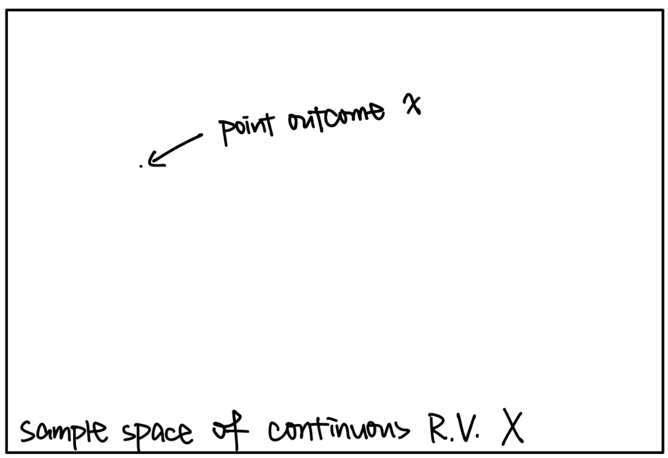
Supplementary Handout for Dis 6: Continuous Probability Distributions

1 Density Functions

- When we examined discrete probability distributions, we said that a discrete probability distribution should describe point probability $P(x)$ at all possible x outcome values.
- Ideally, we would like to provide a similar definition for continuous probability distributions.
- But there's a problem: when random variable X is continuous, **point probability equals to 0 at every single point** ($P(X = x) = 0$ for all x).
  - Intuition 1: A continuous random variable has uncountable number of values, so if each outcome value has probability $\varepsilon > 0$, then the sum of all probabilities would equal to ∞ instead of 1.
  - Intuition 2: Say that the rectangle below represents the sample space for a continuous random variable. The probability of hitting a point within the rectangle is the area of the point divided by the area of the rectangle. However, a point has area $= 0$, so the point probability $= 0$.



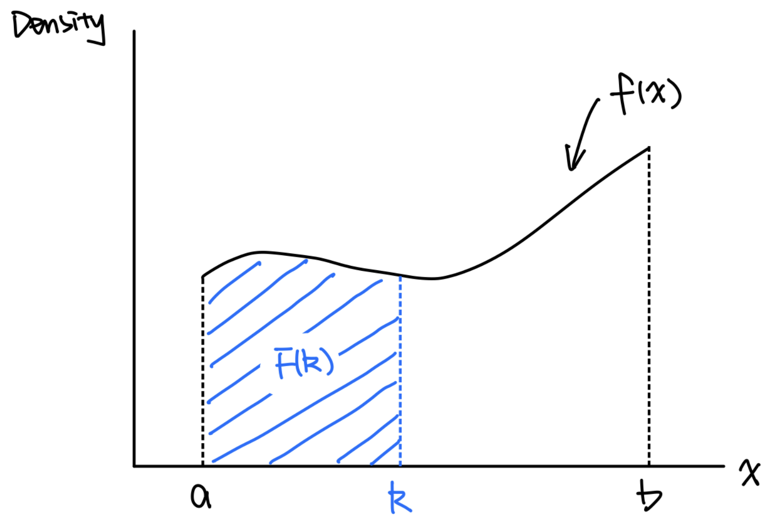
- So our definition for a continuous probability distribution needs to be modified (slightly).
- Solution: describe **density** of x instead of probability.

Definition 1 (Probability density function (PDF)). A function $f(x)$ is called a probability density function (PDF) over $a \leq x \leq b$ if it satisfies the following two criteria:

1. $f(x) \geq 0$ for all x between a and b, and
2. Total area under the curve of $f(x)$ between a and b is 1.

Definition 2 (Cumulative density function (CDF)). A cumulative density function (CDF) describes probability up to a point x. That is, CDF $F(x) = P(X \leq x)$ for random variable X.

Exercise.



1. Label $f(x)$ and $F(k)$ on the graph above.
   See the labelled graph above.
2. In order for $f(x)$ to be a PDF, what additional requirement is needed?
   Two requirements are needed for $f(x)$ to be a PDF: $f(x) \geq 0$ for all x between a and b, and the total area between a and b equals to 1.
   The first requirement is already satisfied given that $f(x)$ lies above the horizontal axis. Thus, the additional requirement needed is that $F(b) = P(X \leq b) = 1$.

- With the help of density functions, we can finally define a continuous probability distribution:

Definition 3 (Continuous probability distribution)**.** A continuous probability distribution describes a valid PDF $f(x)$ at all possible x values for a continuous random variable X.

How does this compare with the discrete case?

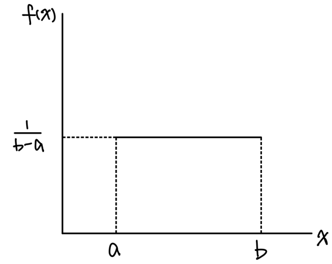
	Discrete Prob Dist	**Continuous Prob Dist**
Describes . . . at all valid x	$P(x)$	$f(x)$
Range of measure for all valid x	$0 \leq P(x) \leq 1$	$f(x) \geq 0$
How to make sure all valid x are covered	$\sum_x P(x) = 1$	$F(b) = \int_a^b f(x)dx = 1$

2 Examples of Continuous Probability Distribution

2.1 Uniform Distribution

- If X is uniformly distributed between point a and b, then $X \sim \text{Uniform}(a, b)$

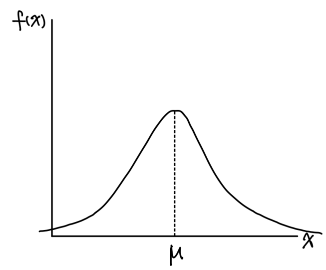
- PDF: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$



- $E(X) = \frac{a+b}{2}$
- $V(X) = \frac{(b-a)^2}{12}$

2.2 Normal Distribution

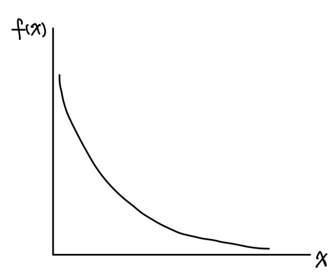
- If X is normally distributed with expected value μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$
- PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $-\infty < x < \infty$



- Usually, for random variable X that follows a normal distribution, it is helpful to standardize the variable so that we are examining a transformed variable with **standard normal distribution** instead.
  - A random variable Z that follows standard normal distribution is denoted as $Z \sim N(0,1)$
  - How to transform X to be standard normal? $\Rightarrow$ Since $X \sim N(\mu, \sigma^2)$, $\frac{X-\mu}{\sigma} = Z \sim N(0,1)$

2.3 Exponential Distribution

- An exponential distribution describes the waiting time until the next "success" event.
- If X is exponentially distributed with success arrival rate λ, then $X \sim \text{exponential}(\lambda)$
- PDF: $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$



- CDF: $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- $E(X) = \frac{1}{\lambda}$
- $\sigma = \sqrt{V(X)} = \frac{1}{\lambda}$