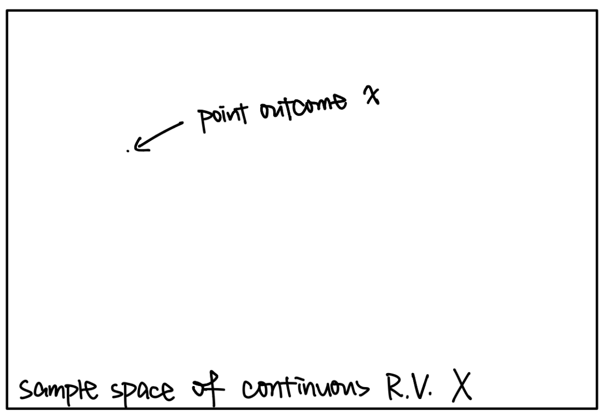
Dis 6: Continuous Probability Distributions

Related textbook chapter: 8
Ch 8 handout and solution offered by Dr. Pac can be accessed here: Handout Solution
This handout incorporates reviews with all exercises from the handout given by Dr. Pac.

1 Density Functions

- When we examined discrete probability distributions, we said that a discrete probability distribution should describe point probability $P(x)$ at all possible x outcome values.
- Ideally, we would like to provide a similar definition for continuous probability distributions.
- But there's a problem: when random variable X is continuous, **point probability equals to 0 at every single point** ($P(X = x) = 0$ for all x).
  - Reason 1: A continuous random variable has uncountable amount of values, so if each outcome value has probability $\varepsilon > 0$, then the sum of all probabilities would equal to ∞ instead of 1.
  - Reason 2: Say that the rectangle below represents the sample space for a continuous random variable. The probability of hitting a point within the rectangle is the area of the point divided by the area of the rectangle. However, a point has area $= 0$, so the point probability $= 0$.



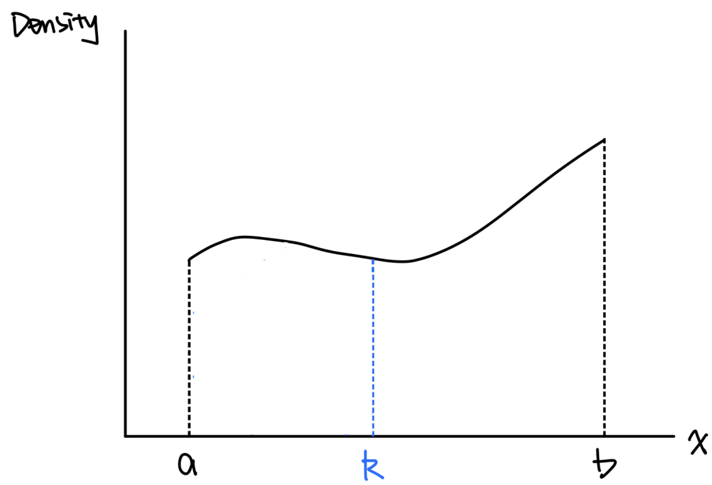
- So our definition for a continuous probability distribution needs to be modified (slightly).
- Solution: describe **density** of x instead of probability.

Definition 1 (Probability density function (PDF)). A function $f(x)$ is called a probability density function (PDF) over $a \leq x \leq b$ if it satisfies the following two criteria:

1. $f(x) \geq 0$ for all x between a and b, and
2. Total area under the curve of $f(x)$ between a and b is 1.

Definition 2 (Cumulative density function (CDF)). A cumulative density function (CDF) describes probability up to a point x. That is, CDF $F(x) = P(X \leq x)$ for random variable X.

Exercise.



1. Label $f(x)$ and $F(k)$ on the graph above.
   See the labelled graph above.
2. In order for $f(x)$ to be a PDF, what additional requirement is needed?
   Two requirements are needed for $f(x)$ to be a PDF: $f(x) \geq 0$ for all x between a and b, and the total area between a and b equals to 1.
   The first requirement is already satisfied given that $f(x)$ lies above the horizontal axis. Thus, the additional requirement needed is that $F(b) = P(X \leq b) = 1$.

- With the help of density functions, we can finally define a continuous probability distribution:

Definition 3 (Continuous probability distribution)**.** A continuous probability distribution describes a valid PDF $f(x)$ at all possible x values for a continuous random variable X.

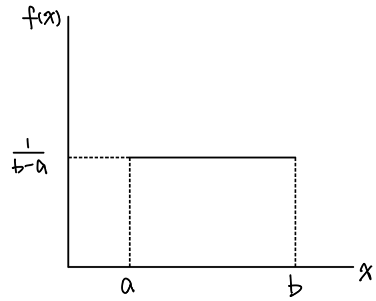
How does this compare with the discrete case?

	Discrete Prob Dist	**Continuous Prob Dist**
Describes ... at all valid x	$P(x)$	$f(x)$
Range of measure for all valid x	$0 \leq P(x) \leq 1$	$f(x) \geq 0$
How to make sure all valid x are covered	$\sum_x P(x) = 1$	$F(b) = \int_a^b f(x)dx = 1$

2 Examples of Continuous Probability Distribution

2.1 Uniform Distribution

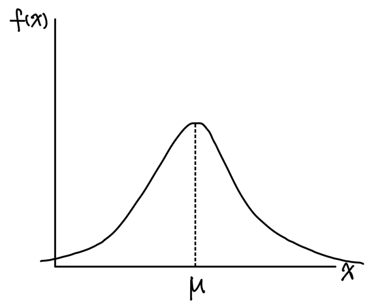
- If X is uniformly distributed between point a and b, then $X \sim \text{Uniform}(a, b)$
- PDF: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$



- $E(X) = \frac{a+b}{2}$
- $V(X) = \frac{(b-a)^2}{12}$

2.2 Normal Distribution

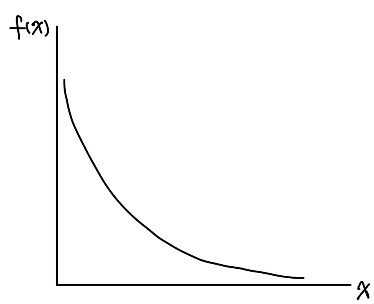
- If X is normally distributed with expected value μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$
- PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $-\infty < x < \infty$



- Usually, for random variable X that follows a normal distribution, it is helpful to standardize the variable so that we are examining a transformed variable with **standard normal distribution** instead.
  - A random variable Z that follows standard normal distribution is denoted as $Z \sim N(0, 1)$
  - How to transform X to be standard normal? $\Rightarrow$ Since $X \sim N(\mu, \sigma^2)$, $\frac{X-\mu}{\sigma} = Z \sim N(0, 1)$

2.3 Exponential Distribution

- An exponential distribution describes the waiting time until the next "success" event.
- If X is exponentially distributed with success arrival rate λ, then $X \sim \text{exponential}(\lambda)$
- PDF: $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$



- CDF: $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- $E(X) = \frac{1}{\lambda}$
- $\sigma = \sqrt{V(X)} = \frac{1}{\lambda}$

3 Exercises

1. The weekly output of a steel mill is a uniformly distributed random variable that lies between 110 and 175 metric tons

   (a) What is the probability the steel mill will produce more than 150 metric tons next week?

   (b) What is the probability the mill will produce between 120 and 160 metric tons next week?

   (c) What is the expected value and variance of the mill's weekly output?

2. An analysis of the amount of interest paid monthly by Visa cardholders is normally distributed with a mean of \$27 and a standard deviation of \$6. (Note: A probability table for a standard normal can be found on the last page of the handout.)

   (a) What proportion of Visa cardholders pay less than \$30 in interest?

   (b) What proportion pay more than \$42 in interest?

   (c) What proportion pay less than \$15 in interest?

(d) What proportion pay interest within one standard deviation of mean? Within two standard deviations? Within three? Do your answers line up with what you'd expect based on the Empirical Rule?

3. Answer the following questions.

   (a) Let $Z \sim N(0,1)$. If $P(Z \leq A) = 0.75$, then what is A?

   (b) Let $X \sim N(3,49)$. If $P(X > D) = 0.25$, then what is D?

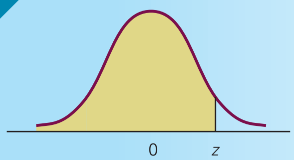
(c) Let $Y \sim N(\mu, 49)$. If $P(Y < 3) = 0.75$, then what is μ?

(d) Let $M \sim N(3, \sigma^2)$. If $P(M > 4) = 0.4$, then what is σ?

4. A cancer drug is produced in one kilogram batches, with an average of six batches produced every hour. Assume the waiting time until the next batch is exponentially distributed.

   (a) What is the probability the production process requires more than 15 minutes to produce the next batch of drugs?

TABLE **3** ***(Continued)***



$P(-\infty < Z < z)$

Z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990