Time Series Analysis

Advanced Assignment

1. When modeling ln Y_t using a time trend model, what is the relationship between exp E_T[ln Y_{T+h}] and $E_T[Y_{T+h}]$ for any forecasting period h? Are these ever the same? Assume that the error terms are normally distributed around a mean of zero.

2. Why does a unit root with a time trend, Yt = d1 + Yt - 1 + Ɛt not depend explicitly on t?

3. The Yule-Walker (YW) equations provide a set of expression that relate the parameters of an AR to the autocovariances of the AR process. This approach uses p + 1 equations to solve for the long-run variance g_0 and the first p autocorrelations. Autocovariances (or autocorrelations) at lags larger than p are then easily computed with a recursive structure starting from the first p autocovariances. The equations are:

Cov[Y_t, Y_t] = Cov[d + $f_1 Y_{t-1}$ +g+ $f_p Y_{t-p}$ + P_t, Y_t] Cov[Y_t,
Y_{t-1}] = Cov[d + $f_1 Y_{t-1}$ +g+ $f_p Y_{t-p}$ + P_t, Y_{t-1}]

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Cov[Y_t, Y_{t-p}] = Cov[d + $f_1 Y_{t-1}$ +g+ $f_p Y_{t-p}$ + P_t, Y_{t-p}]

Excluding the first equation, dividing each row by the long run variance g_0 produces a set of equations that relate the autocorrelations:

$$\rho_1 = \phi_1 + \phi_2\rho_1 + \cdots + \phi_p\rho_{p-1} \\ \vdots \\ \rho_p = \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_p$$

Compute the first three autocorrelations for the AR(2) process as shown below:

$Y_t = 1.4Y_{t-1} - 0.45Y_{t-2}$

4. (Python programming assignment)

a. Download the "S&P Dividend Yield by Month" and the "S&P Dividend Yield by Year". The data can be accessed using the Quandl/NASDAQ data link. (Sample here).

b. Download the series MULTPL/SP500_DIV_YIELD_MONTH and MULTPL/SP500_DIV_YIELD_YEAR.

- Plot and compare the autocorrelation function(ACF) and partial autocorrelation function(PACF) for the monthly and yearly series . Take the log of each series and plot the ACF and PACF of the log series. How are the ACF/PACFs different for the log series and the raw series(without log).
- Perform the Box-Ljung test for the first 5 autocorrelation for each of the 4 series from part a (annual, monthly) * (log, without log). Report the test statistics and p values. What can you conclude based on these observations?
- Perform the ADF test for each of the 4 series from part a (annual, monthly) * (log, without log). Report the test statistics and p values. What can you conclude based on results of these tests?

ANSWERS

Q1. Modeling lnY_{T+h} as a time trend model:

$lnY_{T+h} = \alpha + \beta*(T+h) + \varepsilon_{T+h}$, where $\varepsilon_{T+h} \sim N(0,\sigma^2)$ (mean given to be 0)

Let $X \equiv lnY_{T+h}$

Then taking expectation of X we obtain $E_T[lnY_{T+h}] = E_T[X] = \alpha + \beta*(T+h) + 0$

$\Rightarrow X \sim N(\mu,\sigma^2)$ with $\mu = \alpha + \beta*(T+h)$.

Thus the term exp $E_T[lnY_{T+h}]$ is exp μ.

Now, $E_T[Y_{T+h}] = E_T[\exp X] = \int_{-\infty}^{\infty} e^x f_X(x)dx = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{(x-\mu)^2}{2\sigma^2})dx$

Inside the integrand we have two exponentials, so combining them $\Rightarrow e^x exp(-\frac{(x-\mu)^2}{2\sigma^2}) = exp(x - \frac{(x-\mu)^2}{2\sigma^2})$

Now completing the square in exponent:

We need to rewrite $x - \frac{(x-\mu)^2}{2\sigma^2}$ as a perfect square plus constant:

$$x - \frac{(x-\mu)^2}{2\sigma^2} = \frac{2\sigma^2 x - (x-\mu)^2}{2\sigma^2} = \frac{2\sigma^2 x - x^2 - \mu^2 + 2x\mu}{2\sigma^2}$$

Now, $-x^2 - \mu^2 + 2x\mu + 2\sigma^2 x = -[x^2 - 2(\mu + \sigma^2)x + (\mu + \sigma^2)^2] + (\mu + \sigma^2)^2 - \mu^2$

Computing the term out of square: $(\mu + \sigma^2)^2 - \mu^2 = \sigma^4 + 2\mu\sigma^2$

$$x - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2} + \frac{\sigma^4 + 2\mu\sigma^2}{2\sigma^2} = -\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2} + (\frac{\sigma^2}{2} + \mu)$$

Now, $E_T[\exp X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp(x - \frac{(x-\mu)^2}{2\sigma^2})dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2} + (\frac{\sigma^2}{2} + \mu))dx$

The remaining integral is exactly the integral of a normal density with mean $\mu + \sigma^2$ and variance σ^2:

$\Rightarrow E_T[\exp X] = exp(\mu+\frac{\sigma^2}{2}) * \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2})dx$, as $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2})dx = 1$

$$\Rightarrow \mathbf{E_T[\exp X] = exp(\mu+\frac{\sigma^2}{2})}.$$

On comparing $E_T[\exp X]$ and exp $E_T[lnY_{T+h}]$, **It becomes a comparison between $exp(\mu+\frac{\sigma^2}{2})$ and $exp\mu$**

And essentially,

if $\sigma^2 > 0$ then $exp\mu < exp(\mu+\frac{\sigma^2}{2})$ as exponent function is an increasing function

if $\sigma^2 = 0$, then $exp\mu = exp(\mu+\frac{\sigma^2}{2})$

Hence, if $\sigma^2 > 0$, $\exp \mathrm{E}T\,[lnY_{T+h}] < \mathrm{E}T\,[\exp lnY_{T+h}]$

if $\sigma^2 = 0$, $\exp \mathrm{E}T\,[lnY_{T+h}] = \mathrm{E}T\,[\exp lnY_{T+h}]$

Q2.

An AR(1) process $Y_t = \phi Y_{t-1} + \varepsilon_t$ has a unit root when $\phi = 1$,

In lag operator form that is $(1 - L)Y_t = \varepsilon_t$ and the polynomial 1- z = 0 has root z = 1

A unit root makes the series non stationary as expectation of above AR(1) process is constant, but variance has t in it and is not constant (because of the shocks ε_t).

The expression $Y_t = \phi Y_{t-1} + \varepsilon_t + d_1$ is just adding a constant drift d_1 to each period

$$Y_t = Y_{t-1} + \varepsilon_t + d_1 \Leftrightarrow (1 - L)Y_t = \varepsilon_t + d_1$$

There is no explicit 'βt' term so this does not depend on t explicitly,

Now , to check any other dependence implicitly we iterate the one step equation from some initial Y_0 :

$$Y_1 = Y_0 + \varepsilon_1 + d_1,$$
$$Y_2 = Y_1 + \varepsilon_2 + d_1 = Y_0 + \varepsilon_1 + \varepsilon_2 + 2d_1,$$
$$\ldots\ldots..$$
$$Y_t = Y_0 + td_1 + \sum_{i=1}^{t} \varepsilon_i$$

The term td_1 is deterministic trend in the mean: $\mathrm{E}[Y_t] = Y_0 + td_1$,

The shocks or error terms which are white noise result in the term $\sum_{i=1}^{t} \varepsilon_i$, because of which the variance of Y_t becomes $0 + t\sigma^2$ which is $t\sigma^2$, $\mathrm{Var}(Y_t) = t\sigma^2$ which also depends on t.

Solving the equation shows a dependence on t for the recursed version considering an initial point along with linear dependence on t for both mean and variance.

- The model's form is recursive, not a direct regression on t.
- The **dependence on t** appears only when recursion is unwind. We add d_1 each of the t steps,

giving td_1

- The unit root (ϕ=1) is what makes the effect of each drift and shock permanent— keep carrying it forward—so both the mean and variance grow with t.

Thus, In conclusion:

even without an explicit t in the initial time trend $Y_t = \phi Y_{t-1} + \varepsilon_t$, which is unlike the regression style trend model with explicit t dependence shown as $Y_t = \alpha + \beta t + u_t$, in the unit-root-with-drift formulation, the dependence on t is built into the recursive accumulation of the constant drift d_1.

Adding d_1 every period naturally produces a $t^* d_1$ term once the recursion is solved.

Q3.

For an AR(2) process: $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$ where ε_t is white noise

The Yule-Walker equations for autocorrelations $\rho_k = Corr(Y_t, Y_{t-k})$ are:

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad for\ k \geq 3$$

Given AR(2) process: $Y_t = 1.4Y_{t-1} - 0.45Y_{t-2} + \varepsilon_t$

So for our process ϕ_1= 1.4, ϕ_2= - 0.45

Solving for ρ_1 and ρ_2

Equation 1:

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$$

But ρ_0= 1, so:

$$\rho_1 = 1.4 \times 1 + (-0.45)\rho_1$$

$$\rho_1 + (0.45)\rho_1 = 1.4$$

$$1.45\rho_1 = 1.4$$

$$\rho_1 = \frac{1.4}{1.45} \approx 0.9655$$

Equation 2:

$$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0$$
$$\rho_2 = 1.4 \times 0.9655 + (-0.45) \times 1$$
$$\rho_2 = 1.3517 - 0.45 = 0.9017$$

Solving for ρ_3:

$$\rho_3 = \phi_1\rho_2 + \phi_2\rho_1$$
$$\rho_3 = 1.4 \times 0.9017 + (-0.45) \times 0.9655$$
$$\rho_3 = 1.2624 - 0.4345 = 0.8279$$

Hence the first 3 autocorrelations of the AR(2) process are : **0.9655, 0.9017, 0.8279**

Q4-

Here is the notebook where we have executed all the commands (all interpretations are also written as comments in this notebook)-

time_series.ipynb

1.Plot Comparison: ACF & PACF for Monthly Dividend Yield-

```
# Plot Comparison: ACF & PACF for Monthly Dividend Yield

#1. ACF (Autocorrelation Function)
# Raw & Log Series:

# Slowly decaying ACF with very high autocorrelation at lag 1 and gradually
decreasing values beyond that.

# This is a classic indicator of non-stationarity — specifically a unit root
or persistent trend.

# The similarity between raw and log ACF curves confirms that log
transformation had little impact.

# 2.PACF (Partial Autocorrelation Function)
# Raw & Log Series:
```

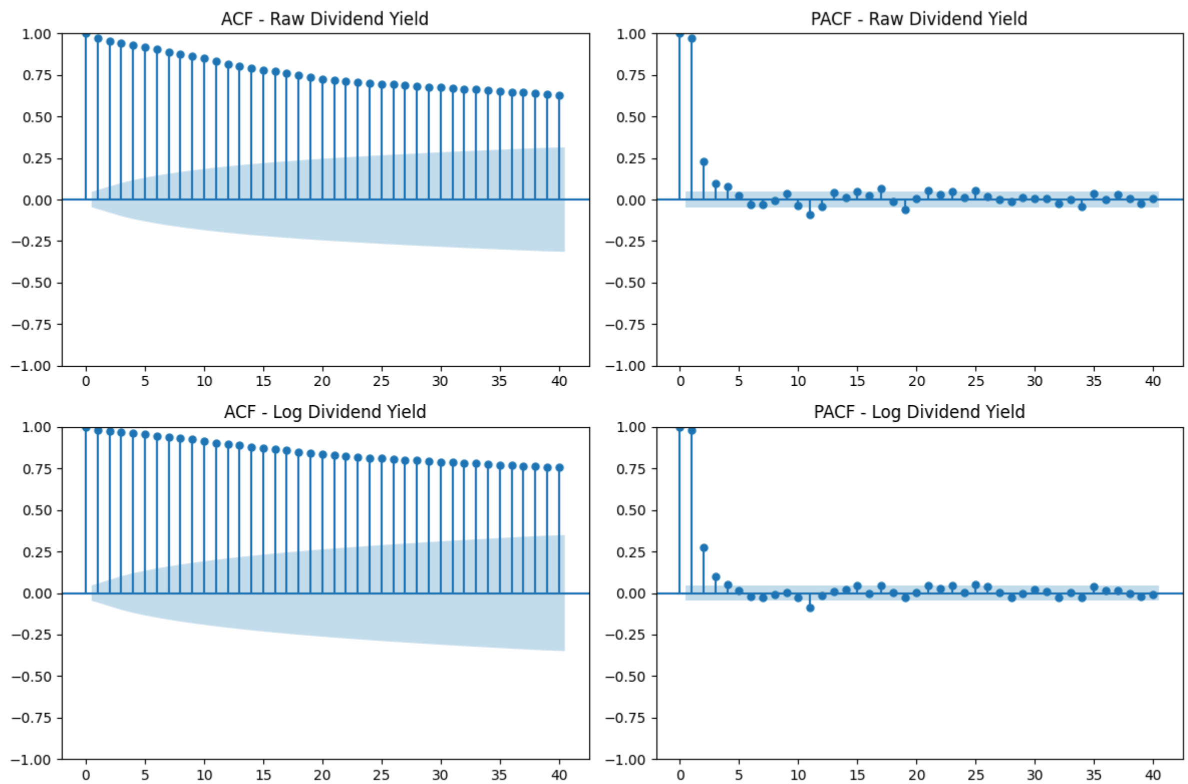
```
# PACF shows a strong spike at lag 1, and smaller spikes at lag 2 (and possibly 3).

# After that, PACF values drop closer to zero.

# This structure suggests:

# Possible AR(1) or AR(2) behavior,

# Once differenced, the series may become stationary and suitable for ARIMA modeling.
```



2.Why log and raw series look similar-

```
# Why Log and Raw Series Look Similar
# Log transformation has minimal impact on values that are already small and positive (like the Dividend_Yield values, which are around 0.05–0.06).

# Since the range of values is narrow, log(x) behaves nearly linearly — so the correlation structure (ACF/PACF) is preserved.
```

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# Hence, both the raw and log series are non-stationary and exhibit strong autocorrelation (slowly decaying ACF, significant PACF spikes).

# Interpretation of Plots
# ACF:
# Both raw and log series show strong persistence (high values) — the ACF tails off slowly, indicating non-stationarity (likely a trend or unit root).

# PACF:
# Both show significant spikes at lags 1–2, suggesting short-term dependency that might be captured by an AR(1) or AR(2) process after differencing.
```

Results of the hypothesis test-

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Box-Ljung Test (lag=5):
Dividend Yield: Statistic = 8133.1721, p-value = 0.0000
Log Dividend Yield: Statistic = 8592.5017, p-value = 0.0000

 Augmented Dickey-Fuller (ADF) Test:
Dividend Yield: ADF Statistic = -2.4718, p-value = 0.1225
Log Dividend Yield: ADF Statistic = -1.3980, p-value = 0.5832
```

```
# Interpretation:

# The null hypothesis of the Box-Ljung test is that the data are independently distributed (i.e., no autocorrelation).

# Since p-values < 0.05, we reject the null hypothesis for both raw and log series.

# Conclusion: Significant autocorrelation exists in both series → the data are not white noise.
```

```
# Interpretation:

# The null hypothesis of the ADF test is that the series has a unit root (i.e., non-stationary).
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# Since p-values > 0.05, we fail to reject the null hypothesis for both series.

# Conclusion: Both raw and log Dividend Yield series are non-stationary.
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# Final Conclusion-
# 1. Autocorrelation is present (Box-Ljung Test).

# 2.Both series are non-stationary (ADF Test).

# 3.Log transformation does not induce stationarity, as ACF/PACF and ADF test show similar results.
```