Sequences

Definition: *Sequence vs Series*

Simply put as possible, a **sequence** is an ordered list of numbers. A **series** is the sum of the terms of a sequence.

For example, the list of numbers $1, 3, 5, 7, 9, \ldots$ is a sequence. The series would be the sum $1+3+5+7+9+\ldots$.

Definition: *Arithmetic Sequence*

An **arithmetic sequence** is a sequence of numbers in which the difference of any two successive members is a constant called the **common difference**, denoted d. The formula for the nth term of an arithmetic sequence is given by

$$\textbf{Explicit: } a_n = a_1 + (n-1)d \qquad\qquad \textbf{Recursive: } a_n = a_{n-1} + d$$

where a_n is the nth term, a_1 is the first term, and d is the common difference.

Examples

1. Given the following arithmetic sequence: $-3, 1, 5, 9, 13, \ldots$

   (a) Find a_1.

   (b) Find the common difference.

   (c) Find the 10th term.

2. Given the following arithmetic sequence: $6, 1, -4, -9, \ldots$.

   (a) Find the a_{10}.

   (b) Find the 100th term.

Definition: *Geometric Sequence*

A **geometric sequence** is a sequence of numbers in which the ratio of any two successive terms is a constant called the **common ratio**, denoted r. The formula for the nth term of a geometric sequence is given by

$$\textbf{Explicit: } a_n = a_1 \cdot r^{n-1} \qquad\qquad \textbf{Recursive: } a_n = a_{n-1} \cdot r$$

Example

1. Given the following geometric sequence: $2, 6, 18, 54, \ldots$

   (a) Find a_1.

   (b) Find the common ratio.

   (c) Find the 21st term.

Examining Sequences

For the following questions, consider the sequence

$$-5, -\frac{9}{2}, -4, -\frac{7}{2}, -3, \ldots$$

1. Is the sequence arithmetic, geometric, or neither? Explain.

2. Find two formulas for the nth term of the sequence.

3. Find the 8th term of the sequence.

Now consider the sequence

$$-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \ldots$$

1. Is the sequence arithmetic, geometric, or neither? Explain.

2. Find two formulas for the nth term of the sequence.

3. Find the 12th term of the sequence.

Series

> **Recall:** *Summation Notation*
>
> Summation notation is a shorthand way to write the sum of a series. The sum of the first n terms of a sequence is denoted by
> $$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$
> where a_i is the ith term of the sequence.

Examples

1. Expand the sum $\displaystyle\sum_{i=3}^{6}(i^2 - 1)$.

2. Write in summation notation:
$$3^3 + 3^4 + \cdots + 3^{14}$$

> **Definition:** *Arithmetic Series*
>
> An **arithmetic series** is the sum of the terms of an arithmetic sequence. The formula for the sum of the first n terms of an arithmetic series is given by
> $$S_n = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}\left(2a_1 + d(n-1)\right)$$
> where S_n is the sum of the first n terms, a_1 is the first term, and a_n is the nth term.

Examples

1. Find the sum of the given sequence:
$$3 + 7 + 11 + \cdots + 47$$

2. An arithmetic sequence has a first term $a_1 = 7$ and fourth term $a_4 = 22$. What value of n is required to obtain $S_n = 3043$?

Definition: *Geometric Series*

A **geometric series** is the sum of the terms of a geometric sequence. The formula for the sum of the first n terms of a geometric series is given by
$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$
where S_n is the sum of the first n terms, a_1 is the first term, and r is the common ratio.

Examples

1. Find the sum of the given sequence:
$$2 + 6 + 18 + \cdots + 4374$$

2. A geometric sequence has a first term $a_1 = 2$ and common ratio $r = -\sqrt{2}$. What value of n is required to obtain $S_n > 300$?

Check Your Understanding

1. How many terms are in the sequence $3, 8, 13, \ldots, 73$?

2. Find the sum of the first 20 terms of the sequence $-3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots$.

3. Find the 18th term of the sequence $-x, \ -x + 3, \ -x + 6, \ -x + 9, \ldots$.

4. Determine a formula for the nth term of the sequence $x, \ \dfrac{x^2}{5}, \ \dfrac{x^3}{25}, \ \dfrac{x^4}{125}, \ldots$

5. Answer the following questions regarding summation notation.

   (a) Evaluate $\displaystyle\sum_{i=1}^{5}(2i - 1)$.

   (b) Write using summation notation
   $$9x + 10x^2 + 11x^3 + 12x^4 + \cdots + 98x^{90}.$$

   (c) Evaluate $\displaystyle\sum_{n=0}^{20}(1 - 7n)$.

   (d) Write using summation notation
   $$4 + 8 + 12 + 16 + \cdots + 40.$$

6. The third term of an arithmetic sequence is 7 and the 10th term is 22.

    (a) Find the first term and common difference.

    (b) Find the sum of the first 20 terms.

7. The 5th term of a geometric sequence is 3 and the 9th term is 81.

    (a) Find the common ratio(s).

    (b) Find the sum(s) of the first 10 terms.