Supplementary Handout for Dis 4: Probability

1 Sample Space, Outcome, and Event

- Say that we roll a six-sided die:
  - The roll could be 1, 2, 3, 4, 5, or 6
  - The set containing all possible rolls is defined as the **sample space**.
  - Each possible roll is an **outcome**.
    - * Outcomes in a sample space should be **exhaustive** – all possible outcomes must be included in the sample space.
    - * Outcomes in a sample space should be **mutually exclusive** – no two outcomes can occur at the same time.
  - An **event** is a collection of one or more simple events (i.e. outcomes) in a sample space.

Exercise. Is the following true or false? Why?

1. The set $\{\text{rolling a } 1, \text{rolling a } 2, \text{rolling a } 3, \text{rolling a } 4\}$ describes the sample space from a fair, six-sided die roll.

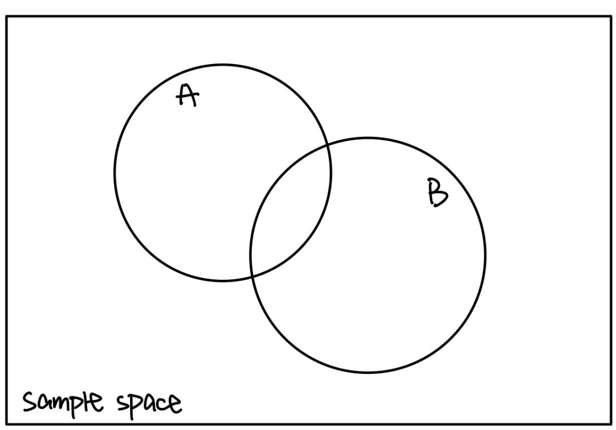
   False. This list is not exhaustive, since rolling a six-sided die also includes outcomes "rolling a 5" and "rolling a 6". The actual sample space is {rolling a 1, rolling a 2, rolling a 3, rolling a 4, rolling a 5, rolling a 6}.
2. "Rolling a 3" is an event.

   True. "Rolling a 3" is also an outcome (also known as a simple event), and one simple event also constitutes as an event.
3. "Rolling an even number" is an event.

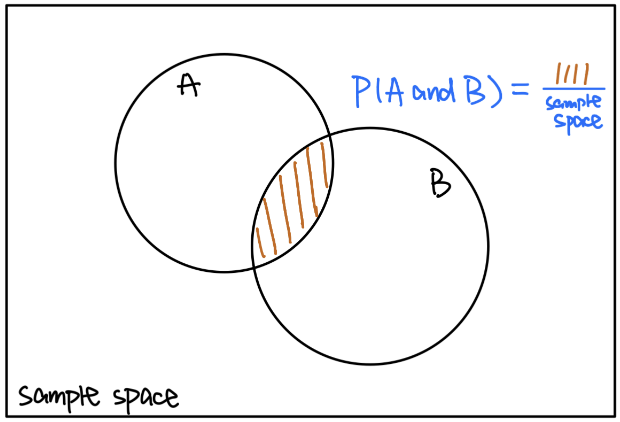
   True. "Rolling an even number" consists of simple events "Rolling a 2", "Rolling a 4", and "Rolling a 6". A collection of simple events is an event.

2 Joint vs. Conditional Probability

- Say that there are two events, denoted by A and B.



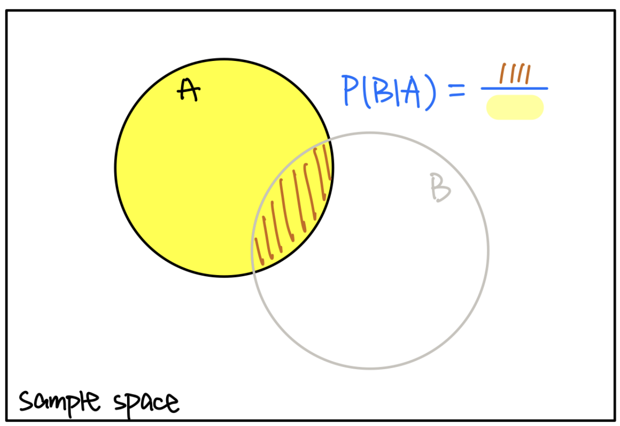
- If one is interested in the probability that event A and B occurs together, then



$$P(A \text{ and } B) = \frac{\text{Area of the line shaded space}}{\text{Area of the entire sample space (rectangle)}}$$

This is because **joint** probability looks at case where event A and B occur together with respect to all the possible outcomes (i.e. the sample space).

- Sometimes though, one is interested in the probability of B occuring conditional on A already occurs. This is depicted as



$$P(B|A) = \frac{\text{Area of the line shaded space}}{\text{Area of event } A} = \frac{P(A \text{ and } B)}{P(A)}$$

This is because **conditional** probability of $P(B|A)$ looks at case where event B occurs with respect to A already occuring.

- One way to visualize the relationship between joint and conditional probability is through the tree diagram (will see this in exercise later).
- Conditional probability gives rise to **Bayes Law**:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\underbrace{P(B|A)P(A) + P(B|A^C)P(A^C)}_{P(A \text{ and } B)+P(A^C \text{ and } B)}}$$

3 Mutually Exclusive vs. Independence

- Two events are **mutually exclusive** if they cannot occur at the same time.

  e.g. When rolling a fair six-sided die, "Rolling a 3" and "Rolling a 4" are mutually exclusive, since you cannot roll both a 3 and a 4 at the same time.

- Two events are **independent** if one event is not impacted by the other.

  e.g. When rolling a fair six-sided die, "Rolling a 3 in the first round" and "Rolling a 3 in the second round" are independent, since the first and second roll does not influence each other.

  - This is why if A and B are independent, $P(B|A) = P(B)$, and $P(A|B) = P(A)$, since conditional on event A does not give us any new information regarding event B, and conditional on event B does not give us any new information regarding event A.

- When looking at joint probability between A and B,
  - If A and B are mutually exclusive, $P(A \text{ and } B) = 0$
  - If A and B are independent, $P(A \text{ and } B) = P(A) \times P(B)$

Exercise. Are the following two events mutually exclusive, independent, both, or neither?

1. "Weather today in Madison at 11:05am is sunny" vs. "Weather today in Madison at 11:05am is rainy"

   Mutually exclusive. Since the weather today in Madison at 11:05am can only be sunny or rainy (or something else), but not both at the same time.

2. "Roll a (fair six-sided) die and get an even number" vs. "Roll a (fair six-sided) die and get 4"

   Neither mutually exclusive nor independent. It is possible that a die roll is both even and is exactly 4, so they are not mutually exclusive. Clearly, when a die roll is 4, then it is absolutely an even number. The two events impact each other, so they are not independent.

3. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 4"

   Independent. Flip a coin does not interact with rolling a die.

4. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 10"

   Both mutually exclusive and independent. It is not possible to have both events occur at the same time (mainly because it's fully impossible to get a 10 from rolling a fair six-sided die), so they are mutually exclusive. On the other hand, the two events do not impact each other (flipping a coin has no impact on the result of rolling a die), so they are also independent.