Lec 3*: Probability

1 Probability

1.1 Sample Space, Outcome, and Event

- Say that we roll a six-sided die:
  - The roll could be 1, 2, 3, 4, 5, or 6
  - The set containing all possible rolls is defined as the **sample space**.
  - Each possible roll is an **outcome**.
    - Outcomes in a sample space should be **exhaustive** – all possible outcomes must be included in the sample space.
    - Outcomes in a sample space should be **mutually exclusive** – no two outcomes can occur at the same time.
  - An **event** is a collection of one or more simple events (i.e. outcomes) in a sample space.

Exercise. Is the following true or false? Why?

1. The set $\{\text{rolling a } 1, \text{rolling a } 2, \text{rolling a } 3, \text{rolling a } 4\}$ describes the sample space from a fair, six-sided die roll.

   False. This list is not exhaustive, since rolling a six-sided die also includes outcomes "rolling a 5" and "rolling a 6". The actual sample space is {rolling a 1, rolling a 2, rolling a 3, rolling a 4, rolling a 5, rolling a 6}.

2. "Rolling a 3" is an event.

   True. "Rolling a 3" is also an outcome (also known as a simple event), and one simple event also constitutes as an event.

3. "Rolling an even number" is an event.

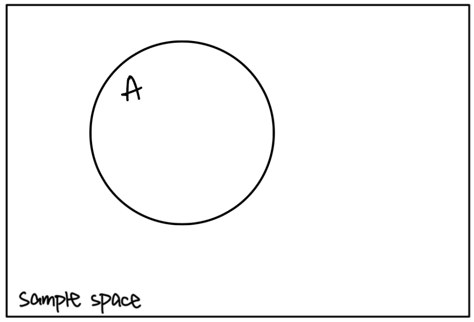
   True. "Rolling an even number" consists of simple events "Rolling a 2", "Rolling a 4", and "Rolling a 6". A collection of simple events is an event.

1.2 How is Probability Calculated?

- Let's continue with the dice rolling example.
  - The sample space is {rolling a 1, rolling a 2, rolling a 3, rolling a 4, rolling a 5, rolling a 6}
- Say that event A is as simple as "Rolling a 1".
- How do we calculate $P(A)$?
- It seems pretty straightforward in this example: "Rolling a 1" is one out of 6 possible outcomes. In reality, if we keeping rolling a die for a large number of times (say, 6 million times), we will see the number 1 being the outcome very close to 1 million times.

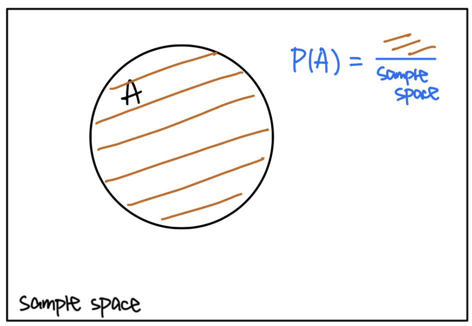
*Some exercise questions are taken from or slightly modified based on Dr. Gregory Pac's Econ 310 discussion handout.

- Thus, $P(A) = \frac{1}{6}$
- **To generalize this**, let's visualize what happened here:



Probability of event A is how often A happens, relatively to all the possible outcomes in the sample space. Thus,

$$P(A) = \frac{\text{Area of A}}{\text{Area of the entire sample space (rectangle)}}$$

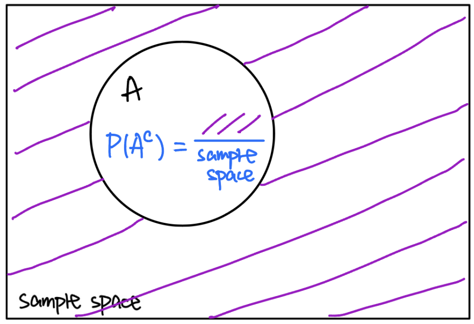


- Sometimes, instead of finding the probability of event A, we need the probability of **the opposite of** A.
  - Call this opposite event A^C, where the superscript C stands for complement.
  - To find $P(A^C)$, notice first that

$$P(A) + P(A^C) = 1$$

  Thus,

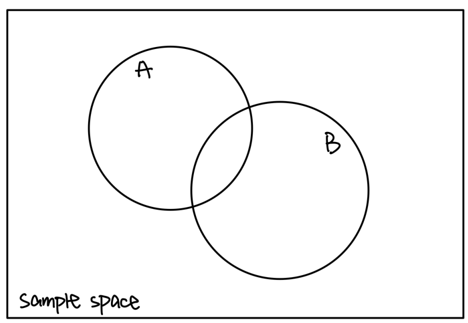
$$P(A^C) = 1 - P(A) = \frac{\text{Area outside of A}}{\text{Area of the entire sample space (rectangle)}}$$



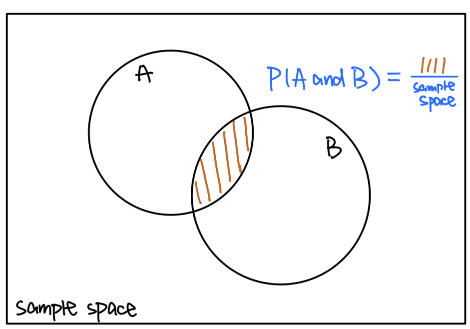
- Some general property of probability:
  - $0 \leq P(A) \leq 1$ for any event A
  - Adding up all possible, mutually exclusive events from the sample space must yield a sum of 1
  - **Complement rule**: $P(A^C) = 1 - P(A)$
  - **Addition rule**: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

1.3 Joint vs. Conditional Probability

- Say that there are two events, denoted by A and B.



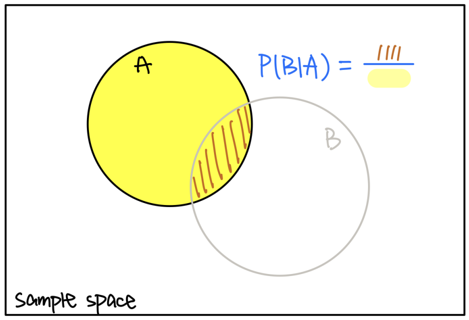
- If one is interested in the probability that event A and B occurs together, then



$$P(A \text{ and } B) = \frac{\text{Area of the line shaded space}}{\text{Area of the entire sample space (rectangle)}}$$

This is because **joint** probability looks at case where event A and B occur together with respect to all the possible outcomes (i.e. the sample space).

- Sometimes though, one is interested in the probability of B occuring conditional on A already occurs. This is depicted as



$$P(B|A) = \frac{\text{Area of the line shaded space}}{\text{Area of event } A} = \frac{P(A \text{ and } B)}{P(A)}$$

This is because **conditional** probability of $P(B|A)$ looks at case where event B occurs with respect to A already occuring.

- One way to visualize the relationship between joint and conditional probability is through the tree diagram (will see this in exercise later).

- Conditional probability gives rise to **Bayes Law**:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\underbrace{P(B|A)P(A) + P(B|A^C)P(A^C)}_{P(A \text{ and } B)+P(A^C \text{ and } B)}}$$

1.4 Mutually Exclusive vs. Independence

- Two events are **mutually exclusive** if they cannot occur at the same time.

  e.g. When rolling a fair six-sided die, "Rolling a 3" and "Rolling a 4" are mutually exclusive, since you cannot roll both a 3 and a 4 at the same time.

- Two events are **independent** if one event is not impacted by the other.

  e.g. When rolling a fair six-sided die, "Rolling a 3 in the first round" and "Rolling a 3 in the second round" are independent, since the first and second roll does not influence each other.

  - This is why if A and B are independent, $P(B|A) = P(B)$, and $P(A|B) = P(A)$, since conditional on event A does not give us any new information regarding event B, and conditional on event B does not give us any new information regarding event A.

- When looking at joint probability between A and B,
  - If A and B are mutually exclusive, $P(A \text{ and } B) = 0$
  - If A and B are independent, $P(A \text{ and } B) = P(A) \times P(B)$

Exercise. Are the following two events mutually exclusive, independent, both, or neither?

1. "Weather today in Madison at 11:05am is sunny" vs. "Weather today in Madison at 11:05am is rainy"

   Mutually exclusive. Since the weather today in Madison at 11:05am can only be sunny or rainy (or something else), but not both at the same time. Additionally, when the weather in Madison today at 11:05am is sunny, it then cannot be that the weather is rainy at this point, so the two events impact each other, so they are not independent.

2. "Roll a (fair six-sided) die and get an even number" vs. "Roll a (fair six-sided) die and get 4"

   Neither mutually exclusive nor independent. It is possible that a die roll is both even and is exactly 4, so they are not mutually exclusive. Clearly, when a die roll is 4, then it is absolutely an even number. The two events impact each other, so they are not independent.

3. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 4"

   Independent. Flip a coin does not interact with rolling a die. On the other hand, both events can occur at the same time, so they are not mutually exclusive.

4. "Flip a coin and get tail" vs. "Roll a (fair six-sided) die and get 10"

   Both mutually exclusive and independent. It is not possible to have both events occur at the same time (mainly because it's fully impossible to get a 10 from rolling a fair six-sided die), so they are mutually exclusive. On the other hand, the two events do not impact each other (flipping a coin has no impact on the result of rolling a die), so they are also independent.

2 Exercises

1. You go to Vegas to play craps; luckily, this class has prepared you to solve the following problems:

   (a) Assuming you rolled a single die and got an even number, what's the probability the number on that die is a two?

   First, since there are six equally likely outcomes in the sample space, we know that:

   $$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

   By the addition rule (note that these events are mutually exclusive), we also know that:

   $$P(\text{even}) = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

   Finally, note that $P(\text{even}|2) = 1$, since a die roll of two ensures the roll must be even. Given this, applying Bayes' Law, we have:

   $$P(2|\text{even}) = \frac{P(2 \text{ and even})}{P(\text{even})} = \frac{P(\text{even}|2)P(2)}{P(\text{even})} = \frac{1 \times \frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

   (b) Rolling two dice, what's the probability of two sixes? What's the probability of no sixes?

Since there are 36 equally likely outcomes in the sample space:

$$P(\text{six on both sides}) = P(6,6) = \frac{1}{36}$$

By the addition rule:

$$\begin{aligned} P(\text{at least one six}) &= P(\text{six on die one}) + P(\text{six on die two}) - P(\text{six on both dice}) \\ &= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36} \end{aligned}$$

By the complement rule:

$$P(\text{no sixes}) = 1 - P(\text{at least one six}) = 1 - \frac{11}{36} = \frac{25}{36}$$

(c) Rolling two dice, what's the probability both dice are even? What's the probability either die is even?
Since rolling an even number on the first die and the second die are independent events, we can apply the simplified version of the multiplication rule:

$$\begin{aligned} P(\text{even on Die 1 and even on Die 2}) &= P(\text{even on Die 1}) \times P(\text{even on Die 2}) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{aligned}$$

By the same logic, the probability of both dice coming up odds is also $\frac{1}{4}$. So, we can conclude that:

$$P(\text{even on Die 1 or even on Die 2}) = 1 - P(\text{odd on Die 1 and odd on Die 2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

The last equation is derived from the fact that the opposite (i.e. complement) event of either die being even is that both dice are odd.

(d) Rolling two dice, what's the probability they sum to seven?
There are six permutations that lead to a seven, so since there are 36 equally likely outcomes in the sample space the addition rule tells us:

$$\begin{aligned} P(\text{two dice sum to 7}) &= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) \\ &= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{6} \end{aligned}$$

The order between the two dice matters. For example, $P(1,6)$ represents that the first die is a 1, and the second die is a 6. This differs from getting 6 on the first die, and 1 on the second die.

2. Consider the following joint probability table:

	A	A^C
B	.15	.25
B^C	.40	.20

(a) What is $P(A \text{ and } B)$? What is $P(A)$? What is $P(B)$?

The table gives us all the joint probabilities, so we know that $P(A \text{ and } B) = 0.15$.

Using the addition rule we can calculate the marginal probability of A:

$$P(A) = P(A \text{ and } B) + P(A \text{ and } B^C) = 0.15 + 0.40 = 0.55$$

Similarly, we can find the marginal probability of B:

$$P(B) = P(A \text{ and } B) + P(A^C \text{ and } B) = 0.15 + 0.25 = 0.40$$

(b) What is $P(A \text{ or } B)$?

We use the addition rule (note that since these events are not mutually exclusive, we must subtract off the joint probability):

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.55 + 0.40 - 0.15 = 0.80$$

(c) What is $P(A|B)$? What is $P(B|A)$?

By the definition of conditional probability,

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0.15}{0.40} = 0.3750$$

Similarly,

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{0.15}{0.55} = 0.2727$$

(d) Are A and B independent?

No, since

$$P(A|B) = 0.3750 \quad \neq \quad 0.55 = P(A)$$

Or, equivalently, since

$$P(B|A) = 0.2727 \quad \neq \quad 0.40 = P(B)$$

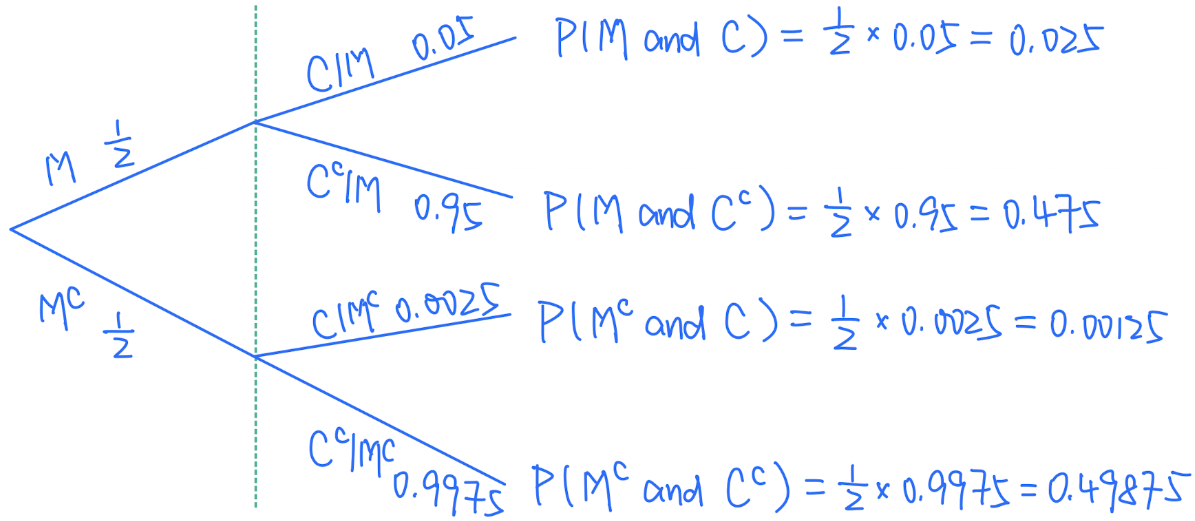
Or, equivalently, since

$$P(A \text{ and } B) = 0.15 \quad \neq \quad 0.22 = 0.55 * 0.40 = P(A) \times P(B)$$

3. Suppose exactly half the population is male. 5% of males and only 0.25% of females are color blind.

(a) Draw a tree diagram to illustrate the sample space.

Denote the event of being a male as M, and the event of being color blind as C. The tree diagram should look like the following:



(b) A person is chosen at random and that person is color blind. What's the probability that the person is a male?

Since the person chosen is color blind, being color blind (C) is the event we condition on. So the question is asking about the conditional probability $P(M|C)$.

Using the definition of conditional probability, we have

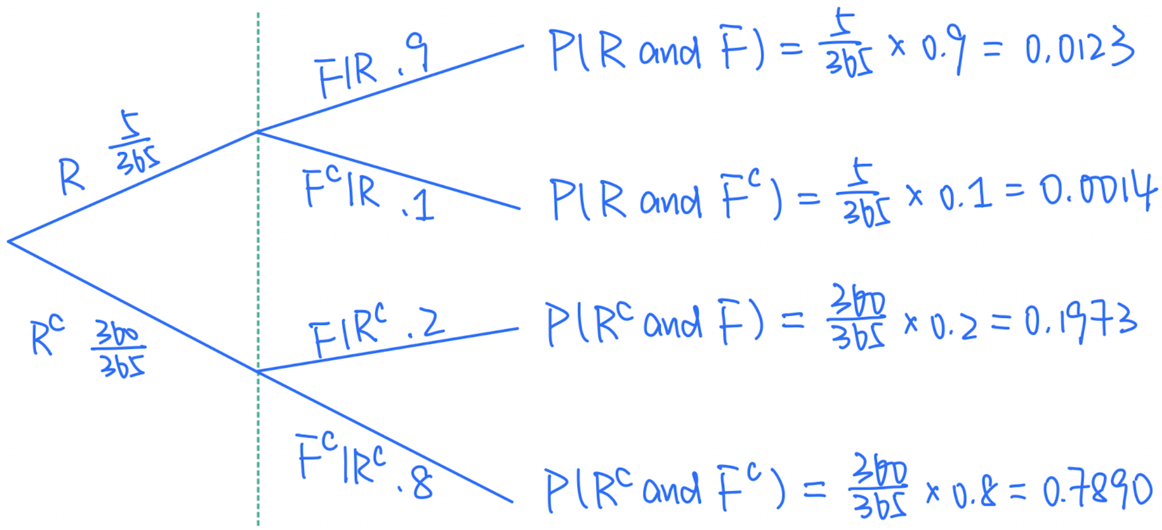
$$\begin{aligned} P(M|C) &= \frac{P(M \text{ and } C)}{P(C)} \\ &= \frac{P(M \text{ and } C)}{P(M \text{ and } C) + P(M^C \text{ and } C)} \\ &= \frac{0.025}{0.025 + 0.00125} = 0.9524 \end{aligned}$$

Thus, when a person is color blind, the probability that the person is a male is 0.9524.

4. Marie is getting married tomorrow at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 20% of the time. Assume there are 365 days in a year.

   (a) Draw a tree diagram to illustrate the sample space.

   Denote the event of raining as R, and the event of the weatherman forecasting that it will rain as F. The tree diagram should look like the following:



(b) What is the probability it will rain tomorrow for Marie's wedding?

Since the weatherman already predicted rain for tomorrow, the probability that the question is asking should condition on the event F; that is, the question asks for $P(R|F)$.

Using the conditional probability formula, we have

$$\begin{aligned} P(R|F) &= \frac{P(R \text{ and } F)}{P(F)} \\ &= \frac{P(R \text{ and } F)}{P(R \text{ and } F) + P(R^C \text{ and } F)} \\ &= \frac{0.0123}{0.0123 + 0.1973} = 0.0587 \end{aligned}$$

Thus, given that the weatherman predicted rain for tomorrow, the probability that it will rain tomorrow is 0.0587.