Dis 11: Inference about a Population

Related textbook chapter: 12
Ch 12 handout and solution offered by Dr. Pac can be accessed here: Handout Solution
This handout incorporates reviews with all exercises from the handout given by Dr. Pac.

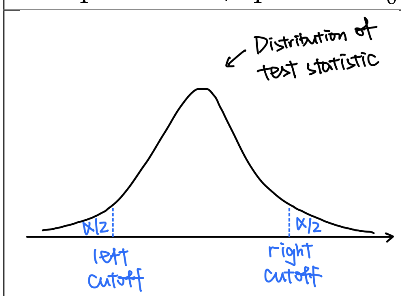
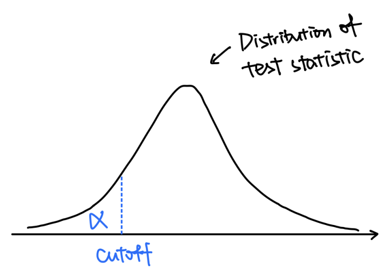
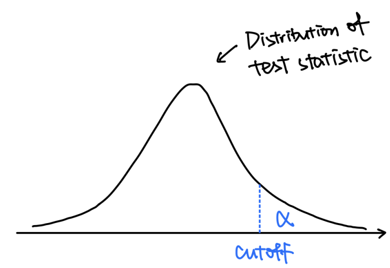
1 Motivation

- Two weeks ago, we learned how to perform hypothesis testing regarding population mean μ, assuming that population standard deviation σ parameter is known.
- The technique we learned can be extended to testing other population parameters obtained from one single population, and we are going to focus on three extensions this week:
  1. Testing μ, but remove the assumption that σ is known
  2. Testing σ^2
  3. Testing p (proportion of success from a binomial experiment)

2 General Approach

2.1 How is the test statistic & rejection region method generally applied?

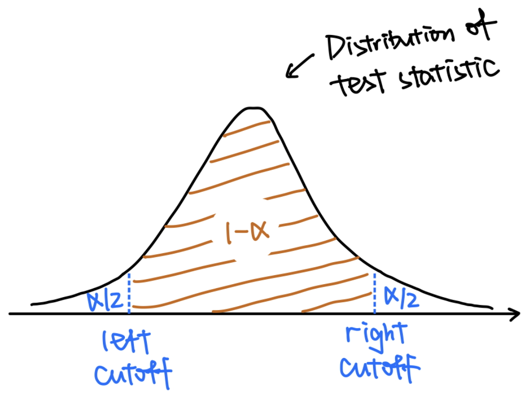
1. Does the test statistic follow (exactly or approximately) some known distribution? And what is it?
2. Based on the test (two-tailed or one-tailed), select the appropriate tail of the distribution for rejection using the specified significance level. This constitutes of the rejection region.
3. Calculate the test statistic with the given sample, and see if it falls within the rejection region.
   - If it falls within the rejection region, then we reject H_0 at the specified α significance level;
   - If it doesn't fall within the rejection region, then we fail to reject H_0 at α significance level.

<table>
<tr><th>Two-tailed Test</th><th>Left-tailed Test</th><th>Right-tailed Test</th></tr>
<tr><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $\neq$ parameter$_0$</td><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $<$ parameter$_0$</td><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $>$ parameter$_0$</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Reject H_0 if
test statistic $<$ left cutoff, or if
test statistic $>$ right cutoff</td><td>Reject H_0 if test statistic $<$ cutoff</td><td>Reject H_0 if test statistic $>$ cutoff</td></tr>
</table>

2.2 How to construct a confidence interval in general?

1. Based on the distribution of the test statistic, one can set probability of drawing sample statistic across multiple samples to be the confidence level $(1-\alpha)$ by

$$P\left(\text{left cutoff} \leq \text{test statistic} \leq \text{right cutoff}\right) = 1-\alpha$$



2. Shuffle some terms around to rewrite the above equation as

$$P\left(LB \leq \text{parameter} \leq UB\right) = 1-\alpha$$

Then the confidence interval of $(1-\alpha)$ confidence level is $[LB, UB]$

Note: In the special case where test statistic $= \frac{\text{statistic}-\text{parameter}}{se(\text{statistic})}$, step 2 is achieved through the following procedure:

$$\begin{aligned}
P\left(\text{left cutoff} \leq \frac{\text{statistic}-\text{parameter}}{se(\text{statistic})} \leq \text{right cutoff}\right) &= 1-\alpha \\
P\left(\text{left cutoff} \times se(\text{statistic}) \leq \text{statistic} - \text{parameter} \leq \text{right cutoff} \times se(\text{statistic})\right) &= 1-\alpha \\
P\left(\text{left cutoff} \times se(\text{statistic}) - \text{statistic} \leq -\text{parameter} \leq \text{right cutoff} \times se(\text{statistic}) - \text{statistic}\right) &= 1-\alpha \\
P\left(\text{statistic} - \text{left cutoff} \times se(\text{statistic}) \geq \text{parameter} \geq \text{statistic} - \text{right cutoff} \times se(\text{statistic})\right) &= 1-\alpha
\end{aligned}$$

which implies that, in this special case, the confidence interval of $(1-\alpha)$ confidence level is

$$[\text{statistic} - \text{right cutoff} \times se(\text{statistic}), \quad \text{statistic} - \text{left cutoff} \times se(\text{statistic})]$$

- Using the constructed confidence interval of confidence level $(1-\alpha)$, one can perform a **two-tailed** test under significance level α:
  - If parameter_0 is NOT contained within the $(1-\alpha)$ confidence interval, then we reject H_0 at α significance level;
  - If parameter_0 is contained within the $(1-\alpha)$ confidence interval, then we fail to reject H_0 at α significance level.

3 Inferences on Three Parameters from a Single Population

3.1 Inference on μ, when σ is unknown

- Last week, we looked at how to perform hypothesis testing (inference) on μ using $\bar{X}$, while assuming that the population standard deviation of X (i.e. σ) is known.
  - Recall: if σ is known, then

$$\text{test statistic} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1^2)$$

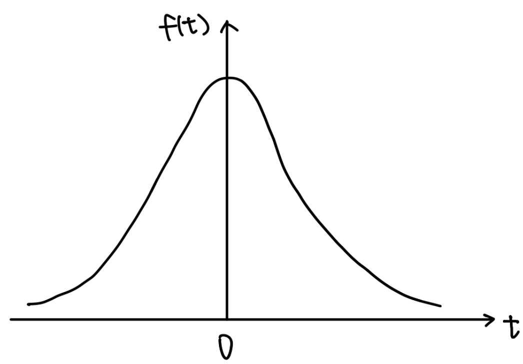
    (If X is already normally distributed, then test statistic is exactly normally distributed; otherwise, as long as $n \geq 30$, then CLT implies that test statistic is approximately normally distributed.)

- However, often in practice, σ is unknown.
  - To address this problem, one might think about substituting σ with unbiased sample estimate s.
  - Replacing σ with s introduces some problem though: s is an estimated object, instead of something that's known for certain (like σ).
  - So the new test statistic $\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ follows a different distribution. One figured out that this new distribution is called **student-t distribution**:

$$\text{test statistic} = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

    where $n - 1$ is the degree of freedom (DOF).

    (Regardless of how X is distributed, the test statistic with estimated s always follows student-t distribution.)



  - Since this falls under the special case that test statistic is standardized by subtracting parameter and then divided by standard error of the statistic, the $(1 - \alpha)$ confidence interval is

$$\left[\bar{X} - t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}, \quad \bar{X} + t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}\right]$$

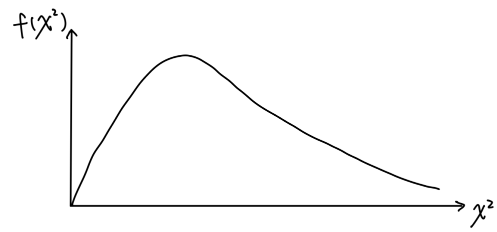
[Go to Exercise 1 & 2]

3.2 Inference on σ^2

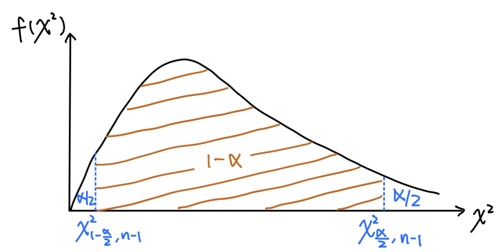
- Since we are assuming away from knowing σ with certainty, one might be interested in conducting hypothesis testing on population standard deviation / population variance.
  - Since variance = standard deviation^2, let's just always perform the test on variance.
- The test statistic used for testing variance follows a **Chi-squared distribution**:

$$\text{test statistic} = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

  where $n-1$ is the DOF.



- To construct the confidence interval with $(1-\alpha)$ confidence level, we need to find the relevant cutoff values that yield middle portion probability of $(1-\alpha)$:



  Hence,

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{\frac{\alpha}{2},n-1}\right) = 1-\alpha$$
$$P\left(\frac{\chi^2_{1-\frac{\alpha}{2},n-1}}{(n-1)s^2} \le \frac{1}{\sigma^2} \le \frac{\chi^2_{\frac{\alpha}{2},n-1}}{(n-1)s^2}\right) = 1-\alpha$$
$$P\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right) = 1-\alpha$$

  The confidence interval with $(1-\alpha)$ confidence level for σ^2 is constructed to be

$$\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}},\quad \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$$

[Go to Exercise 3]

3.3 Inference on p (proportion of success from a binomial experiment)

- Recall from Dis 8 that a binomial $X \overset{a}{\sim} N(np, np(1-p))$ if the following conditions both hold:
  1. $np \geq 5$, and
  2. $n(1-p) \geq 5$

  This implies that the sample success proportion $\hat{p} \overset{a}{\sim} N\left(p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right)$.

- To perform hypothesis testing, we can check, under the sample proportion $\hat{p}$ from the given sample, if we can first approximate $\hat{p}$ as a normally distributed variable:
  1. $n\hat{p} \geq 5$, and
  2. $n(1-\hat{p}) \geq 5$

  If both conditions hold, we will use the standardized version of $\hat{p}$ as our test statistic, so that the test statistic follows (approximately) a standard normal distribution $N(0, 1^2)$. That is,

$$\text{test statistic} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \overset{a}{\sim} N(0, 1^2)$$

- This also falls under the special case that test statistic is standardized by subtracting parameter and then divided by standard error of the statistic. So the $(1-\alpha)$ confidence interval is

$$\left[\hat{p} - Z_{\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \quad \hat{p} + Z_{\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

[Go to Exercise 4]

4 Exercises

1. A researcher would like to estimate the average weight of American middle school students. She collects a sample of size 51 and calculates a sample mean of 120. The standard deviation is 15.

   (a) Assuming the standard deviation given above is the true population standard deviation, construct an 80 percent confidence interval for the population mean.

(b) Assuming the standard deviation above is the sample standard deviation, construct an 80 percent confidence interval for the population mean.

(c) Which of the above confidence intervals is wider? Does this match up with what you would expect? Why?

2. A citrus fruit dryer randomly sampled 10 observations, finding a sample mean drying time of 103 and a sample standard deviation of 17.

   (a) Is there sufficient evidence to conclude the population mean drying time is less than 110 with a 10% significance level?

(b) Repeat part (a) assuming that you know that the population standard deviation is $\sigma = 17$ (for this part, assume that citrus fruit drying time follows a normal distribution).

(c) Did you draw the same conclusion in the previous two parts? If there was a difference, explain why.

3. A company produces machined engine parts that are supposed to have diameter variance no larger than 0.2 cm (centimeters). A random sample of 31 parts yields a sample variance of 0.3 cm.

   (a) Using a 5% significance level, test whether the variance is larger than 0.2 cm.

(b) Construct 99% confidence interval for σ^2.

4. A poll asks a simple random sample of 100 Madison residents who makes better pizza: Roman Candle or Glass Nickel. They find that 55% of their sample prefers Roman Candle.

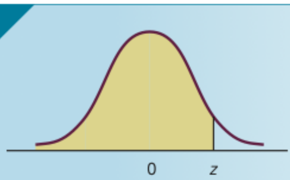
   (a) Test whether more than half of Madison residents prefer Roman Candle pizza to Glass Nickel, using a 5% size of test.

   (b) What was the distribution of your test statistic in the previous part? Explain your reasoning.

   (c) Construct a 95% confidence interval for the population proportion of Madison residents who prefer Roman Candle pizza to Glass Nickel.

Probability table for a standard normal distribution ($z \geq 0$)

TABLE **3** ***(Continued)***

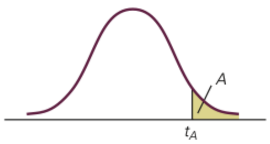


$P(-\infty < Z < z)$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Probability table for a t-distribution

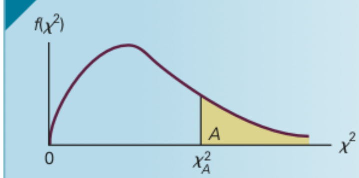
TABLE **4**
Critical Values of the Student *t* Distribution



Degrees of Freedom	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.306	1.690	2.030	2.438	2.724
40	1.303	1.684	2.021	2.423	2.704
45	1.301	1.679	2.014	2.412	2.690
50	1.299	1.676	2.009	2.403	2.678
55	1.297	1.673	2.004	2.396	2.668
60	1.296	1.671	2.000	2.390	2.660
65	1.295	1.669	1.997	2.385	2.654
70	1.294	1.667	1.994	2.381	2.648
75	1.293	1.665	1.992	2.377	2.643
80	1.292	1.664	1.990	2.374	2.639
85	1.292	1.663	1.988	2.371	2.635
90	1.291	1.662	1.987	2.368	2.632
95	1.291	1.661	1.985	2.366	2.629
100	1.290	1.660	1.984	2.364	2.626
110	1.289	1.659	1.982	2.361	2.621
120	1.289	1.658	1.980	2.358	2.617
130	1.288	1.657	1.978	2.355	2.614
140	1.288	1.656	1.977	2.353	2.611
150	1.287	1.655	1.976	2.351	2.609
160	1.287	1.654	1.975	2.350	2.607
170	1.287	1.654	1.974	2.348	2.605
180	1.286	1.653	1.973	2.347	2.603
190	1.286	1.653	1.973	2.346	2.602
200	1.286	1.653	1.972	2.345	2.601
∞	1.282	1.645	1.960	2.326	2.576

Probability table for a χ^2-distribution

TABLE **5** **Critical Values of the χ^2 Distribution**



Degrees of Freedom	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000039	0.000157	0.000982	0.00393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.6
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	40.3	43.8	47.0	50.9	53.7
40	20.7	22.2	24.4	26.5	29.1	51.8	55.8	59.3	63.7	66.8
50	28.0	29.7	32.4	34.8	37.7	63.2	67.5	71.4	76.2	79.5
60	35.5	37.5	40.5	43.2	46.5	74.4	79.1	83.3	88.4	92.0
70	43.3	45.4	48.8	51.7	55.3	85.5	90.5	95.0	100	104
80	51.2	53.5	57.2	60.4	64.3	96.6	102	107	112	116
90	59.2	61.8	65.6	69.1	73.3	108	113	118	124	128
100	67.3	70.1	74.2	77.9	82.4	118	124	130	136	140