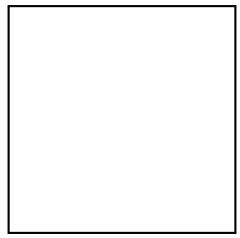
Infinite Series

Investigation 1

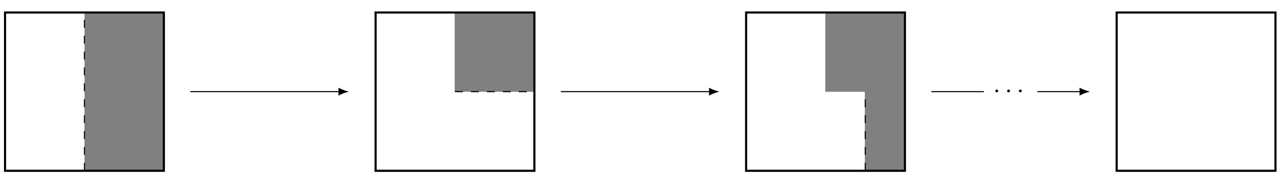
Consider the large box below. With a pen or pencil, cut the box in half and shade one of the two resulting halves. Now, cut the remaining half in half and shade one of the two resulting quarters. Continue this process indefinitely.



Consider the following questions:

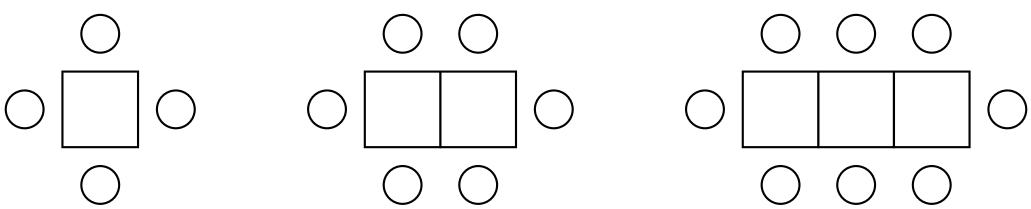
1. How many times can this process be repeated?
2. As we continue to make more and more shaded regions, what happens to the area of each new shaded region?
3. What is the total area of all the shaded regions? Does this total area have a finite value?

Now repeat this process again, but this time, alternate shading and unshading each half region. What can we say about the total area of all the shaded regions now?



Investigation 2

Consider the sequence of lunch tables surrounded by chairs below. Assume that this sequence continues indefinitely.



Consider the following questions:

1. Find a formula for the number of chairs needed for the nth table.
2. Find a formula for the total number of chairs needed for the first n tables.
3. What happens to the total number of chairs as n becomes very large?

Definition: *Sum of an Infinite Series*

The sum of an infinite series is the limit of the sequence of partial sums. If the sequence of partial sums converges to a finite value, then the infinite series is said to **converge**.

$$S = \lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} a_i$$

If the sequence of partial sums does not converge to a finite value, then the infinite series is said to **diverge**.

Arithmetic

If a_n is an arithmetic sequence, then the sum

$$\sum_{i=1}^{\infty} a_i$$

is always **divergent** unless $a_n = 0$ for all n.

Geometric

If a_n is a geometric sequence, then the sum

$$\sum_{i=1}^{\infty} a_i = \frac{a}{1-r}$$

if $|r| < 1$ and diverges otherwise.

Examples

Determine if the sum converges or diverges. If the sum converges, find the value of the sum.

1. $12 + 8 + 4 + \cdots$

2. $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \cdots$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{15} \cdot \left(\frac{5}{3}\right)^k$

4. $\displaystyle\sum_{n=1}^{\infty} 5n$

5. $24 - 12 + 6 - 3 + \cdots$

6. $\displaystyle\sum_{i=3}^{\infty} 4374 \left(-\frac{1}{3}\right)^{i-1}$

Other Examples

1. A geometric series has a sum of 21 and a first term of 3. Find the common ratio.

2. Consider the geometric series $\sum_{n=0}^{\infty} 8\left(\frac{x-1}{3}\right)^n$. For what value(s) of x does the series have a sum?

Check Your Understanding

Questions

1. A geometric series has a sum of 15 and a common ratio of 0.5. Find the first term.

2. Determine if the series converges or diverges. If the series converges, find the sum.

   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

   (b) $\sum_{n=1}^{\infty} \frac{2^n}{5^2}$

3. A geometric sequence has a 4th term of 16 and a 7th term of 64. Find the sum of the series, if it exists.

4. A geoemtric sequence has a 3rd term of $-\frac{1}{8}$ and a 6th term of $\frac{1}{64}$. Find the sum of the series, if it exists.

5. A ball is dropped from a height of 6 feet. Each time the ball bounces, it rebounds to 60% of the height from which it fell. What is the total vertical distance the ball travels?

Answers: 1. 10 2.(a) Converges to 1 (b) Diverges 3. Diverges 4. 2/3 5. 24