Lec 4*: Random Variables; Population Distributions

1 Motivation

- Last lecture, we talked about probability, which formally discusses the likelihood of an event
- Tying this back to data:
  - Say that you have access to the population data
  - One way for us to present the population data is to list off every single observation from the population
  - Now, with probability, an alternative way is to list off the unique values from the population, and then use probability to describe the frequency – or the likelihood – of hitting a specific value
    - This alternative way of describing data gives us a **distribution**
    - Knowing the distribution of the population data gives us information on how likely certain values will be included in a sample
      $\Rightarrow$ helps with our end goal of making inference
- In this lecture, we will examine two types of distribution: **discrete** and **continuous**
- But before getting there, we'll start off talking about **random variables**, which will simplify our discussion on distributions in a bit

2 Random Variables

- A random variable **assigns a number to each outcome** of an experiment.

  e.g. Let X be a random variable recording the outcome of rolling a fair, six-sided die, then

$$\begin{aligned} \text{Rolling a 1} &\quad\rightarrow\quad X = 1 \\ \text{Rolling a 2} &\quad\rightarrow\quad X = 2 \\ &\cdots \\ \text{Rolling a 6} &\quad\rightarrow\quad X = 6 \end{aligned}$$

- Notice that each outcome has been assigned a number:
  - When all possible outcomes are listed, this is equivalent to the **sample space** from our probability discussion
  - When all possible outcomes are listed, these describe the set of possible values that the **population** data can take on
- Now, if someone describes the associated probabilities at each of the values of a random variable (using a table / formula / graph / something else), then we have a **probability distribution**.

  e.g. Continue with the example of rolling a fair, six-sided die. The following describes a probability distribution:

$$P(X = x) = \frac{1}{6} \quad \text{where } x \in \{1, 2, 3, 4, 5, 6\}$$

*Some exercise questions are taken from or slightly modified based on Dr. Gregory Pac's Econ 310 discussion handout.

- With a probability distribution, we have the probability described at any possible value in the **population** data. This means that we can calculate **parameters** using a probability distribution:

Parameter Name	**Notation**	**Formula**	**Shortcut**
Expected value (mean)	$E(X) = \mu = \mu_X$	$\sum_x xP(x)$	-
Variance	$V(X) = \sigma^2 = \sigma_X^2$	$\sum_x (x-\mu)^2 P(x)$	$E(X^2)$ $-[E(X)]^2$
Standard deviation	$\sigma = \sigma_X$	$\sqrt{V(X)}$	-

- When there is more than one probability distribution (say, distributions for random variables X and Y), then **parameters** on the relationship between two random variables can be expressed:

Parameter Name	**Notation**	**Formula**	**Shortcut**
Covariance	$Cov(X,Y) = \sigma_{XY}$	$\sum_x \sum_y (x-\mu_X)(y-\mu_Y)P(x,y)$	$E(XY)$ $-E(X)E(Y)$
Correlation (of coefficient)	$Corr(X,Y) = \rho_{XY}$	$\frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$	-

- Common parameter operations, when the random variable is transformed in some way:
  (Let X and Y be random variables, a, b, c, d be constants.)
  * Expected value (mean):
    · $E(c) = c$
    · $E(aX + b) = aE(X) + b$
    · $E(X + Y) = E(X) + E(Y)$
  * Variance:
    · $V(c) = 0$
    · $V(aX + b) = a^2 V(X)$
    · $V(X \pm Y) = V(X) + V(Y) \pm 2Cov(X,Y)$
  * Covariance:
    · $Cov(a,b) = 0$
    · $Cov(X,X) = V(X)$
    · $Cov(aX + b, cY + d) = acCov(X,Y)$

- There are two types of probability distributions. Each associated with a specific type of random variable:

1. **Discrete probability distribution**:
   Generated by a **discrete random variable**, which means that numbers assigned to the random variable are countable.
2. **Continuous probability distribution**:
   Generated by a **continuous random variable**, which means that numbers assigned to the random variable are NOT countable.

Aside: what does it mean to be countable?

(A pretty loose definition:) as long as you can sequentially count all the numbers assigned – even though it might take forever – then such series of numbers is considered as countable.

Exercise. Is the following random variable discrete or continuous?

1. $X =$ whether the result from a fair coin flip is head or not

   X is a discrete random variable. When the outcome is head, $X = 1$; when the outcome is not head (i.e. tail), $X = 0$. These two numbers are certainly countable.

2. $X =$ amount of time it takes for a student to complete a 60-minute exam

   X is a continuous random variable. The outcome is assigned to anything between 0 minute and 60 minutes, but if someone tries to count all possible time between 0 and 60 minutes, they won't be able to.

   More explicitly, say that someone wants to count from 0 to 60 minutes with 0.01 minute increment. However, there are time between 0.00 minute and 0.01 minute (such as 0.003 minute) that will be left uncounted. If this person then lower the increment to 0.001, the same argument would apply since some even smaller time would be left uncounted.

   The point being that, regardless of how small the time increment is made for counting, there always exists some even smaller time increment. Thus, one can never sequentially count all potential outcomes, which is why X is considered as a continuous random variable.

3. $X =$ the number of rolls it takes to get a 6 from rolling a six-sided die

   X is a discrete random variable. The outcome is $1, 2, 3, 4, \ldots$. While it might be the case that it will take you infinite number of rolls to get a 6, as long as you keep counting, then you can sequentially count all outcomes. Thus, numbers assigned to X are countable, which means X is discrete.

   Notice the difference between this and the second exercise. In this exercise, the increment between all potential outcomes is always 1, so as long as you keep counting, all outcomes could be counted. In the second exercise, even if you have infinite amount of time, the time increment used for counting can always be made smaller, meaning that there is just no way for you to sequentially count all the time between 0 and 60 minutes to begin with.

3 Discrete Probability Distributions

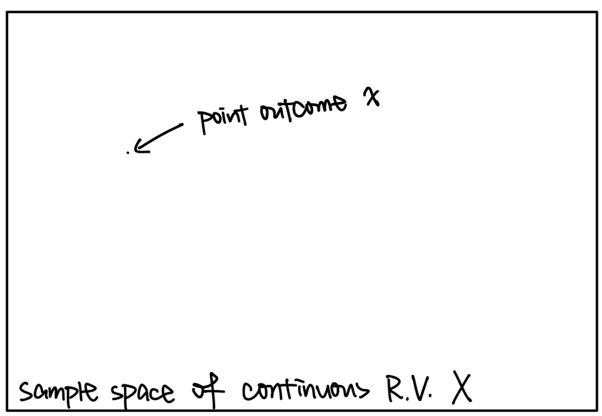
- A discrete probability distribution needs to describe the probability $P(x)$ at all possible x values that a discrete random variable X can take on.
- What are some conditions that a discrete probability distribution needs to satisfy?
  1. $0 \leq P(x) \leq 1$ for all x
  2. $\sum_x P(x) = 1$
- One important example of discrete probability distribution: **Binomial distribution**
  - **Binomial distribution** is a distribution of success among n trials

- Random variable $X \sim \text{Binomial}(n, p)$ if the following holds:
    - There are fixed number (n) of trials.
    - Each trial has two outcomes: success, and failure.
    - $P(\text{success}) = p$ is constant across all trials.
    - Trials are independent.
- Once $X \sim \text{Binomial}(n, p)$ is established, then
    - $P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \dots, n$
    - $E(X) = np$
    - $V(X) = np(1-p)$

4 Continuous Probability Distributions

4.1 Density Functions

- When we examined discrete probability distributions, we said that a discrete probability distribution should describe point probability $P(x)$ at all possible x outcome values.
- Ideally, we would like to provide a similar definition for continuous probability distributions.
- But there's a problem: when random variable X is continuous, **point probability equals to 0 at every single point** ($P(X = x) = 0$ for all x).
    - Reason 1: A continuous random variable has uncountable amount of values, so if each outcome value has probability $\varepsilon > 0$, then the sum of all probabilities would equal to ∞ instead of 1.
    - Reason 2: Say that the rectangle below represents the sample space for a continuous random variable. The probability of hitting a point within the rectangle is the area of the point divided by the area of the rectangle. However, a point has area $= 0$, so the point probability $= 0$.



- So our definition for a continuous probability distribution needs to be modified (slightly).
- Solution: describe **density** of x instead of probability.

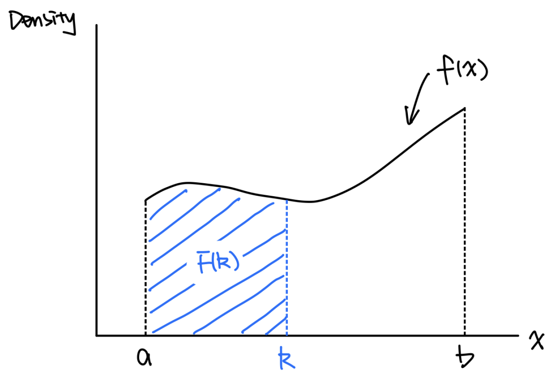
Definition 1 (Probability density function (PDF))**.** A function $f(x)$ is called a probability density function (PDF) over $a \leq x \leq b$ if it satisfies the following two criteria:

1. $f(x) \geq 0$ for all x between a and b, and

2. Total area under the curve of $f(x)$ between a and b is 1.

Definition 2 (Cumulative density function (CDF)). A cumulative density function (CDF) describes probability up to a point x. That is, CDF $F(x) = P(X \leq x)$ for random variable X.

Example.



- In this graph, $f(x)$ is the density function for random variable X, and area under the curve describes the cumulative density. For example, the cumulative density up until point k is $F(k)$, which is the total area up until point k.
- The cumulative density describes probability up to a point. Therefore,

$$P(X \leq k) = F(k) = \text{Area under } f(x) \text{ up until point } k$$

- To make sure the density function can properly describe all outcomes, we need to make sure of the following two things:
  1. it's possible to calculate probability $\Rightarrow$ it's possible to calculate area under the density function $\Rightarrow f(x) \geq 0$
  2. total probability should sum up to 1 $\Rightarrow F(b) = 1$

- With the help of density functions, we can finally define a continuous probability distribution:

Definition 3 (Continuous probability distribution). A continuous probability distribution describes a valid PDF $f(x)$ at all possible x values for a continuous random variable X.

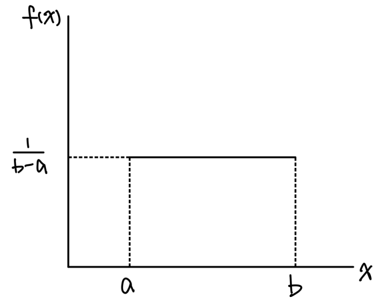
How does this compare with the discrete case?

	Discrete Prob Dist	**Continuous Prob Dist**
Describes . . . at all valid x	$P(x)$	$f(x)$
Range of measure for all valid x	$0 \leq P(x) \leq 1$	$f(x) \geq 0$
How to make sure all valid x are covered	$\sum_x P(x) = 1$	$F(b) = \int_a^b f(x)dx = 1$

4.2 Examples of Continuous Probability Distribution

4.2.1 Uniform Distribution

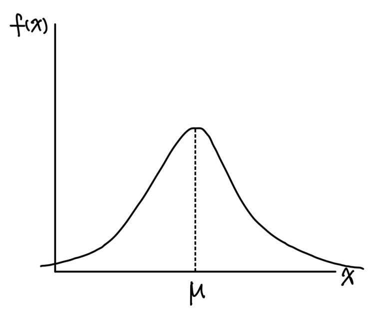
- If X is uniformly distributed between point a and b, then $X \sim \text{Uniform}(a, b)$
- PDF: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$



- $E(X) = \frac{a+b}{2}$
- $V(X) = \frac{(b-a)^2}{12}$

4.2.2 Normal Distribution

- If X is normally distributed with expected value μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$
- PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $-\infty < x < \infty$



- Usually, for random variable X that follows a normal distribution, it is helpful to standardize the variable so that we are examining a transformed variable with **standard normal distribution** instead.
  - A random variable Z that follows standard normal distribution is denoted as $Z \sim N(0, 1)$
  - How to transform X to be standard normal? $\Rightarrow$ Since $X \sim N(\mu, \sigma^2)$, $\frac{X-\mu}{\sigma} = Z \sim N(0, 1)$

5 Exercises

1. The table below contains the joint distribution of the random variables X and Y representing the percentage return for Xenon Incorporated and Yellow Company:

		Y	
		0.0	0.5
X	0.0	0.4	0.1
	0.5	0.1	0.4

(a) Find the population mean, variance, and standard deviation of X and Y.

We can calculate the marginal probabilities for X using the joint probability table:

$$P(X = 0.0) = P(X = 0.0, Y = 0.0) + P(X = 0.0, Y = 0.5) = 0.4 + 0.1 = 0.5$$
$$P(X = 0.5) = P(X = 0.5, Y = 0.0) + P(X = 0.5, Y = 0.5) = 0.1 + 0.4 = 0.5$$

Then we use these to calculate the population mean:

$$\begin{aligned} E(X) &= \sum_x xP(x) \\ &= 0.0 \times P(X = 0.0) + 0.5 \times P(X = 0.5) \\ &= 0.0 \times 0.5 + 0.5 \times 0.5 = 0.25 \end{aligned}$$

Similarly, we can find $E(X^2)$:

$$\begin{aligned} E(X^2) &= \sum_x x^2 P(x) \\ &= 0.0^2 \times P(X = 0.0) + 0.5^2 \times P(X = 0.5) \\ &= 0.0^2 \times 0.5 + 0.5^2 \times 0.5 = 0.125 \end{aligned}$$

By the shortcut method, the variance of X is

$$V(X) = E(X^2) - [E(X)]^2 = 0.125 - (0.25)^2 = 0.0625$$

Taking the squared root of $V(X)$ gives us the standard deviation of X as $\sigma_X = \sqrt{V(X)} = \sqrt{0.0625} = 0.25$

Since the marginal distribution of Y is the same as for X, it has the same mean, variance, and standard deviation as X.

(b) Find the population covariance and correlation coefficient between X and Y

To calculate the covariance via the shortcut method, we first need $E(XY)$:

$$\begin{aligned} E(XY) &= \sum_x \sum_y xyP(x, y) \\ &= 0.0 \times 0.0 \times P(0.0, 0.0) + 0.0 \times 0.5 \times P(0.0, 0.5) \\ &\quad + 0.5 \times 0.0 \times P(0.5, 0.0) + 0.5 \times 0.5 \times P(0.5, 0.5) \\ &= 0.0 \times 0.0 \times 0.4 + 0.0 \times 0.5 \times 0.1 \\ &\quad + 0.5 \times 0.0 \times 0.1 + 0.5 \times 0.5 \times 0.4 \\ &= 0.1 \end{aligned}$$

By the shortcut method, the covariance of X and Y is

$$\sigma_{XY} = Cov(X,Y) = E(XY) - E(X)E(Y) = 0.1 - 0.25 \times 0.25 = 0.0375$$

From (a), we know that $\sigma_X = \sigma_Y = 0.25$. Thus, the correlation of coefficient is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.0375}{0.25 \times 0.25} = 0.6$$

(c) Let $Z = \frac{1}{2}X + \frac{1}{2}Y$ represent the return on a 50/50 mix of the two assets. What are $E(Z)$ and $V(Z)$?

Using the properties of expectation:

$$\begin{aligned} E(Z) &= E\left(\frac{1}{2}X + \frac{1}{2}Y\right) \\ &= \frac{1}{2}E(X) + \frac{1}{2}E(Y) = \frac{1}{2} \times 0.25 + \frac{1}{2} \times 0.25 = 0.25 \end{aligned}$$

And by properties of variance:

$$\begin{aligned} V(Z) &= V\left(\frac{1}{2}X + \frac{1}{2}Y\right) \\ &= V\left(\frac{1}{2}X\right) + V\left(\frac{1}{2}Y\right) + 2Cov\left(\frac{1}{2}X, \frac{1}{2}Y\right) \\ &= \frac{1}{4}V(X) + \frac{1}{4}V(Y) + 2 \times \frac{1}{2} \times \frac{1}{2}Cov(X,Y) \\ &= \frac{1}{4} \times 0.0625 + \frac{1}{4} \times 0.0625 + 2 \times \frac{1}{2} \times \frac{1}{2} \times 0.0375 = 0.05 \end{aligned}$$

(d) Suppose a shift in the market changes the return for Xenon Incorporated, the new return is $X^* = 3X + 0.10$. What is the mean and variance of Xenon Incorporated's return after this market shift?

By the properties of expectation:

$$E(X^*) = E(3X + 0.10) = 3E(X) + 0.10 = 3 \times 0.25 + 0.10 = 0.85$$

And by the properties of variance:

$$V(X^*) = V(3X + 0.10) = 9V(X) = 9 \times 0.0625 = 0.5625$$

2. The table below contains the joint distribution of the random variables X_1 and X_2, which represent pass/fail grades on two quizzes:

		X_2	
		0	1
X_1	0	0.30	0.40
	1	0.10	0.20

(a) What is $E(X_1)$?

First, we calculate the marginal probabilities of X_1 using the joint probability table:

$$P(X_1 = 0) = P(X_1 = 0, X_2 = 0) + P(X_1 = 0, X_2 = 1) = 0.30 + 0.40 = 0.7$$
$$P(X_1 = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 1, X_2 = 1) = 0.10 + 0.20 = 0.3$$

Then we use these to calculate the expected value:

$$\begin{aligned} E(X_1) &= \sum_{x_1} x_1 P(x_1) = 0 \times P(X_1 = 0) + 1 \times P(X_1 = 1) \\ &= 0 \times 0.7 + 1 \times 0.3 = 0.3 \end{aligned}$$

(b) What is $E(X_1|X_2 = 1)$?

To calculate the conditional expectation, we next need to know the probability distribution conditional on $X_2 = 1$. We get this by dividing the joint probabilities in the second column of the table by the marginal probability of $X_2 = 1$:

$$\begin{aligned} P(X_1 = 0|X_2 = 1) &= \frac{P(X_1 = 0, X_2 = 1)}{P(X_2 = 1)} = \frac{P(X_1 = 0, X_2 = 1)}{P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 1)} \\ &= \frac{0.4}{0.4 + 0.2} = \frac{2}{3} \\ P(X_1 = 1|X_2 = 1) &= \frac{P(X_1 = 1, X_2 = 1)}{P(X_2 = 1)} = \frac{P(X_1 = 1, X_2 = 1)}{P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 1)} \\ &= \frac{0.2}{0.4 + 0.2} = \frac{1}{3} \end{aligned}$$

We use these to calculate the conditional expectation:

$$\begin{aligned} E(X_1|X_2 = 1) &= \sum_{x_1} x_1 P(x_1|X_2 = 1) \\ &= 0 \times P(X_1 = 0|X_2 = 1) + 1 \times P(X_1 = 1|X_2 = 1) \\ &= 0 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{1}{3} \end{aligned}$$

3. To help market their latest blockbuster sports franchise, Croquet 2015, EA Sports decides to distribute a free demo of their video game. Suppose each customer who plays the demo buys the full game with probability 0.8. For this problem, you may find it helpful to reference the probability tables on the last page.

   (a) Let X_n be the total number of sales after n customers have played the demo. What distribution does X_n have? Does it seem likely that the conditions of this distribution are satisfied? (Regardless of your answer, for the rest of the problem you may assume the conditions are satisfied.)

   Because it's the number of successes in n trials, this has the basic setup of a binomial random variable. But in order for the binomial distribution to yield valid answers, we must check whether the conditions for this distribution are satisfied.

   - First, are there fixed amount of trials?
     Yes, the number of trials $= n$ always.

- Second, does each trial results in a success or a failure?
  Yes, in this case a success occurs when a demo-playing customer purchases the full game.
- Third, is the probability of success constant across all trials?
  Yes, in this case the probability of success is $p = 0.8$, and this probability remains the same for all trials.
- Finally, is each trial's outcome independent of the outcome of every other trial?
  This one is debatable. Notice that while probability of success stays at 0.8, it doesn't mean that trials need to be independent. Say that two customers before you played the demo, one has purchased the game, and the other didn't. If your decision on purchasing the game is influenced by both consumers, and their degree of impact on you is the same, then you'll still end up purchasing the game with a probability of 0.8, but your decision is no longer independent of other trials.

(b) What are the expected value and variance of X_1?

One could solve this problem exactly as we've been doing up until now, using the definition of expected value and variance:

$$E(X_1) = 0 \times 0.2 + 1 \times 0.8 = 0.8$$
$$V(X_1) = (0 - 0.8)^2 \times 0.2 + (1 - 0.8)^2 \times 0.8 = 0.16$$

But it's much easier to make use of the fact that this is a binomial distribution with $n = 1$ and $p = 0.8$:

$$E(X_1) = np = 1 \times 0.8 = 0.8$$
$$V(X_1) = np(1 - p) = 1 \times 0.8 \times (1 - .8) = 0.16$$

(c) What is the expected value and variance of X_5?

Noting that it's a binomial random variable with $n = 5$ and $p = 0.8$, we have:

$$E(X_5) = np = 5 \times 0.8 = 4$$
$$V(X_5) = np(1 - p) = 5 \times 0.8 \times 0.2 = 0.8$$

(d) Assuming $n = 5$, what is the probability at least 1 customer buys the full game? What is the probability exactly 5 customers buy the full game?

For the probability at least 1 customer buys the full game, it's easiest to begin with the complement rule and notice that the only outcome that satisfies $X_5 \leq 0$ is when $X_5 = 0$:

$$P(X_5 > 0) = 1 - P(X_5 \leq 0)$$

Now, if you look up the value of $P(X_5 \leq 0)$ from the binomial table, you'll find that $P(X_5 \leq 0) = 0.0003$. So we have

$$P(X_5 > 0) = 1 - 0.0003 = 0.9997$$

Alternatively, recall that the binomial probability distribution for a binomial random variable is:

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

this means that

$$P(X_5 \leq 0) = P(X_5 = 0) = \frac{5!}{0!(5-0)!}(0.8)^0(1-0.8)^{5-0} = 0.00032$$

So we obtain

$$P(X_5 > 0) = 1 - 0.00032 = 0.99968$$

For the probability exactly five customers buy the full game, we can get this directly using the binomial formula:

$$P(X_5 = 5) = \frac{5!}{5!(5-5)!}(0.8)^5(1-0.8)^{5-5} = 0.32768$$

Or, if you prefer to use the binomial table, we get:

$$P(X_5 = 5) = 1 - P(X_5 \leq 4) = 1 - 0.6723 = 0.3277$$

(You might notice that the probability you get from using the binomial formula is a little bit different from the one from using the binomial table. This is mainly the result of rounding error, so it's okay.)

(e) Assuming $n = 5$, what is the probability that the number of customers who buy the full game lies between 1 and 4 (including both 1 and 4)?

We've already done most of the work, since by the complement rule:

$$P(1 \leq X_5 \leq 4) = P(X_5 > 0) - P(X_5 = 5) = 0.672$$

Alternatively, notice that

$$P(1 \leq X_5 \leq 4) = P(X_5 \leq 4) - P(X_5 \leq 0)$$

Using the binomial table, we find that $P(X_5 \leq 4) = 0.6723$, and that $P(X_5 \leq 0) = 0.0003$, so this yields the same result.

TABLE **1** **Binomial Probabilities**

Tabulated values are $P(X \leq k) = \sum_{x=0}^{k} p(x_i)$. (Values are rounded to four decimal places.)

***n* = 5**

	p														
k	**0.01**	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
0	0.9510	0.7738	0.5905	0.3277	0.2373	0.1681	0.0778	0.0313	0.0102	0.0024	0.0010	0.0003	0.0000	0.0000	0.0000
1	0.9990	0.9774	0.9185	0.7373	0.6328	0.5282	0.3370	0.1875	0.0870	0.0308	0.0156	0.0067	0.0005	0.0000	0.0000
2	1.0000	0.9988	0.9914	0.9421	0.8965	0.8369	0.6826	0.5000	0.3174	0.1631	0.1035	0.0579	0.0086	0.0012	0.0000
3	1.0000	1.0000	0.9995	0.9933	0.9844	0.9692	0.9130	0.8125	0.6630	0.4718	0.3672	0.2627	0.0815	0.0226	0.0010
4	1.0000	1.0000	1.0000	0.9997	0.9990	0.9976	0.9898	0.9688	0.9222	0.8319	0.7627	0.6723	0.4095	0.2262	0.0490

4. The weekly output of a steel mill is a uniformly distributed random variable that lies between 110 and 175 metric tons

   (a) What is the probability the steel mill will produce more than 150 metric tons next week?
   Notice that this is a uniform distribution with $a = 110$ and $b = 175$, so the PDF is

$$f(x) = \frac{1}{b-a} = \frac{1}{175-110} = \frac{1}{65}$$

   It would be perfectly valid to answer this question by integrating the PDF from 150 to the top of the sample space (175), but since this is a uniform distribution, the area under the PDF is a rectangle, so it's easier to calculate the probability by multiplying width times height. In this case:

$$P(150 < \text{output} < 175) = (175 - 150) \times \frac{1}{65} = 0.3846$$

   (b) What is the probability the mill will produce between 120 and 160 metric tons next week?
   Since between 120 and 160 metric tons is still within the range $[a, b] = [110, 175]$ that the uniform distribution is defined on, we don't have to worry about any weird out-of-bound problem. Hence,

$$P(120 < \text{output} < 160) = (160 - 120) \times \frac{1}{65} = 0.6154$$

   (c) What is the expected value and variance of the mill's weekly output?
   Using the formulas for the expected value and variance of a uniformly distributed random variable, we obtain

$$E(X) = \frac{a+b}{2} = \frac{110+175}{2} = 142.5$$
$$V(X) = \frac{(b-a)^2}{12} = \frac{(175-110)^2}{12} = 352.083$$

5. An analysis of the amount of interest paid monthly by Visa cardholders is normally distributed with a mean of \$27 and a standard deviation of \$6. (Note: A probability table for a standard normal can be found on the last page of the handout.)

    (a) What proportion of Visa cardholders pay less than \$30 in interest?

    We want $P(X \leq 30)$, where $X \sim N(27, 6^2)$. In order to make headway, we must first convert the question into one involving a standard normal. This is done by subtracting the mean and dividing by the standard deviation:

$$P(X \leq 30) = P\left(\frac{X-\mu}{\sigma} \leq \frac{30-\mu}{\sigma}\right) = P\left(\frac{X-27}{6} \leq \frac{30-27}{6}\right) = P(Z \leq 0.50)$$

    Now we have a probability that we can find using our standard normal table. Looking up the probability when $z = 0.50$, we find

$$P(X \leq 30) = P(Z \leq 0.50) = 0.6915$$

z	0.00	0.01	0.02	0.03	0.04	0.(
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7

    (b) What proportion pay more than \$42 in interest?

    The strategy is similar to before. The only difference is that we must use the complement rule this time to flip the inequality:

$$\begin{aligned} P(X > 42) &= P\left(\frac{X-27}{6} > \frac{42-27}{6}\right) = P(Z > 2.5) \\ &= 1 - P(Z \leq 2.5) = 1 - 0.9938 = 0.0062 \end{aligned}$$

2.2	0.9861	0.9864	0.9868	0.9871	0.987
2.3	0.9893	0.9896	0.9898	0.9901	0.990
2.4	0.9918	0.9920	0.9922	0.9925	0.992
2.5	0.9938	0.9940	0.9941	0.9943	0.994
2.6	0.9953	0.9955	0.9956	0.9957	0.995

    (c) What proportion pay less than \$15 in interest?

    Again, we begin by standardizing the normal distribution:

$$P(X \leq 15) = P\left(\frac{X-27}{6} \leq \frac{15-27}{6}\right) = P(Z \leq -2)$$

    Now the extra twist is that you are only provided with the probability table for $z \geq 0$. To make headway, we must make use of the fact that the standard normal distribution is symmetric:

$$P(Z \leq -2) = P(Z \geq 2)$$

This gives us

$$P(Z \leq -2) = P(Z \geq 2) = 1 - P(Z < 2) = 1 - P(Z \leq 2)$$

(Point probability is 0, so adding equality sign doesn't matter)

$$= 1 - 0.9772 = 0.0228$$

1.9	0.9713	0.9719	0.9726	0.9732	0.97
2.0	0.9772	0.9778	0.9783	0.9788	0.97
2.1	0.9821	0.9826	0.9830	0.9834	0.98
2.2	0.9861	0.9864	0.9868	0.9871	0.98

6. Answer the following questions.

   (a) Let $Z \sim N(0, 1)$. If $P(Z \leq A) = 0.75$, then what is A?

   Doing an inverse look-up of 0.75 on our probability table, we see that $A = 0.675$. (Since the answer lies between two cells on the table, an answer of 0.67 or 0.68 is also fine.)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

   (b) Let $X \sim N(3, 49)$. If $P(X > D) = 0.25$, then what is D?

   Standardizing and using the complement rule, we can write this as

$$P(X > D) = P\left(Z > \frac{D-3}{7}\right) = 1 - P\left(Z \leq \frac{D-3}{7}\right) = 0.25$$

   This implies that

$$P\left(Z \leq \frac{D-3}{7}\right) = 0.75$$

   Since an inverse lookup of 0.75 probability yields $z = 0.675$, it must be that

$$\frac{D-3}{7} = 0.675 \quad \Rightarrow \quad D = 7.725$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

(c) Let $Y \sim N(\mu, 49)$. If $P(Y < 3) = 0.75$, then what is μ?
Standardizing, we know that:

$$P(Y < 3) = P(Y \leq 3) = P\left(Z \leq \frac{3-\mu}{7}\right) = 0.75$$

Since an inverse lookup of 0.75 probability yields $z = 0.675$, we have

$$\frac{3-\mu}{7} = 0.675 \quad \Rightarrow \quad \mu = -1.725$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

(d) Let $M \sim N(3, \sigma^2)$. If $P(M > 4) = 0.4$, then what is σ?
Standardizing and using the complement rule, we know that:

$$P(M > 4) = P\left(Z > \frac{4-3}{\sigma}\right) = 1 - P\left(Z \leq \frac{4-3}{\sigma}\right) = 0.4$$

This means that

$$P\left(Z \leq \frac{4-3}{\sigma}\right) = 0.6$$

Since an inverse lookup of 0.6 probability yields $z = 0.255$, it must be that

$$\frac{4-3}{\sigma} = 0.255 \quad \Rightarrow \quad \sigma = 3.9216$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517