Name: ______________________ Grade: ______/200

Answer the questions in the spaces provided on the following pages. If you run out of room for an answer, continue on the back of the page. Show **all** your work to be able to receive full credit on any question. **YOU ARE A MISSILE**

For each statement below, circle whether they are true or false. As a reminder, a statement that is not always true is considered false. No work is needed for the following problems.

1. True **FALSE** If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent. [3]
2. **TRUE** False If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent. [3]
3. True **FALSE** If $\sum |a_n|$ is divergent, then $\sum a_n$ is divergent. [3]
4. **TRUE** False The ratio test can be used to determine whether $\sum 1/n!$ converges. [3]
5. True **FALSE** The ratio test can be used to determine whether $\sum 1/n^3$ converges. [3]
6. **TRUE** False If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. [3]
7. **TRUE** False If $-1 < \alpha < 1$, then $\sum \alpha^n$ is convergent. [3]
8. True **FALSE** If $a_n > 0$ and $\lim_{n\to\infty} (a_{n+1}/a_n) = 1$, then $\sum a_n$ converges. [3]
9. **TRUE** False $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ [3]
10. **TRUE** False If a series converges, then the sequence of its terms converges. [3]
11. **TRUE** False If a series converges, then the sequence of its partial sums converge. [3]
12. True **FALSE** A series can be convergent without its sequence being convergent. [3]
13. True **FALSE** If $a_n > 0$ and $b_n > 0$, $\sum b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ is convergent. [3]
14. True **FALSE** $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ [3]
15. **TRUE** False The alternating harmonic series is conditionally convergent. [3]

16. Determine whether the following series are convergent or divergent. Show all work to recieve full credit. An answer of convergent or divergent alone will yeild no credit.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ [10]

Solution: Let $b_n = \dfrac{n}{n^3} \geq \dfrac{\ln n}{n^3} = a_n$ for all n. Then $\sum b_n = \sum \dfrac{1}{n^2}$ which is a convergent p-series with $p = 2 > 1$. Therefore, $\sum a_n$ is convergent by the direct comparison test.

(b) $\displaystyle\sum_{i=1}^{\infty} \left(\sqrt[i]{i} - 1\right)^i$ [10]

Solution: Using the root test:

$$\lim_{i\to\infty} \sqrt[i]{\left(\sqrt[i]{i} - 1\right)^i} = \lim_{i\to\infty} \sqrt[i]{i} - 1 = 1 - 1 = 0 < 1.$$

Therefore (absolutely) convergent by the root test.

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} e^{\left(\frac{2}{n+1}\right)}$ [10]

Solution:

$$\lim_{n\to\infty} e^{\frac{2}{n+1}} = e^0 = 1.$$

Therefore divergent by the divergence test.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{3\sqrt{k} + 2k}{2k^{1.5} + k^2}$ [10]

Solution: If $f(x) = \dfrac{3\sqrt{x} + 2x}{2x^{1.5} + x^2}$, then f is positive, continuous and decreasing. Now,

$$\int_1^{\infty} \frac{3\sqrt{x} + 2x}{2x^{1.5} + x^2} = \lim_{b\to\infty} \left[\ln\left|2b^{1.5} + b^2\right| - \ln(3)\right] = \infty \therefore \text{Divegent.}$$

Thus $\sum a_n$ is divergent by the integral test.

17. Find the sum of the following convergent series. [10]

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-4)^n}$

(b) $\displaystyle\sum_{i=2}^{\infty} \left[\frac{1}{i^2} - \frac{1}{(i+2)^2}\right]$

18. Find the raidus of convergence for the given power series. [5]

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n$$

19. Find the interval of convergence for the given power series. [10]

$$\sum_{n=2}^{\infty} \frac{n}{n+1}\left(\frac{x}{4}\right)^{n-1}$$

20. Consider the function given by [15]

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^{n+1}}{n+1}.$$

Answer the following.

(a) Find a series for $f'(x)$. Then, find its interval of convergence.

(b) Write the series $\displaystyle\int_{-2}^{-1.3} f(x)dx$. You do not need to find the interval of convergence.

(c) Show that the maximum error associated with the approximation of S_7 for the series obtained in part (b) is less than $\dfrac{1}{3125}$.

21. Consider the following functions. Using any means, construct a power series for the given function at the specified center. Leave your answer in summation notation. Simplify completely. [30]

(a) $h(x) = 3xe^{-\frac{x^2}{3}}$, $c = 0$

(b) $r(x) = \dfrac{1}{1+x}$, $c = 3$

(c) $t(x) = \sin 3x \cos 3x$, $c = 0$

22. Let f be the function given by $f(x) = \dfrac{1}{1-2x}$ and let $P_3(x)$ be the third degree MacLaurin Polynomial for f.

(a) Find $P_3(x)$. [10]

(b) Use the Lagrange Error Bound to show that $\left| f\left(\frac{1}{6}\right) - P_3\left(\frac{1}{6}\right) \right| \le \dfrac{1}{10}$. [15]

23. The function ϕ has a Taylor series about $x = 2$ that converges for all x in its interval of convergence. The nth derivative of ϕ at $x = 2$ is given by

$$\phi^{(n)}(2) = \frac{(n+1)!}{3^n}$$

for $n \geq 1$. Assume that $\phi^{(n)}(x)$ is increasing for all $x \in \mathbb{R}$.

(a) Find an expression for the Lagrange Error bound associated with the approximation of $\phi(1)$ using the nth degree Taylor polynomial centered about $x = 2$. [10]

(b) How many terms are needed for the error to be less than $\frac{1}{1200}$? [10]

Extra Credit (10pts): Consider the following series.

$$\sum_{n=0}^{\infty} \frac{n! \cdot (n+1)! \cdot 3^n}{\left(1 \cdot 3 \cdot 6 \cdot \dots \cdot (3n)\right)^2}.$$

Determine the convergence or divergence of the series. Show all work.