Bifurcation analysis of delay equations using software for ODEs

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Material available at: https://github.com/scarabel/IFAC-TDS-2021
http://cdlab.uniud.it/software

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Test delay equations

Logistic Delay Differential Equation (DDE):

$$y'(t) = r\,y(t)\,(1 - y(t-1))\,, \qquad t \geq 0,\ r > 0$$

A cannibalism model formulated as Renewal Equations (RE):

$$y(t) = \frac{\gamma}{2}\int_{a_0}^{\tau} y(t-a)\,\mathrm{e}^{-y(t-a)}\,\mathrm{d}a, \qquad t > 0,\ \gamma > 0$$

Note:

- delay (bounded)
- both point delays and distributed delays
- dependence on parameters: r,γ
- history function: $y_t(\theta) := y(t+\theta), \qquad \theta \in [-\tau, 0],$

The abstract Cauchy problem

Setting $v(t) := y_t(\cdot)$, we can write a DE as an abstract equation

$$\frac{d}{dt}v(t) = \mathcal{A}v(t), \qquad t \geq 0$$

for some initial condition $v(0) = \psi$ and

$$\mathcal{A}\psi = \psi', \qquad \psi \in D(\mathcal{A})$$

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For a DDE $y'(t) = f(y_t)$, we have

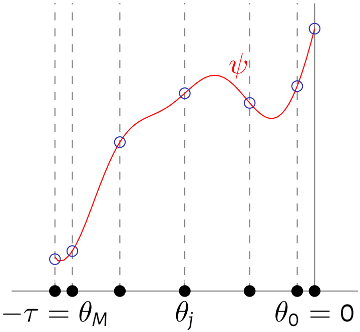
$$D(\mathcal{A}) = \{\psi, \psi' \in C([-\tau, 0]) \colon \psi'(0) = f(\psi)\}$$

For a RE $y(t) = f(y_t)$, we have

$$D(\mathcal{A}) = \{\psi, \psi' \in L^1([-\tau, 0]) \colon \psi(0) = f(\psi)\}$$

From infinite dimension to finite dimension

via pseudospectral methods



Mesh of $M+1$ nodes in $[-\tau, 0]$

$$-\tau = \theta_M < \cdots < \theta_0 = 0$$

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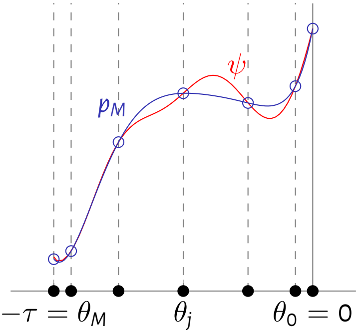

Mesh of $M+1$ nodes in $[-\tau, 0]$

$$-\tau = \theta_M < \cdots < \theta_0 = 0$$

function in Y $\approx$ polynomial of degree M

$$\psi(\theta) \approx p_M(\theta) = \sum_{j=0}^{M} \ell_j(\theta)\psi(\theta_j)$$

with $\ell_j(\theta)$ Lagrange polynomials:

$$\ell_j(\theta) = \prod_{k\neq j} \frac{\theta - \theta_k}{\theta_j - \theta_k}$$

Delay differential equations

Abstract equation in C

$$\dot{v}(t) = \mathcal{A}(v(t))$$

$$v(t) \in D(\mathcal{A}) = \{\psi'(0) = f(\psi)\}$$

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$\Rightarrow$

ODE in $\mathbb{R}^{M+1}$

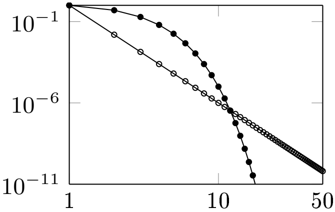
$$\dot{v}_0 = f(p_M(v_0, \ldots, v_M))$$

$$\dot{v}_k = \sum_{j=0}^{M} d_{kj} v_j, \; k = 1, \ldots, M$$

- system of $M+1$ ODEs, with $v_k(t) \approx v(t)(\theta_k)$
- linear part for differentiation independent of the specific equation, $d_{kj} = \ell_j'(\theta_k)$
- the right-hand side f appears only in the equation for v_0

Equilibria and characteristic roots

- one-to-one correspondence of equilibria: $\bar{y} \leftrightarrow (\bar{y}, \cdots, \bar{y})$
- Chebyshev nodes (roots or extrema) ensure spectral convergence: $M = 10 - -20$ typically gives satisfactory accuracy



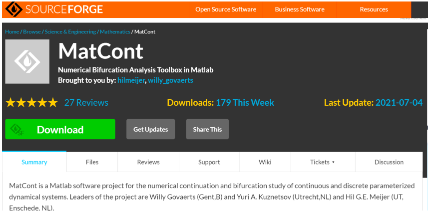
- ODE system can be studied with established software

Bifurcation analysis and MatCont

Bifurcation analysis — how the dynamical long-term properties depend on parameters

MatCont — a software package for the bifurcation analysis of ODEs

- MATLAB package, available on sourceforge.net



Bifurcation analysis and MatCont

Bifurcation analysis — how the dynamical long-term properties depend on parameters

MatCont — a software package for the bifurcation analysis of ODEs

- MATLAB package, available on sourceforge.net
- numerical continuation of equilibria and periodic orbits with respect to parameters
- other packages for ODEs, e.g. AUTO, but an important point is the possibility of varying the dimension of the system
- dde-biftool is a similar MATLAB package for DDEs with discrete delays

MatCont example: logistic DDE

$$y'(t) = r\, y_t(0)\, (1 - y_t(-1))\,, \qquad t \geq 0,\ r > 0$$

With $v_k \approx y_t(\theta_k)$, the approximating ODE reads

$$\begin{aligned} \dot{v}_0(t) &= r\, v_0(t)\, (1 - v_M(t)) \\ \dot{v}_k(t) &= \sum_{j=0}^{M} d_{kj} v_j(t),\ k = 1, \ldots, M \end{aligned}$$

Renewal equations and the integrated state

Given a renewal equation, $y(t) = f(y_t)$, $t > 0$, consider

$$v(t)(\theta) := \int_0^\theta y_t(s)\,\mathrm{d}s, \qquad \theta \in [-\tau, 0]$$

and the operator $\mathcal{A}\psi = \psi'$, $\psi \in D(\mathcal{A}) = \{\psi \in L^1 \colon \psi' \in NBV,\ \psi(0) = 0\}$.

Abstract equation in NBV

$$\dot{v}(t) = \mathcal{A}v(t) - f(\mathcal{A}v(t))$$

$$v(t) \in D(\mathcal{A}) = \{\psi(0) = 0\}$$

$\Rightarrow$

ODE in $\mathbb{R}^M$

$$\dot{v}_k = \sum_{j=0}^{M} d_{kj} v_j - f(p_M'), \ k = 1, \dots, M$$

$$p_M = p_M(0, v_1, \dots, v_M)$$

- $y_t \in L^1 \Rightarrow v(t) \in AC$
- algebraic condition for v_0 replaced by nonlinear perturbation

MatCont example: a RE for cannibalism

$$y(t) = \frac{\gamma}{2}\int_{a_0}^{\tau} y_t(-a)\,\mathrm{e}^{-y_t(-a)}\,\mathrm{d}a, \qquad t > 0,\ \gamma > 0$$

With $v_k \approx \int_0^{\theta_k} y_t(s)\,\mathrm{d}s$ and $p_M = p_M(0, v_1, \ldots, v_M)$, the approximating ODE reads

$$\dot{v}_k(t) = \sum_{j=0}^{M} d_{kj} v_j(t) - \frac{\gamma}{2}\int_{a_0}^{\tau} p_M'(-a)\,\mathrm{e}^{-p_M'(-a)}\,\mathrm{d}a,\ k = 1, \ldots, M$$

MatCont example: a RE for cannibalism

$$y(t) = \frac{\gamma}{2} \int_{a_0}^{\tau} y_t(-a)\, \mathrm{e}^{-y_t(-a)}\, \mathrm{d}a, \qquad t > 0,\ \gamma > 0$$

With $v_k \approx \int_0^{\theta_k} y_t(s)\,\mathrm{d}s$ and $p_M = p_M(0, v_1, \dots, v_M)$, the approximating ODE reads

$$\begin{aligned}
\dot{v}_k(t) &= \sum_{j=0}^{M} d_{kj} v_j(t) - \frac{\gamma}{2} \int_{a_0}^{\tau} p_M'(-a)\, \mathrm{e}^{-p_M'(-a)}\, \mathrm{d}a,\ k = 1, \dots, M \\
&= \sum_{j=0}^{M} d_{kj} v_j(t) - \frac{\gamma}{2} \sum_{j=1}^{M} w_j p_M'(-a_j)\, \mathrm{e}^{-p_M'(-a_j)}\, \mathrm{d}a
\end{aligned}$$

for a_j, w_j nodes and weights for the quadrature formula.

Relevant references

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