Name: ______________________________ Grade: ______/200

> Answer the questions in the spaces provided on the following pages. If you run out of room for an answer, continue on the back of the page. Show **all** your work to be able to receive full credit on any question. **YOU ARE A MISSILE**

For each statement below, circle whether they are true or false. As a reminder, a statement that is not always true is considered false. No work is needed for the following problems.

1. True False If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent. [3]

2. True False If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent. [3]

Fix

3. True False If $\sum |a_n|$ is divergent, then $\sum \cancel{|}a_n\cancel{|}$ is divergent. [3]

4. True False The ratio test can be used to determine whether $\sum 1/n!$ converges. [3]

5. True False The ratio test can be used to determine whether $\sum 1/n^3$ converges. [3]

6. True False If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. [3]

7. True False If $-1 < \alpha < 1$, then $\sum \alpha^n$ is convergent. [3]

8. True False If $a_n > 0$ and $\lim_{n\to\infty}(a_{n+1}/a_n) = 1$, then $\sum a_n$ converges. [3]

9. True False $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ [3]

10. True False If a series converges, then the sequence of its terms converges. [3]

11. True False If a series converges, then the sequence of its partial sums converge. [3]

12. True False A series can be convergent without its sequence being convergent. [3]

13. True False If $a_n > 0$ and $b_n > 0$, $\sum b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ is convergent. [3]

14. True False $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ [3]

15. True False The alternating harmonic series is conditionally convergent. [3]

16. Determine whether the following series are convergent or divergent. Show all work to recieve full credit. An answer of convergent or divergent alone will yeild no credit.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ [10]

$b_n = \frac{n}{n^3} \geq \frac{\ln n}{n^3}$

$\therefore \sum b_n$ conv p-series w/ $p=2$

$\therefore \sum a_n$ conv by DCT

(b) $\displaystyle\sum_{i=1}^{\infty} \left(\sqrt[i]{i} - 1\right)^i$ [10]

$\lim_{i\to\infty} \sqrt[i]{\left(\sqrt[i]{i} - 1\right)^i} = \lim_{i\to\infty}\left(\sqrt[i]{i} - 1\right) = 1-1=0 \Rightarrow$ conv by root test

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} e^{\left(\frac{2}{n+1}\right)}$ [10]

$\lim_{n\to\infty} e^{\left(\frac{2}{n+1}\right)} = e^0 = 1 \Rightarrow$ Div by D.T.

change

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan(k)}{k^{1.5}}$ [10]

$b_n = \frac{\pi/2}{k^{1.5}} \geq a_n$

17. Find the sum of the following convergent series. [10]

(a) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-4)^n}$ $= \sum 3\left(-\frac{3}{4}\right)^n$

$= \frac{3}{1+3/4}$

$= \frac{12}{7}$

(b) $\sum_{i=2}^{\infty} \left[\frac{1}{i^2} - \frac{1}{(i+2)^2}\right]$

$= \left(\frac{1}{2^2} - \cancel{\frac{1}{4^2}}\right) + \left(\frac{1}{3^2} - \cancel{\frac{1}{5^2}}\right) + \left(\cancel{\frac{1}{4^2}} - \cancel{\frac{1}{6^2}}\right) + \cancel{\cdots}$

$= \frac{1}{4} + \frac{1}{9}$

$= \frac{13}{36}$

18. Find the raidus of convergence for the given power series. [5]

$\sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n$

$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^n}{(n+1)!} \cdot \frac{n!}{2^n (x-1)^n} \right| = \lim_{n\to\infty} \frac{\sim}{n+1} = 0$

ROC $= \infty$

19. Find the interval of convergence for the given power series. [10]

$\sum_{n=2}^{\infty} \frac{n}{n+1}\left(\frac{x}{4}\right)^{n-1}$

$\lim_{n\to\infty} \left| \frac{n+1}{n+2}\left(\frac{x}{4}\right)^n \cdot \left(\frac{4}{x}\right)^{n-1} \frac{n}{n+1} \right| = \left|\frac{x}{4}\right| < 1 \rightarrow -4 < x < 4$

$x = -4$: $\sum \frac{(-1)^{n-1} n}{n+1}$ div by DT

$x = 4$: $\sum \frac{n}{n+1}$ div by DT

$\therefore$ IOC: $-4 < x < 4$

20. Consider the function given by [15]

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^{n+1}}{n+1}.$$

Answer the following.

(a) Find a series for $f'(x)$. Then, find its interval of convergence.

$f'(x) = \sum (-1)^n (x+2)^n$

IOC: $-3 < x < -1$

(b) Write the series $\int_{-2}^{-1.3} f(x)dx$. You do not need to find the interval of convergence.

$$\int_{-2}^{-1.3} f(x)dx = \sum \frac{(-1)^n (x+2)^{n+2}}{(n+1)(n+2)} \Bigg|_{-2}^{-1.3}$$

$$= \sum \frac{(-1)^n (.7)^{n+2}}{(n+1)(n+2)}$$

(c) Show that the maximum error associated with the approximation of S_7 for the series obtained in part (b) is less than $\cancel{\frac{29}{500}}$.

$\frac{1}{3125}$

ASRT: $|E| \le a_8 = \frac{.7^{10}}{9 \cdot 10} = 0.000313\ldots$

21. Consider the following functions. Using any means, construct a power series for the given function at the specified center. Leave your answer in summation notation. Simplify completely. [30]

(a) $h(x) = 3xe^{-\frac{x^2}{3}}$, $c = 0$

$$= 3x \sum \frac{(-x^2/3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{n-1} \cdot n!}$$

(b) $r(x) = \dfrac{1}{1+x}$, $c = 3$

$$= \frac{1}{1 + (x-3+3)}$$

$$= \frac{1}{4} \sum \left(-\left(\frac{x-3}{4}\right)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{4^{n+1}}$$

(c) $t(x) = \sin 3x \cos 3x$, $c = 0$

$$= \frac{1}{2} \sin(6x)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (6x)^{2n+1}}{(2n+1)!}$$

$$= \sum \frac{(-1)^n \, 6^{2n+1} \, x^{2n+1}}{2 \cdot (2n+1)!}$$

22. Let f be the function given by $f(x) = \dfrac{1}{1-2x}$ and let $P_3(x)$ be the third degree MacLaurin Polynomial for f.

(a) Find $P_3(x)$. [10]

$f = \sum (2x)^n$

$= 1 + 2x + 4x^2 + 8x^3 \; (+ 16x^4)$

$f' \quad \dfrac{2}{(\quad)^2}$

$f'' \quad \dfrac{2^2 \cdot 2}{(\quad)^3}$

$f''' \quad 2^3 \; 2 \cdot 3$

$f^{(4)}(x) = \dfrac{2^4 4!}{(1-2x)^5}$

$f^{(4)} = \dfrac{2^4 4!}{(1-2x)^5}$

(b) Use the Lagrange Error Bound to show that $\left| f\left(\frac{1}{6}\right) - P_3\left(\frac{1}{6}\right) \right| \le \frac{1}{1800} \cdot \frac{1}{10}$ [15]

$\dfrac{\left(\frac{1}{6} - 0\right)^4}{4!} \max \left| f^{(4)}(z) \right|$

$\dfrac{1}{3^n}$

$\dfrac{\left(\frac{1}{6}\right)^4}{\cancel{4!}} \left(\dfrac{\cancel{2^4}\,\cancel{4!}}{\left(1 - 2\left(\frac{1}{6}\right)\right)^5} \right) \rightarrow \left(\dfrac{2}{3}\right)^5 \qquad \dfrac{3}{25} =$

$\left(1 - \frac{1}{3}\right)$

$= \dfrac{1}{\cdot 2^4} =$

23. The function ϕ has a Taylor series about $x = 2$ that converges for all x in it's interval of convergence. The nth derivative of ϕ at $x = 2$ is given by

$$\phi^{(n)}(2) = \frac{(n+1)!}{3^n}$$

mention inc.

for $n \geq 1$.

(a) Find an expression for the Lagrange Error bound associated with the approximation of $\phi(3)$ using the nth degree Taylor polynomial centered about $x = 2$. [10]

1

$$|\phi(3) - P_n(3)| \le \frac{(3-2)^{n+1}}{(n+1)!} \max|\phi^{(n+1)}(2)|$$

$$\le \frac{\cancel{|(1-2)|^{n+1}}}{\cancel{(n+1)!}} \frac{\cancel{(n+2)!}}{3^{n+1}}$$

$$\le \frac{n+2}{3^{n+1}}$$

(b) How many terms are needed for the error to be less than $\dfrac{1}{1200}$? [10]

$$\frac{n+2}{3^{n+1}} < \frac{1}{1200}$$

$$n = 8$$

Extra Credit (10pts): Consider the following series.

$$\sum_{n=0}^{\infty} \frac{n! \cdot (n+1)! \cdot 3^n}{\left(1 \cdot 3 \cdot 6 \cdot \cdots \cdot (3n)\right)^2}.$$

Determine the convergence or divergence of the series. Show all work.

$$\lim_{n\to\infty} \left| \frac{(n+1)\cancel{!}\,(n+2)\cancel{!}\, 3^{\cancel{n+1}}}{(\cancel{1 \cdot 3 \cdots (3n)}(3n+3))^2} \cdot \frac{\cancel{(1 \cdot 3 \cdots (3n))^2}}{\cancel{n!}\,(\cancel{n+1})!\, \cancel{3^n}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+2)(3)}{(3n+3)(3n+3)} = \frac{3}{9} = \frac{1}{3} < 1$$

$\Rightarrow$ conv by Ratio test