Supplementary Handout for Dis 11: Inference about a Population

1 Motivation

- Last week, we learned how to perform hypothesis testing regarding population mean μ.
- The technique we learned can be extended to testing other population parameters obtained from one single population, and we are going to focus on three extensions this week:
  1. Testing μ, but remove the assumption that σ is known
  2. Testing σ^2
  3. Testing p (proportion of success from a binomial experiment)

2 General Approach

- Before we talk about each specific extension, let's generalize two out of the three testing methods that we learned from last week: test statistic & rejection region, and confidence interval method.

2.1 How is the test statistic & rejection region method generally applied?

1. Does the test statistic follow (exactly or approximately) some known distribution? And what is the distribution?
2. Based on the test (two-tailed or one-tailed), select the appropriate tail of the distribution for rejection using the specified significance level. This constitutes of the rejection region.
3. Calculate the test statistic with the given sample, and see if it falls within the rejection region.
   - If it falls within the rejection region, then we reject H_0 at the specified α significance level;
   - If it doesn't fall within the rejection region, then we fail to reject H_0 at the specified α significance level.

<table>
<tr><th>Two-tailed Test</th><th>Left-tailed Test</th><th>Right-tailed Test</th></tr>
<tr><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $\neq$ parameter$_0$</td><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $<$ parameter$_0$</td><td>H_0 : parameter $=$ parameter$_0$
H_1 : parameter $>$ parameter$_0$</td></tr>
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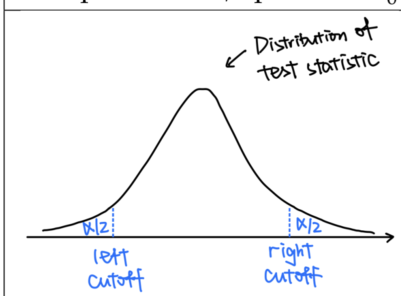

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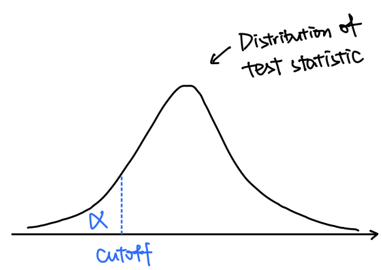

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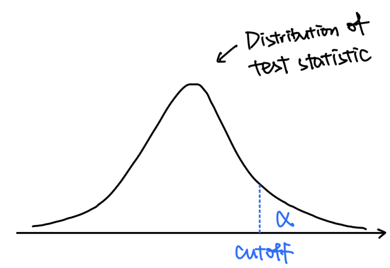
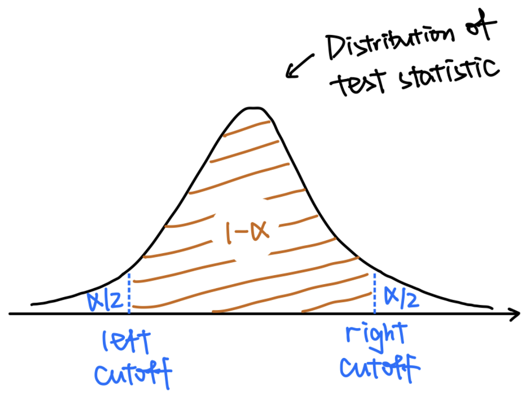


</td></tr>
<tr><td>Reject H_0 if test statistic $<$ left cutoff, or if test statistic $>$ right cutoff</td><td>Reject H_0 if test statistic $<$ cutoff</td><td>Reject H_0 if test statistic $>$ cutoff</td></tr>
</table>

2.2 How to construct a confidence interval in general?

1. Based on the distribution of the test statistic, one can set probability of drawing sample statistic across multiple samples to be the confidence level $(1-\alpha)$ by

$$P\left(\text{left cutoff} \leq \text{test statistic} \leq \text{right cutoff}\right) = 1-\alpha$$



2. Shuffle some terms around to rewrite the above equation as

$$P\left(LB \leq \text{parameter} \leq UB\right) = 1-\alpha$$

Then the confidence interval of $(1-\alpha)$ confidence level is $[LB, UB]$

Note: In the special case where test statistic $= \frac{\text{statistic}-\text{parameter}}{se(\text{statistic})}$, step 2 is achieved through the following procedure:

$$\begin{aligned}
P\left(\text{left cutoff} \leq \frac{\text{statistic}-\text{parameter}}{se(\text{statistic})} \leq \text{right cutoff}\right) &= 1-\alpha \\
P\left(\text{left cutoff} \times se(\text{statistic}) \leq \text{statistic} - \text{parameter} \leq \text{right cutoff} \times se(\text{statistic})\right) &= 1-\alpha \\
P\left(\text{left cutoff} \times se(\text{statistic}) - \text{statistic} \leq -\text{parameter} \leq \text{right cutoff} \times se(\text{statistic}) - \text{statistic}\right) &= 1-\alpha \\
P\left(\text{statistic} - \text{left cutoff} \times se(\text{statistic}) \geq \text{parameter} \geq \text{statistic} - \text{right cutoff} \times se(\text{statistic})\right) &= 1-\alpha
\end{aligned}$$

which implies that, in this special case, the confidence interval of $(1-\alpha)$ confidence level is

$$[\text{statistic} - \text{right cutoff} \times se(\text{statistic}), \quad \text{statistic} - \text{left cutoff} \times se(\text{statistic})]$$

- Using the constructed confidence interval of confidence level $(1-\alpha)$, one can perform a **two-tailed** test under significance level α:
  - If parameter_0 is NOT contained within the $(1-\alpha)$ confidence interval, then we reject H_0 at α significance level;
  - If parameter_0 is contained within the $(1-\alpha)$ confidence interval, then we fail to reject H_0 at α significance level.

3 Inferences on Three Parameters from a Single Population

3.1 Inference on μ, when σ is unknown

- Last week, we looked at how to perform hypothesis testing (inference) on μ using $\bar{X}$, while assuming that the population standard deviation of X (i.e. σ) is known.
  - Recall: if σ is known, then

$$\text{test statistic} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1^2)$$

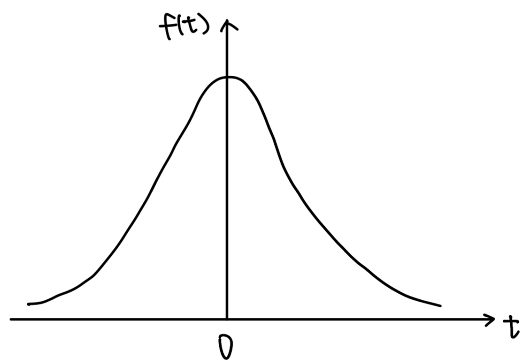
    (If X is already normally distributed, then test statistic is exactly normally distributed; otherwise, as long as $n \geq 30$, then CLT implies that test statistic is approximately normally distributed.)

- However, often in practice, σ is unknown.
  - To address this problem, one might think about substituting σ with unbiased sample estimate s.
  - Replacing σ with s introduces some problem though: s is an estimated object, instead of something that's known for certain (like σ).
  - So the new test statistic $\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ follows a different distirbution. One figured out that this new distribution is called **student-t distribution**:

$$\text{test statistic} = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

    where $n-1$ is the degree of freedom (DOF).

    (Regardless of how X is distributed, the test statistic with estimated s always follows student-t distribution.)



  - Since this falls under the special case that test statistic is standardized by subtracting parameter and then devided by standard error of the statistic, the $(1-\alpha)$ confidence interval is

$$\left[\bar{X} - t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}, \quad \bar{X} + t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}\right]$$

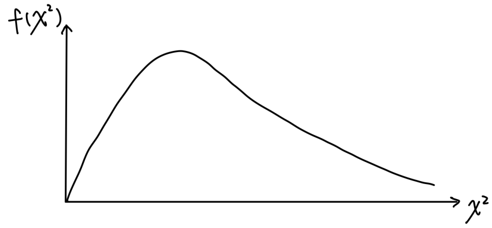
[Go to Exercise 1 & 2]

3.2 Inference on σ^2

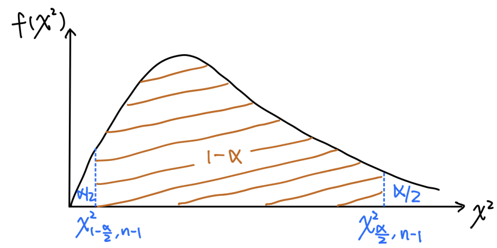
- Since we are assuming away from knowing σ with certainty, one might be interested in conducting hypothesis testing on population standard deviation / population variance.
  - Since variance = standard deviation^2, let's just always perform the test on variance.
- The test statistic used for testing variance follows a **Chi-squared distribution**:

$$\text{test statistic} = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

  where $n-1$ is the DOF.



- To construct the confidence interval with $(1-\alpha)$ confidence level, we need to find the relevant cutoff values that yield middle portion probability of $(1-\alpha)$:



  Hence,

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{\frac{\alpha}{2},n-1}\right) = 1-\alpha$$
$$P\left(\frac{\chi^2_{1-\frac{\alpha}{2},n-1}}{(n-1)s^2} \leq \frac{1}{\sigma^2} \leq \frac{\chi^2_{\frac{\alpha}{2},n-1}}{(n-1)s^2}\right) = 1-\alpha$$
$$P\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right) = 1-\alpha$$

  The confidence interval with $(1-\alpha)$ confidence level for σ^2 is constructed to be

$$\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}},\quad \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$$

[Go to Exercise 3]

3.3 Inference on p (proportion of success from a binomial experiment)

- Recall from Dis 8 that a binomial $X \overset{a}{\sim} N(np, np(1-p))$ if the following conditions both hold:

  1. $np \geq 5$, and
  2. $n(1-p) \geq 5$

  This implies that the sample success proportion $\hat{p} \overset{a}{\sim} N\left(p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right)$.

- To perform hypothesis testing, we can check, under the sample proportion $\hat{p}$ from the given sample, if we can first approximate $\hat{p}$ as a normally distributed variable:

  1. $n\hat{p} \geq 5$, and
  2. $n(1-\hat{p}) \geq 5$

  If both conditions hold, we will use the standardized version of $\hat{p}$ as our test statistic, so that the test statistic follows (approximately) a standard normal distribution $N(0, 1^2)$. That is,

$$\text{test statistic} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \overset{a}{\sim} N(0, 1^2)$$

- This also falls under the special case that test statistic is standardized by subtracting parameter and then devided by standard error of the statistic. So the $(1-\alpha)$ confidence interval is

$$\left[\hat{p} - Z_{\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \quad \hat{p} + Z_{\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

[Go to Exercise 4]