

## 703 Problem Set 4

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26. To prove that  $f : X \rightarrow \mathbb{R}$  is continuous, we want to show that for any  $x, y \in X$ ,  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  if  $d(x, y) < \delta$ .

Since  $d$  is a metric, by triangle inequality,

$$\begin{aligned}d(a, x) &\leq d(a, y) + d(x, y) \\d(a, x) - d(a, y) &\leq d(x, y)\end{aligned}$$

Similarly,

$$\begin{aligned}d(a, y) &\leq d(a, x) + d(x, y) \\d(a, y) - d(a, x) &\leq d(x, y)\end{aligned}$$

Since  $d(a, x) - d(a, y) \leq d(x, y)$  and  $d(a, y) - d(a, x) \leq d(x, y)$  both hold, we have

$$|d(a, x) - d(a, y)| \leq d(x, y)$$

Since  $f(x) = d(a, x)$ , the above inequality is equivalent to  $|f(x) - f(y)| \leq d(x, y)$ .

Thus, for  $d(x, y) < \delta$ , set  $\varepsilon = \delta$ , then  $\forall \varepsilon > 0$ ,  $d(x, y) < \delta$  implies  $|f(x) - f(y)| < \varepsilon$ , proving that  $f(x)$  is continuous.