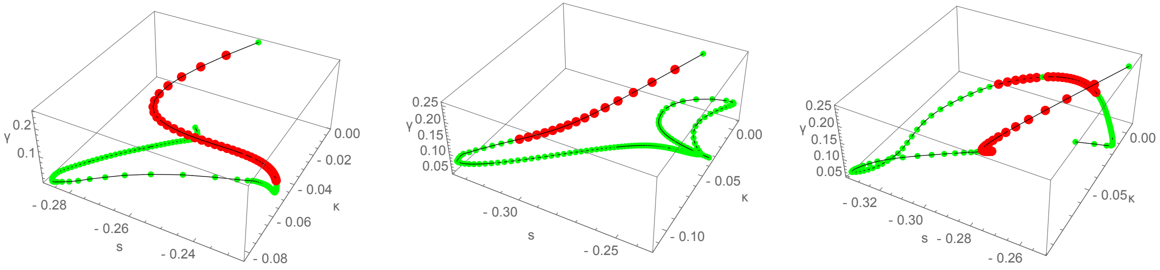
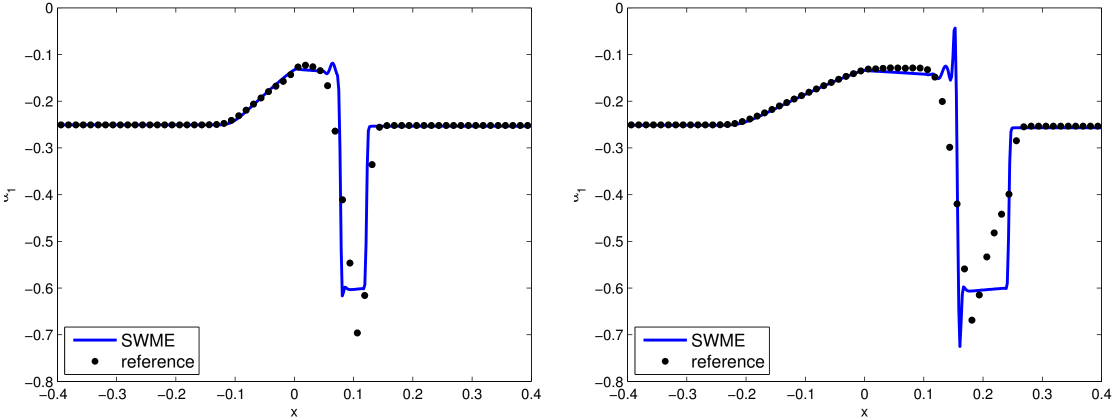


Figure: Hyperbolic breakdown (red) for $x_1 = -0.5$; $x_2 = 0$; $x_3 = 0.5$.

Hyperbolicity breakdown

Solution looses hyperbolicity directly after the first time step.

Instability



(a) α_1, $t = 0.05$ (b) α_1, $t = 0.1$

Figure: Unstable dam break simulation of SWME for $N = 3$.

Hyperbolic regularization

Idea

- Change system matrix to obtain hyperbolicity
- Preserve structure and conservation of mass

SWME to HSWME [JK, Rominger, 2020]

- Linearization around $(h, u_m, \alpha_1, \alpha_2, \ldots, \alpha_M) = (h, u_m, \alpha_1, 0, \ldots, 0)$
- hyperbolic for all $N \in \mathbb{N}$

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_H(\boldsymbol{u}_M)\partial_x \boldsymbol{u}_M = \boldsymbol{S}(\boldsymbol{u}_M)$$

Example $M = 2$:

Variable vector

$$\boldsymbol{u}_M = \left(h, hu_m, h\alpha_1, h\alpha_2\right)^T \in \mathbb{R}^4$$

$$\boldsymbol{A}_H(\boldsymbol{u}_M) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} & 2u_m & \frac{2\alpha_1}{3} & 0 \\ -2\alpha_1 u_m & 2\alpha_1 & u_m & \frac{3\alpha_1}{5} \\ -\frac{2}{3}\alpha_1^2 & 0 & -\frac{\alpha_1}{3} & u_m \end{pmatrix}$$

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_H(\boldsymbol{u}_M)\partial_x \boldsymbol{u}_M = \boldsymbol{S}(\boldsymbol{u}_M)$$

general M:

Variable vector

$$\boldsymbol{u}_M = (h, hu_m, h\alpha_1, h\alpha_2)^T \in \mathbb{R}^{M+2}$$

$$\boldsymbol{A}_H(\boldsymbol{u}_M) = \begin{pmatrix} & 1 & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{pmatrix}$$

Hyperbolic Shallow Water Moment Equations [JK, Rominger, 2020]

Theorem

The eigenvalues of the system matrix $\boldsymbol{A}_H(\boldsymbol{u}_M) \in \mathbb{R}^{(M+2)\times(M+2)}$ *are the real numbers*

$$
\begin{aligned}
\lambda_{1,2} &= u_m \pm \sqrt{gh + \alpha_1^2} \\
\lambda_{i+2} &= u_m + c_i \cdot \alpha_1, \quad i = 1, \dots, M
\end{aligned}
$$

with $c_i \in \mathbb{R}$.
The HSWME system is thus globally hyperbolic.

Remarks:

- Analytical form of characteristic polynomial [JK, Rominger, 2020]
- General hyperbolicity proof [Huang, JK, Yong, 2022]
- Explicit characteristic polynomial [JK, submitted]

Hyperbolic Shallow Water Moment Equations [JK, Rominger, 2020]

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with $c_i \in \mathbb{R}$.
The HSWME system is thus globally hyperbolic.

Remarks:

- Analytical form of characteristic polynomial [JK, Rominger, 2020]
- General hyperbolicity proof [Huang, JK, Yong, 2022]
- Explicit characteristic polynomial [JK, submitted]

not the unique hyperbolic system!

SWME system matrix

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0,$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -u_m + gh - \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} & 2u_m & \frac{2}{3}\alpha_1 & \cdots & \frac{2}{2N+1}\alpha_N \\ -2u_m\alpha_1 - \sum_{j,k=1}^{N} A_{1jk}\alpha_j\alpha_k & 2\alpha_1 & & & \\ \vdots & \vdots & & \mathcal{A} & \\ -2u_m\alpha_N - \sum_{j,k=1}^{N} A_{Njk}\alpha_j\alpha_k & 2\alpha_N & & & \end{pmatrix},$$

with block matrix $\mathcal{A} \in \mathbb{R}^{N\times N}, \mathcal{A}_{i,l} = \sum_{j=1}^{N} \left(B_{ilj} + 2A_{ijl}\right)\alpha_j + u_m\delta_{i,l}$

with Kronecker delta $\delta_{i,j}$ and A, B, coefficients defined in [Torrilhon, Kowalski, 2019].

HSWME

- Linearization around $(h, hu_m, h\alpha_1, 0, \dots, 0)$

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}(\boldsymbol{u}_N)\partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\boldsymbol{A}(\boldsymbol{u}_N) = \begin{pmatrix} & 1 & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{pmatrix}$$

β-HSWME

- HSWME plus additional parameters for different eigenvalues

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}(\boldsymbol{u}_N)\partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\boldsymbol{A}(\boldsymbol{u}_N) = \begin{pmatrix} & 1 & & & & \\ -u_m^2 + gh - \frac{\alpha_1^2}{3} & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{2N^2-N-1}{2N^2+N-1}\alpha_1 & u_m \end{pmatrix}$$

SWLME

- Keep first 2 equations exactly; neglect other higher order products, $\alpha_i\alpha_j \approx 0$ for $i > 1$

$$\partial_t \boldsymbol{u}_N + \boldsymbol{A}(\boldsymbol{u}_N)\partial_x \boldsymbol{u}_N = \boldsymbol{S}(\boldsymbol{u}_N), \quad \boldsymbol{u}_N = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$$

$$\boldsymbol{A}(\boldsymbol{u}_N) = \begin{pmatrix} & 1 & & & & \\ -u_m^2 + gh - \sum\limits_{i=1}^{N} \frac{3\alpha_1^2}{2i+1} & 2u_m & \frac{2}{3}\alpha_1 & \dots & \dots & \frac{2}{2N+1}\alpha_N \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & & & \\ -2u_m\alpha_2 & 2\alpha_2 & & u_m & & \\ \vdots & \vdots & & & \ddots & \\ -2u_m\alpha_N & 2\alpha_N & & & & u_m \end{pmatrix}$$

SWLME hyperbolicity [Pimentel, JK, 2022]

Theorem

The eigenvalues of the system matrix $\boldsymbol{A}_L(\boldsymbol{u}_M) \in \mathbb{R}^{(M+2)\times(M+2)}$ are the real numbers

$$\begin{aligned}\lambda_{1,2} &= u_m \pm \sqrt{gh + \sum_{i=1}^{M} \frac{3\alpha_i^2}{2i+1}} \\ \lambda_{i+2} &= u_m, \quad i = 1, \ldots, M.\end{aligned}$$

The SWLME system is thus globally hyperbolic.

Primitive regularization [JK, submitted]

New idea:

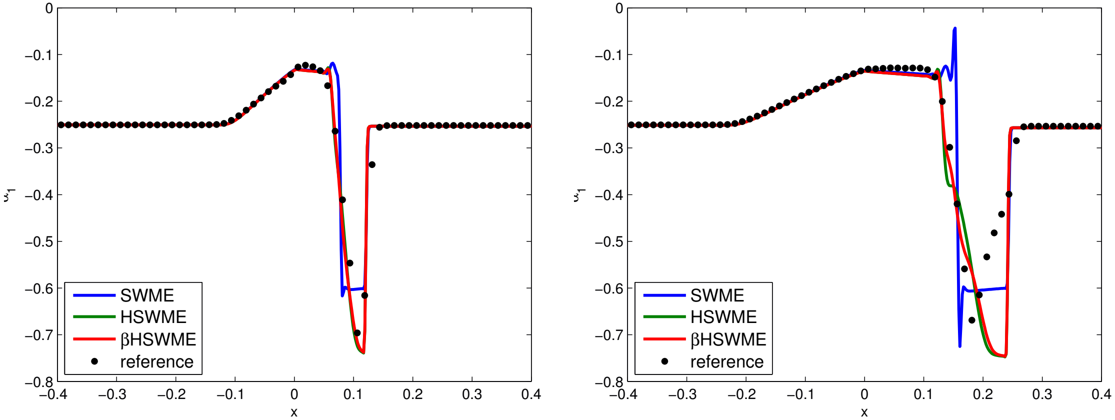
- trafo to primitive variables, linearize last M eqns $(h, u_m, \alpha_1, 0, \ldots, 0)$, trafo back

$$
A = \begin{pmatrix}
 & 1 & & & & & & \\
-u_m^2 + gh - \sum\limits_{i=1}^{N} \frac{\alpha_1^2}{2i+1} & 2u_m & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 & \cdots & \frac{2}{2i+1}\alpha_i & \cdots & \frac{2}{2N+1}\alpha_N \\
-2u_m - \frac{3}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & & & \\
-u_m\alpha_2 - \frac{4}{7}\alpha_1\alpha_3 - \frac{2}{3}\alpha_1^2 & \alpha_2 & \frac{1}{3}\alpha_1 & u_m & \ddots & & & \\
\vdots & \vdots & & \ddots & \ddots & \frac{i+1}{2i+1}\alpha_1 & & \\
-u_m\alpha_i - \frac{i-1}{2i-1}\alpha_1\alpha_{i-1} - \frac{i+1}{2i+1}\alpha_i\alpha_{i+1} & \alpha_i & & & \frac{i-1}{2i-1}\alpha_1 & u_m & \ddots & \\
\vdots & \vdots & & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\
-u_m\alpha_N - \frac{N-1}{2N-1}\alpha_1\alpha_N & \alpha_N & & & & & \frac{N-1}{2N-1}\alpha_1 & u_m
\end{pmatrix}
$$

Hyperbolic SWME models

- Hyperbolic Shallow Water Moment Equations [JK, Rominger, 2020]
- Shallow Water Linearized Moment Equations [JK, Pimentel-Garcia, 2022]
- Primitive variable regularization [JK, submitted]
- axisymmetric quasi-2D [Verbiest, JK, 2025] and 2D [Bauerle et al., 2025]

2.3 accuracy

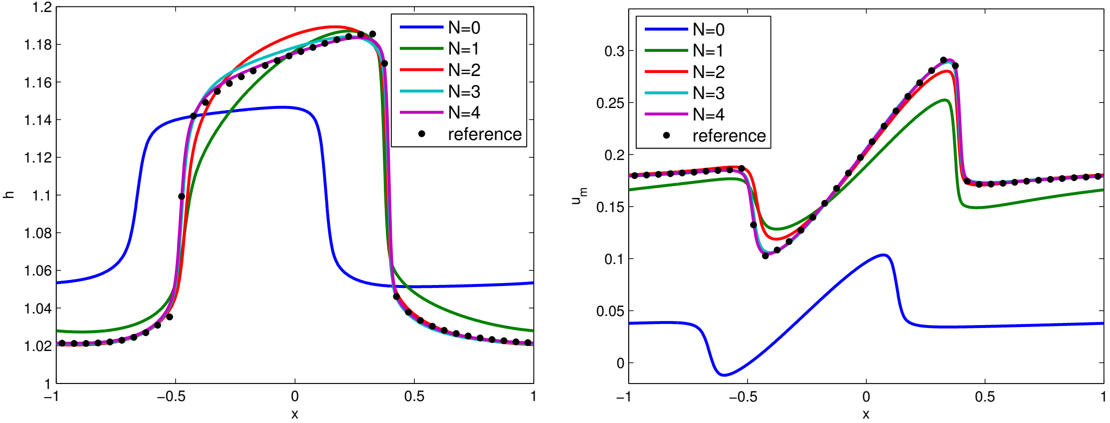


(a) α_1, $t = 0.05$

(b) α_1, $t = 0.1$

Figure: Now stable dam break simulation of HSWME, β-HSWME for $N = 3$.

Smooth test case, HSWME



(a) HSWME, h

(b) HSWME, u_m

Figure: Smooth test case for HSWME for varying N.

Smooth test case, convergence

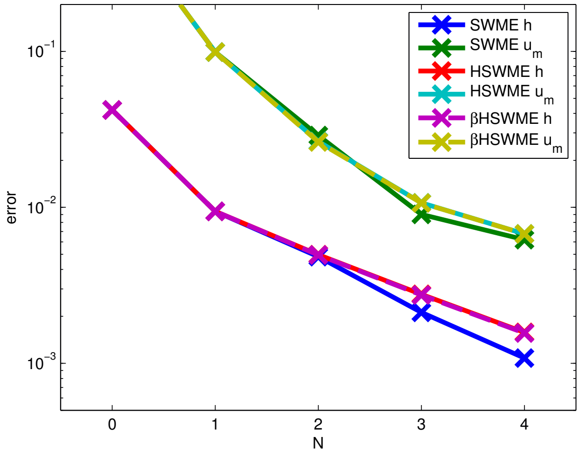
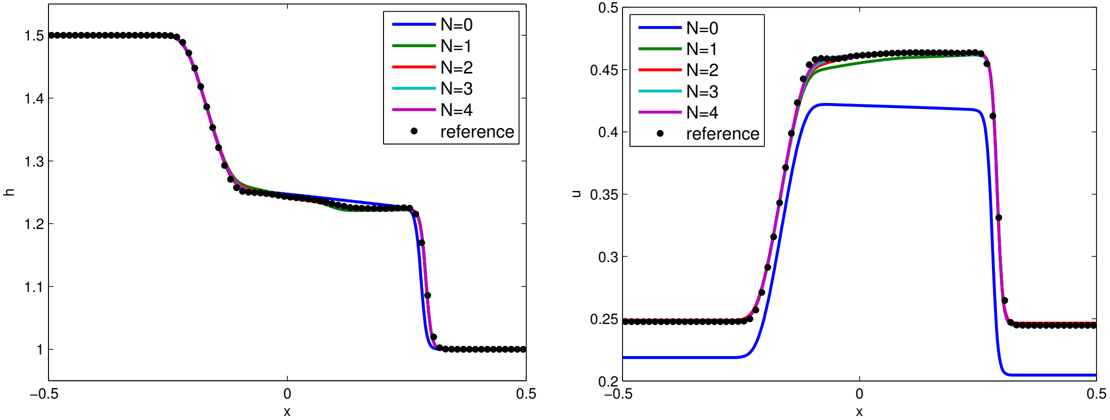


Figure: Error convergence of smooth test case.

Dam break test case, HSWME



(a) HSWME, h_m (b) HSWME, u_m

Figure: Dam break test case for HSWME for varying N.

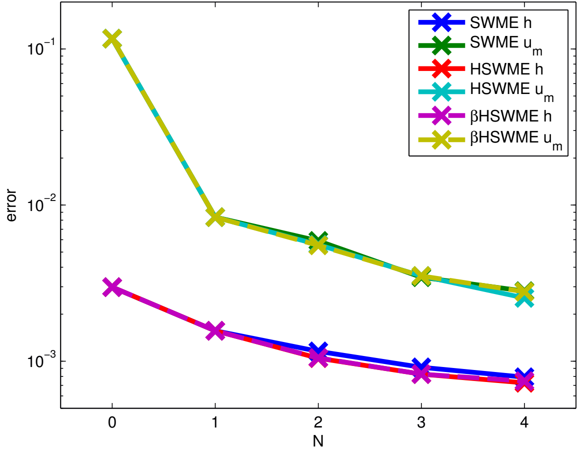


Figure: Error convergence of dam break test case.

2.4 stability

Linear stability

We consider the PDEs

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) + A\frac{\partial}{\partial x}\mathbf{u}(t,x) = -\frac{1}{\tau}B\mathbf{u}.$$

Definition

A PDE system is called linearly stable for a linearisation if possible wave solutions of the form $\mathbf{u}(t,x) = \mathbf{U}e^{-(\kappa x - \omega t)}$ are damped in time, i.e. $Im(\omega) < 0$.

We assume linearisation around some equilibrium or steady state and use a wave ansatz:

$$\mathbf{u}(t,x) = \mathbf{U}e^{-(\kappa x - \omega t)},$$

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) = -i\omega\mathbf{u}(t,x), \qquad \frac{\partial}{\partial x}\mathbf{u}(t,x) = i\kappa\mathbf{u}(t,x)$$

Example 1: Relaxation system

We consider the relaxation system

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) = -\frac{1}{\tau}B\mathbf{u}(t,x),$$

Inserting the wave ansatz:

$$-i\omega\mathbf{u}(t,x) = -\frac{1}{\tau}B\mathbf{u}(t,x),$$

$$\left(-\frac{i}{\tau}B - \omega I\right)\mathbf{u}(t,x) = 0.$$

ω is the solution of an eigenvalue problem:

$$\omega = \mathrm{EV}\left(-\frac{i}{\tau}B\right) = -\frac{i}{\tau}\mathrm{EV}(B).$$

For stability, the eigenvalues of B have to meet condition $Re(\mathrm{EV}(B)) > 0$

Example 2: Hyperbolic PDE System

We consider the hyperbolic PDE system

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) + A\frac{\partial}{\partial x}\mathbf{u}(t,x) = 0,$$

Inserting the wave ansatz:

$$-i\omega\mathbf{u}(t,x) + i\kappa A\mathbf{u}(t,x) = 0,$$

which in turn leads to the condition

$$(\kappa A - \omega I)\,\mathbf{u}(t,x) = 0.$$

Therefore, ω is the solution of an eigenvalue problem:

$$\omega = \mathrm{EV}(\kappa A) = \kappa \mathrm{EV}(A).$$

For stability, the eigenvalues of A have to all be real; otherwise complex conjugated unstable eigenvalues would exist. This leads to the condition $Im(\mathrm{EV}(A)) = 0$, i.e., hyperbolicity.

Example 3: Hyperbolic Relaxation System

We consider the hyperbolic relaxation system

$$\frac{\partial}{\partial t}\mathbf{u}(t,x) + A\frac{\partial}{\partial x}\mathbf{u}(t,x) = -\frac{1}{\tau}B\mathbf{u}.$$

Inserting the wave ansatz:

$$-i\omega\mathbf{u}(t,x) + i\kappa\mathbf{u}(t,x) = -\frac{1}{\tau}B\mathbf{u}(t,x),$$

which in turn leads to the condition

$$\left(\kappa A - \frac{i}{\tau}B - \omega I\right)\mathbf{u}(t,x) = 0.$$

Therefore, ω is the solution of an eigenvalue problem:

$$\omega = \mathrm{EV}\left(\kappa A - \frac{i}{\tau}B\right).$$

stability is not clear a priori.

2.4 equilibria

Equilibrium manifolds of Shallow Water Moment Equations

Model:

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_M \partial_x \boldsymbol{u}_M = \boldsymbol{S}(\boldsymbol{u}_M), \quad \boldsymbol{u}_M \in \mathbb{R}^{M+2}$$

$$\boldsymbol{u}_M = (h, hu_m, h\alpha_1, h\alpha_2, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2}$$

Friction term:

$$S = -\frac{\nu}{\lambda}\begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}, \quad S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{N}\alpha_j\right) - \frac{\nu}{h}\sum_{j=1}^{N} C_{ij}\alpha_j$$

Newtonian fluid: slip length λ and viscosity ν

Definition (Equilibrium manifold)

Friction terms vanish in equilibrium: $\mathcal{E} = \{\boldsymbol{u}_M : \boldsymbol{S}(\boldsymbol{u}_M) = \boldsymbol{0}\}$

Water-at-rest equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M} \alpha_j\right) - \frac{\nu}{h}\sum_{j=1}^{M} C_{ij}\alpha_j$$

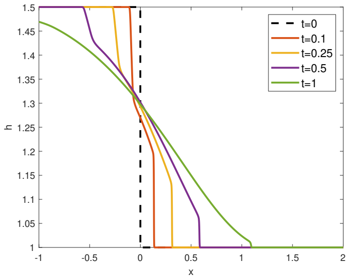
Water-at-rest is in equilibrium

$$\mathcal{E} = \{\boldsymbol{u}_M : u_m = \alpha_1 = \ldots = \alpha_M = 0\}$$

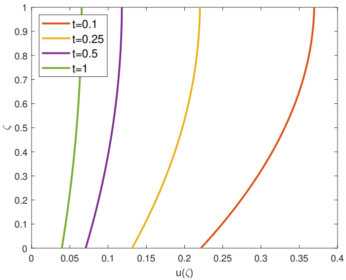
$$\Rightarrow u(t,x,z) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x)\phi_i\left(\frac{z-h_b}{h}\right) = 0$$

Water-at-rest convergence for $\lambda = 1$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the water-at-rest equilibrium with time

Constant-velocity equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M}\alpha_j\right) - \frac{\nu}{h}\sum_{j=1}^{M} C_{ij}\alpha_j$$

If $\lambda \gg h$ (perfect slip limit)

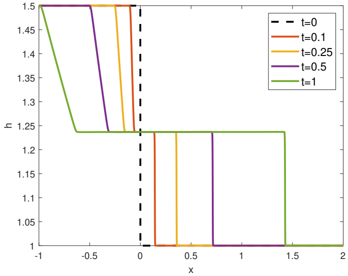
$$S_i = -\frac{\nu}{h}\sum_{j=1}^{M} C_{ij}\alpha_j$$

Constant-velocity is in equilibrium

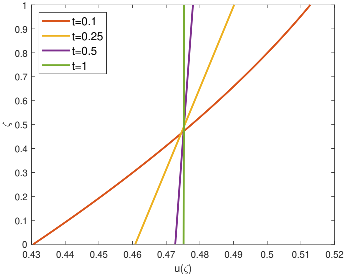
$$\mathcal{E} = \{\boldsymbol{u}_M : \alpha_1 = \ldots = \alpha_N = 0\}$$

$$\Rightarrow u(t,x,z) = u_m(t,x) + \sum_{i=1}^{M}\alpha_i(t,x)\phi_i\left(\frac{z-h_b}{h}\right) = u_m(t,x)$$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the constant-velocity equilibrium with time

Bottom-at-rest equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M} \alpha_j\right) - \frac{\nu}{h}\sum_{j=1}^{M} C_{ij}\alpha_j$$

If $\lambda \ll h$ (no-slip limit)

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M} \alpha_j\right)$$

Bottom-at-rest is in equilibrium

$$\mathcal{E} = \{\boldsymbol{u}_M : u_m + \sum_{j=1}^{M} \alpha_j = 0\}$$

Bottom-at-rest equilibrium

Friction term:

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M}\alpha_j\right) - \frac{\nu}{h}\sum_{j=1}^{M} C_{ij}\alpha_j$$

If $\lambda \ll h$ (no-slip limit)

$$S_i = -\frac{\nu}{\lambda}(2i+1)\left(u_m + \sum_{j=1}^{M}\alpha_j\right)$$

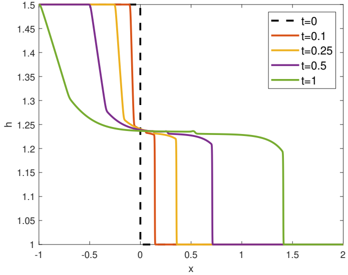
Bottom-at-rest is in equilibrium

$$\mathcal{E} = \{\boldsymbol{u}_M : u_m + \sum_{j=1}^{M}\alpha_j = 0\}$$

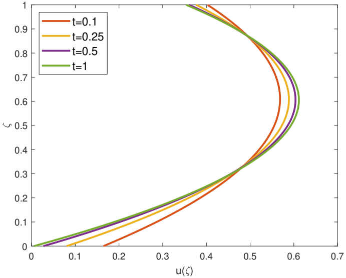
$$\Rightarrow u(t,x,h_b) = u_m(t,x) + \sum_{i=1}^{M}\alpha_i(t,x)\phi_i\left(\frac{h_b - h_b}{h}\right) = u_m(t,x) + \sum_{i=1}^{M}\alpha_i(t,x) = 0$$

Bottom-at-rest convergence for $\lambda = 10^{-3}$

water height h



velocity profile $u(z)$, $x = 0$



Model is converging to the bottom-at-rest equilibrium with time

Equilibrium stability analysis [Yong, 1999]

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_M \partial_x \boldsymbol{u}_M = \boldsymbol{S}\left(\boldsymbol{u}_M\right)$$

Equilibrium stability

- system is stable for small perturbation around equilibrium
- relaxation back towards equilibrium
- instabilities may or may not cause numerical problems

Structural stability conditions [Yong, 1999]

(**I**): For any $U \in \mathcal{E}$, the Jacobian $S_U(U)$ can be manipulated by an invertible $n \times n$ matrix $P = P(U)$ and an invertible $r \times r$ $(0 < r \leq n)$ matrix $\hat{T}(U)$ such that

$$P(U)S_U(U) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{T}(U) \end{bmatrix} P(U), \quad \forall\ U \in \mathcal{E}.$$

(**II**): There exists a positive definite symmetrizer $A_0 = A_0(U)$ of the coefficient matrix $A(U)$ such that

$$A_0(U)A(U) = A^T(U)A_0(U), \quad \forall\ U \in G.$$

(**III**): On the equilibrium manifold $\mathcal{E}$, the coefficient matrix and the source term are coupled as

$$A_0(U)S_U(U) + S_U^T(U)A_0(U) \preceq -P^T(U) \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} P(U), \quad \forall\ U \in \mathcal{E}.$$

Equilibrium stability analysis

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_M \partial_x \boldsymbol{u}_M = \boldsymbol{S}\left(\boldsymbol{u}_M\right)$$

Structural stability conditions [Yong, 1999]

1. source term jacobian is invertible
2. transport term is hyperbolic
3. coupling between source and transport term

Equilibrium stability analysis of SWME [Huang et al., 2022]

1. water-at-rest is stable
2. constant-velocity is stable
3. bottom-at-rest can be *unstable*

Equilibrium stability analysis

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_M \partial_x \boldsymbol{u}_M = \boldsymbol{S}\left(\boldsymbol{u}_M\right)$$

Structural stability conditions [Yong, 1999]

1. source term jacobian is invertible
2. transport term is hyperbolic
3. coupling between source and transport term

Equilibrium stability analysis of SWME [Huang et al., 2022]

1. water-at-rest is stable
2. constant-velocity is stable
3. bottom-at-rest can be *unstable*

We observed no instabilities in numerical simulations

2.6 steady states

Steady states of Shallow water equations

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(flat bottom $\partial_x b = 0$ and zero friction); the steady state fulfills

$$\partial_x \left(hu_m\right) = 0, \quad \partial_x \left(hu_m^2 + \frac{1}{2}gh^2\right) = 0.$$

Rankine-Hugoniot conditions from a given state $(h_0, h_0 u_{m,0})$ to a state (h, hu_m):

$$\frac{h}{h_0} = -\frac{1}{2} + \frac{1}{2} \cdot \sqrt{1 + 8Fr^2},$$

where Fr is the Froude number for the given state defined by

$$Fr = \frac{u_{m,0}}{\sqrt{gh_0}}.$$

Steady states of Shallow water moment equations ($M = 1$)

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \begin{pmatrix} 0 \\ gh\partial_x b \\ 0 \end{pmatrix} - \frac{\nu}{\lambda} P,$$

For flat bottom $\partial_x b = 0$ and zero friction, the steady state fulfills

$$\begin{aligned} \partial_x (hu_m) &= 0, \\ \partial_x \left(hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha^2 \right) &= 0, \\ \partial_x (2hu_m\alpha) &= u_m \partial_x (h\alpha), \end{aligned}$$

We obtain $hu_m = const$ and $hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha^2 = const$ and

$$u_m = 0 \quad \text{or} \quad \frac{\alpha}{h} = const.$$

Steady states of Shallow water moment equations ($M = 1$)

Rankine-Hugoniot conditions from a given state $(h_0, h_0 u_{m,0}, h_0\alpha_0)$ to a state $(h, hu_m, h\alpha)$:

$$(h - h_0)\left[-\frac{u_{m,0}^2}{gh_0} + \frac{1}{2}\left(\left(\frac{h}{h_0}\right)^2 + \left(\frac{h}{h_0}\right)\right) + \frac{1}{3}\frac{\alpha_0^2}{gh_0}\left(\left(\frac{h}{h_0}\right)^3 + \left(\frac{h}{h_0}\right)^2 + \left(\frac{h}{h_0}\right)\right)\right] = 0.$$

dimensionless flow numbers:

$$Fr = \frac{u_{m,0}}{\sqrt{gh_0}}, \qquad M\alpha = \frac{\alpha_0}{u_{m,0}},$$

use $y = \frac{h}{h_0}$:

$$h = h_0 \quad \vee \quad -Fr^2 + \frac{1}{2}\left(y^2 + y\right) + \frac{1}{3}M\alpha^2 Fr^2\left(y^3 + y^2 + y\right) = 0.$$

third order polynomial with two parameters Fr and $M\alpha$.

Steady states of Shallow water moment equations ($M = 2$)

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix}$$

No analytical solution possible.

Linearised SWME [Pimentel, JK, 2022] (example: $M = 2$)

$$
\boldsymbol{A}(\boldsymbol{u}_M) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
gh - u_m^2 - \frac{\alpha_1^2}{3} - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\
-2\alpha_1 u_m - \frac{4}{5}\alpha_1\alpha_2 & 2\alpha_1 & u_m + \alpha_2 & \frac{3\alpha_1}{5} \\
-\frac{2}{3}\alpha_1^2 - 2u_m\alpha_2 - \frac{2}{7}\alpha_2^2 & 2\alpha_2 & -\frac{\alpha_1}{3} & u_m + \frac{3\alpha_2}{7}
\end{pmatrix}
$$

SWLME idea:

In higher-order equations, assume near equilibrium: $\alpha_i = \mathcal{O}(\epsilon)$, neglect terms $\mathcal{O}(\epsilon^2)$

$$
\boldsymbol{A}_L(\boldsymbol{u}_M) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{\alpha_1^2}{3} - u_m^2 + gh - \frac{\alpha_2^2}{5} & 2u_m & \frac{2\alpha_1}{3} & \frac{2\alpha_2}{5} \\
-2u_m\alpha_1 & 2\alpha_1 & u_m & 0 \\
-2u_m\alpha_2 & 2\alpha_2 & 0 & u_m
\end{pmatrix}
$$

SWLME [Pimentel, JK, 2022]

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \ldots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & u_m & & \\ & & & \ddots & \\ & & & & u_m \end{pmatrix} \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix}$$

Alternatively:

$$\partial_t h + \partial_x (hu_m) = 0,$$

$$\partial_t (hu_m) + \partial_x \left(hu_m^2 + h\sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} + \frac{1}{2}gh^2 \right) = -gh\partial_x b,$$

$$\partial_t (h\alpha_i) + \partial_x (2hu_m\alpha_i) = u_m \partial_x (h\alpha_i),$$

SWLME vs SWME

SWLME

$$\partial_t h + \partial_x \left(h u_m\right) = 0,$$

$$\partial_t(h u_m) + \partial_x \left(h u_m^2 + h \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} + \frac{1}{2} g h^2 \right) = -g h \partial_x b,$$

$$\partial_t(h\alpha_i) + \partial_x(2 h u_m \alpha_i) = u_m \partial_x(h\alpha_i),$$

SWME

$$\partial_t h + \partial_x \left(h u_m\right) = 0,$$

$$\partial_t(h u_m) + \partial_x \left(h u_m^2 + \sum_{i=1}^{N} \frac{h\alpha_i^2}{2i+1} + \tfrac{1}{2} g h^2 \right) = -g h \partial_x b,$$

$$\partial_t(h\alpha_i) + \partial_x \left(2 h u_m \alpha_i + \mathfrak{A}_i \right) = u_m \partial_x \left(h\alpha_i\right) - \mathfrak{B}_i,$$

SWME

$$\partial_t h + \partial_x (h u_m) = 0,$$

$$\partial_t (h u_m) + \partial_x \left(h u_m^2 + \sum_{i=1}^{N} \frac{h\alpha_i^2}{2i+1} + \tfrac{1}{2} g h^2 \right) = -gh\partial_x b,$$

$$\partial_t (h\alpha_i) + \partial_x \left(2hu_m\alpha_i + \mathfrak{A}_i \right) = u_m \partial_x (h\alpha_i) - \mathfrak{B}_i,$$

where $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are

$$\mathfrak{A}_i = h \sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k, \qquad \mathfrak{B}_i = \sum_{j,k=1}^{N} B_{ijk}\alpha_k\partial_x (h\alpha_j),$$

and the coefficients are

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k d\zeta, \qquad B_{ijk} = (2i+1)\int_0^1 \phi_i' \left(\int_0^\zeta \phi_j d\zeta \right) \phi_k d\zeta.$$

SWLME steady states

$$
\begin{aligned}
\partial_x \left(h u_m \right) &= 0 \\
\partial_x \left(h u_m^2 + \frac{1}{2} g h^2 + \frac{1}{3} h \alpha_1^2 + \ldots + \frac{1}{2N+1} h \alpha_N^2 \right) &= 0 \\
\partial_x \left(2 h u_m \alpha_i \right) &= u_m \partial_x \left(h \alpha_i \right)
\end{aligned}
$$

$$
\begin{aligned}
h u_m &= const, \\
u_m = 0 \text{ or } \frac{\alpha_i}{h} &= const, \text{ for } i = 1, \ldots, N.
\end{aligned}
$$

dimensionless flow numbers for each variable and writing $y = \frac{h}{h_0}$

$$
Fr = \frac{u_{m,0}}{\sqrt{g h_0}}, \qquad (M\alpha)_i = \frac{\alpha_{i,0}}{u_{m,0}}, \quad \text{for } i = 1, \ldots, N,
$$

$$
h = h_0 \ \vee \ -Fr^2 + \frac{1}{2} \left(y^2 + y \right) + \sum_{i=1}^{N} \frac{1}{2i+1} \left(M\alpha \right)_i^2 Fr^2 \left(y^3 + y^2 + y \right) = 0.
$$

2.7 entropy

Entropy equation for Shallow Water Equations

From standard Shallow Water Equations

$$\begin{aligned}
(\mathrm{C}): \quad & \partial_t h + \partial_x (h u_m) = 0 \\
(\mathrm{M1}): \quad & \partial_t (h u_m) + \partial_x \left(h u_m^2 + \frac{1}{2} g h^2 \right) = -g h \partial_x b.
\end{aligned}$$

Derive energy equation

$$(\mathrm{E}): \quad \partial_t \left(\frac{h u_m^2}{2} + g \frac{h^2}{2} + g h b \right) + \partial_x \left(\frac{h u_m^3}{2} + g h u_m (h + b) \right) = 0$$

where the total energy $\frac{1}{2} h u_m^2 + \frac{1}{2} g h^2 + g h b$ is the entropy and $\frac{h u_m^3}{2} + g h u_m (h + b)$ is the entropy flux.

Derivation of entropy equation for Shallow Water Equations (1)

standard Shallow Water Equations

$$\begin{aligned} \text{(C)}: & \qquad \partial_t h + \partial_x \left(h u_m \right) = 0 \\ \text{(M1)}: & \qquad \partial_t \left(h u_m \right) + \partial_x \left(h u_m^2 + \frac{1}{2} g h^2 \right) = -g h \partial_x b. \end{aligned}$$

modified momentum balance

$$\text{(M2)}: \qquad \partial_t \left(h u_m \right) + \partial_x \left(h u_m^2 \right) + g h \partial_x (h + b) = 0.$$

$\text{(M2)} - u_m \cdot \text{(C)}$ to get

$$\text{(A)}: \qquad h \partial_t u_m + h u_m \partial_x u_m + g h \partial_x (h + b) = 0.$$

Derivation of entropy equation for Shallow Water Equations (2)

modified momentum balance

$$(\mathrm{M2}): \qquad \partial_t \left(hu_m\right) + \partial_x \left(hu_m^2\right) + gh\partial_x(h+b) = 0.$$

$(\mathrm{M2}) - u_m \cdot (\mathrm{C})$ to get

$$(\mathrm{A}): \qquad h\partial_t u_m + hu_m\partial_x u_m + gh\partial_x(h+b) = 0.$$

average $(\mathrm{S}) = \frac{1}{2}(\mathrm{A}) + \frac{1}{2}(\mathrm{M2})$

$$(\mathrm{S}): \qquad \frac{1}{2}\Big(\partial_t(hu_m) + h\partial_t u_m\Big) + \frac{1}{2}\Big(\partial_x \left(hu_m^2\right) + hu_m\partial_x u_m\Big) + gh\partial_x(h+b) = 0.$$

Derivation of entropy equation for Shallow Water Equations (3)

$$(\mathrm{S}): \qquad \frac{1}{2}\Big(\partial_t(hu_m) + h\partial_t u_m\Big) + \frac{1}{2}\Big(\partial_x\left(hu_m^2\right) + hu_m\partial_x u_m\Big) + gh\partial_x(h+b) = 0.$$

equation for the kinetic energy (K) by multiplying u_m to (S), then product rule

$$(\mathrm{K}): \qquad \partial_t\left(\frac{hu_m^2}{2}\right) + \partial_x\left(\frac{hu_m^3}{2}\right) + ghu_m\partial_x(h+b) = 0,$$

where the term $k = \frac{1}{2}hu_m^2$ is the kinetic energy.

standard Shallow Water Equations

$$(\mathrm{C}): \qquad \partial_t h + \partial_x (h u_m) = 0$$

Compute $g(h+b)\cdot(\mathrm{C})$:

$$(\mathrm{P}): \qquad \partial_t \left(g\frac{h^2}{2} + ghb \right) + g(h+b)\partial_x(hu_m) = 0,$$

where $p = \frac{1}{2}gh^2 + ghb$ denotes the potential energy.

Derivation of entropy equation for Shallow Water Equations (5)

$$(\mathrm{P}): \qquad \partial_t \left(g\frac{h^2}{2} + ghb \right) + g(h+b)\partial_x(hu_m) = 0,$$

$$(\mathrm{K}): \qquad \partial_t \left(\frac{hu_m^2}{2} \right) + \partial_x \left(\frac{hu_m^3}{2} \right) + ghu_m\partial_x(h+b) = 0,$$

$(\mathrm{E}) = (\mathrm{P}) + (\mathrm{K})$ is the total energy equation

$$(\mathrm{E}): \qquad \partial_t \left(\frac{hu_m^2}{2} + g\frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{hu_m^3}{2} + ghu_m(h+b) \right) = 0$$

where the total energy is $k + p = \frac{1}{2}hu_m^2 + \frac{1}{2}gh^2 + ghb$

Derivation of entropy equation for SWLME

SWLME

$$
\begin{aligned}
\partial_t h + \partial_x (h u_m) &= 0,\\
\partial_t (h u_m) + \partial_x \left(h u_m^2 + h \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} + \frac{1}{2} g h^2 \right) &= -g h \partial_x b,\\
\partial_t (h\alpha_i) + \partial_x (2 h u_m \alpha_i) &= u_m \partial_x (h\alpha_i),
\end{aligned}
$$

$$
\partial_t \left(\frac{h u_m^2}{2} + \frac{h}{2} \sum_{i=1}^{N} \frac{\alpha_i^2}{2i+1} + g\frac{h^2}{2} + ghb \right) + \partial_x \left(\frac{h u_m^3}{2} + \frac{3 h u_m}{2} \sum_{i=1}^{N} \frac{\alpha_i^2}{2i+1} + g h u_m (h+b) \right) = 0,
$$

where we denote the total energy by

$$
e = k_\alpha + p = \frac{h u_m^2}{2} + \frac{h}{2} \sum_{i=1}^{N} \frac{\alpha_i^2}{2i+1} + g\frac{h^2}{2} + ghb.
$$

Derivation of entropy equation for SWME (1)

SWME

$$\partial_t h + \partial_x (h u_m) = 0,$$
$$\partial_t (h u_m) + \partial_x \left(h u_m^2 + \sum_{i=1}^{N} \frac{h \alpha_i^2}{2i+1} + \frac{1}{2} g h^2 \right) = -g h \partial_x b,$$
$$\partial_t (h \alpha_i) + \partial_x \left(2 h u_m \alpha_i + \mathfrak{A}_i \right) = u_m \partial_x (h \alpha_i) - \mathfrak{B}_i,$$

where $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are

$$\mathfrak{A}_i = h \sum_{j,k=1}^{N} A_{ijk} \alpha_j \alpha_k, \qquad \mathfrak{B}_i = \sum_{j,k=1}^{N} B_{ijk} \alpha_k \partial_x (h \alpha_j),$$

(SWLME is SWME with $\mathfrak{A}_i = \mathfrak{B}_i = 0$.)

SWME

$$\partial_t h + \partial_x \left(h u_m \right) = 0,$$

$$\partial_t (h u_m) + \partial_x \left(h u_m^2 + \sum_{i=1}^{N} \frac{h \alpha_i^2}{2i+1} + \frac{1}{2} g h^2 \right) = -g h \partial_x b,$$

$$\partial_t (h \alpha_i) + \partial_x \left(2 h u_m \alpha_i + \mathfrak{A}_i \right) = u_m \partial_x \left(h \alpha_i \right) - \mathfrak{B}_i,$$

$$\partial_t \left(\frac{h u_m^2}{2} + \frac{h}{2} \sum_{i=1}^{N} \frac{\alpha_i^2}{2i+1} + g \frac{h^2}{2} + g h b \right) + \partial_x \left(\frac{h u_m^3}{2} + \frac{3 h u_m}{2} \sum_{i=1}^{N} \frac{\alpha_i^2}{2i+1} + g h u_m (h + b) + \widehat{\mathcal{Q}} \right) = 0,$$

with

$$\widehat{\mathcal{Q}} = \sum_{i,j,k=1}^{N} \left(\widetilde{A}_{ijk} + \widetilde{B}_{ijk} \right) h \alpha_i \alpha_j \alpha_k, \quad \widetilde{A}_{ijk} = \frac{A_{ijk}}{2i+1}, \quad \widetilde{B}_{ijk} = \frac{B_{ijk}}{2i+1}$$

and the same (!) total energy

summary

Part 2 Summary

1 repetition

- Shallow Water Moment Models

2 analysis

- conservation
- hyperbolicity
- stability and equilibria
- steady states
- entropy