1 Derivative (10+10+10)

(a), (b): From Taylor expansion, we could write:

$$f(x+h) = f(x) + hf^{(1)}(x) + \frac{h^2}{2}f^{(2)}(x) + \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) + ... \tag{1}$$

$$f(x-h) = f(x) - hf^{(1)}(x) + \frac{h^2}{2}f^{(2)}(x) - \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) + ... \tag{2}$$

$$f^{(2)}(x) = (f(x+h) + f(x-h) - 2f(x))/h^2 - \frac{h^2}{12}f^{(4)}(x) \tag{3}$$

So the error is : $-\frac{h^2}{12}f^{(4)}(x)$

(c):

$$f(x+2h) = f(x) + 2hf^{(1)}(x) + \frac{4h^2}{2}f^{(2)}(x) + \frac{8h^3}{6}f^{(3)}(x) + \frac{16h^4}{24}f^{(4)}(x) + \frac{32h^5}{5!}f^{(5)}(x) + \frac{64h^6}{6!}f^{(6)}(x) \tag{4}$$

$$f(x-2h) = f(x) - 2hf^{(1)}(x) + \frac{4h^2}{2}f^{(2)}(x) - \frac{8h^3}{6}f^{(3)}(x) + \frac{16h^4}{24}f^{(4)}(x) - \frac{32h^5}{5!}f^{(5)}(x) + \frac{64h^6}{6!}f^{(6)}(x) \tag{5}$$

Calculate (1) + (2) and (4) + (5) separately,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f^{(2)}(x) + \frac{h^4}{12}f^{(4)} + \frac{h^6}{360}f^{(6)}(x) \tag{6}$$

$$f(x+2h) + f(x-2h) = 2f(x) + 4h^2 f^{(2)}(x) + \frac{4h^4}{3}f^{(4)}(x) + \frac{8h^6}{45}f^{(6)}(x) \tag{7}$$

(7)-16×(6):

$$f(x+2h) + f(x-2h) - 16f(x+h) - 16f(x-h) = -30f(x) - 12f^{(2)}(x) + \frac{2h^6}{15}f^{(6)}(x) \tag{8}$$

$$f^{(2)}(x) = \frac{1}{12h^2}(-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h) + \frac{h^2}{90}f^{(6)}(x) \tag{9}$$

2 Interpolation and extrapolation (15+15)

This part can be found on lecture note 3. Basic formula are needed.

3 LU decomposition (10+10+20)

(a): LU decomposition is a matrix decomposition method factors a matrix A as a product of a lower triangular matrix L and an upper triangular matrix U. $A = LU$. If we want to solve x from the linear equations in matrix form:

$$Ax = b \tag{10}$$

Suppose we already got the LU decomposition form of matrix A. Then we can solve the problem in two steps:

$$Ly = b \tag{11}$$

$$Ux = y \tag{12}$$

In both equations above we only need to dealing wtih triangular matrices, which can be solved directly by forward and backward substitution without using the Gaussian elimination process.

(b): Assuming the diagonal elements in lower triangular matrix are all 1.

$$L = \begin{bmatrix} 1 & & \\ L_{21} & 1 & \\ L_{31} & L_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ & U_{22} & U_{23} \\ & & u_{33} \end{bmatrix} \tag{13}$$

By solving this equation, we could get all elements in L and U

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & \frac{3}{4} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 3 \\ & -3 & -4 \\ & & -5 \end{bmatrix} \tag{14}$$

You can also assume the diagonal elements in upper triangular matrix are all 1. By doing so, the solution is :

$$L = \begin{bmatrix} 1 & & \\ 2 & -3 & \\ 3 & -4 & -\frac{8}{3} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 3 \\ & 1 & \frac{4}{3} \\ & & 1 \end{bmatrix} \tag{15}$$