Seismic Waves

Abstract

We discuss the different waves that propagate through the Earth and the speeds at which they do so. We then create ray path figures to show how different waves propagate through different parts of the Earth and at what angles they do so using integrals which can be computed numerically. We then calculate how long it takes for P and S waves to get from Durham to Tokyo.

1 Introduction

The Earth is mostly a solid ball, where waves propagate inside it. Earthquakes cause some of these largest waves, which can be detected by seismographs. There are two types of seismic waves P (Primary) and S (Secondary). P waves arrive at the detector first and are longitudinal waves so the vibrations are along the same direction as the direction of travel, they travel at speed V_p. S waves arrive at the detector second and are transverse waves which means the vibrations are at right angles to the direction of travel, they travel at speed V_s, there are no S waves in liquids, only in P waves. P and S waves both satisfy the wave equation:

$$\begin{aligned} \frac{1}{V^2}\frac{\partial^2\phi}{\partial t^2} &= \boldsymbol{\nabla}^2\phi \\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \end{aligned} \tag{1}$$

Where V, equal to V_p or V_s is the speed of propagation of the waves. There are 4 main layers to the Earth:

- The crust which is between 3 km and 50 km deep.
- The mantel which is 2850 km thick and is solid.
- The outer core which is 2265 km deep and is liquid.
- The inner core which is 1215 km deep and is made of solid iron nickel and other metals.

2 Ray Theory

P and S waves satisfy the wave equation (1). A solution to this equation is **Q1**

$$\phi = A\exp\big(-i\omega(t - \mathbf{k}.\mathbf{x})\big), \tag{2}$$

where $\mathbf{k}$ is a constant vector and $\mathbf{k}.\mathbf{x} = k_x x + k_y y + k_z z$. This is a solution as substituting (2) into (1) gives

$$\begin{aligned}
\frac{1}{V^2}\frac{\partial^2\ A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big)}{\partial t^2} &= \frac{\partial^2\ A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big)}{\partial x^2} + \frac{\partial^2\ A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big)}{\partial y^2} \\
&\quad + \frac{\partial^2\ A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big)}{\partial z^2} \\
-\frac{1}{V^2}\omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big) &= -k_x^2\omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big) - k_y^2\omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big) \\
&\quad -k_z^2\omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big) \\
\frac{1}{V^2}\omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big) &= \omega^2 A\exp\big(-i\omega(t-\mathbf{k}.\mathbf{x})\big)(k_x^2+k_y^2+k_z^2) \\
\frac{1}{V^2} &= k_x^2+k_y^2+k_z^2 \\
&= \mathbf{k}^2 \\
\frac{1}{V} &= \mathbf{k}. \qquad (3)
\end{aligned}$$

Therefore $\mathbf{k}$ is related to V by (3). You can see that V is the speed of the wave because $\mathbf{k}$ is $\frac{time}{length}$ and V is $\frac{1}{\mathbf{k}}$ therefore V is $\frac{length}{time}$ which is speed, so therefore V is the speed of the wave.

The period of a wave is $T = \frac{2\pi}{\omega}$. If the speed of the wave doesn't vary a lot over distances of the order of the wavelength $\lambda = \frac{V}{T}$ we can computer approximate solutions of the form

$$\phi = A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big), \qquad (4)$$

where A and T are both real functions.

Q2

Substituting (4) into (1) leads to

$$\begin{aligned}
\frac{1}{V^2}\frac{\partial^2\ A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big)}{\partial t^2} &= \boldsymbol{\nabla}^2 A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big) \\
-\frac{1}{V^2}\omega^2 A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x})\big) \\
&\quad -i\omega\boldsymbol{\nabla}T(\mathbf{x})A(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big)) \\
&= \boldsymbol{\nabla}\exp\big(-i\omega(t+T(\mathbf{x}))\big)(\boldsymbol{\nabla}A(\mathbf{x}) - i\omega\boldsymbol{\nabla}T(\mathbf{x})A(\mathbf{x})) \\
&= \exp\big(-i\omega(t+T(\mathbf{x}))\big)(\boldsymbol{\nabla}^2 A(\mathbf{x}) - i\omega\boldsymbol{\nabla}^2 T(\mathbf{x})A(\mathbf{x})) \\
&\quad -i\omega\boldsymbol{\nabla}T(\mathbf{x})\boldsymbol{\nabla}A(\mathbf{x})) - i\omega T(\mathbf{x})\exp\big(-i\omega(t+T(\mathbf{x}))\big)(A(\mathbf{x}) \\
&\quad -i\omega\boldsymbol{\nabla}T(\mathbf{x})A(\mathbf{x})) \\
-\frac{1}{V^2}\omega^2 A(\mathbf{x}) &= \boldsymbol{\nabla}^2 A(\mathbf{x}) - i\omega\boldsymbol{\nabla}^2 B(\mathbf{x})A(\mathbf{x}) - 2i\omega\boldsymbol{\nabla}T(\mathbf{x})\boldsymbol{\nabla}A(\mathbf{x}) \\
&\quad -\omega^2\boldsymbol{\nabla}T(\mathbf{x})\cdot\boldsymbol{\nabla}T(\mathbf{x})A(\mathbf{x}). \qquad (5)
\end{aligned}$$

The real and imaginary parts of (5) using $A(\mathbf{x}) = A$ and $T(\mathbf{x}) = T$ are

$$-\frac{\omega^2}{V^2}A = \boldsymbol{\nabla}^2 A - \omega^2 A\boldsymbol{\nabla}T\cdot\boldsymbol{\nabla}T, \qquad (6)$$

$$0 = -\omega A\boldsymbol{\nabla}^2 T - 2\omega\boldsymbol{\nabla}T\boldsymbol{\nabla}A. \qquad (7)$$

Dividing (6) by $-A\omega^2$ we obtain

$$-\frac{\boldsymbol{\nabla}^2 A}{A\omega^2} + |\boldsymbol{\nabla} T|^2 = \frac{1}{V^2}. \tag{8}$$

Our ansatz (4) assumes that V changes very little over a distance of the wavelength λ. Using a Taylor expansion, we can write

$$\frac{\mathrm{d}^2 A(x)}{\mathrm{d}x^2} \approx \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{\mathrm{d}x^2}, \tag{9}$$

and take $\mathrm{d}x = \lambda$ so that

$$\begin{aligned} -\frac{\boldsymbol{\nabla}^2 A}{A\omega^2} &\approx \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{\lambda^2} \frac{\tau^2}{A(x)4\pi^2} \\ &= \frac{A(x+\mathrm{d}x) + A(x-\mathrm{d}x) - 2A(x)}{A(x)} \frac{1}{4\pi^2 V^2} \ll \frac{1}{V^2}. \end{aligned} \tag{10}$$

where τ is the period of the wave. We use the fact that $V = \lambda/\tau$ and we can neglect the first term in (8) if A is nearly constant over a distance $\mathrm{d}x = \lambda$. Then (8) simplifies to

$$|\boldsymbol{\nabla} T|^2 = \frac{1}{V^2}. \tag{11}$$

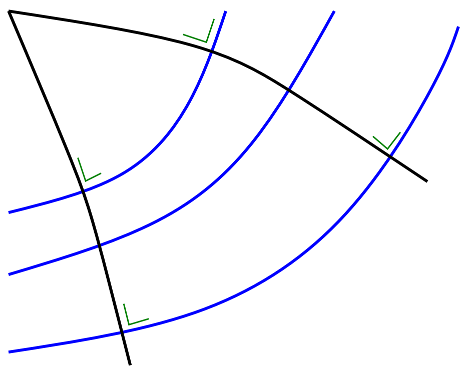

Figure 1: Wave-front (blue) and ray path (black).

In (4) $T(\mathbf{x}) = T_0$, where T_0 is a constant, defines a surface corresponding to a wave front. This is the surface corresponding to all parts of the wave which have travelled a time T_0 since a reference point, usually taken as the source of the wave. Ray paths are the lines which are perpendicular to those surfaces at all points.

The derivatives of T in the direction parallel to the wave-fronts is zero (as T is constant on it). Equivalently

$$\boldsymbol{\nabla} T \cdot \mathbf{q}_{\mathrm{wf}} = 0, \tag{12}$$

where $\mathbf{q}_{\mathrm{wf}}$ is a vector tangent/parallel to the wave-front.

Indeed: we can take a system of coordinates around a point $\mathbf{r}$ such that x and y are parallel to the wave-front and z perpendicular to it: $\partial T/\partial x = \partial T/\partial y = 0$. We have

$$\boldsymbol{\nabla} T \cdot \mathbf{q}_{\mathrm{wv}} = (\partial T/\partial x, \partial T/\partial y, \partial T/\partial z) \cdot (q_x, q_y, 0) = 0, \tag{13}$$

and as the result is independent of the system of coordinates, (12) holds.

A ray path can be parametrised as $\mathbf{r}(s)$ where s is a parameter satisfying $|\mathrm{d}\mathbf{r}/\mathrm{d}s| = 1$, and s is measured in the same units as $\mathbf{r}$. The vector $\mathrm{d}\mathbf{r}/\mathrm{d}s$ is tangent to a ray path: if we consider two close points $\mathbf{r}(s+\delta s)$ and $\mathbf{r}(s)$, the vector joining them is

$$\mathbf{b} = \mathbf{r}(s+\delta s) - \mathbf{r}(s) = \delta s \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} + \mathcal{O}(\mathrm{d}s^2)\,. \tag{14}$$

If we take δs very small then $\mathbf{b}$ is parallel to $\mathrm{d}\mathbf{r}/\mathrm{d}s$. We also notice that

$$T(\mathbf{r}+\mathbf{q}) - T(\mathbf{r}) \approx \mathbf{q}\cdot\boldsymbol{\nabla} T\,, \tag{15}$$

and if $\mathbf{q}$ is infinitesimal and parallel to the wave-front (i.e. proportional to $\mathbf{q}_{\mathrm{wf}}$), the right-hand side is zero as $\boldsymbol{\nabla} T$ is perpendicular to the wave-front. If $\mathbf{q}_{\mathrm{rp}}$ is parallel to the ray path, then $T(\mathbf{r}+\mathbf{q_{rp}}) - T(\mathbf{r})$ is the time taken by the wave to travel from $\mathbf{r}$ to $\mathbf{r}+\mathbf{q_{rp}}$ which is $|\mathbf{q_{rp}}|/V$ and so $|\mathbf{q_{rp}}|/V = \mathbf{q_{rp}}\cdot\boldsymbol{\nabla} T = |\mathbf{q_{rp}}||\boldsymbol{\nabla} T|$. We thus have

$$V|\boldsymbol{\nabla} T| = 1\,. \tag{16}$$

As $V\boldsymbol{\nabla} T$ and $\mathrm{d}\mathbf{r}/\mathrm{d}s$ are vectors perpendicular to the wave front and of norm 1 we have

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = V\boldsymbol{\nabla} T\,, \tag{17}$$

as one can always orient s so that the equality holds with a positive sign.

The time δT needed to go from a point s on the path to a close point $s+\delta s$ on the same path is $\delta T = T(s+\delta s) - T(s) = \delta s\frac{dT}{ds}$. As $\delta s/\delta T = V$, $\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{1}{V} = u$ where we have defined $u = 1/V$ which is called the slowness by geophysicists.

We now take the gradient of $\frac{\mathrm{d}T}{\mathrm{d}s} = u$ from both sides, we get

$$\frac{\mathrm{d}\boldsymbol{\nabla} T}{\mathrm{d}s} = \nabla u\,. \tag{18}$$

And inserting (17) into (18) we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[u\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right] = \nabla u\,. \tag{19}$$

Knowing the underlying structure of the Earth, implies knowing $u(\mathbf{r})$ and we can solve (19) for $\mathbf{r}(s)$ corresponding to the ray followed by the wave.

We assume that the Earth is radially symmetric and that the speed of seismic waves only depends on the radial coordinate r measured from the centre of the Earth. **Q3**

Computing

$$\frac{d}{ds}\left[\mathbf{r}\times\left(u\frac{d\mathbf{r}}{ds}\right)\right], \tag{20}$$

using (19) we get

$$\begin{aligned}\frac{d}{ds}\left[\mathbf{r}\times\left(u\frac{d\mathbf{r}}{ds}\right)\right] &= \frac{d\mathbf{r}}{ds}\times u\frac{d\mathbf{r}}{ds} + r\times\frac{d}{ds}(u\frac{d\mathbf{r}}{ds})\\ &= \frac{d\mathbf{r}}{ds}\times u\frac{d\mathbf{r}}{ds} + r\times\boldsymbol{\nabla} u\\ &= 0,\end{aligned} \tag{21}$$

this is because the cross product of a vector with itself is 0 and u and r are parallel to each other therefore the whole equation is 0. $\mathbf{r}\times\left(u\frac{d\mathbf{r}}{ds}\right)$ is a constant because of (20) equalling 0.

We then write

$$\left|\mathbf{r} \times u\frac{d\mathbf{r}}{ds}\right| = |\mathbf{K}| = p = ru\sin(\theta_i), \tag{22}$$

where θ_i is the angle between the ray path direction $\frac{d\mathbf{r}}{ds}$ and the radial direction. Notice that p, called the ray parameter, is a constant. In other words, for any given path, p will remain the same for all times, even when the wave changes type.

When $\theta_i = \pi/2$, the ray path is perpendicular to the radial direction meaning it has reached its lowest point.

We use (16) to evaluate $c(\mathbf{r})^{-2} = u(\mathbf{r})^2$ by computing

$$\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T = \left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \theta}\right)^2 + \frac{1}{r^2\sin(\theta^2)}\left(\frac{\partial T}{\partial \phi}\right)^2 = u(\mathbf{r})^2. \tag{23}$$

For a radially symmetric Earth, waves travel along a great circle and the last term is zero. We then seek solution for T of the form

$$T(r,\theta) = f(\theta) + g(r), \tag{24}$$

and substitute it into (23) to get

$$r^2\left[\frac{dg(r)}{dr}\right]^2 - r^2u(r)^2 = -\left[\frac{df(\theta)}{d\theta}\right]^2 = -k^2. \tag{25}$$

Both sides must be equal to the same constant because if you varied r on the left hand side it wouldn't change the right hand side because it is in terms of θ and if we varied θ on the right hand side it wouldn't change the left hand side because it's in terms of r, therefore both sides must be equal to the same constant which we call $-k^2$. As $df/d\theta = k$ we have

$$f(\theta) = \int_0^\theta k d\phi = k\theta. \tag{26}$$

From the left hand side of (25)

$$\frac{dg(r)}{dr} = \pm\left(u(r)^2 - \frac{k^2}{r^2}\right)^{1/2}. \tag{27}$$

and we have

$$T(r,\theta) = k\theta \pm \int_0^r \left(u(x)^2 - \frac{k^2}{x^2}\right)^{1/2} dx. \tag{28}$$

Q4

We can use (17) and the fact that $V = \frac{1}{u}$ to show that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\boldsymbol{\nabla} T}{u}. \tag{29}$$

The gradient ($\boldsymbol{\nabla}$) in polar coordinates is

$$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\right]. \tag{30}$$

Therefore

$$\begin{aligned} u(r)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} &= \boldsymbol{\nabla} T = \left[\frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial T}{\partial \theta}, 0\right] \\ &= \left[\pm\frac{1}{r}\left(r^2u(r)^2 - k^2\right)^{1/2}, \frac{k}{r}, 0\right]. \end{aligned} \tag{31}$$

Therefore we can conclude that

$$\frac{d\mathbf{r}}{ds} = \left[\pm\frac{1}{ru(r)}\left(r^2u(r)^2 - k^2\right)^{1/2}, \frac{k}{u(r)r}, 0\right]. \tag{32}$$

As $|d\mathbf{r}/ds| = 1$, the angle θ between the incident ray path and the radial direction is

$$\sin(\theta) = \frac{k}{ru(r)}, \tag{33}$$

and so $k = ru(r)\sin(\theta) = p$ (see (22)). Then $T(r,\theta)$ can be obtained by computing the integral

$$T(r,\theta) = p\theta \pm \int_{r_s}^{r} \frac{\sqrt{x^2u(x)^2 - p^2}}{x} dx. \tag{34}$$

with r_s the radius at the starting point. As T is a travelling time it must increase and so we take the $+$ sign for ascending waves, $dr > 0$, and the $-$ sign for descending waves, $dr < 0$.

It can be shown [1] that the wave travelling time along the path is a local minimum (Fermat's principle). Mathematically:

$$\frac{\partial T}{\partial p} = \theta \pm \int_{r_s}^{r} \frac{-p}{x\sqrt{r^2u(x)^2 - p^2}} dx = 0 \tag{35}$$

and so

$$\theta = \pm p \int_{r_s}^{r} \frac{1}{x\sqrt{r^2u(x)^2 - p^2}} dx. \tag{36}$$

Substituting this back into (34):

$$\begin{aligned} T(r,\theta) &= \pm \int_{r_s}^{r} \left[\frac{p^2}{x\sqrt{x^2u(x)^2 - p^2}} + \frac{\sqrt{x^2u(x)^2 - p^2}}{x}\right] dx & (37)\\ &= \pm \int_{r_s}^{r} \frac{x^2u(x)^2}{x\sqrt{x^2u(x)^2 - p^2}} dx. & (38) \end{aligned}$$

The total angle of propagation between 2 points can be determined from (36):

$$\Delta = \int_{r_s}^{r} \frac{p}{x\sqrt{x^2u(x)^2 - p^2}} dx. \tag{39}$$

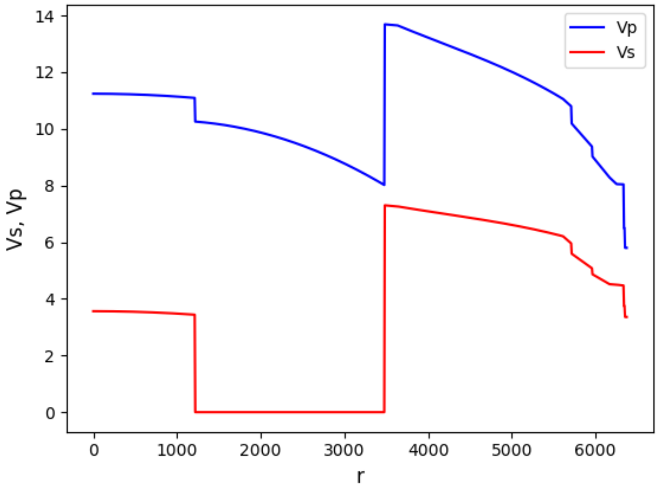

Figure 2: A graph to show the speeds (in km/s) of primary and secondary waves as a function of r (in km).

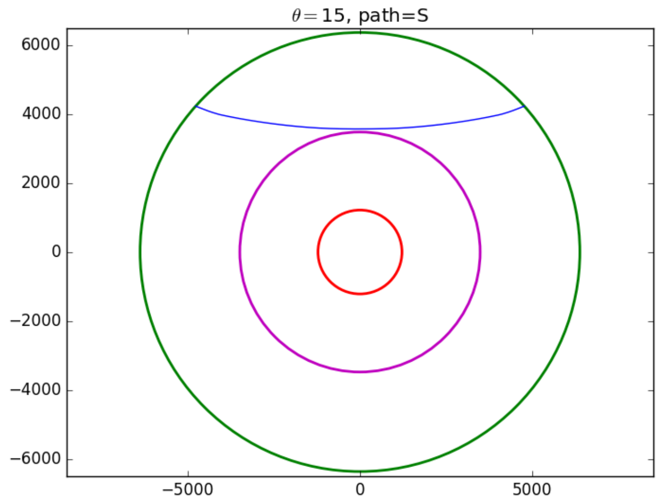

Figure 3: A ray path for a S wave propagating in the crust where $\theta = 15$.

Figure (2) shows us the speeds of P and S waves as they go through the Earth at different depths. Where as Figure (3) shows us the ray path for a S wave propagating in the crust at a certain angle. The other circles on the figure show the different layers of the Earth. Therefore the S wave is only going through the crust as the ray path only goes through the first ring of the Earth which is the crust.
Figures (4) to (11) show the paths of different waves coming in at different angles. They also show different parts of the Earth, where inside the red circle is the inner core, inside the purple circle is the outer core, inside the green circle is the mantel and the green circle is the surface of the Earth.

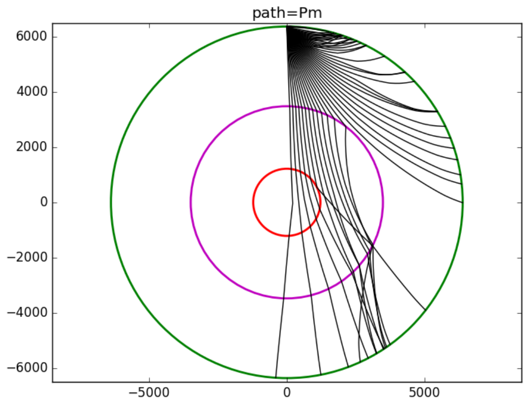

Figure 4: A ray path (in km) for P waves propagating through the mantel.

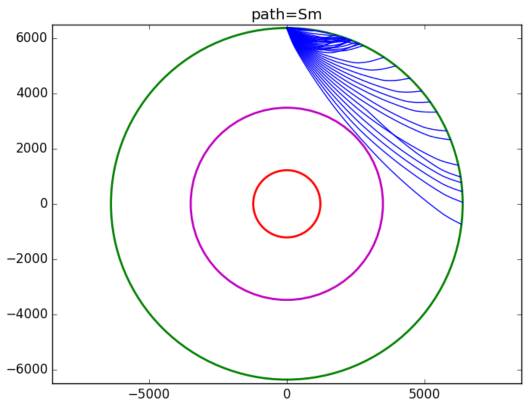

Figure 5: A ray path (in km) for S waves propagating through the mantel.

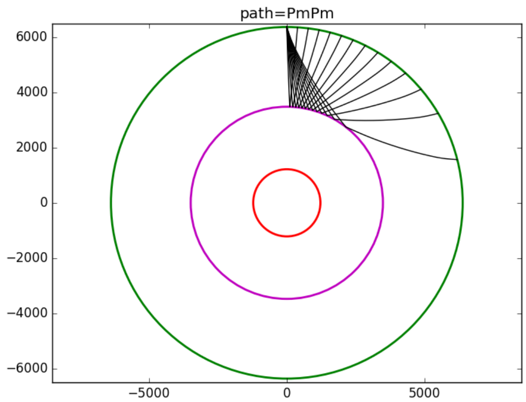

Figure 6: A ray path (in km) for P waves bouncing on the top of the outer core as P waves.

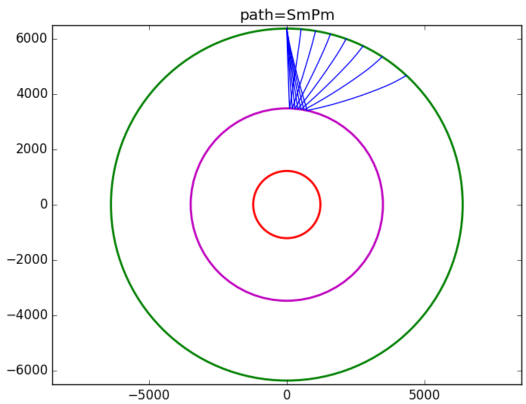

Figure 7: A ray path (in km) for S waves bouncing on the top of the outer core as P waves.

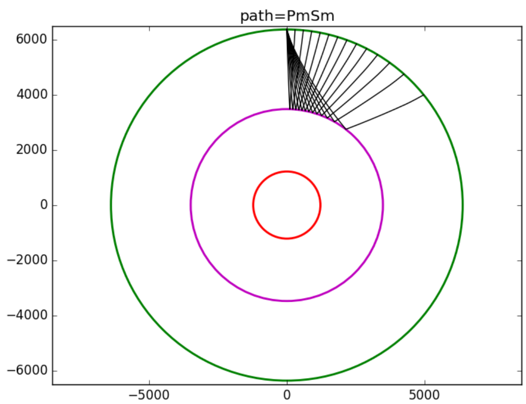

Figure 8: A ray path (in km) for P waves bouncing on the top of the outer core as S waves.

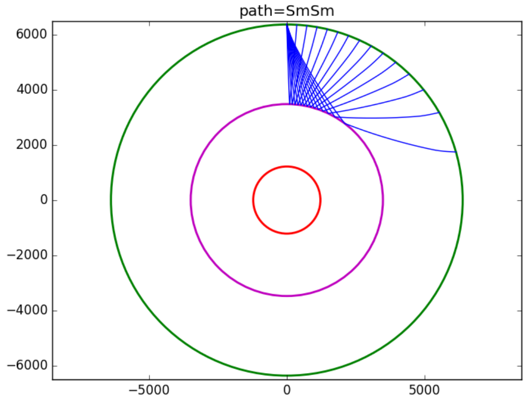

Figure 9: A ray path (in km) for S waves bouncing on the top of the outer core as S waves.

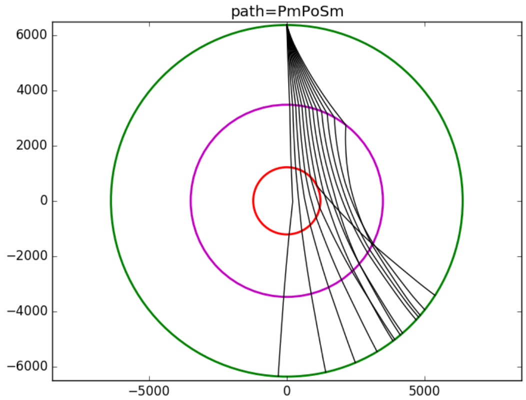

Figure 10: A ray path (in km) for P wavse propagating through the mantel, transmitted to the core and then transmitted as S waves in the mantel.

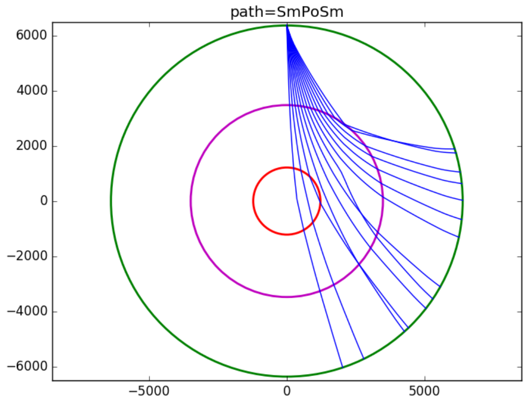

Figure 11: A ray path (in km) for S wavse propagating through the mantel, transmitted to the core as P waves and then transmitted as S waves in the mantel.

Q5

Durham has coordinates 54.7753°N and 1.5849°W, Japan has the coordinates 35.689°N and 139.6917°E so the angle between Durham and Japan is 137.584447787°. Which to 1 decimal place is 137.6° For a P wave travelling through the mantel (see Figure 4) where $\delta = 137.584447787°$ the table below shows how T is calculated to half a degree:

$\theta(°)$	T(s)	$\delta(°)$
5.00	1180	145.9
5.42	1167	138.6
5.45	**1165**	**137.6**
5.47	1164	136.9
6.00	1148	127.9
6.11	1165	136.3
6.12	**1168**	**137.6**
6.16	1178	142.6
7.00	1198	152.6

Therefore the quickest time for a P wave to get from Durham to Japan is 1165 seconds.
For a P wave travelling through the mantel, transmitted to the outer core and then transmitted as an S wave in the mantel (see Figure 10) where $\delta = 137.584447787°$ the table below shows how T is calculated to half a degree:

$\theta(°)$	T(s)	$\delta(°)$
5.00	1389	141.7
5.24	1382	137.7
5.26	**1381**	**137.6**
5.33	1381	137.1
7.00	1402	146.1
8.00	1389	140.6
8.65	**1380**	**137.6**
9.00	1377	136.6

For a S wave propagating through the mantel, transmitted to the outer core as a P wave and then transmitted as an S wave in the mantel (see Figure 11) where $\delta = 137.584447787°$ the table below shows how T is calculated to half a degree:

$\theta(°)$	T(s)	$\delta(°)$
3.00	1592	134.6
3.57	1595	134.0
3.58	**1603**	**137.6**
3.61	1615	143.8
4.18	1604	138.7
4.20	**1602**	**137.6**
4.22	1601	137.3
5.00	1580	129.7

Therefore the quickest time for a S wave travelling through the mantel is 1380 seconds.

Different points on the surface of the Earth receive waves at different times depending on how long it takes them to get there, what type of wave they are and what they have to travel through. Therefore looking at figures (4) to (11) there are places where there are no lines, where the amplitude is 0 and places where lots of lines are, where the amplitude is higher. However, at some points where waves arrive that have taken multiple different paths the amplitude will be effected by constructive waves and some by destructive waves. The amplitude will therefore be greater/ smaller than first expected. **Q6**

3 Conclusion

We have discussed the different seismic waves that propagate through the Earth, such as P (Primary) and S (Secondary) waves. Using ray path figures we have shown that P and S waves propagate through different parts of the Earth at different speeds. We have seen that the way in which the waves travel through the Earth is dependent on the angle at which they enter the Earth, such angles can be calculated using integrals which can be computed numerically. We have also been able to see that using ray path traces it is possible to get the time it takes for an Earthquake at one point on the Earth's surface to be felt at another point.

References

[1] Nick Rawlinson *Lecture notes on Seismology* `http://rses.anu.edu.au/~nick/teaching.html`