Dis 12: Inference about Two Populations

Related textbook chapter: 13
Ch 13 handout and solution offered by Dr. Pac can be accessed here: Handout Solution
This handout incorporates reviews with all exercises from the handout given by Dr. Pac.

1 Motivation

- Last discussion, we talked about how to test parameters obtained from one population.
- This week, we will look at how to test parameters obtained from two different populations.
- One immediate question: how to test something like

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

where both sides of the equation contain something unknown (μ_1 on the left hand side, μ_2 on the right hand side)?

  - Solution: we can rewrite the above hypotheses as the following:

$$H_0 : \mu_1 - \mu_2 = 0$$
$$H_1 : \mu_1 - \mu_2 \neq 0$$

    Now the left hand side is something unknown related to the population parameters & waiting to be tested, and the right hand side is a concrete number to test the unknown against.

- Similar to last discussion about the one population case, we will discuss how to test / compare the following three sets of population parameters:
  1. The population means ($\mu_1 - \mu_2$)
  2. The population variances ($\frac{\sigma_1^2}{\sigma_2^2}$)
  3. The population success proportions ($p_1 - p_2$, from two binomial experiments)

2 General Approach

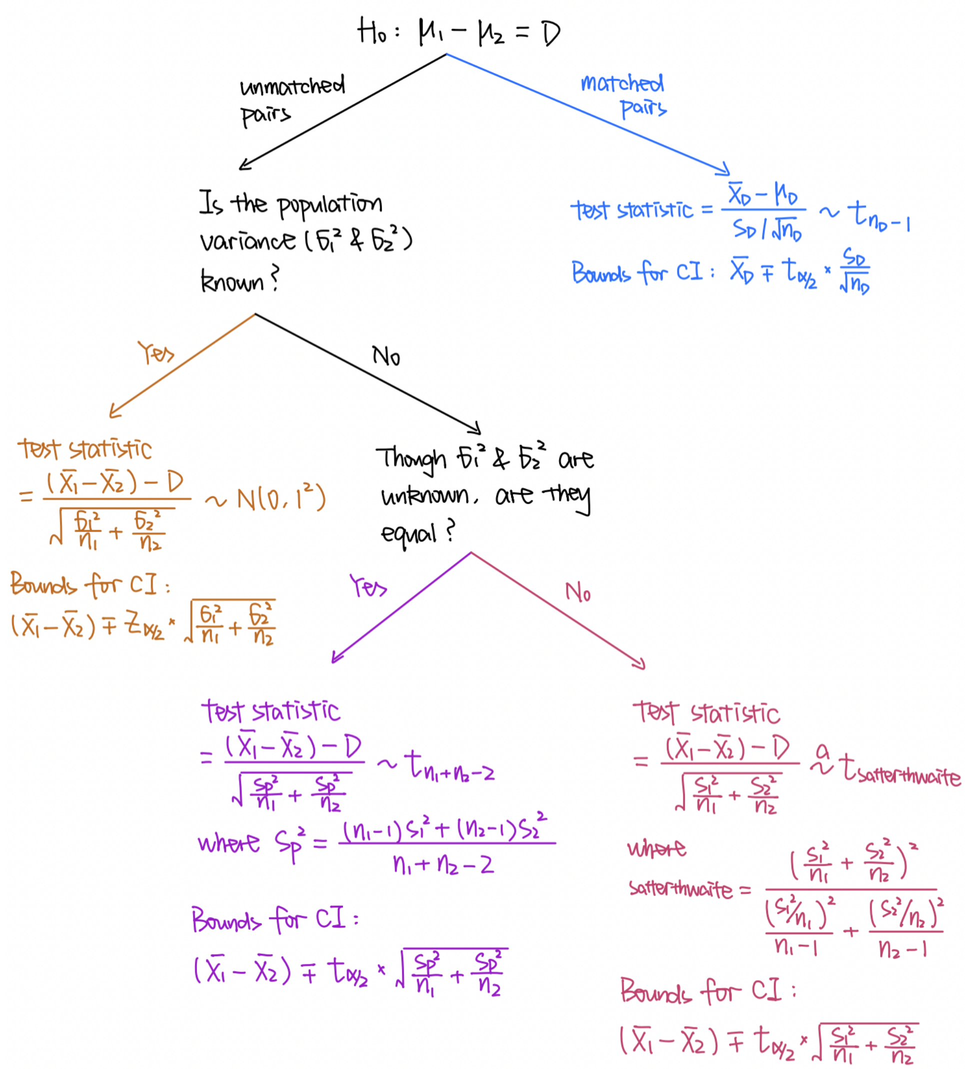
- The general testing approach that we will take is very similar to what we talked about last discussion:
  - If using the rejection region & test statistic method:
    1. Find out what distribution the test statistic follows.
    2. Set up significance level & the appropriate sizes for the tail-ends of the distribution. Find cutoff values to construct rejection region.
    3. Calculate test statistic from the given sample, and see if it falls within the rejection region.
       * If test statistic falls within the rejection region, then we reject H_0 under the specified significance level.

* If test statistic doesn't fall within the rejection region, then we fail to reject H_0 under the specified significance level.

- If using the confidence interval method:
  1. Find out what distribution the test statistic follows.
  2. Construct $(1-\alpha)$ level of confidence interval based on the confidence level $(1-\alpha)$ and the associated cutoffs from the distribution.
  3. Check if the hypothesized null falls within the $(1-\alpha)$ confidence interval:
     * If the hypothesized null falls within the $(1-\alpha)$ confidence interval, then we fail to reject H_0 at α significance level.
     * If the hypothesized null doesn't fall within the $(1-\alpha)$ confidence interval, then we reject H_0 at α significance level.

3 Inference about Two Populations

3.1 About the difference between two population means ($\mu_1 - \mu_2$)



3.2 About two population variances ($\frac{\sigma_1^2}{\sigma_2^2}$)

$$H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$$

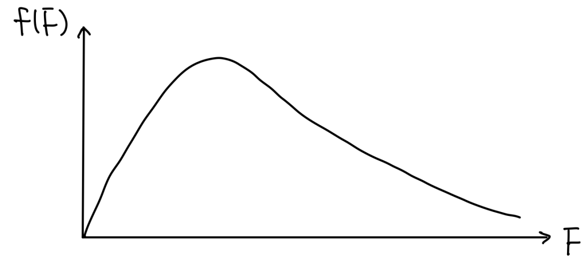
$$\downarrow$$

numerator DOF (ν_1)

denominator DOF (ν_2)

$$\text{test statistic} = \frac{s_1^2}{s_2^2} \sim F_{n_1-1,\, n_2-1}$$

- We are introduced to a new distribution: **F distribution**
  - F distribution usually looks like the following:



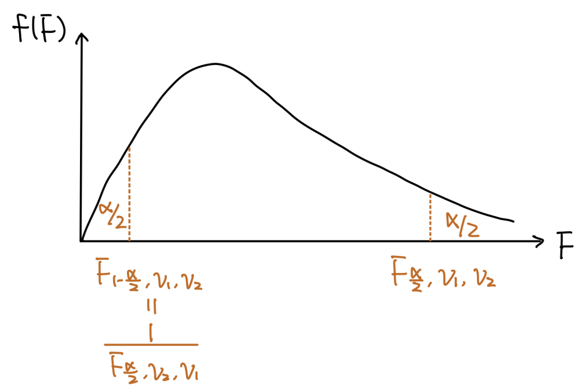
    It looks very similar to the Chi-squared distribution. (In fact, F distribution is actually obtained by dividing a Chi-squared distribution by a different Chi-squared distribution.)

  - A F distribution is denoted as the following:

$$F_{\nu_1,\nu_2}$$

    Here, ν_1 is the numerator degree of freedom, and ν_2 is the denominator degree of freedom.

- The F distribution table is usually given for a right tail area only – how does one find the appropriate left tail cutoff value, if we are performing a two-tail test?



Solution: the left tail cutoff value can be found as right tail cutoff value from a slightly different F distribution (switching the numerator and denominator degree of freedom)

$$F_{1-\frac{\alpha}{2},\nu_1,\nu_2} = \frac{1}{F_{\frac{\alpha}{2},\nu_2,\nu_1}}$$

3.3 About the difference between two success proportions ($p_1 - p_2$)

$H_0 : p_1 - p_2 = D$

$\downarrow$

Does $\hat{p}_1 - \hat{p}_2$ follow a normal distribution?

i.e. Check if ALL of the following hold:

1. $n_1 \hat{p}_1 \geq 5$
2. $n_1(1-\hat{p}_1) \geq 5$
3. $n_2 \hat{p}_2 \geq 5$
4. $n_2(1-\hat{p}_2) \geq 5$

No (one / more fail(s))

Yes (ALL 4 hold)

Does $D = 0$?

Yes ($H_0 : p_1 - p_2 = 0$)

test statistic

$$= \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \overset{a}{\sim} N(0, 1^2)$$

where $\hat{p} = \dfrac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$

Bounds for CI:

$$(\hat{p}_1 - \hat{p}_2) \mp z_{\alpha/2} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

No ($H_0 : p_1 - p_2 = D$, where $D \neq 0$)

test statistic

$$= \frac{(\hat{p}_1 - \hat{p}_2) - D}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \overset{a}{\sim} N(0, 1^2)$$

Bounds for CI:

$$(\hat{p}_1 - \hat{p}_2) \mp z_{\alpha/2} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

4 Exercises

1. You are conducting a poll for a hat company in preparation for the upcoming holiday season. In your samples of 296 men and 282 women, 26 percent of men and 34 percent of women say they plan to purchase a hat.

   (a) Using a 5% significance level, test whether women are more likely than men to purchase a hat.
   Define w subscript for women, and m subscript for men.

      i. Hypotheses:

$$H_0 : p_w - p_m = 0$$
$$H_1 : p_w - p_m > 0$$

ii. Check distribution:

$$\begin{aligned}
n_w \hat{p}_w &= 282 \times 0.34 = 95.88 \geq 5 \\
n_w(1-\hat{p}_w) &= 282 \times (1-0.34) = 186.12 \geq 5 \\
n_m \hat{p}_m &= 296 \times 0.26 = 76.96 \geq 5 \\
n_m(1-\hat{p}_m) &= 296 \times (1-0.26) = 219.04 \geq 5
\end{aligned}$$

Since the above four conditions are satisfied, $\hat{p}_w - \hat{p}_m \overset{a}{\sim} N\left(0, \left(\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_w}+\frac{1}{n_m}\right)}\right)^2\right)$.

The distribution has mean 0 since the null hypothesis defaults the true difference between ps to be 0. Additionally,

$$\hat{p} = \frac{\hat{p}_w n_w + \hat{p}_m n_m}{n_w + n_m}$$

The test statistic is the standardized version of $\hat{p}_w - \hat{p}_m$:

$$\text{test statistic} = \frac{(\hat{p}_w - \hat{p}_m) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_w}+\frac{1}{n_m}\right)}} \overset{a}{\sim} N(0, 1^2)$$

iii. Find rejection region:
Since test statistic follows (approximately) a normal distribution, and our alternative hypothesis permits rejection when we are far in the right side of the distribution, with $\alpha = 5\%$, our rejection region is when test statistic > 1.645.
Checking the value of test statistic for this specific sample:

$$\begin{aligned}
\hat{p} &= \frac{\hat{p}_w n_w + \hat{p}_m n_m}{n_w + n_m} = 0.30 \\
\text{test statistic} &= \frac{(\hat{p}_w - \hat{p}_m) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_w}+\frac{1}{n_m}\right)}} = 2.10
\end{aligned}$$

Since test statistic > 1.645 is satisfied, we reject the null hypothesis at 5% significance level. Thus, we conclude that women are more likely than men to purchase a hat at 5% significance level.

(b) Construct and interpret a 95% confidence interval for the difference between the proportion of women and men who plan to purchase a hat.

In this case, the formula for the confidence interval is:

$$(\hat{p}_w - \hat{p}_m) \mp z_{\alpha/2}\sqrt{\frac{\hat{p}_w(1-\hat{p}_w)}{n_w} + \frac{\hat{p}_m(1-\hat{p}_m)}{n_m}}$$

Plugging the values from this problem into the formula, we get:

$$(0.34 - 0.26) \mp 1.96 \times \sqrt{\frac{0.34(1-0.34)}{282} + \frac{0.26(1-0.26)}{296}}$$

This confidence interval estimator contains the true difference between the proportion of women and men who plan to buy hats 95% of the time. For these samples, we get a confidence interval estimate of $[0.005, 0.155]$.

(c) If women are more than 5 percentage points more likely to purchase a hat than men, then the store will run a special female targeted promotion. Would you advise them to run the promotion? (Use a 5% significance level.)

i. Hypotheses:

$$H_0 : p_w - p_m = 0.05$$
$$H_1 : p_w - p_m > 0.05$$

ii. Check distribution:

$$\begin{aligned} n_w \hat{p}_w &= 282 \times 0.34 = 95.88 \geq 5 \\ n_w(1-\hat{p}_w) &= 282 \times (1-0.34) = 186.12 \geq 5 \\ n_m \hat{p}_m &= 296 \times 0.26 = 76.96 \geq 5 \\ n_m(1-\hat{p}_m) &= 296 \times (1-0.26) = 219.04 \geq 5 \end{aligned}$$

Since the above four conditions are satisfied, $\hat{p}_w - \hat{p}_m \overset{a}{\sim} N\left(0.05, \left(\sqrt{\frac{\hat{p}_w(1-\hat{p}_w)}{n_w} + \frac{\hat{p}_m(1-\hat{p}_m)}{n_m}}\right)^2\right)$.

The distribution has mean 0.05 since the null hypothesis defaults the true difference between ps to be 0.05.

The test statistic is the standardized version of $\hat{p}_w - \hat{p}_m$:

$$\text{test statistic} = \frac{(\hat{p}_w - \hat{p}_m) - 0.05}{\sqrt{\frac{\hat{p}_w(1-\hat{p}_w)}{n_w} + \frac{\hat{p}_m(1-\hat{p}_m)}{n_m}}} \overset{a}{\sim} N(0, 1^2)$$

iii. Find rejection region:

Since test statistic follows (approximately) a normal distribution, and our alternative hypothesis permits rejection when we are far in the right side of the distribution, with $\alpha = 5\%$, our rejection region is when test statistic > 1.645.

Checking the value of test statistic for this specific sample:

$$\text{test statistic} = \frac{(\hat{p}_w - \hat{p}_m) - 0}{\sqrt{\frac{\hat{p}_w(1-\hat{p}_w)}{n_w} + \frac{\hat{p}_m(1-\hat{p}_m)}{n_m}}} = 0.789$$

Since test statistic > 1.645 is NOT satisfied, we fail to reject the null hypothesis at 5% significance level. Thus, we would not be able to advise the store to run the promotion.

2. Continuing with the previous example, the hat company now wants to get a sense of how many hats men and women will purchase. So they commission a new survey, this time asking respondents how many hats they plan to purchase. They survey 200 men and 200 women, with men planning to purchase an average of 2.2 hats and women purchasing to purchase an average of 1.8 hats in your sample. You estimate a sample standard deviation of 2.3 hats for men and 2.0 hats for women.

(a) Assuming the population variances are unequal, test whether the mean number of planned hat purchases are equal for men and women. (Use a 5% significance level.)

Define w subscript for women, and m subscript for men.

i. Hypotheses:

$$H_0 : \mu_w - \mu_m = 0$$
$$H_1 : \mu_w - \mu_m \neq 0$$

ii. Check distribution:

$$\text{test statistic} = \frac{(\bar{X}_w - \bar{X}_m) - 0}{\sqrt{\frac{s_w^2}{n_w} + \frac{s_m^2}{n_m}}} \sim t_{\text{satterthwaite}}$$

where

$$\text{satterthwaite} = \frac{\left(s_w^2/n_w + s_m^2/n_m\right)^2}{\frac{(s_w^2/n_w)^2}{n_w-1} + \frac{(s_m^2/n_m)^2}{n_m-1}} = 390$$

and satterthwaite is the degree of freedom (DOF) for the t-distribution.
Given that DOF $= 390 > 200$, we can consider t-distribution essentially as standard normal distribution in this case.

iii. Find rejection region:
Since test statistic follows (essentially) a normal distribution (since DOF is large enough), and our alternative hypothesis permits rejection when we are far in the left or right side of the distribution, with $\alpha = 5\%$, our rejection region is when $|\text{test statistic}| > 1.96$.
Checking the value of test statistic for this specific sample:

$$\text{test statistic} = \frac{(\bar{X}_w - \bar{X}_m) - 0}{\sqrt{\frac{s_w^2}{n_w} + \frac{s_m^2}{n_m}}} = -1.86$$

Since $|\text{test statistic}| > 1.96$ is NOT satisfied, we fail to reject the null hypothesis at 5% significance level. Thus, we cannot conclude that the mean number of planned hat purchases are different for men and women.

(b) Construct and interpret a 95% confidence interval for the difference between the mean planned number of hat purchases for women and men.

In this case, the formula for the confidence interval is:

$$(\bar{X}_w - \bar{X}_m) \mp t_{\alpha/2}\sqrt{\frac{s_w^2}{n_w} + \frac{s_m^2}{n_m}}$$

Plugging the values from this problem into the formula, we get:

$$(1.8 - 2.2) \mp 1.96 \times \sqrt{\frac{2.0^2}{200} + \frac{2.3^2}{200}}$$

This confidence interval estimator contains the true difference between the mean planned number of hat purchases for women and men 95% of the time. For these samples, we get a confidence interval estimate of $[-0.822, 0.022]$.

(c) Repeat previous two questions while assuming equal population variance.

i. Hypotheses:

$$H_0 : \mu_w - \mu_m = 0$$
$$H_1 : \mu_w - \mu_m \neq 0$$

ii. Check distribution:

$$\text{test statistic} = \frac{(\bar{X}_w - \bar{X}_m) - 0}{\sqrt{\frac{s_p^2}{n_w} + \frac{s_p^2}{n_m}}} \sim t_{n_w+n_m-2}$$

where

$$s_p^2 = \frac{(n_w - 1)s_w^2 + (n_m - 1)s_m^2}{n_w + n_m - 2} = 4.645$$

Given that the degree of freedom $= n_w + n_m - 2 = 200 + 200 - 2 = 398 > 200$, we can consider t-distribution essentially as standard normal distribution in this case.

iii. Find rejection region:
Since test statistic follows (essentially) a normal distribution (since DOF is large enough), and our alternative hypothesis permits rejection when we are far in the left or right side of the distribution, with $\alpha = 5\%$, our rejection region is when $|\text{test statistic}| > 1.96$.
Checking the value of test statistic for this specific sample:

$$\text{test statistic} = \frac{(\bar{X}_w - \bar{X}_m) - 0}{\sqrt{\frac{s_p^2}{n_w} + \frac{s_p^2}{n_m}}} = -1.856$$

Since $|\text{test statistic}| > 1.96$ is NOT satisfied, we fail to reject the null hypothesis at 5% significance level. Thus, when we assume that population variance is equal, we still cannot conclude that the mean number of planned hat purchases are different for men and women.

To construct confidence interval, the formula for the confidence interval is now:

$$(\bar{X}_w - \bar{X}_m) \mp t_{\alpha/2}\sqrt{\frac{s_p^2}{n_w} + \frac{s_p^2}{n_m}}$$

Plugging the values from this problem into the formula, we get:

$$(1.8 - 2.2) \mp 1.96 \times \sqrt{\frac{4.645}{200} + \frac{4.645}{200}}$$

So the confidence interval estimate is $[-0.822, 0.022]$.

(d) Using a 5% significance level, test whether the population variance of planned hat purchases for men and women are equal.

i. Hypotheses:

$$H_0 : \frac{\sigma_w^2}{\sigma_m^2} = 1$$
$$H_1 : \frac{\sigma_w^2}{\sigma_m^2} \neq 1$$

ii. Check distribution:

$$\text{test statistic} = \frac{s_w^2}{s_m^2} \sim F_{n_w-1,n_m-1}$$

where $n_w - 1 = 200 - 1 = 199$ is the numerator degree of freedom (DOF), and $n_m - 1 = 200 - 1 = 199$ is the denominator DOF.

iii. Find rejection region:
Since test statistic follows a F-distribution, and our alternative hypothesis permits rejection when we are far in the left or right side of the distribution, with $\alpha = 5\%$, by looking up the F-distribution table with right tail area of $2.5\% = 0.025$, our rejection region becomes

$$\text{test statistic} < 0.76 \quad \text{or} \quad \text{test statistic} > 1.32$$

Checking the value of test statistic for this specific sample:

$$\text{test statistic} = \frac{s_w^2}{s_m^2} = 0.7561$$

Since test statistic < 0.76, we reject the null hypothesis at 5% significance level. Thus, we conclude that the variance of planned hat purchases for men and women are not equal.

3. Now suppose that the survey in Exercise 2, rather than being a simple random sample of men and a simple random sample of women, the survey had a paired design. In particular, the data collection was paired using the respondents' zip codes. As before, you have data for 200 women and 200 men. Using this data, you compute the paired mean difference and standard deviation, which are: $\bar{X}_D = -0.4$, and $s_D = 2$.

   (a) Construct a hypothesis test to determine whether these populations have different means. (Use a 5% significance level.)

   i. Hypotheses:

$$H_0 : \mu_D = 0$$
$$H_1 : \mu_D \neq 0$$

   ii. Check distribution:

$$\text{test statistic} = \frac{\bar{X}_D - \mu_D}{s_D/\sqrt{n_D}} \sim t_{n_D-1}$$

   where $n_D - 1 = 200 - 1 = 199$ is the degree of freedom (DOF) for the t-distribution.

   iii. Find rejection region:

Since test statistic follows a t-distribution (DOF is not large enough to approximate the t-distribution as standard normal), and our alternative hypothesis permits rejection when we are far in the left or right side of the distribution, with $\alpha = 5\%$, our rejection region is when $|\text{test statistic}| > 1.972$.
Checking the value of test statistic for this specific sample:

$$\text{test statistic} = \frac{\bar{X}_D - \mu_D}{s_D / \sqrt{n_D}} = -2.828$$

Since $|\text{test statistic}| > 1.972$ is satisfied, we reject the null hypothesis at 5% significance level. Thus, we conclude with a 5% significance level that the mean number of planned hat purchases are different for men and women.

(b) Construct and interpret a 95% confidence interval for the difference between the mean planned number of hat purchases for women and men.

In this case, the formula for the confidence interval is:

$$\bar{X}_D \mp t_{\alpha/2} \frac{s_D}{\sqrt{n_D}}$$

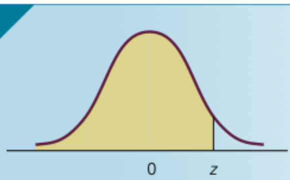
Plugging the values from this problem into the formula, we get:

$$-0.4 \mp 1.972 \times \frac{2}{\sqrt{200}}$$

This confidence interval estimator contains the true difference between the mean planned number of hat purchases for women and men 95% of the time. For these samples, we get a confidence interval estimate of $[-0.679, -0.121]$.

Probability table for a standard normal distribution ($z \geq 0$)

TABLE 3 *(Continued)*

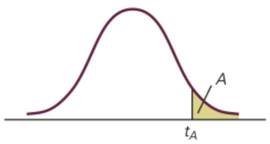


$P(-\infty < Z < z)$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Probability table for a t-distribution

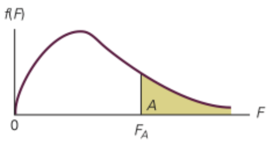
TABLE 4
Critical Values of the Student *t* Distribution



Degrees of Freedom	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.306	1.690	2.030	2.438	2.724
40	1.303	1.684	2.021	2.423	2.704
45	1.301	1.679	2.014	2.412	2.690
50	1.299	1.676	2.009	2.403	2.678
55	1.297	1.673	2.004	2.396	2.668
60	1.296	1.671	2.000	2.390	2.660
65	1.295	1.669	1.997	2.385	2.654
70	1.294	1.667	1.994	2.381	2.648
75	1.293	1.665	1.992	2.377	2.643
80	1.292	1.664	1.990	2.374	2.639
85	1.292	1.663	1.988	2.371	2.635
90	1.291	1.662	1.987	2.368	2.632
95	1.291	1.661	1.985	2.366	2.629
100	1.290	1.660	1.984	2.364	2.626
110	1.289	1.659	1.982	2.361	2.621
120	1.289	1.658	1.980	2.358	2.617
130	1.288	1.657	1.978	2.355	2.614
140	1.288	1.656	1.977	2.353	2.611
150	1.287	1.655	1.976	2.351	2.609
160	1.287	1.654	1.975	2.350	2.607
170	1.287	1.654	1.974	2.348	2.605
180	1.286	1.653	1.973	2.347	2.603
190	1.286	1.653	1.973	2.346	2.602
200	1.286	1.653	1.972	2.345	2.601
∞	1.282	1.645	1.960	2.326	2.576

Probability table for a F-distribution, with $A = 0.025$

TABLE **6(b)** **Values of the *F*-Distribution: *A* = .025**



ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	648	799	864	900	922	937	948	957	963	969	973	977	980	983	985	987	989	990	992	993
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	14.3	14.3	14.3	14.2	14.2	14.2	14.2	14.2
4	12.2	10.6	10.0	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.71	8.68	8.66	8.63	8.61	8.59	8.58	8.56
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34	6.33
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18	5.17
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48	4.47
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02	4.00
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68	3.67
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44	3.42
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24	3.23
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09	3.07
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96	2.95
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.84
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77	2.76
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.68
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63	2.62
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58	2.56
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53	2.51
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48	2.46
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41	2.39
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35	2.33
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29	2.28
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25	2.23
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21	2.20
35	5.48	4.11	3.52	3.18	2.96	2.80	2.68	2.58	2.50	2.44	2.39	2.34	2.30	2.27	2.23	2.21	2.18	2.16	2.14	2.12
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07
45	5.38	4.01	3.42	3.09	2.86	2.70	2.58	2.49	2.41	2.35	2.29	2.25	2.21	2.17	2.14	2.11	2.09	2.07	2.04	2.03
50	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03	2.01	1.99
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96	1.94
70	5.25	3.89	3.31	2.97	2.75	2.59	2.47	2.38	2.30	2.24	2.18	2.14	2.10	2.06	2.03	2.00	1.97	1.95	1.93	1.91
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28	2.21	2.16	2.11	2.07	2.03	2.00	1.97	1.95	1.92	1.90	1.88
90	5.20	3.84	3.26	2.93	2.71	2.55	2.43	2.34	2.26	2.19	2.14	2.09	2.05	2.02	1.98	1.95	1.93	1.91	1.88	1.86
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89	1.87	1.85
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.10	2.05	2.01	1.98	1.94	1.92	1.89	1.87	1.84	1.82
140	5.13	3.79	3.21	2.88	2.66	2.50	2.38	2.28	2.21	2.14	2.09	2.04	2.00	1.96	1.93	1.90	1.87	1.85	1.83	1.81
160	5.12	3.78	3.20	2.87	2.65	2.49	2.37	2.27	2.19	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80
180	5.11	3.77	3.19	2.86	2.64	2.48	2.36	2.26	2.19	2.12	2.07	2.02	1.98	1.94	1.91	1.88	1.85	1.83	1.81	1.79
200	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82	1.80	1.78
∞	5.03	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.95	1.90	1.87	1.83	1.80	1.78	1.75	1.73	1.71

NUMERATOR DEGREES OF FREEDOM (ν_1); DENOMINATOR DEGREES OF FREEDOM (ν_2)

NUMERATOR DEGREES OF FREEDOM

DENOMINATOR DEGREES OF FREEDOM ν_2 \ ν_1	22	24	26	28	30	35	40	45	50	60	70	80	90	100	120	140	160	180	200	∞
1	995	997	999	1000	1001	1004	1006	1007	1008	1010	1011	1012	1013	1013	1014	1015	1015	1015	1016	1018
2	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5
3	14.1	14.1	14.1	14.1	14.1	14.1	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0	13.9	13.9	13.9	13.9	13.9	13.9
4	8.53	8.51	8.49	8.48	8.46	8.43	8.41	8.39	8.38	8.36	8.35	8.33	8.33	8.32	8.31	8.30	8.30	8.29	8.29	8.26
5	6.30	6.28	6.26	6.24	6.23	6.20	6.18	6.16	6.14	6.12	6.11	6.10	6.09	6.08	6.07	6.06	6.06	6.05	6.05	6.02
6	5.14	5.12	5.10	5.08	5.07	5.04	5.01	4.99	4.98	4.96	4.94	4.93	4.92	4.92	4.90	4.90	4.89	4.89	4.88	4.85
7	4.44	4.41	4.39	4.38	4.36	4.33	4.31	4.29	4.28	4.25	4.24	4.23	4.22	4.21	4.20	4.19	4.18	4.18	4.18	4.14
8	3.97	3.95	3.93	3.91	3.89	3.86	3.84	3.82	3.81	3.78	3.77	3.76	3.75	3.74	3.73	3.72	3.71	3.71	3.70	3.67
9	3.64	3.61	3.59	3.58	3.56	3.53	3.51	3.49	3.47	3.45	3.43	3.42	3.41	3.40	3.39	3.38	3.38	3.37	3.37	3.33
10	3.39	3.37	3.34	3.33	3.31	3.28	3.26	3.24	3.22	3.20	3.18	3.17	3.16	3.15	3.14	3.13	3.13	3.12	3.12	3.08
11	3.20	3.17	3.15	3.13	3.12	3.09	3.06	3.04	3.03	3.00	2.99	2.97	2.96	2.96	2.94	2.94	2.93	2.92	2.92	2.88
12	3.04	3.02	3.00	2.98	2.96	2.93	2.91	2.89	2.87	2.85	2.83	2.82	2.81	2.80	2.79	2.78	2.77	2.77	2.76	2.73
13	2.92	2.89	2.87	2.85	2.84	2.80	2.78	2.76	2.74	2.72	2.70	2.69	2.68	2.67	2.66	2.65	2.64	2.64	2.63	2.60
14	2.81	2.79	2.77	2.75	2.73	2.70	2.67	2.65	2.64	2.61	2.60	2.58	2.57	2.56	2.55	2.54	2.54	2.53	2.53	2.49
15	2.73	2.70	2.68	2.66	2.64	2.61	2.59	2.56	2.55	2.52	2.51	2.49	2.48	2.47	2.46	2.45	2.44	2.44	2.44	2.40
16	2.65	2.63	2.60	2.58	2.57	2.53	2.51	2.49	2.47	2.45	2.43	2.42	2.40	2.40	2.38	2.37	2.37	2.36	2.36	2.32
17	2.59	2.56	2.54	2.52	2.50	2.47	2.44	2.42	2.41	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.30	2.29	2.29	2.25
18	2.53	2.50	2.48	2.46	2.44	2.41	2.38	2.36	2.35	2.32	2.30	2.29	2.28	2.27	2.26	2.25	2.24	2.23	2.23	2.19
19	2.48	2.45	2.43	2.41	2.39	2.36	2.33	2.31	2.30	2.27	2.25	2.24	2.23	2.22	2.20	2.19	2.19	2.18	2.18	2.13
20	2.43	2.41	2.39	2.37	2.35	2.31	2.29	2.27	2.25	2.22	2.20	2.19	2.18	2.17	2.16	2.15	2.14	2.13	2.13	2.09
22	2.36	2.33	2.31	2.29	2.27	2.24	2.21	2.19	2.17	2.14	2.13	2.11	2.10	2.09	2.08	2.07	2.06	2.05	2.05	2.00
24	2.30	2.27	2.25	2.23	2.21	2.17	2.15	2.12	2.11	2.08	2.06	2.05	2.03	2.02	2.01	2.00	1.99	1.99	1.98	1.94
26	2.24	2.22	2.19	2.17	2.16	2.12	2.09	2.07	2.05	2.03	2.01	1.99	1.98	1.97	1.95	1.94	1.94	1.93	1.92	1.88
28	2.20	2.17	2.15	2.13	2.11	2.08	2.05	2.03	2.01	1.98	1.96	1.94	1.93	1.92	1.91	1.90	1.89	1.88	1.88	1.83
30	2.16	2.14	2.11	2.09	2.07	2.04	2.01	1.99	1.97	1.94	1.92	1.90	1.89	1.88	1.87	1.86	1.85	1.84	1.84	1.79
35	2.09	2.06	2.04	2.02	2.00	1.96	1.93	1.91	1.89	1.86	1.84	1.82	1.81	1.80	1.79	1.77	1.77	1.76	1.75	1.70
40	2.03	2.01	1.98	1.96	1.94	1.90	1.88	1.85	1.83	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70	1.70	1.69	1.64
45	1.99	1.96	1.94	1.92	1.90	1.86	1.83	1.81	1.79	1.76	1.74	1.72	1.70	1.69	1.68	1.66	1.66	1.65	1.64	1.59
50	1.96	1.93	1.91	1.89	1.87	1.83	1.80	1.77	1.75	1.72	1.70	1.68	1.67	1.66	1.64	1.63	1.62	1.61	1.60	1.55
60	1.91	1.88	1.86	1.83	1.82	1.78	1.74	1.72	1.70	1.67	1.64	1.63	1.61	1.60	1.58	1.57	1.56	1.55	1.54	1.48
70	1.88	1.85	1.82	1.80	1.78	1.74	1.71	1.68	1.66	1.63	1.60	1.59	1.57	1.56	1.54	1.53	1.52	1.51	1.50	1.44
80	1.85	1.82	1.79	1.77	1.75	1.71	1.68	1.65	1.63	1.60	1.57	1.55	1.54	1.53	1.51	1.49	1.48	1.47	1.47	1.40
90	1.83	1.80	1.77	1.75	1.73	1.69	1.66	1.63	1.61	1.58	1.55	1.53	1.52	1.50	1.48	1.47	1.46	1.45	1.44	1.37
100	1.81	1.78	1.76	1.74	1.71	1.67	1.64	1.61	1.59	1.56	1.53	1.51	1.50	1.48	1.46	1.45	1.44	1.43	1.42	1.35
120	1.79	1.76	1.73	1.71	1.69	1.65	1.61	1.59	1.56	1.53	1.50	1.48	1.47	1.45	1.43	1.42	1.41	1.40	1.39	1.31
140	1.77	1.74	1.72	1.69	1.67	1.63	1.60	1.57	1.55	1.51	1.48	1.46	1.45	1.43	1.41	1.39	1.38	1.37	1.36	1.28
160	1.76	1.73	1.70	1.68	1.66	1.62	1.58	1.55	1.53	1.50	1.47	1.45	1.43	1.42	1.39	1.38	1.36	1.35	1.35	1.26
180	1.75	1.72	1.69	1.67	1.65	1.61	1.57	1.54	1.52	1.48	1.46	1.43	1.42	1.40	1.38	1.36	1.35	1.34	1.33	1.25
200	1.74	1.71	1.68	1.66	1.64	1.60	1.56	1.53	1.51	1.47	1.45	1.42	1.41	1.39	1.37	1.35	1.34	1.33	1.32	1.23
∞	1.67	1.64	1.61	1.59	1.57	1.52	1.49	1.46	1.43	1.39	1.36	1.33	1.31	1.30	1.27	1.25	1.23	1.22	1.21	1.00