Lec 3*: Probability

1 Probability

1.1 Sample Space, Outcome, and Event

- Say that we roll a six-sided die:
  - The roll could be 1, 2, 3, 4, 5, or 6
  - The set containing all possible rolls is defined as the **sample space**.
  - Each possible roll is an **outcome**.
    - Outcomes in a sample space should be **exhaustive** – all possible outcomes must be included in the sample space.
    - Outcomes in a sample space should be **mutually exclusive** – no two outcomes can occur at the same time.
  - An **event** is a collection of one or more simple events (i.e. outcomes) in a sample space.

Exercise. Is the following true or false? Why?

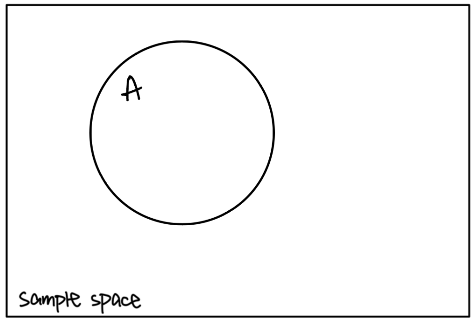
1. The set $\{\text{rolling a } 1, \text{rolling a } 2, \text{rolling a } 3, \text{rolling a } 4\}$ describes the sample space from a fair, six-sided die roll.
2. "Rolling a 3" is an event.
3. "Rolling an even number" is an event.

1.2 How is Probability Calculated?

- Let's continue with the dice rolling example.
  - The sample space is {rolling a 1, rolling a 2, rolling a 3, rolling a 4, rolling a 5, rolling a 6}
- Say that event A is as simple as "Rolling a 1".
- How do we calculate $P(A)$?

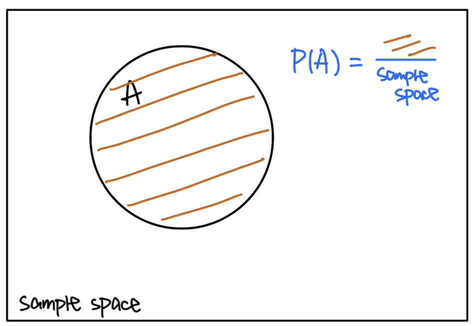
*Some exercise questions are taken from or slightly modified based on Dr. Gregory Pac's Econ 310 discussion handout.

- It seems pretty straightforward in this example: "Rolling a 1" is one out of 6 possible outcomes. In reality, if we keeping rolling a die for a large number of times (say, 6 million times), we will see the number 1 being the outcome very close to 1 million times.
- Thus, $P(A) = \frac{1}{6}$
- **To generalize this**, let's visualize what happened here:



Probability of event A is how often A happens, relatively to all the possible outcomes in the sample space. Thus,

$$P(A) = \frac{\text{Area of A}}{\text{Area of the entire sample space (rectangle)}}$$

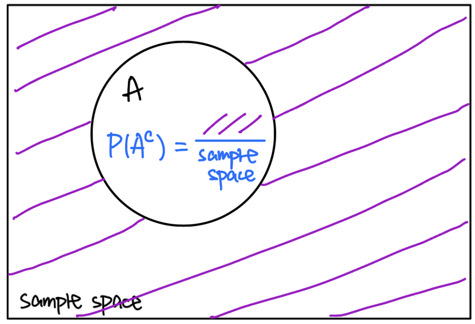


- Sometimes, instead of finding the probability of event A, we need the probability of **the opposite of** A.
  - Call this opposite event A^C, where the superscript C stands for complement.
  - To find $P(A^C)$, notice first that

    $$P(A) + P(A^C) = 1$$

    Thus,

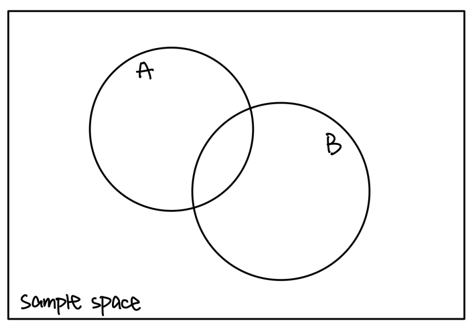
    $$P(A^C) = 1 - P(A) = \frac{\text{Area outside of A}}{\text{Area of the entire sample space (rectangle)}}$$



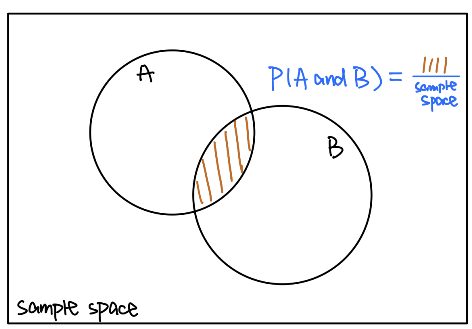
- Some general property of probability:
  - $0 \le P(A) \le 1$ for any event A
  - Adding up all possible, mutually exclusive events from the sample space must yield a sum of 1
  - **Complement rule**: $P(A^C) = 1 - P(A)$
  - **Addition rule**: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

1.3 Joint vs. Conditional Probability

- Say that there are two events, denoted by A and B.



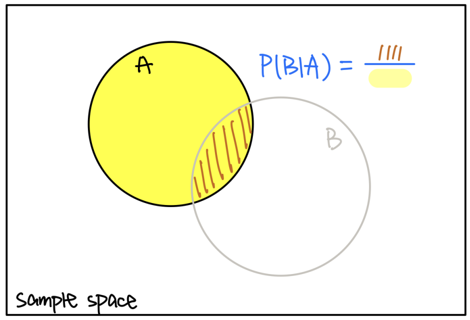
- If one is interested in the probability that event A and B occurs together, then



$$P(A \text{ and } B) = \frac{\text{Area of the line shaded space}}{\text{Area of the entire sample space (rectangle)}}$$

This is because **joint** probability looks at case where event A and B occur together with respect to all the possible outcomes (i.e. the sample space).

- Sometimes though, one is interested in the probability of B occuring conditional on A already occurs. This is depicted as



$$P(B|A) = \frac{\text{Area of the line shaded space}}{\text{Area of event } A} = \frac{P(A \text{ and } B)}{P(A)}$$

This is because **conditional** probability of $P(B|A)$ looks at case where event B occurs with respect to A already occuring.

- One way to visualize the relationship between joint and conditional probability is through the tree diagram (will see this in exercise later).
- Conditional probability gives rise to **Bayes Law**:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\underbrace{P(B|A)P(A) + P(B|A^C)P(A^C)}_{P(A \text{ and } B) + P(A^C \text{ and } B)}}$$

1.4 Mutually Exclusive vs. Independence

- Two events are **mutually exclusive** if they cannot occur at the same time.

  e.g. When rolling a fair six-sided die, "Rolling a 3" and "Rolling a 4" are mutually exclusive, since you cannot roll both a 3 and a 4 at the same time.

- Two events are **independent** if one event is not impacted by the other.

  e.g. When rolling a fair six-sided die, "Rolling a 3 in the first round" and "Rolling a 3 in the second round" are independent, since the first and second roll does not influence each other.

  - This is why if A and B are independent, $P(B|A) = P(B)$, and $P(A|B) = P(A)$, since conditional on event A does not give us any new information regarding event B, and conditional on event B does not give us any new information regarding event A.

- When looking at joint probability between A and B,
  - If A and B are mutually exclusive, $P(A \text{ and } B) =$
  - If A and B are independent, $P(A \text{ and } B) =$

Exercise. Are the following two events mutually exclusive, independent, both, or neither?

1. "Weather today in Madison at 11:05am is sunny" vs. "Weather today in Madison at 11:05am is rainy"
2. "Roll a (fair six-sided) die and get an even number" vs. "Roll a (fair six-sided) die and get 4"
3. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 4"
4. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 10"

2 Exercises

1. You go to Vegas to play craps; luckily, this class has prepared you to solve the following problems:

   (a) Assuming you rolled a single die and got an even number, what's the probability the number on that die is a two?

(b) Rolling two dice, what's the probability of two sixes? What's the probability of no sixes?

(c) Rolling two dice, what's the probability both dice are even? What's the probability either die is even?

(d) Rolling two dice, what's the probability they sum to seven?

2. Consider the following joint probability table:

	A	A^C
B	.15	.25
B^C	.40	.20

(a) What is $P(A \text{ and } B)$? What is $P(A)$? What is $P(B)$?

(b) What is $P(A \text{ or } B)$?

(c) What is $P(A|B)$? What is $P(B|A)$?

(d) Are A and B independent?

3. Suppose exactly half the population is male. 5% of males and only 0.25% of females are color blind.

   (a) Draw a tree diagram to illustrate the sample space.

   (b) A person is chosen at random and that person is color blind. What's the probability that the person is a male?

4. Marie is getting married tomorrow at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 20% of the time. Assume there are 365 days in a year.

   (a) Draw a tree diagram to illustrate the sample space.

   (b) What is the probability it will rain tomorrow for Marie's wedding?