703 Problem Set 4

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26. To prove that $f : X \to \mathbb{R}$ is continuous, we want to show that for any $x, y \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $d(x, y) < \delta$.

    Since d is a metric, by triangle inequality,

    $$\begin{aligned} d(a,x) &\leq d(a,y) + d(x,y) \\ d(a,x) - d(a,y) &\leq d(x,y) \end{aligned}$$

    Similarly,

    $$\begin{aligned} d(a,y) &\leq d(a,x) + d(x,y) \\ d(a,y) - d(a,x) &\leq d(x,y) \end{aligned}$$

    Since $d(a,x) - d(a,y) \leq d(x,y)$ and $d(a,y) - d(a,x) \leq d(x,y)$ both hold, we have

    $$|d(a,x) - d(a,y)| \leq d(x,y)$$

    Since $f(x) = d(a,x)$, the above inequality is equivalent to $|f(x) - f(y)| \leq d(x,y)$.

    Thus, for $d(x,y) < \delta$, set $\varepsilon = \delta$, then $\forall \varepsilon > 0$, $d(x,y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$, proving that $f(x)$ is continuous.