Linear Algebra Assignment

Case Study - 1

1. a) Mean return $= \frac{1}{n}\sum_{i=1}^{n} x_i$

Asset A : $\frac{1}{3}(0.10 + 0.05 + 0.15) = \frac{1}{3}(0.3) = 0.1$

Asset B : $\frac{1}{3}(0.2 + 0.1 + 0.25) = \frac{1}{3}(0.55) = 0.183$

Asset C : $\frac{1}{3}(0.15 + 0.05 + 0.2) = \frac{1}{3}(0.4) = 0.133$

b) Centering the data.

→ Subtract mean from it

	Asset A $x - 0.1$	Asset B $y - 0.183$	Asset C $z - 0.133$
T_1	0	0.017	0.017
T_2	-0.05	-0.083	-0.083
T_3	0.05	0.067	0.067

c) Covariance matrix.

This is a sample taken from a set of data, So use

$$Cov(x,y) = \frac{\sum(x-\bar{x}_i)(y-\bar{y}_i)}{n-1}$$

$$Cov(x,x) = \frac{(0)^2 + (-0.05)^2 + (0.05)^2}{2} = 0.0025$$

$$Cov(y,y) = \frac{(0.017)^2 + (-0.083)^2 + (0.067)^2}{2} = 0.00583$$

$$Cov(z,z) = Cov(y,y) = 0.00583$$

$$Cov(x,y) = Cov(y,x) = \frac{0 + (0.05)(0.083) + (0.05)(0.067)}{2}$$

$$= 0.00375$$

$$Cov(y,z) = Cov(z,y) = 0.00583$$

$$Cov(x,z) = Cov(z,x) = 0.00375$$

$$\begin{bmatrix} 0.0025 & 0.00375 & 0.00375 \\ 0.00375 & 0.00583 & 0.00583 \\ 0.00375 & 0.00583 & 0.00583 \end{bmatrix}$$

2. Singular Value Decomposition

$A = U\Sigma V$ $\qquad\qquad |A - \lambda I| = 0.$

$$\begin{vmatrix} 0.0025-\lambda & 0.00375 & 0.00375 \\ 0.00375 & 0.00583-\lambda & 0.00583 \\ 0.00375 & 0.00583 & 0.00583-\lambda \end{vmatrix} = 0.$$

$$(0.0025-\lambda)\left[(0.00583-\lambda)^2 - (0.00583)^2\right]$$
$$-0.0375\left[(0.00375)(0.00583-\lambda-0.00583\right]$$
$$+0.00375\left[0.00375\left(0.00583-(0.0583-\lambda)\right)\right]$$

$$= (0.0025-\lambda)\left[\lambda^2 - 2\lambda(0.00583)\right]$$
$$-(0.00375)^2[-\lambda] + (0.00375)^2\lambda$$

$$= \lambda(0.0025-\lambda)(\lambda-0.01166) + 0.0028\lambda\times10^{-2}$$

$$= \lambda\left[-\lambda^2 + \lambda(0.01166+0.0025) - (0.0025)(0.01166)\right.$$
$$\left. + 10^{-2}\times 0.0028\right].$$

$$= \lambda\left[-\lambda^2 + 0.01416\lambda + \cancel{0.002771}\right] = 0.$$
-1.025×10^{-6}.

$$= \cancel{\lambda(\lambda-0.06)(\lambda-0.046)}.$$
$$\lambda(\cancel{\lambda-0.014})(\cancel{\lambda-8.16\times10^{-5}}) = 0.$$
$$\lambda(\lambda-0.014)(\lambda-7.276\times10^{-5}) = 0.$$

$\therefore$ Eigen values $= 0,\ 0.014,\ 7.276\times10^{-5}$.

Sum of eigen values $= 0.014073$.

Trace of the matrix $= 0.0025 + 0.00583 + 0.00583$

$= 0.0141.$

$\Rightarrow$ Sum of eigen values = Trace of matrix.

Calculation of eigen vectors.

$$(A-\lambda I)x = 0.$$

For $\lambda = 0 \Rightarrow Ax = 0.$

$$\begin{bmatrix} 0.0025 & 0.00375 & 0.00375 \\ 0.00375 & 0.00583 & 0.00583 \\ 0.00375 & 0.00583 & 0.00583 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2.5x_1 + 3.75x_2 + 3.75x_3 = 0.$$
$$3.75x_1 + 5.83x_2 + 5.83x_3 = 0.$$

$$\Rightarrow (x_2+x_3) = \frac{2.5x_1}{3.75} = \frac{3.75x_1}{5.83} \Rightarrow x_1 = 0.$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

For $\lambda = 0.014 \Rightarrow (A - 0.014I)x = 0.$

$$(2.5-14)x_1 + 3.75x_2 + 3.75x_3 = 0. \quad -①$$
$$3.75x_1 + (5.83-14)x_2 + 5.83x_3 = 0. \quad -②$$
$$3.75x_1 + 5.83x_2 + (5.83-14)x_3 = 0 \quad -③.$$

$$② - ③ \Rightarrow -14x_2 + 14x_3 = 0 \Rightarrow x_2 = x_3$$

$$\frac{-11.5}{3.75}x_1 = -2x_2 \Rightarrow x_1 = 0.6521. \qquad v_2 = \begin{bmatrix} 0.6521 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 7.276 \times 10^{-5}$

$$\begin{bmatrix} 2.5-0.07 & 3.75 & 3.75 \\ 3.75 & 5.83-0.07 & 5.83 \\ 3.75 & 5.83 & 5.83-0.07 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$② - ③ \Rightarrow -0.07x_2 + 0.07x_3 = 0 \Rightarrow x_2 = x_3.$$

$$2.49x_1 + 7.5x_2 = 0 \Rightarrow x_1 = -3.01x_2. \qquad v_3 = \begin{bmatrix} -3.01 \\ 1 \\ 1 \end{bmatrix}$$

Normalized eigen vectors.

$$v_1 = \begin{bmatrix} 0 \\ 0.7071 \\ -0.7071 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.419 \\ 0.642 \\ 0.642 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -0.905 \\ 0.3 \\ 0.3 \end{bmatrix}$$

2. PROPORTION OF VARIANCE :-

PC1 : ($\lambda_1 \approx 0.0141$) → eigenvector 1

$$\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} = 0.9958 \Rightarrow 99.58\%$$

PC2 : ($\lambda_2 \approx 0.00006$) → eigenvector 2

$$\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} = 0.0042 = 0.42\%$$

PC3 : ($\lambda_3 = 0$) → eigenvector 3

↳ Proportion of variance = 0%

∴ Variance in : PC1 >> PC2 > PC3

∴ Most of the movement among the three assets is driven by a single factor (PC1)

③ INTERPRETATION :-

ⓐ SIGNIFICANCE OF PRINCIPAL COMPONENTS IN TERMS OF MARKET TRENDS

↳ First Principal Component (PC1) → DOMINANT MARKET FACTOR

- Since 99.5% variability of all assets comes from just this one Principal component, this tells us that nearly every fluctuation in the data is explained by this single dimension. Thus, the assets react to the same underlying cause, called the market factor. When the market factor is up, all three move together.
- The values of the entries for A, B, C are all positive [$v_1 = (0.416, 0.643, 0.643)$] hence if this factor goes up, the returns of all three ~~are~~ assets are above their average at the same time; when it's down, they are all below their average.
- This could come from a general market sentiment or a macroeconomic condition.

↳ Second Principal Component (PC2) : (~0.42% total variance)

- This captures a tiny part in the total movement of the assets. Basically, there is a small difference in the movement of A v/s that of B and C. This rarely matters ~~on~~ in practical terms.
- $V_2 = (0.9093, -0.2943, -0.2943)$; this signifies that PC2 is slightly more weighted towards A than B and C.
- This may reflect a more asset specific influence or niche factor that operates independently of the broader market trend.

↳ Third principal component (PC3): (0% of total variance)

- Has exactly 0 variance direction.
  $v3 = (0, 1, -1)\frac{1}{\sqrt{2}}$, which could mean buy B and sell C in equal amounts, because they move in almost the exact same way. This combination has no fluctuation, hence the variance is 0.
- B and C ~~as~~ are basically duplicates in this data, offering no extra diversification if you hold them both.

(h) How do these components help understand asset behaviour?

- Identifying common drivers-
PC1 shows that all three assets move mostly together, driven by a shared market or economic factor.
- Detecting minor differences-
PC2 suggests a small difference in how asset A behaves compared to B and C, which might show up during specific events-within the sector.
- Spotting Redundancy-
PC3 reveals that B and C are nearly identical.
Holding both adds little diversification since they react the same way to market changes.

(i) Recommendations for Investment strategy or Risk Management :-

- AVOID HOLDING SIMILAR ASSETS :-
Since B and C behave in almost the same way, they offer no diversification relative to each other. So, in order to spread out the risk, the firm should consider more diverse assets, that have lower correlations, and relate ~~different~~ differently to the Principal components.

- FOCUS ON HEDGING THE RISK FROM THE DOMINANT MARKET FACTOR :-
Since PC1 captures almost all the movement ~~acro~~ across the assets, it represents the main source of risk in the portfolio.
To mange this, the firm can hedge against the specific risk, by using derivatives or taking offsetting positions that reduce exposure to broad market swings. Doing this can reduce most of the portfolio's overall volatility, without needing to hedge every individual asset separately.

- USE PC2 FOR SMALL STRATEGIC ADJUSTMENTS :-
Since PC2 reflects small differences between how A reacts and how assets B and C react, during specific situations, if the firm expects short term changes, they can take advantageous positions in these small shifts.

- SMARTER PORTFOLIO BUILDING :-
Invest mainly along PC1 to capture broad market trends, with smaller positions in PC2 to tap into asset-specific opportunities.

- SIMPLIFIED RISK TRACKING :-
Since PC1 drives most of the risk, monitoring key market indicators tied to it is enough for effective risk mangement.

Linear Algebra Assignment

Case Study 2

Given :-

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^T \quad v_i \in R^m \Rightarrow A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & & & \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix}$$

Aim → To minimize $L = \| \sum_{i=1}^{n} x_i v_i - w \|_2$

(a) To prove :- Problem can be solved directly if A is a diagonal matrix.

If $A = [v_1 \ v_2 \ \cdots \ v_n]$ and $x \in R^n \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\Rightarrow L = \|Ax - w\|_2$$

When A is a diagonal matrix :-

$$A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & & & d_n \end{bmatrix}$$

$A \in R^{m \times n}$

☆ Each diagonal entry d_i corresponds to the scaling of the coefficient x_i, and all non-diagonal entries are zero.

$$\Rightarrow Ax = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_m x_m \end{bmatrix}$$

$$\Rightarrow L = \sqrt{\sum_{i=1}^{m} (d_i x_i - w_i)^2}$$

To minimize $L \Rightarrow \frac{dL_i}{dx_i} = 0 \Rightarrow \frac{d\left[(d_i x_i - w_i)^2\right]}{dx_i} = 0$

$$\Rightarrow 2(d_i x_i - w_i)(d_i) = 0$$

$$\Rightarrow \boxed{x_i = \frac{w_i}{d_i}} \quad (\text{if } d_i \neq 0)$$

$$\Rightarrow x = \begin{bmatrix} \frac{w_1}{d_1} \\ \frac{w_2}{d_2} \\ \vdots \\ \frac{w_m}{d_m} \end{bmatrix}$$

$\Rightarrow$ solution is direct as it involves only division when A is diagonal.

☆ Special cases:-

① if $d_i = 0$ and $w_i \neq 0$ $\Rightarrow$ no solution exists.

② if $d_i = 0$ and $w_i = 0$ $\Rightarrow$ ∞ solutions.

Hence, For a diagonal matrix A, the problem reduces to minimizing independent quadratic terms and sol is given by :- $x_i = \frac{w_i}{d_i}$

(b) Geometric interpretation of SVD (singular value decomposition) :-

$$A_{m \times n} = U \Sigma V^T$$

where → U is an m×n orthogonal matrix

→ Σ is an m×n diagonal matrix containing singular values of A.

→ V is an n×n orthogonal matrix ($VV^T = I$)

☆ Geometrically :-

① Transformation of a vector :-

→ Any vector $x \in R^n$ is first rotated by V^T.

→ Then it is stretched along the axes defined by the singular values in Σ.

→ Finally, it is rotated again by U.

☆ Meaning of singular values of a matrix :-

① Geometric interpretation :-

→ Singular values represent how much a matrix A stretches or compresses a vector when transforming it.

→ If x is a vector, applying A to x scales it by a factor of σ_i along the corresponding singular vector.

(b) ~~Main~~ Objective :- minimize $\|Ax - w\|$ → least squares eq^n — (1)

Assumption :- $A = U \Sigma V^T$

V^T → orthogonal matrix $(n \times n)$

U → orthogonal matrix $(m \times m)$

Σ → diagonal matrix $(m \times n)$

Using SVD, we write eq (1) as :-

$\Rightarrow Ax - w = U \Sigma V^T x - w$

$\Rightarrow$ Since U is orthonormal, we can write :-

$$w = U(U^T w) \qquad \{\text{As } UU^T = I_{m \times m}\}$$

$$\Rightarrow Ax - w = U[\Sigma V^T x - U^T w]$$

$$\Rightarrow \|Ax - w\| = \|U(\Sigma V^T x - U^T w)\|$$

Because multiplying by an orthonormal matrix U does not change the Euclidean norm, we get

$$\|Ax - w\| = \|\Sigma V^T x - U^T w\|$$

Thus :- original problem is equivalent to

$$\min_{x \in \mathbb{R}^n} \|\Sigma V^T x - U^T w\|$$

And $\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & & \sigma_r \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

where $\sigma_1, \dots, \sigma_r$ are the non-zero singular values (and $r \le \min\{m, n\}$).

$\Rightarrow \Sigma V^T x$ is essentially a vector in $\mathbb{R}^m$ whose first r components involve linear combinations of $x_1, \dots, x_n$ scaled by $\sigma_1, \dots, \sigma_r$, and whose remaining components (if $m > r$) are all zero.

Let $y = V^T x$ (another orthonormal transformation from x), and let $z = U^T w$. Then:-

$$\min \| \Sigma V^T x - U^T w \| = \min \| \Sigma y - z \|$$

And Σ is diagonal in its first r rows/columns (where r is the rank), which means this norm expands to:

$$\Rightarrow \| \Sigma y - z \|^2 = \sum_{i=1}^{r} (\sigma_i y_i - z_i)^2 + \sum_{i=r+1}^{m} (0 - z_i)^2 =$$

$$\sum_{i=1}^{r} (\sigma_i y_i - z_i)^2 + \sum_{i=r+1}^{m} z_i^2$$

① The first r coordinates behave exactly like the "diagonal" case (each $\sigma_i y_i$ must approximate z_i)

② The remaining $m-r$ coordinates corresponds to rows of Σ that are zero, so they simply contribute $\| z_{r+1:m} \|^2$ which cannot be reduced by changing y.

Solving for y :- Minimizing each squared term $(\sigma_i y_i - z_i)^2$ independently for $i = 1, \dots, r$. Taking the derivative gives :-

$$\sigma_i^2 y_i - \sigma_i z_i = 0 \Rightarrow y_i^* = \begin{cases} \frac{z_i}{\sigma_i}, & \text{if } \sigma_i \neq 0 \\ \text{arbitrary}, & \text{if } \sigma_i = 0 \end{cases}$$

But by definition of the singular values, $\sigma_i \neq 0$ for $1 \leq i \leq r$. For $i > r$, $\sigma_i = 0$ indeed, but those terms do not appear in the sum (they are effectively 0 for the product)

Hence

$$y_i^* = \frac{z_i}{\sigma_i} \text{ (for } i = 1, \dots, r), \; y_i^* \text{ can be anything for } i > r,$$

though typically we take $y_{r+1} = \dots = y_n = 0$ to get the minimal norm solution.

Since $y = V^T x$, we recover

$$x^* = Vy^* = V \begin{pmatrix} z_1/\sigma_1 \\ \vdots \\ z_r/\sigma_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

And $z_i = U^T w$

(c) given n data points $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n) \in R^2$

Aim → to find best fitting linear eq^n (this is linear regression)

$\Rightarrow y = ax + b$

So, we want to minimize L :-

$$L = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \quad \text{— (1)}$$

Converting this to a least squares problem :-

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \in R^{n \times 2}$$

$$x = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$\Rightarrow L = ||Ax - y||^2$

$$\Rightarrow \quad Ax = y$$

$$\Rightarrow \quad A^T A x = A^T y$$

$$\Rightarrow \quad \boxed{x = (A^T A)^{-1} A^T y}$$

→ This yields the best-fit values of a and b, making this a special case of general least squares formulation.

(d) Polynomial Regression :-

$$y = ax^2 + bx + c$$

We want to minimize :- $L = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2$ — ①

Converting this to a least squares problem :-

$$A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \in R^{n \times 3}$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$ → unknowns we have to find.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

eq ① becomes :- $L = \|Ax - y\|^2$

$$\Rightarrow \quad A^T A x = A^T y$$

$$\Rightarrow \quad \boxed{x = (A^T A)^{-1} A^T y}$$