

# Supplementary Handout for Dis 8: Sampling Distributions

## 1 Motivation

- Prior to this week, we discussed random variables, and the possible probability distributions that such random variables could follow.
- As we learned from Dis 5 and 6,
  - A random variable assign a number to each possible outcome.
  - A discrete probability distribution describes the point probability at all possible values for a discrete random variable.
  - A continuous probability distribution describes the density (PDF) at all possible values for a continuous random variable.

Thus, these measures are related to the population.

- However, in reality, what we get to work with is often the sample data, which means we need to relate statistics obtained from samples to the population ( $\Rightarrow$  process of statistical inference).
- This is why we need to look at the distribution of sample statistics, i.e. **sampling distributions**

## 2 Examples of Sampling Distribution

### 2.1 Sampling distribution of the mean

- Statistic of interest:  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ , obtained from simple random sampling
- How  $\bar{X}$  is distributed depends on the distribution of  $X_i$ :
  - If each  $X_i$  is normally distributed, then  $\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2)$  with certainty
  - If each  $X_i$  is NOT normally distributed, we might be able to approximate  $\bar{X}$  using a normal distribution (i.e.  $\bar{X} \stackrel{a}{\sim} N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2)$ ) based on central limit theorem.

**Definition 1 (Central limit theorem (CLT)).** The mean of a random variable drawn from any population is approximately normal for a sufficiently large sample size.

In practice, we use  $n \geq 30$  as the cutoff:

- \* For non-normally distributed  $X_i$ , if  $n \geq 30$ , then CLT can be invoked, and  $\bar{X} \stackrel{a}{\sim} N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2)$
  - \* For non-normally distributed  $X_i$ , if  $n < 30$ , then CLT cannot be invoked, so the distribution of  $\bar{X}$  is undetermined.
- To summarize, for random variable  $X$ , **the sampling distribution of the mean is the following:**

	<b>X is normally distributed</b>	<b>X is NOT normally distributed</b>
<b>Sample size is small (<math>n &lt; 30</math>)</b>		
<b>Sample size is large (<math>n \geq 30</math>)</b>		

- What is  $\mu_{\bar{X}}$  and  $\sigma_{\bar{X}}^2$ ?
  - $\mu_{\bar{X}}$  is the expected value of  $\bar{X}$ :

$$\mu_{\bar{X}} = E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \times n \times \mu_X = \mu_X$$

- $\sigma_{\bar{X}}^2$  is the variance of  $\bar{X}$ , and it depends on the population size:
  - \* If population size is infinitely large (in practice, if  $N \geq 20n$ ),

$$\sigma_{\bar{X}}^2 = V(\bar{X}) = V\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^n V(X_i) = \frac{1}{n^2} \times n \times \sigma_X^2 = \frac{\sigma_X^2}{n}$$

- \* If population size is not infinitely large (in practice, if  $N < 20n$ ), then  $\sigma_{\bar{X}}^2$  needs to be adjusted:

- **Finite population correction factor:** an adjustment applied to the **standard deviation** of sample mean (i.e.  $\sigma_{\bar{X}}$ ), where the correction factor equals to

$$\sqrt{\frac{N-n}{N-1}}$$

- Thus, the standard deviation of sample mean is

$$\sigma_{\bar{X}} = \sqrt{\frac{\sigma_X^2}{n}} \cdot \sqrt{\frac{N-n}{N-1}}$$

which means that the variance of the sample mean is

$$\sigma_{\bar{X}}^2 = (\sigma_{\bar{X}})^2 = \frac{\sigma_X^2}{n} \cdot \frac{N-n}{N-1}$$

(The above part in red color has been adjusted to clarify the definition of finite population correction factor.)

## 2.2 Sampling distribution of the proportion (from a binomial experiment)

- Say that we have a random variable  $X \sim \text{Binomial}(n, p)$  recording the number of successes in  $n$  trials where the probability of success in each trial is  $p$ .
- Turns out, under certain conditions,  $X$  can be well approximated by a normal distribution.

Conditions for normal approximation of a binomial random variable  $X$ :

1.  $np \geq 5$ , and
2.  $n(1-p) \geq 5$

If the aforementioned conditions are satisfied, then

$$X \stackrel{a}{\sim} N(\mu_X, \sigma_X^2)$$

where, based on binomial distribution properties,

$$\begin{aligned}\mu_X &= E(X) = np \\ \sigma_X^2 &= V(X) = np(1 - p)\end{aligned}$$

Aside: A binomial  $X$  is a discrete random variable. However, the approximation approximates  $X \stackrel{a}{\sim} N(np, np(1 - p))$ , which is a continuous distribution.

Thus, a **correction factor for continuity** is needed when calculating probability using the normal approximation.

Exercise. Accounting for the correction factor for continuity, how should the following probabilities be expressed for a binomial random variable  $X$ ?

1.  $P(X = 3) =$

2.  $P(X \geq 3) =$

3.  $P(X > 3) =$

4.  $P(X \leq 3) =$

5.  $P(X < 3) =$

- Why is this needed?  $\Rightarrow$  helps us approximate the sampling distribution of the proportion!
  - As long as a binomial distributed  $X$  can be approximated using a normal distribution (i.e.  $np \geq 5$  and  $n(1 - p) \geq 5$ ), then the proportion of successes ( $\hat{p}$ ) can be approximated using a normal distribution:

$$\hat{p} = \frac{X}{n} \stackrel{a}{\sim} N(\mu_{\hat{p}}, \sigma_{\hat{p}}^2)$$

- What is  $\mu_{\hat{p}}$  and  $\sigma_{\hat{p}}^2$ ?

$$\begin{aligned}\mu_{\hat{p}} &= E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = p \\ \sigma_{\hat{p}}^2 &= V(\hat{p}) = V\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 V(X) = \frac{p(1 - p)}{n}\end{aligned}$$

## 2.3 Sampling distribution of the difference between two means

- Statistic of interest:  $\bar{X} - \bar{Y}$ , where  $X \sim N(\mu_X, \sigma_X^2)$ , and  $Y \sim N(\mu_Y, \sigma_Y^2)$ , and  $X$  is independent of  $Y$
- From subsection 2.1, assuming that the population sizes are sufficiently large, we know that

$$\bar{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n_X}\right) \quad \bar{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{n_Y}\right)$$

- Since the sum of two normal distributions is still a normal distribution, we have

$$\bar{X} - \bar{Y} \sim N(\mu_{\bar{X}-\bar{Y}}, \sigma_{\bar{X}-\bar{Y}}^2)$$

where

$$\begin{aligned}\mu_{\bar{X}-\bar{Y}} &= E(\bar{X} - \bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_X - \mu_Y \\ \sigma_{\bar{X}-\bar{Y}}^2 &= V(\bar{X} - \bar{Y}) = V(\bar{X}) + V(\bar{Y}) - 2 \underbrace{Cov(\bar{X}, \bar{Y})}_{=0 \text{ by indep}} = \frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}\end{aligned}$$