Supplementary Handout for Dis 5: Random Variables; Discrete Probability Distributions

1 Random Variables

- A random variable **assigns a number to each outcome** of an experiment.

  e.g. Let X be a random variable recording the outcome of rolling a fair, six-sided die, then

$$\begin{aligned} \text{Rolling a 1} &\rightarrow X = 1 \\ \text{Rolling a 2} &\rightarrow X = 2 \\ &\cdots \\ \text{Rolling a 6} &\rightarrow X = 6 \end{aligned}$$

- There are two types of random variables:
  1. **Discrete random variable**: numbers assigned to the random variable are countable.
  2. **Continuous random variable**: numbers assigned to the random variable are NOT countable.

Aside: what does it mean to be countable?

(A pretty loose definition:) as long as you can sequentially count all the numbers assigned – even though it might take forever – then such series of numbers is considered as countable.

Exercise. Is the following random variable discrete or continuous?

1. $X =$ whether the result from a fair coin flip is head or not

   X is a discrete random variable. When the outcome is head, $X = 1$; when the outcome is not head (i.e. tail), $X = 0$. These two numbers are certainly countable.
2. $X =$ amount of time it takes for a student to complete a 60-minute exam

   X is a continuous random variable. The outcome is assigned to anything between 0 minute and 60 minutes, but if someone tries to count all possible time between 0 and 60 minutes, they won't be able to.

   More explicitly, say that someone wants to count from 0 to 60 minutes with 0.01 minute increment. However, there are time between 0.00 minute and 0.01 minute (such as 0.003 minute) that will be left uncounted. If this person then lower the increment to 0.001, the same argument would apply since some even smaller time would be left uncounted.

   The point being that, regardless of how small the time increment is made for counting, there always exists some even smaller time increment. Thus, one can never sequentially count all potential outcomes, which is why X is considered as a continuous random variable.
3. $X =$ the number of rolls it takes to get a 6 from rolling a six-sided die

   X is a discrete random variable. The outcome is $1, 2, 3, 4, \ldots$. While it might be the case that it will take you infinite number of rolls to get a 6, as long as you keep counting, then you can sequentially count all outcomes. Thus, numbers assigned to X are countable, which means X is discrete.

Notice the difference between this and the second exercise. In this exercise, the increment between all potential outcomes is always 1, so as long as you keep counting, all outcomes could be counted. In the second exercise, even if you have infinite amount of time, the time increment used for counting can always be made smaller, meaning that there is just no way for you to sequentially count all the time between 0 and 60 minutes to begin with.

2 Discrete Probability Distributions

- **Probability distribution**: a table / formula / graph that describes the values of a random variable and the associated probabilities (at these values).
  - In the case of **discrete probability distribution** for discrete random variable X, we need some way to describe $P(x)$ at all possible x values.
    * Conditions that a discrete probability distribution needs to satisfy:
      1. $0 \leq P(x) \leq 1$ for all x
      2. $\sum_x P(x) = 1$
  - If we already know that some other random variable Y yields outcome y, and want to describe all possible probabilities related to x after y outcome (that is, $P(X = x|Y = y)$ for all x), then **discrete conditional probability distribution** is appropriate.
    * Conditions that a discrete conditional probability distribution needs to satisfy:
      1. $0 \leq P(x|y) \leq 1$ for all x
      2. $\sum_x P(x|y) = 1$
- Since a probability distribution describes the probability at all possible outcomes, this is a representation of the **population**.
  - Thus, one can use a probability distribution to write down the calculation of some **parameters**:

Parameter Name	**Notation**	**Formula**	**Shortcut**
Expected value (mean)	$E(X) = \mu = \mu_X$	$\sum_x xP(x)$	-
Variance	$V(X) = \sigma^2 = \sigma_X^2$	$\sum_x (x-\mu)^2 P(x)$	$E(X^2)$ $-[E(X)]^2$
Standard deviation	$\sigma = \sigma_X$	$\sqrt{V(X)}$	-

  - When there is more than one probability distribution (say, distributions for random variables X and Y), then **parameters** on the relationship between two random variables can be expressed:

Parameter Name	**Notation**	**Formula**	**Shortcut**
Covariance	$Cov(X,Y) = \sigma_{XY}$	$\sum_x \sum_y (x-\mu_X)(y-\mu_Y)P(x,y)$	$E(XY)$ $-E(X)E(Y)$
Correlation (of coefficient)	$Corr(X,Y) = \rho_{XY}$	$\frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$	-

- Common parameter operations, when the random variable is transformed in some way: (Let X and Y be random variables, a, b, c, d be constants.)
  - Expected value (mean):
    - $E(c) = c$
    - $E(aX + b) = aE(X) + b$
    - $E(X + Y) = E(X) + E(Y)$
  - Variance:
    - $V(c) = 0$
    - $V(aX + b) = a^2 V(X)$
    - $V(X \pm Y) = V(X) + V(Y) \pm 2Cov(X, Y)$
  - Covariance:
    - $Cov(a, b) = 0$
    - $Cov(X, X) = V(X)$
    - $Cov(aX + b, cY + d) = acCov(X, Y)$

- Two examples of discrete probability distribution:
  1. **Binomial distribution**: distribution of success among n trials
     - Random variable $X \sim \text{Binomial}(n, p)$ if the following holds:
       - There are fixed number (n) of trials.
       - Each trial has two outcomes: success, and failure.
       - $P(\text{success}) = p$ is constant across all trials.
       - Trials are independent.
     - Once $X \sim \text{Binomial}(n, p)$ is established, then
       - $P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$
       - $E(X) = np$
       - $V(X) = np(1-p)$
  2. **Poisson distribution**: distribution of success within a fixed time period / fixed interval, with success arriving at rate $\mu > 0$
     - Random variable $X \sim \text{Poisson}(\mu)$ if the following holds:
       - Number of success in any interval is independent of the number of success in any other interval.
       - Probability of a success in any equal-size interval is constant.
       - Probability of a success is proportional to the size of the interval.
       - Probability of more than one success in an interval approaches 0 as the interval becomes smaller.
     - Once $X \sim \text{Poisson}(\mu)$ is established, then
       - $P(x) = \frac{e^{-\mu} \mu^x}{x!}$ for $x = 0, 1, 2, \ldots$
       - $E(X) = \mu$
       - $V(X) = \mu$