Minimal-Perimeter Polyominoes: Chains, Roots, and Algorithms

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Abstract. A polyomino is a set of edge-connected squares on the square lattice. We investigate the combinatorial and geometric properties of minimal-perimeter polyominoes. We explore the behavior of minimal-perimeter polyominoes when they are "inflated," i.e., expanded by all empty cells neighboring them, and show that inflating all minimal-perimeter polyominoes of a given area create the set of all minimal-perimeter polyominoes of some larger area. We characterize the roots of the infinite chains of sets of minimal-perimeter polyominoes which are created by inflating polyominoes of another set of minimal-perimeter polyominoes, and show that inflating any polyomino for a sufficient amount of times results in a minimal-perimeter polyomino. In addition, we devise two efficient algorithms for counting the number of minimal-perimeter polyominoes of a given area, compare the algorithms and analyze their running times, and provide the counts of polyominoes which they produce.

1 Introduction

A polyomino is a set of edge-connected squares on the square lattice. The study of polyominoes began in the 1950s, where it was studied in parallel as topics in combinatorics [7] and in statistical physics [5]. Typically, polyominoes are enumerated by their area, i.e., the number of lattice cells they contain. The number of polyominoes with area n is usually denoted in the literature by $A(n)$. No formula for $A(n)$ is known as of today, but there exist a few algorithms for computing values of $A(n)$, such as those of Redelmeier [13] and Jensen [8]. Using the latter algorithm, values of $A(n)$ were computed up to $n = 56$. The existence of the *growth constant* of $A(n)$, namely, $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$, was shown by Klarner [9]. More than 30 years later, Madras [11] showed that $\lim_{n\to\infty} A(n+1)/A(n)$ exists and, thus, is equal to λ. The best known lower and upper bounds on λ are 4.0025 [4] and 4.6496 [10], resp. The currently best *estimate* of λ is 4.0625696 ± 0.0000005 [8].

Some works studied polyominoes by their *perimeter*. The perimeter of a polyomino is the set of empty (unoccupied) cells adjacent to the polyomino. (Sometimes, when the meaning is clear from the context, we used the term "perimeter"

Work on this paper by both authors has been supported in part by ISF Grant 575/15 and by BSF (joint with NSF) Grant 2017684.

S. P. Pal and A. Vijayakumar (Eds.): CALDAM 2019, LNCS 11394, pp. 109–123, 2019.
https://doi.org/10.1007/978-3-030-11509-8_10

to denote the *number* of perimeter cells.) This definition originated in statistical physics, where the numbers of polyominoes with certain areas and perimeters are used for modeling physical phenomena, such as percolation processes. Asinowski et al. [2] recently provided formulae for the number of polyominoes whose perimeter is close to the maximum possible. Here, we complement their work and study *minimal-perimeter polyominoes*, that is, polyominoes with the minimum possible perimeter size for their area.

Several works discussed the minimum perimeter of a polyomino with area n. Wang and Wang [17] were the first to address the subject, showing a sequence of polyominoes with increasing areas, whose perimeter is minimal. Altshuler et at. [1] later characterized all polyominoes with *maximal* area for a given perimeter. (As we explain below, there is a close relation between minimal-perimeter (for a fixed area) and maximal-area (for a fixed perimeter) polyominoes.) Minimal-perimeter polyominoes also appear in certain game-theoretic problems, where formulae for minimum perimeter of polyominoes of area n [14], or, similarly, for polyiamonds and polyhexes (shapes on the planar triangular and hexagonal lattices, resp.) [6], were developed.

In a recent work [3], we addressed the question of how many minimal-perimeter polyominoes of a given area exist. In that work, we defined the inflated version of a polyomino to be the union of the polyomino and the set of its perimeter cells. We showed that this operation induces a bijection between the set of minimal-perimeter polyominoes of a given area, and the set of the minimal-perimeter polyominoes of the area of the inflated polyominoes. By inflating repeatedly polyominoes in such a set, we obtain an infinite chain of sets of minimal-perimeter polyominoes, all of which have the same cardinality. We call these sequences "inflation chains." A natural interesting question is "For which areas n, the set of minimal-perimeter polyominoes of area n cannot be generated by inflating a set of polyominoes of a smaller area $n' < n$?" We call such areas n "inflation-chains roots" and give a full characterization of these values. Inspired by inflation chains, we also show that any polyomino becomes a minimal-perimeter polyomino after a finite amount of inflation steps.

We also investigate the question of how many minimal-perimeter polyominoes of area n exist for a general value of n (not necessarily an inflation-chain root). The inflation chains provide a partial answer to this question. If n is not a root, we reduce the problem to computing the number of minimal-perimeter polyominoes of area n', for some $n' < n$. However, the question remains open for inflation-chain roots. Hence, we present two algorithms for counting minimal-perimeter polyominoes of area n. Both algorithms are based on some geometric properties of minimal-perimeter polyominoes.

2 Preliminaries

As mentioned above, this work answers a few questions raised in an earlier work on minimal-perimeter polyominoes [3]. Thus, we provide here the needed definitions and results, and refer the reader to the previous paper for more details.

Let Q be a polyomino. The perimeter of Q, denoted by $\mathcal{P}(Q)$, is the set of all empty cells adjacent to Q. The border of Q, denoted by $\mathcal{B}(Q)$, is the set of all cells of Q which have an adjacent empty cell. The inflation of Q is defined as $I(Q) = Q \cup \mathcal{P}(Q)$, namely, inflating Q is expanding it by adding to it its perimeter. Figure 1 illustrates these concepts.

A minimal-perimeter polyomino has the minimum perimeter size (number of perimeter cells) out of all polyominoes of the same area. Let M_n denote the set of all minimal-perimeter polyominoes with area n, and $\epsilon(n)$ denote the minimum perimeter of a polyomino with area n. Sieben [14] showed that $\epsilon(n) = \lceil 2 + \sqrt{8n-4} \rceil$.

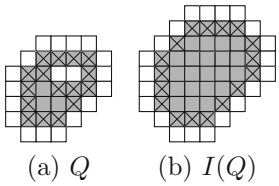


Fig. 1. An example of a polyomino Q and its inflated polyomino $I(Q)$.

Following our early terminology [3], a cell of Type (d) is a polyomino cell with exactly two adjacent empty cells in two opposite sides (□■□), and a cell of Type (z) is an empty cell with exactly two adjacent occupied cells in two opposite sides (■□■).

Here are a few previous results which we will need later in the current paper.

Lemma 1 [3, Corollary 1]**.** *For any n, we have $\epsilon(n + \epsilon(n)) = \epsilon(n) + 4$.*

Theorem 1 [3, Theorem 2]**.** *For a polyomino Q with no holes and no patterns of Types (d) and (z), we have $|\mathcal{P}(Q)| = |\mathcal{B}(Q)| + 4$.*

Lemma 1 and Theorem 1 are given merely for using them in proofs of some claims. However, the next theorem is interesting by its own.

Theorem 2 [3, Theorem 4]**.** *For all $n \geq 3$, we have $\{I(Q) \mid Q \in M_n\} = M_{n+\epsilon(n)}$.*

This theorem shows the existence of inflation chains which we discuss in the next section.

3 Inflation Chains: Roots and Convergence

In this section we address the question of what numbers are the roots of the inflation chains. In addition, we show that inflating repeatedly any polyomino for a sufficient amount of times results in a minimal-perimeter polyomino.

3.1 Roots of Inflation Chains

Recall that $\epsilon(n)$ is defined as the minimum perimeter of polyominoes of area n. This function is not one-to-one, thus, it does not have an inverse function. To overcome this, we define a pseudo-inverse function $\epsilon^{-1}(p) = \min_{n \in \mathbb{N}} \{n \mid \epsilon(n) = p\}$, that is, the minimum area of minimal-perimeter polyominoes with perimeter p.

Theorem 3. *An integer n is an inflation-chain root if and only if $n = \epsilon^{-1}(p)$ for some p.*

Proof. Notice that the function $\epsilon(n)$ behaves like a step function (see Fig. 3), thus, for any perimeter p, let us define the values $n_b^p = \epsilon^{-1}(p)$ and $n_e^p = \epsilon^{-1}(p+1)-1$. (These are the begin and end of the 'step' correspond to the perimeter p). Note that for any polyomino Q, such that $|Q| \in [n_b^p, n_e^p]$, the area of $I(Q)$ is $|Q|+p$. Therefore, any area between $n_b^p + p$ and $n_e^p + p$ (inclusive) is not an inflation-chain root. Hence, any inflation-chain root must be between n_e^p+p and $n_b^{p+1}+p+1$ (that is, between the end of the inflation of the step of p and the beginning of the inflation of the step of $p+1$). By the definitions of n_e^p and n_b^{p+1}, we have that $n_b^{p+1} + p + 1 = n_e^p + p + 2$, thus, the area $n_e^p + p + 1$ is included neither in the sequence ending at $n_e^p + p$, nor by the sequence beginning at $n_b^{p+1} + p + 1$. Therefore, the area $n_e^p + p + 1$ is an inflation-chain root for all p.

From Lemma 1, we know that $\epsilon(n_e^p + p) = p + 4$ and $\epsilon(n_e^p + p + 2) = p + 5$, thus, $n_e^p + p + 1$ equals either n_e^{p+4} or n_b^{p+5}. Altshuler et al. [1] characterized the set of minimal-perimeter polyominoes with *maximal* area and divided it into four classes. These classes essentially consist of square-like polyominoes, aligned with the diagonals, and differ by the 'corners' which contain one or two cells (see Fig. 2). This set is closed under inflation, thus, the inflation-chain root cannot be n_e^{p+4}, and so we are left with n_b^{p+5} as the root. This means that for any p, the value n_b^p is an inflation-chain root.

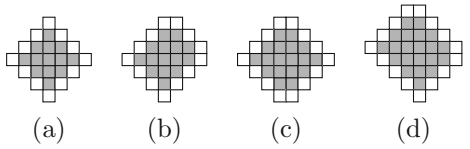


Fig. 2. Representative minimal-perimeter polyominoes with maximal area.

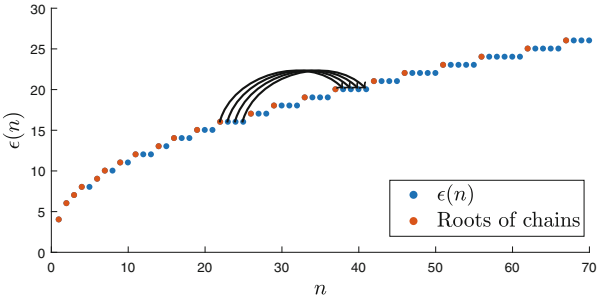


Fig. 3. Values of $\epsilon(n)$. Inflation-chains roots are colored in red. The arrows show the effect of the inflation operation on minimal-perimeter polyominoes. (Color figure online)

Theorem 3 is illustrated in Fig. 3. Altshuler et al. [1] also provided a formula for the values of n_e^p. However, their formula is a function of the area: Given an

area k, the function provides the largest $n \leq k$, such that $n = n_e^p$ for some p. We provide an equivalent formula, however, as a function of p.

Theorem 4. $\epsilon^{-1}(p) = \left\lfloor \frac{(p-3)^2}{8} + \frac{3}{2} \right\rfloor$.

Proof. $\epsilon^{-1}(p) = \min_{n\in\mathbb{N}} \{n \mid \epsilon(n) = p\} = \min_{n\in\mathbb{N}} \left\{n \mid \left\lceil 2 + \sqrt{8n-4} \right\rceil \geq p\right\} = \min_{n\in\mathbb{N}} \left\{n \mid \sqrt{8n-4} > p-3\right\} = \min_{n\in\mathbb{N}} \left\{n \mid n > \frac{(p-3)^2}{8} + \frac{1}{2}\right\} = \left\lfloor \frac{(p-3)^2}{8} + \frac{3}{2} \right\rfloor$.

3.2 Convergence of Inflation

We now discuss the structure of an inflated polyomino, and show that after a sufficient number of inflations, it will obey the formula $|\mathcal{P}(Q)| = |\mathcal{B}(Q)|+4$. Using this, together with Theorem 3, we show that inflating any polyomino (or any set of disconnected lattice cells for that matter), converges to a minimal-perimeter polyomino after a finite amount of inflation steps.

We begin with defining the operator $I^k(Q)$ $(k > 0)$ to be Q inflated k times:

$$I^k(Q) = Q \cup \{c \mid \mathrm{Dist}(c, Q) \leq k\},$$

where $\mathrm{Dist}(c, Q)$ is the *Manhattan distance* from a cell c to a polyomino Q. We will use the notation $Q^k = I^k(Q)$ for brevity. Let $R(Q)$ denote the *diameter* of Q, i.e., the maximal horizontal or vertical distance (L^∞) between two cells of Q. The following lemma shows that some geometric features of a polyomino disappear after inflating it enough times. The proof is omitted here due to space limitations, and is given in the full version of the paper.

Lemma 2. *For any $k > R(Q)$, the polyomino Q^k does not contains (i) holes; (ii) cells of Type (d); and (iii) patterns of Type (z).*

Corollary 1. *After $k = R(Q)$ inflations, we will have $\left|\mathcal{P}(Q^k)\right| = \left|\mathcal{B}(Q^k)\right| + 4$.*

Proof. This follows at once from Lemma 2 and Theorem 1.

Consider a polyomino Q, and define the function $\phi(Q) = \epsilon^{-1}(|\mathcal{P}(Q)|) - |Q|$. When $\phi(Q) \geq 0$, it counts the cells that should be added to Q, with no change to its perimeter, in order to make it a min.-perimeter polyomino. In particular, if $\phi(Q) = 0$, then Q is a min.-perimeter polyomino. If $\phi(Q) < 0$, then Q is also a min.-perimeter polyomino, and $-\phi(Q)$ cells can be removed from Q while keeping the result a minimal-perimeter polyomino (without changing the perimeter).

Lemma 3. *For any value of p, we have $\epsilon^{-1}(p+4) - \epsilon^{-1}(p) = p - 1$.*

Proof. Let Q be a min.-perimeter polyomino with area $n_b^p = \epsilon^{-1}(p)$. The area of $I(Q)$ is $n_b^p + p$, thus, by Lemma 1, $\mathcal{P}(I(Q)) = p+4$. The area n_b^{p+4} is an inflation-chain root, thus, the area of $I(Q)$ cannot be n_b^{p+4}. Except n_b^{p+4}, polyominoes of all other areas between n_b^{p+4} and n_e^{p+4} are created by inflating min.-perimeter polyominoes with perimeter p. The polyomino Q is of area n_b^p, i.e., the area

of $I(Q)$ must be the min. area from $\left[n_b^{p+4}, n_e^{p+4}\right]$ which is not an inflation-chain root. Hence, the area of $I(Q)$ is $n_b^{p+4}+1$. We now equate the two expressions for the area of $I(Q)$: $n_b^p + p = n_b^{p+4} + 1$. I.e., $n_b^{p+4} - n_b^p = p - 1$. The claim follows.

Lemma 4. *If* $|\mathcal{P}(I(Q))| = |\mathcal{P}(Q)| + 4$, *then* $\phi(I(Q)) = \phi(Q) - 1$.

Proof. $\phi(I(Q)) = \epsilon^{-1}(|\mathcal{P}(I(Q))|) - |I(Q)| = \epsilon^{-1}(|\mathcal{P}(Q)| + 4) - (|Q| + |\mathcal{P}(Q)|)$
$= \epsilon^{-1}(|\mathcal{P}(Q)|) + |\mathcal{P}(Q)| - 1 - |Q| - |\mathcal{P}(Q)| = \epsilon^{-1}(|\mathcal{P}(Q)|) - |Q| - 1 = \phi(Q) - 1$.

Corollary 2. (Convergence of inflation). *After a finite amount of inflation steps, any polyomino becomes a minimal-perimeter polyomino.*

Proof. Consider a polyomino Q. By Corollary 1, after $R(Q)$ inflation steps, we will have that $\left|\mathcal{P}(Q^{R(Q)})\right| = \left|\mathcal{B}(Q^{R(Q)})\right| + 4$. After each step, the new border is the previous perimeter. Thus, after $R(Q)$ steps, the perimeter will increase by 4 with each additional inflation, and Lemma 4 will hold, i.e., any additional step will decrease the value of $\phi(Q^{R(Q)})$ by 1. After $R(Q)$ steps, $\phi(Q^{R(Q)})$ will have some finite value, φ, thus, after φ more steps, we will necessarily have $\phi(Q^{R(Q)+\varphi}) = 0$. Since a polyomino is a minimal-perimeter polyomino if $\phi(Q) \leq 0$, this implies that Q will become a minimal-perimeter polyomino after $R(Q) + \phi(Q^{R(Q)})$ steps.

4 Counting Minimal-Perimeter Polyominoes

The second question which we address is what is the number of minimal-perimeter polyominoes of area n, i.e., what is $|M_n|$. Notice that by using Theorem 2, this question can be reduced to counting the number of minimal-perimeter polyominoes only for inflation-chains roots. An area n is not an inflation-chain root if there exists some n', such that $n = n' + \epsilon(n')$, i.e., if there exist minimal-perimeter polyominoes of area n', that, after inflation, become polyominoes of area n. Lemma 1 tells us that if such n' exists, then $\epsilon(n) = \epsilon(n') + 4$, thus, we can easily check if such n' exists and if so, reduce the problem to area n'.

We were not able to provide a closed-form formula for $|M_n|$. However, we present here two algorithms which compute $|M_n|$ by using the geometric structure of min.-perimeter polyominoes. The running times of both algorithms are polynomial with n. We discuss the properties that guide the algorithms, present, analyze, and compare the algorithms, and provide the produced polyomino counts.

4.1 First Algorithm: Diagonal Counting

The first algorithm builds the polyominoes under consideration diagonal by diagonal.

A *diagonally-convex* polyomino is a polyomino whose cells along every diagonal (both descending [top-left to bottom-right] and ascending [bottom-left to top-right]) are consecutive without gaps. The first algorithm is based solely on the following theorem. The proof of this theorem is omitted here and is given in the full version of the paper.

Theorem 5. *Any minimal-perimeter polyomino is diagonally-convex.*

Based on Theorem 5, we devise an algorithm which counts minimal-perimeter polyominoes of area n. The algorithm builds the polyomino one diagonal at a time while maintaining the current perimeter. Here are the main details of the algorithm.

Building Polyominoes. The algorithm builds polyominoes along *descending* diagonals, one diagonal at a time. In the first iteration, we iterate through all possible sizes of the most extreme diagonal. In each of the following iterations, we iterate through both all possible sizes of the current diagonal and all possible positions of the current diagonal with respect to the previous one. The algorithm stops if the area of the polyomino reaches a target value n, or if the perimeter size exceeds $\epsilon(n)$. A generated polyomino is counted if the area of the polyomino is n, and its perimeter is $\epsilon(n)$.

Maintaining the Perimeter. Throughout the execution of the algorithm, we maintain the current polyomino perimeter. When a new diagonal is introduced, we add the number of newly-created perimeter cells. The potential number of perimeter cells created by a diagonal of length ℓ is $2(\ell+1)$. From this amount, we subtract the number of cells already occupied (by either polyomino or perimeter cells), as well as the number of perimeter cells eliminated by the new diagonal. All these computations are performed in constant time, given the locations of the first and last cells of the new diagonal and of the last two diagonals.

Keeping the Polyomino Connected. While building the polyomino, we keep track of the disconnected sets of cells. These cells must be connected by the next diagonal, thus, we make sure that this indeed happens. This is illustrated in Fig. 4.

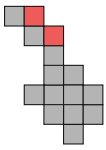


Fig. 4. A partial polyomino. The next diagonal must occupy the red cells in order to keep the polyomino connected. (Color figure online)

Narrowing the Range of Diagonals. Using the fact that Theorem 5 holds for diagonals in both directions, we can narrow the range of possible diagonals to only those that do not create a nonconvex *ascending* diagonal. This does not change the asymptotic complexity of the algorithm, but in practice it speeds up the algorithm substantially.

Memoization. Like many other recursive algorithms, this algorithm can be accelerated using memoization. In each partial state of the polyomino, the number of possible completions of the polyomino to a minimal-perimeter polyomino of area n depends on the current area n', the current perimeter p', and the locations of the first and last cells of the last two diagonals (s_0, t_0, s_1, t_1). The last two diagonals can be represented in a canonical form, for example, with the last diagonal starting at 0, and, thus, each partial polyomino will be represented by

the tuple $(n', p', t_0 - s_0, s_1 - s_0, t_1 - s_0)$. Using this representation, one can store the number of possible completions of a partial polyomino, and look it up when a partial polyomino with the same signature is built.

Complexity Analysis. Consider the graph of states of the algorithm, in which each node stores a state s represented by its memoizing tuple, and there is a directed edge $s_1 \to s_2$ if the algorithms performs a recursive call from state s_1 to state s_2. The time spent in a single node s for the first time is $O(d_{out}(s))$, i.e., the number of edges outgoing from s, and $O(1)$ if s was already memoized. Since we use memoization, any outgoing edge needs to be traversed only once, thus, the amount of times a state, s, is called is at most $d_{in}(s)$. Therefore, the total time complexity is $\sum_s O(1 + d_{out} + d_{in}) = O(V + E)$, where V, and E are the number of edges and vertices in the graph respectively. This means that the total complexity of the computation is the same as the complexity of the computation graph.

Since we look for minimal-perimeter polyominoes, the formula for $\epsilon(n)$ dictates that $p = \Theta(\sqrt{n})$ in the root call to the algorithm. Recall that each state is represented by a 5-tuple, in which the first entry is the area of the polyomino (which is $O(n)$), the second entry is its perimeter (which is $O(\sqrt{n})$), and the last three entries represent the first and last cells along diagonals. Since any added diagonal increase the perimeter (or preserve it), the length of each diagonal is bounded from above by the perimeter we seek to achieve, otherwise, the perimeter of the whole polyomino will be greater than the target perimeter. Therefore, the last three entries are also $O(\sqrt{n})$. We conclude that the total number of nodes is $O(n^3)$.

In each node, the algorithm considers all possible sizes of the next diagonal (which length is at most the size of the perimeter, i.e., it is $O(\sqrt{n})$, and all starting points of the next diagonal with respect to the current diagonal. Since each diagonal must be connected to the previous one, the starting point parameter is also $O(\sqrt{n})$. In total, we have $O(n)$ outgoing edges from each node, thus the total number of edges is $O(n^4)$. The total complexity of the graph, and hence, the total time complexity of the algorithm, is $O(n^4)$. (Note that this is a pseudo-polynomial complexity, since the only input to the algorithm is the *number* n, which can be specified by using $\log n$ bits.)

4.2 Second Algorithm: Bulk Counting

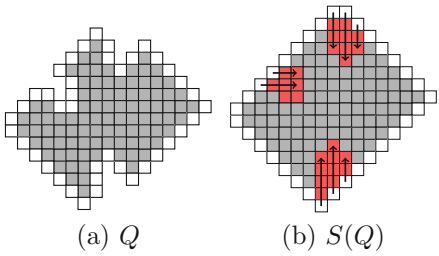
Our second approach for counting minimal-perimeter polyominoes is based on the fact that every such polyomino can be divided into an "octagonal" centerpiece and "caps." We refer to the main eight directions by the terms North, West, South, East, and their combinations.

Settings. Let Q be a minimal-perimeter polyomino. In addition, let c_x, c_y denote, resp., the x- and y-coordinates of a lattice cell c. Finally, let D_{ne} denote the sum of the x- and y-coordinates of the most extreme cells of Q in the north-east direction, that is, $D_{ne} = \max_{c\in Q}\{c_x + c_y\}$. Similarly, $D_{nw} = \min_{c\in Q}\{c_x - c_y\}$, $D_{sw} = \min_{c\in Q}\{c_x + c_y\}$, and $D_{se} = \max_{c\in Q}\{c_x - c_y\}$. Using a construction similar to that of Altshuler et al. [1, Theorem 7], we define $S(Q)$, the squaring of Q, as the set of cells found in the 45°-oriented bounding box of Q, that is,

$$S(Q) = \{c \mid (D_{sw} \le c_x + c_y \le D_{ne}) \wedge (D_{nw} \le c_x - c_y \le D_{se})\}.$$

This construction is illustrated in Fig. 5.

Notice that each perimeter cell of $S(Q)$ can be mapped uniquely to a perimeter cell of Q, therefore, $|\mathcal{P}(S(Q))| \le |\mathcal{P}(Q)|$. In addition to this, since, obviously, $Q \subset S(Q)$, the monotonicity of $\epsilon(n)$ implies that if Q is a minimal-perimeter polyomino, then $|\mathcal{P}(S(Q))| = |\mathcal{P}(Q)|$.



(a) Q (b) $S(Q)$

Fig. 5. The squaring construction. Arrows indicate correspondence between newly-created perimeter cells of $S(Q)$ and destroyed perimeter cells of Q.

It is easy to verify that if an entire diagonal is removed from a squared polyomino, its perimeter is reduced by 1. Let d be the length of the smallest diagonal of $S(Q)$. Then, the minimum perimeter of a polyomino of area $|S(Q)| - d$ is at most $|\mathcal{P}(S(Q))| - 1$. Thus, the area of a minimal-perimeter polyomino Q is at least $|S(Q)| - d + 1$. In order to get a feeling of the decomposition, notice that $d = O(\sqrt{|S(Q)|})$, thus, we may say that the shape of a minimal-perimeter polyomino is always similar to that of a squared polyomino.

In the sequel, consider Q as a minimal-perimeter polyomino. We now describe constructively the decomposition of Q into a "centerpiece" and "caps" (see Fig. 6). Consider the highest row of Q that spans the entire $S(Q)$ (the highest row that contains grey cells in Fig. 6(a)). Cells above this row will be marked as 'caps.' The same process is applied to the lowest row, the rightmost column, and the leftmost column of Q. Without the caps in all directions, we are left with a truncated version of $S(Q)$, which in general looks like an octagon, as seen in Fig. 6(a). (Obviously, the octagon can be degenerate in some directions. For example, Fig. 6(b) shows a triangular "octagon.") Note that since the area of Q is at least $|S(Q)| - d + 1$ (recall that d is the length of the smallest diagonal of Q), at most d rows or columns can be marked as caps (as seen in Fig. 6(b)), and so we are sure to be left with some cells marked as centerpiece. Taking a closer look at the centerpiece, we see that the only place where it differs from $S(Q)$ is in the truncated parts. It is easy to see that filling the truncated parts back to form $S(Q)$ would not change the perimeter size. Instead of a linear sequence of

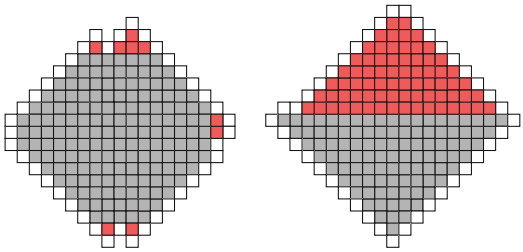


Fig. 6. An example of the decomposition of two minimal-perimeter polyomino to the centerpiece (gray), and caps (red). (Color figure online)

perimeter cells, there is a triangle in $S(Q)$, but the size of the perimeter of the triangle is exactly the same as the size of the perimeter of the centerpiece, thus the size of the perimeter of the centerpiece is equal to the size of the perimeter of $S(Q)$, which we already know to be identical to the size of the perimeter of Q.

In conclusion, any minimal perimeter polymino Q can be decomposed into a centerpiece (with the same perimeter as Q) and caps which do not change the perimeter. Looking at this process backwards, we can construct all the minimal-perimeter polyominoes of area n by iterating over all possible centerpieces with perimeter $\epsilon(n)$, and counting all possible combinations of 'caps' that make, together with the centerpieces, polyominoes of area n.

The Algorithm. Given an integer, n we want to compute the value of $|M_n|$. Based on the previous section, any minimal-perimeter polyomino of area n, can be decomposed to a centerpiece with perimeter $\epsilon(n)$ and caps, thus, by iterating over all the possible centerpieces with perimeter $\epsilon(n)$, and counting all the possible completions of these centerpieces to a polyomino of area n we can compute the value of $|M_n|$. Due to space limitations, a full discussion of the representation of the centerpieces and of how they are traversed is omitted here. Formally, let B_p be the set of all centerpieces with perimeter p, and given a certain centerpiece b, let $C(b)$ be the number of ways to add to b caps with total area of $n-|b|$. Then, we have that

$$|M_n| = \sum_{b \in B_{\epsilon(n)}} C(b). \tag{1}$$

However, we need to be cautious with the definition of $C(b)$. If we allow any cap to be counted in $C(b)$, we may count some polyominoes more than once. Therefore, we define the notion of "proper caps." In a *proper cap*, the first layer does not span the entire length of the centerpiece edge it is attached to (see Fig. 7 for an illustration). By counting only proper caps, polyominoes are counted only once since they are counted only through the completions of the maximal centerpiece (in terms of containment).

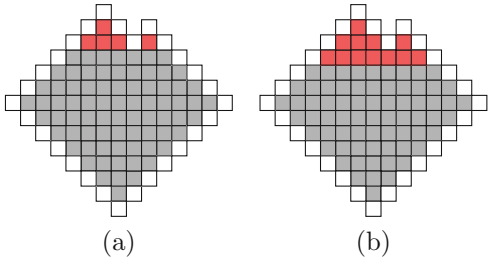


Fig. 7. Caps: (a) proper; (b) nonproper.

Counting Caps. Caps of a minimal-perimeter polyomino consist of cells added on top of the flat edges of the centerpiece without altering the perimeter of the centerpiece. By Theorem 5, we deduce that each layer of the cap must be properly "contained" within the previous level (i.e., as in  and not as in ). For example, consider a cell c in the north cap. The cells below c to its right and to its left must also belong to the polyomino, otherwise, the diagonal connecting c to the centerpiece will have a gap, in contradiction with Theorem 5.

These caps can be represented by Motzkin paths. A Motzkin path of length n is a grid path starting at $(0, 0)$ and ending at $(0, n)$, where each step (represented by a vector) belongs to the set $\{(1, -1), (1, 0), (1, 1)\}$, that is, right-down, right, or right-up, and the path never goes below the x-axis. The number of Motzkin paths of length n is known as the nth Motzkin number. We can map every cell in the envelop of the cap to a step in a Motzkin path. Consider a cap of the north edge. An occupied cell with an empty cell to its left (i.e., a cell which starts a new sequence of steps) corresponds to a right-up step; an occupied cell with an occupied cell to its left corresponds to a right step; and an empty cell with an occupied cell to its left corresponds to a right-down step. This correspondence is shown in Fig. 8. Notice that a cap of length k is mapped to a Motzkin path of length $k + 1$.

Denote by $\mathrm{Mz}(k)$ the kth Motzkin number. It is well known that

$$\mathrm{Mz}(k) = \mathrm{Mz}(k-1) + \sum_{i=0}^{k-2} \mathrm{Mz}(i)\mathrm{Mz}(k-i-2).$$

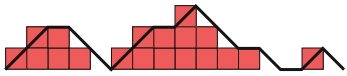


Fig. 8. An example of a cap and the corresponding Motzkin path.

The first element of the recursion corresponds to a path starting with a right step; The rest of the path is itself a Motzkin path of length $(n - 1)$. The other element of the recursion (the summation) corresponds to Motzkin paths that start with a right-up step. Such a path must return to the x-axis at some point (either in the middle or at the end of the path), and the recursion sums over the possible locations where the path returns to the x-axis for the first time.

In our algorithm, we count the caps that fit on an edge of length k, and that contain r cells. These caps correspond to Motzkin paths of length $k+1$, while the area below the path (and above the x-axis) is equal to r. Denote by $\mathrm{Mz}(k,r)$ the number of Motzkin paths of length k and of area of r below them. Several works address the area below Motzkin paths (e.g., [12,15,16]), however, all works of which we are aware study the total area under *all* Motzkin paths of length k, but do not provide a formula for $\mathrm{Mz}(k,r)$.

Lemma 5.

$$\mathrm{Mz}(k,r) = \mathrm{Mz}(k-1,r) + \sum_{i=0}^{k-2} \sum_{j=0}^{r-i-1} \mathrm{Mz}(i,j)\mathrm{Mz}(k-2-i, r-i-1-j).$$

The proof of Lemma 5 is given in the full version of the paper.

Counting only proper caps is simple when we think of it in terms of Motzkin paths. We want to count only "proper paths," i.e., to avoid paths that start with a right-up step and do not touch the x-axis until the last step. This kind of paths is counted in the last element of the recursive formula for Motzkin paths. Thus, the formula for proper Motzkin paths is

$$\widetilde{\mathrm{Mz}}(k,r) = \mathrm{Mz}(k-1,r) + \sum_{i=0}^{k-3} \sum_{j=0}^{r-i-1} \mathrm{Mz}(i,j)\mathrm{Mz}(k-2-i, r-i-1-j),$$

where $\widetilde{\mathrm{Mz}}(k,r)$ is the number of proper Motzkin path with length k and area r.

With this formula at hand, we can devise a formula for $C(b)$. For a specific centerpiece b, our aim is to complete b to a polyomino of area n, thus, we need to add to it $n-|b|$ cells. Let d_i (for $1 \le i \le 4$) be the lengths of the four "flat" edges of b. Then, $C(b) = \sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1+n_2+n_3+n_4=n-|b|}} \prod_{i=1}^{4} \widetilde{\mathrm{Mz}}(d_i-1, n_i)$. That is, we sum over all partitions of the $n-|b|$ additional cells into the four caps.

Complexity Analysis. Due to space limitations, we provide here only a sketch of the complexity analysis. In order to facilitate the analysis, we divide the algorithm into three layers, perform each one of them separately, and store the results in a re-usable memory. First, we compute all relevant values of $\widetilde{\mathrm{Mz}}(n,k)$. Second, we compute all relevant values of $C(b)$. Finally, we use the stored data and compute $|M_n|$ according to Eq. 1. In fact, we use this layered analysis only in order to bound from above the time complexity of the algorithm. In practice, we perform a lazy evaluation, that is, we compute values of each function on demand, and memoize the results.

Computing the values of $\widetilde{\mathrm{Mz}}(n,k)$ takes $O(n^3)$ time. The computation of $C(b)$ takes $O(n^4)$ time. Finally, the computation of $|M_n|$ takes $O(n^{2.5})$ time. The three steps are independent, thus, the total time complexity is $O(n^4)$. The full details are given in the full version of the paper.

4.3 Counts of Minimum-Perimeter Polyominoes

The "Bulk Counting" algorithm computes the first 1,000 values of $|M_n|$ in about one hour on a 2.8 GHz, single processor, 16 GB RAM machine. The counts are shown in Table 1 and are plotted in Fig. 9. A detailed discussion of these counts is provided in the full version of the paper.

Table 1. Values in the four columns of $|M_n|$. Each value in the table is $\left|M_{\epsilon^{-1}(p)}\right|$ (an inflation-chain root), where p is the sum of the number in the left column and the number in the header of the column.

Perimeter	+0	+1	+2	+3
4	1	0	4(2)†	4
8	9	4	22	28
12	52	28	106	124
16	206	124	392	456
20	719	456	1,276	1,464
24	2,206	1,464	3,738	4,252
28	6,226	4,252	10,180	11,468
32	16,368	11,468	25,984	29,112
36	40,751	29,112	63,116	70,268
40	96,691	70,268	146,714	162,584
44	220,574	162,584	328,702	362,672
48	485,846	362,672	712,736	783,508
52	1,038,209	783,508	1,502,024	1,645,536
56	2,159,068	1,645,536	3,084,988	3,369,832
60	4,382,886	3,369,832	6,192,586	6,745,780
64	8,703,908	6,745,780	12,173,336	13,228,364
68	16,945,143	13,228,364	23,480,084	25,457,184
72	32,393,530	25,457,184	44,504,628	48,152,268
76	60,899,182	48,152,268	83,011,376	89,642,052
80	112,731,886	89,642,052	152,546,840	164,439,468
84	205,712,776	164,439,468	276,483,464	297,542,816
88	370,405,262	297,542,816	494,690,890	531,547,196
92	658,692,094	531,547,196	874,505,946	938,293,884
96	1,157,749,216	938,293,884	1,528,537,056	1,637,794,712

†The inflation-chain root of perimeter 6 is area 2, and $|M_2| = 2$. However, when M_2 is inflated, the next element in the chain is M_8, but $|M_8| = 4$. This exception is caused by the two shapes having two diagonally-placed cells  and ) which have a minimal perimeter but do not count as polyominoes. Due to this anomaly, Theorem 2 applies only for $n \geq 3$.

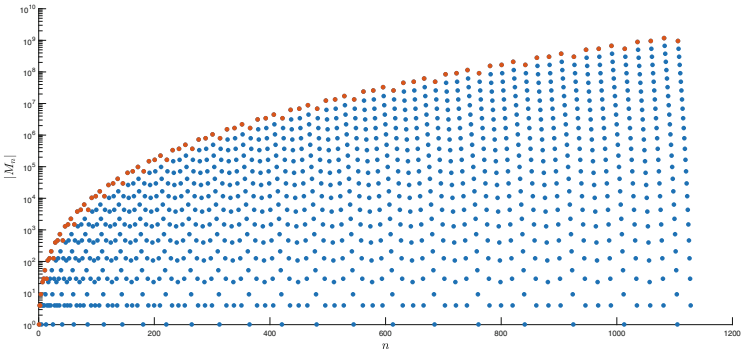


Fig. 9. Values of $|M_n|$. Inflation-chain roots are colored in red. (Color figure online)

5 Conclusion

In this paper, we have shown some results concerning minimal-perimeter polyominoes. First, we characterized the roots of the inflation chains of polyominoes. Second, we showed that any polyomino, when inflated enough times, becomes a minimal-perimeter polyomino. This opens the opportunity to tie between minimal-perimeter polyominoes and all polyominoes by dividing polyominoes into classes, where all members in a class eventually inflate into the same minimal-perimeter polyomino. Characterizing this class may, in turn, shed some light on the total number of polyominoes. Third, we provided two algorithms for computing $|M_n|$. Finding a closed formula for $|M_n|$ remains an open problem.

An interesting open problem is the behavior of $|M_n|$ when one considers only inflation-chain roots. Figure 9 shows that $|M_n|$ increases monotonically with a decreasing slope. Does $|M_n|$ have an asymptotic slope? And if so, what is it?

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