Lecture Transcription

Course: EE334 2026
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Exponential Random Variables

The lecture begins by establishing the properties of exponential random variables. Let

$$X \sim \mathrm{Exp}(\lambda).$$

Probability Density Function (PDF)

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Cumulative Distribution Function (CDF)

$$F_X(x) = \Pr(X \leq x) = 1 - e^{-\lambda x}.$$

$$\Pr(X \geq x) = e^{-\lambda x}.$$

Expectation

$$E[X] = \frac{1}{\lambda}.$$

Memoryless Property

The most important property of the exponential distribution is memorylessness:

$$\Pr(X > t + s \mid X > t) = e^{-\lambda s}.$$

This implies that the future probability does not depend on how much time has already elapsed.

The continuous-time Markov chains require exponential holding times because the time spent in each state must be memoryless.

Poisson Process

A random process $\{A(t) \mid t \geq 0\}$ taking non-negative integer values is called a Poisson process with rate λ.

$A(t)$ represents the cumulative number of events up to time t.

- $\alpha(t)$: Arrival process
- $\beta(t)$: Departure process

$$A(t) - A(s)$$

represents the number of events in the interval (s, t).

Independent Increments

Events occurring in disjoint intervals are independent.

$$\Pr\{A(t+\tau) - A(t) = n\} = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}.$$

Inter-Arrival Times

Let

$$\tau_n = t_n - t_{n-1}.$$

Then

$$\tau_n \sim \mathrm{Exp}(\lambda)$$

and the inter-arrival times are independent and identically distributed.

Small Interval Approximation ($\delta \to 0$)

For a small interval δ:

$$\Pr\{A(t+\delta) - A(t) = 0\} = 1 - \lambda\delta + o(\delta),$$

$$\Pr\{A(t+\delta) - A(t) = 1\} = \lambda\delta + o(\delta),$$

$$\Pr\{A(t+\delta) - A(t) \geq 2\} = o(\delta).$$

Using Taylor expansion,

$$e^{-\lambda\delta} = 1 - \lambda\delta + \frac{(\lambda\delta)^2}{2!} - \dots$$

which verifies the above approximations.

Merging

If multiple independent Poisson processes with rates $\lambda_1, \lambda_2, \dots, \lambda_n$ are merged, the resulting process is Poisson with rate:

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Splitting

If a Poisson process of rate λ is split with probability p_i, then the resulting process has rate:

$$\mu_i = \lambda p_i.$$

The M/M/1 Queue

The M/M/1 queue consists of:

- Poisson arrivals with rate λ
- Exponential service times with rate μ
- Single server

Service time distribution:

$$f_{S_n}(s) = \mu e^{-\mu s}.$$

Memorylessness:

$$\Pr(S_n > r + t \mid S_n > t) = \Pr(S_n > r).$$

Let $N(t)$ denote the number of customers in the system. $N(t)$ forms a continuous-time Markov chain.

Sampling at small intervals δ:

$$N_k = N(k\delta).$$

Transition Probabilities

$$P_{00} = 1 - \lambda\delta + o(\delta),$$

$$P_{i,i+1} = \lambda\delta + o(\delta),$$

$$P_{i,i-1} = \mu\delta + o(\delta),$$

$$P_{ii} = 1 - \lambda\delta - \mu\delta + o(\delta).$$

Steady-State Analysis

Steady-State Probabilities

Let ϕ_n denote the steady-state probability of being in state n.

Balance equation:

$$\phi_n(\lambda\delta + o(\delta)) = \phi_{n+1}(\mu\delta + o(\delta)).$$

As $\delta \to 0$:

$$\phi_{n+1} = \frac{\lambda}{\mu}\phi_n.$$

Define traffic intensity:

$$\rho = \frac{\lambda}{\mu}.$$

Thus,

$$\phi_n = \rho^n \phi_0.$$

Normalization:

$$\sum_{n=0}^{\infty} \phi_n = 1.$$

$$\phi_0 \sum_{n=0}^{\infty} \rho^n = 1.$$

$$\phi_0 \frac{1}{1-\rho} = 1.$$

$$\phi_0 = 1 - \rho.$$

Final distribution:

$$\phi_n = \rho^n (1-\rho), \quad \rho < 1.$$

Average Number of People in the System (N)

The average number of people in the system is defined as the expected value:

$$N = E[n].$$

Using the steady-state probability

$$\phi_n = \rho^n (1-\rho),$$

we compute:

$$N = \sum_{n=0}^{\infty} n\phi_n = \sum_{n=0}^{\infty} n\rho^n (1-\rho).$$

$$N = (1-\rho) \sum_{n=0}^{\infty} n\rho^n.$$

$\rho^n = \rho \cdot \rho^{n-1}$:

$$N = \rho(1-\rho) \sum_{n=0}^{\infty} n\rho^{n-1}.$$

we know that

$$\frac{d}{d\rho}(\rho^n) = n\rho^{n-1}.$$

Thus,

$$N = \rho(1-\rho) \sum_{n=0}^{\infty} \frac{d}{d\rho}(\rho^n).$$

$$N = \rho(1-\rho) \frac{d}{d\rho} \left(\sum_{n=0}^{\infty} \rho^n \right).$$

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}, \quad |\rho| < 1.$$

Therefore,

$$N = \rho(1-\rho) \frac{d}{d\rho} \left(\frac{1}{1-\rho} \right).$$

$$\frac{d}{d\rho} \left(\frac{1}{1-\rho} \right) = \frac{1}{(1-\rho)^2}.$$

Hence,

$$N = \rho(1-\rho) \frac{1}{(1-\rho)^2}.$$

$$N = \frac{\rho}{1-\rho}.$$

Substituting $\rho = \frac{\lambda}{\mu}$:

$$N = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda/\mu}{(\mu-\lambda)/\mu} = \boxed{\frac{\lambda}{\mu-\lambda}}.$$

Average Time in System (T)

Using Little's Theorem:

$$N = \lambda T.$$

Therefore,

$$T = \frac{N}{\lambda} = \frac{\frac{\lambda}{\mu-\lambda}}{\lambda} = \boxed{\frac{1}{\mu-\lambda}}.$$

Average Waiting Time in Queue (W)

The total time in system is:

$$T = W + \frac{1}{\mu}.$$

Thus,

$$W = T - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu}.$$

Taking common denominator:

$$W = \frac{\mu - (\mu - \lambda)}{\mu(\mu - \lambda)} = \boxed{\frac{\lambda}{\mu(\mu - \lambda)}}.$$

Average Number of People in Queue (N_q)

Applying Little's Theorem to the queue:

$$N_q = \lambda W.$$

$$N_q = \lambda \left(\frac{\lambda}{\mu(\mu - \lambda)} \right) = \frac{\lambda^2}{\mu(\mu - \lambda)}.$$

Rewriting in terms of $\rho = \frac{\lambda}{\mu}$:

$$N_q = \frac{(\lambda/\mu)^2}{1 - (\lambda/\mu)} = \boxed{\frac{\rho^2}{1 - \rho}}.$$