# Constraining properties of anisotropic flow in heavy-ion collisions with multiparticle cumulants

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**Abstract**

In heavy ion physics, Quark Gluon Plasma (QGP) is created after a collision. It was shown that it properties can be constrained by the measurements of anisotropic flow. Traditionally anisotropic flow harmonics are estimated by using multivariate cumulants. In this project new multivariate cumulants for anisotropic flow analysis have been introduced, their mathematical and statistical properties have been investigated in detail and tested with carefully designed toy monte carlo studies. These cumulants provide new and independent information about QGP properties when compared to the ones used so far.

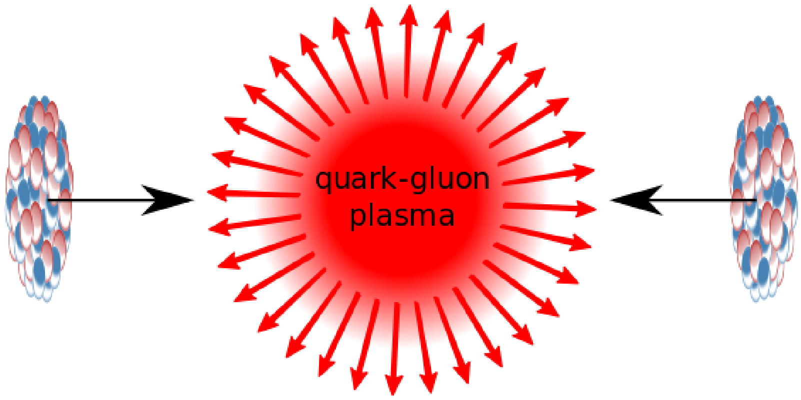


Figure 1: Production of matter at very high temperatures and densities. Two beams of lead nuclei crash to transform in this state. We call it Quark Gluon Plasma and we are looking to describe its behavior. Do we need local interactions and Quantrumchromodynamics? or can we study the collective distribution?

# 1 Introduction

Hadrons are bound states of quarks. Most of the time they appear as mesons (pair quark-antiquark) and baryons (three quarks). If we add energy to mesons, at some point they will decompose producing another meson. We see that at normal conditions (below the Hagedorn temperature of approximately 130-140 MeV), it's impossible to isolate pairs of quarks color-anticolor. This is called color confinement. At hotter conditions this pattern changes, quantum chromodynamics (QCD) predicts that hadrons dissociate to a plasma of quark and gluons (QGP); where they move freely over distances large in comparison to the typical size of a hadron (in the order of fm).

In the QCD phase diagram of Fig. (2) we can see two main regions; the confinement (Hadron gas) and deconfinement (QGP). The boundary area between these two is a topic of current research and it can be either a smooth crossover, or phase transition of first or second order. For instance, in the last decades, new studies describing its expected characteristics emerged; the phase coexistence and location of a hypothetical critical point [7]. Astrophysicists find the conditions which existed in the region at low temperatures and high chemical potential $\mu_B$, the conditions which can be found in the core of neutron stars. Matter inside them exhibits characteristics of the deconfined phase, which is interpreted as evidence for the presence of quark-matter cores [6]. At the upper left, region of high temperatures and low $\mu_B$, we find the conditions just few microseconds after the Big Bang. Early universe was filled with QGP, then went trough a smooth transition to the confinement (under 175 MeV).

Physicists produce QGP at colliders. High energy heavy ion collisions create a fireball that reaches the QGP region and simulates the early universe. Used nuclei must have large mass and energies, because their product has the sufficient density and temperature to become QGP. Momenta and energies of resulting particles in a magnetic field are then detected and so the reconstruction of QGP, as it expands and cools, can be done. In 2000, the first production of QGP was announced at CERN; the Super Proton Synchrotron (SPS) fired very-high-energy beams of lead ions into gold or lead targets. Collisions created temperatures 100,000 times hotter than the centre of the Sun, and the highest energy densities (3–4 GeV per cubic femtometer) [8]. The contemporary experiment is ALICE; it was specifically designed as a dedicated heavy-ion experiment, with the primary goal to

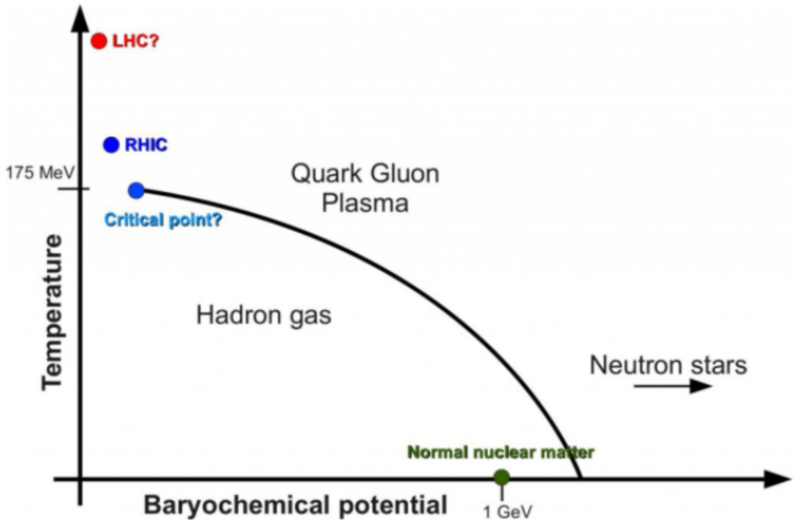


Figure 2: Phase diagram of Quantumchromodynamics. We distinguish two main areas; the hadronic and Quark gluon plasma regions. In the first only exists pairs quark-antiquark (confinement), in the second quarks and gluons move freely over distances typical of a size of a hadron [2].

analyse the QGP. Fig. (3) shows the detector layout.

Describing QGP with only QCD is very difficult; one needs to solve the non-linearity of the gluon interaction, the strong coupling, the dynamical many body system and confinement. Surprisingly, QGP behaves like an almost perfect liquid [9]. This constraints the QCD; we don't need long scattering formulas to describe particle interactions anymore; we surpass this challenge by using effective theories, like viscous hydrodynamics. As consequence of it we can describe QGP with the macroscopic thermodynamic properties, such as the equation of state (EoS), temperature and order of the phase transition, transport coefficients and so on. We need then the transport coefficients such as shear viscosity $\eta$, bulk viscosity $\zeta$, heat conductivity $\lambda$, etc. Let's take a look to the equation of state:

$$P = (e, n) \tag{1}$$

which expresses the pressure P as a function of energy density $e$ and baryon density $n$. We obtain it by doing numerical simulations of QCD on the lattice. It does not only describe expansion and collective flow of matter but also provides important information in the intermediate stage for other phenomena.

In non-central heavy-ion collisions the initial volume of the interacting system is anisotropic in coordinate space (see Fig. 4). The general characteristic of a flow is that any anisotropy in the coordinate space also transforms in the momentum space; moving a bottle of water will move it's molecules and change their momentum, while moving a bottle of gas will do generally nothing, it remains isotropic. Quarks and Gluons play here the role of molecules in QGP. Clearly, this is an indirect probe of the viscosity.

The first important focus of this paper is to describe the probability density function (p.d.f) of the transmitted anisotropy in momentum space $f(\varphi)$. We can always start doing the most general expansion, a Fourier series:

$$f(\varphi) = \int_0^{2\pi} \frac{1}{2\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} v_n \cos\left[ n(\varphi - \Psi_n) \right] \right] \tag{2}$$

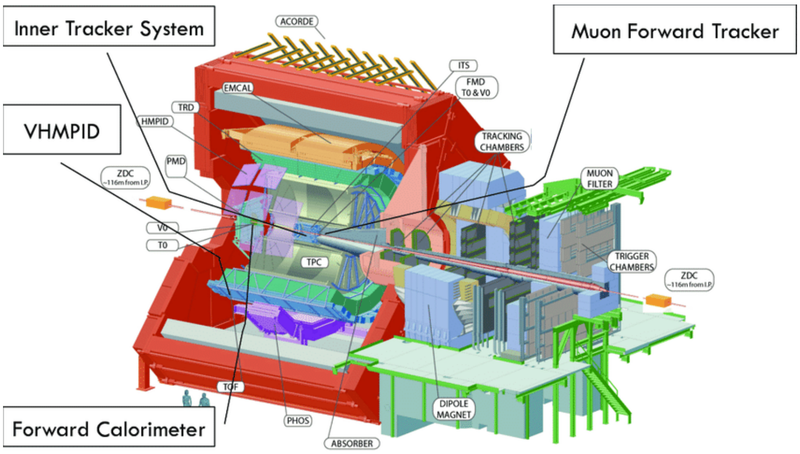


Figure 3: Schematic description of the ALICE experiment. Collision happens at the center of the cylinder (Inner tracking system). For more information about the components see the paper [15].

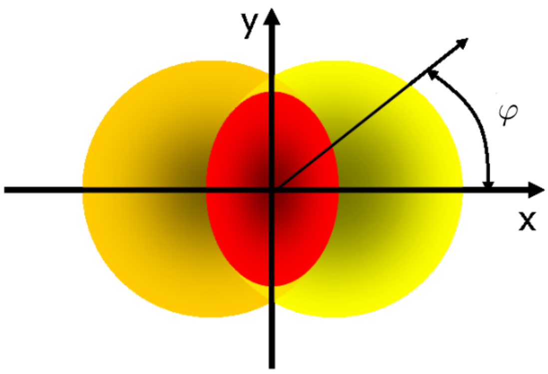


Figure 4: Coordinate space anisotropy of the initial volume of the interacting system (red) created in heavy-ion collisions [2].

where $\varphi$ is the azimuthal angle whose sample space is the interval $[0, 2\pi\rangle$, $v_n$ are anisotropic flow harmonics and $\Psi_n$ corresponding symmetry planes. $v_n$ determines the form of distribution; $v_1$ is directed, $v_2$ elliptic and $v_3$ triangular flow. Due to the collision geometry, the dominant harmonic in non-central collisions is the elliptic flow $v_2$. It quantifies how the system responds to the initial spatial ellipsoidal anisotropy. In this project we considerate up to eight harmonics.

Mathematically can be demonstrated that $v_n$ has the following equivalence (Appendix A.1):

$$v_n = \langle \cos[n(\varphi - \Psi_n)] \rangle \tag{3}$$

Where the average value of function $a(x)$ is defined the following way:

$$\langle a(x) \rangle \equiv \int_0^{2\pi} a(x)f(x)dx \tag{4}$$

Eq. (3) includes also harmonic symmetry planes $\Psi_n$, that are different in every event, and can be difficult to detect at colliders, only easily measurable quantities, are the angles and momenta.

If anisotropic flow is the dominant source of correlations among produced particles at a collision, they are emitted independently and are only correlated to a common symmetry plane. This is why we can factorize the joint p.d.f. for any number of particles n into a product of individual p.d.f's $f_{\varphi_i}(\varphi_i)$, that have the same form of Eq. (2):

$$f(\varphi_1, ..., \varphi_n) = f_{\varphi_1}(\varphi_1)...f_{\varphi_n}(\varphi_n) \tag{5}$$

Of course with adding more particles and multiplying their p.d.f to the joint p.d.f. one can find the correlation of a very large number of them.

Two-particle azimuthal correlation is the simplest one and can be expressed as the square of flow harmonics (Appendix A.2):

$$\langle \cos[n(\varphi_1 - \varphi_2)] \rangle = v_n^2 \tag{6}$$

The main problem of Eq. (6) is that the two azimuthal angles $\varphi_1$ and $\varphi_2$ must be different, otherwise there are trivial contributions from self-correlations, which are large and equal to 1. We can solve this using nested loop algorithms; calculating the correlation of first particle relative to each one of the rest, then doing the same for second particle and so on. Obviously this is not an efficient way, takes too much time for a computer to calculate all the loops and it's impossible with the amount of particles created in heavy-ion collisions. We can avoid this, if we calculate the Q-vector evaluated in harmonic $n$, $Q_n$, that by definition only needs one pass over all particles, and if we find formulas for particle correlations that can be expressed in terms of Q-vectors.

$$Q_n \equiv \sum_{i=1}^{M} e^{in\varphi_i} \,. \tag{7}$$

$\varphi_i$ is the azimuthal angle of the i-particle, and M is the number of particles. The key point is that all multi-particle azimuthal correlations can be expressed analytically it terms of $Q$-vectors evaluated (in general) different harmonics. For example the two particle correlation is now (Appendix A.3):

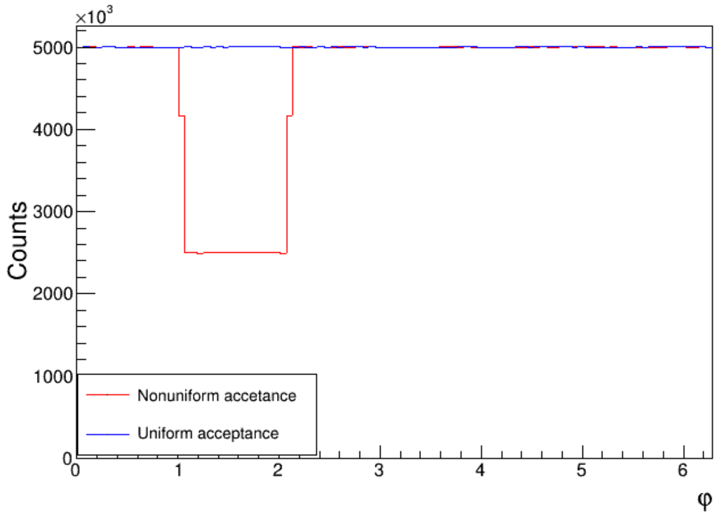


Figure 5: Simulating a detector that have bias in the azimuth angles between 30 and 60°, making them 50% less probable (red histogram). Here we compare this with the uniform acceptance (blue histogram).

$$
\begin{aligned}
\langle 2\rangle &\equiv \langle \cos(n(\varphi_1 - \varphi_2))\rangle \\
&= \frac{1}{\binom{M}{2}2!} \sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{in(\varphi_i - \varphi_j)} \\
&= \frac{1}{\binom{M}{2}2!} \left[|Q_n|^2 - M\right]
\end{aligned}
\tag{8}
$$

Despite the efficiency of a single pass, finding higher order correlation formulas in terms of Q-vectors became very complicated. Four, five and six correlations will gradually have way longer expressions. One clever way to avoid them is using recursion algorithms; the seventh and higher correlations can be calculated in terms of the lower order ones recursively [4].

## 2 Generic Framework

In this chapter we introduce a framework to simulate Pb-Pb collisions and calculate particle correlations using computational algorithms. Later in chapter 5, we explore our new proposed cumulant with the help of this framework.

At the LHC, azimuthal angles can be measured with high precision. New correlation techniques use them to estimate the flow amplitudes $v_n$ and the symmetry planes $\Psi_n$. For instance, the connection between these three is described in the following formula [3]:

$$
\left\langle e^{i(n_1\varphi_1 + \cdots + n_m\varphi_m)} \right\rangle = v_{n_1} \cdots v_{n_m} e^{i(n_1\Psi_{n_1} + \cdots + n_m\Psi_{n_m})}\,, \tag{9}
$$

The average goes over all distinct tuples of $m$ different azimuthal angles $\varphi$ reconstructed in the same event. A set $n_1, \ldots, n_m$ consists of $m$ non-zero integers. In each event, measuring symmetry planes alone is very problematic, that's why we want to eliminate

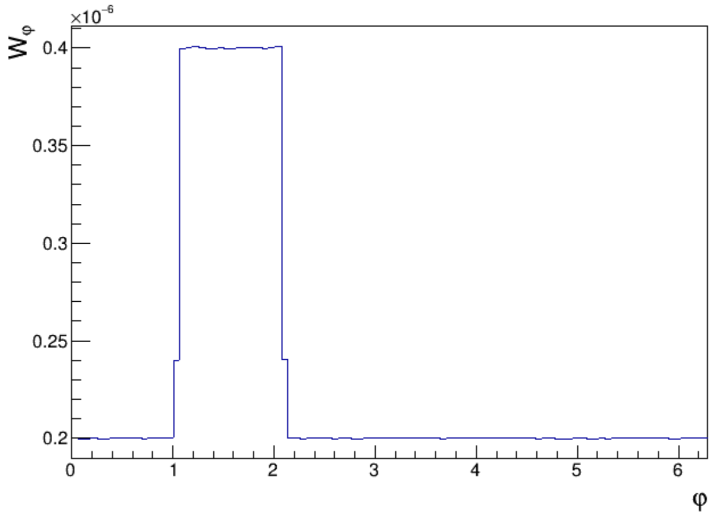


Figure 6: Weights obtained inverting the histogram of nonuniform acceptance from Fig. (2). We don't need to normalize it because the definition in Eq. (11) does it for us.

their contribution. In the case of an idealized geometry, all symmetry planes $\Psi_n$ coincide. If we carefully choose the values in this set, (e.g. by taking value of each integer equal number of times with positive and negative sign), we can then discard the contribution. In other words, since the impact parameter orientation is uncontrolled, the only non-trivial correlations must have azimuthal symmetry [3]:

$$n_1 + \cdots + n_k = 0. \tag{10}$$

Real detection of particle-collisions has some degree of inefficiency. To solve this, we introduce particle weights and we get a new definition for the average m-particle correlation in harmonics $n_1, n_2, ..., n_m$:

$$\begin{aligned} \langle m \rangle_{n_1,n_2,\ldots,n_m} &\equiv \left\langle e^{i(n_1\varphi_1 + n_2\varphi_2 + \cdots + n_m\varphi_m)} \right\rangle \\ &\equiv \frac{\displaystyle\sum_{\substack{k_1,k_2,\ldots,k_m=1 \\ k_1 \neq k_2 \neq \ldots \neq k_m}}^{M} w_{k_1} w_{k_2} \cdots w_{k_m} \, e^{i(n_1\varphi_{k_1} + n_2\varphi_{k_2} + \cdots + n_m\varphi_{k_m})}}{\displaystyle\sum_{\substack{k_1,k_2,\ldots,k_m=1 \\ k_1 \neq k_2 \neq \ldots \neq k_m}}^{M} w_{k_1} w_{k_2} \cdots w_{k_m}} . \end{aligned} \tag{11}$$

M is the multiplicity of an event, $\varphi$ labels the azimuthal angles of the produced particles and $w$ particle weights. The condition removes all possible autocorrelations. Weights compensate, for example, bias in azimuthal angle that the detector can have. Theoretically, if we have the following values of flow harmonics,

$$v_n = 0.04 + n \cdot 0.01, n = 1, 2, ..., 6. \tag{12}$$

and if we use Eq. (9), then we find theoretical value for each multiparticle azimuthal correlation of interest.

To prove our generic framework using weights, we did a toy Monte Carlo simulation for a nonuniform acceptance with just half the probability to get a particle within the azimuth range 30 and 60°.

Firstly we selected the harmonics that resulted in the following input values:

$$\begin{aligned}
\langle 2\rangle &\equiv \langle 2\rangle_{-2,2} = v_2^2 = 3.6\times 10^{-3},\\
\langle 3\rangle &\equiv \langle 3\rangle_{-5,-1,6} = v_1v_5v_6 = 4.5\times 10^{-4},\\
\langle 4\rangle &\equiv \langle 4\rangle_{-3,-2,2,3} = v_2^2v_3^2 = 1.764\times 10^{-5},\\
\langle 5\rangle &\equiv \langle 5\rangle_{-5,-4,3,3,3} = v_3^3v_4v_5 = 2.4696\times 10^{-6},\\
\langle 6\rangle &\equiv \langle 6\rangle_{-2,-2,-1,-1,3,3} = v_1^2v_2^2v_3^2 = 4.41\times 10^{-8},\\
\langle 7\rangle &\equiv \langle 7\rangle_{-6,-5,-1,1,2,3,6} = v_1^2v_2v_3v_5v_6^2 = 9.45\times 10^{-9},\\
\langle 8\rangle &\equiv \langle 8\rangle_{-6,-6,-5,2,3,3,4,5} = v_2v_3^2v_4v_5^2v_6^2 = 1.90512\times 10^{-9}.
\end{aligned} \tag{13}$$

We assumed $10^6$ events. In each one of them the reaction plane $\Psi_{RP}$ was uniformly sampled in the interval $[0,2\pi]$, then 500 azimuth angles from the following p.d.f:

$$f(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{6} v_n \cos\left[n(\varphi - \Psi_{RP})\right]\right] \tag{14}$$

With this, we did two runs over events to fill two histograms, which will play the role of nonuniform and uniform acceptances, respectively. In Fig. (2) we compared both. As we see here, for the filling of first one we forced additionally 0.5 probability to get angles between 30 and 60°. Weights were then obtained inverting this histogram (Fig. 6). Note that the definition in Eq. (11) don't need normalized weights.

Next step was the calculation of correlations. For the first six we used standalone formulas and for the seventh and eighth recursion techniques. These algorithms were obtained in the paper [4]. We did 3 runs. The first for uniform acceptance, second for the nonuniform acceptance and third the nonuniform acceptance that considers weights. Each run had $10^6$ events. In each event we got angles from the p.d.f of Eq. (14) and reaction plane from uniform distribution. Then, we calculated Q vector and correlations. The final results were averages over all events. Fig. (7) shows the 2,3,4,5,6,7 and 8 correlations. Clearly the use of weights like in Eq. (9) corrects values from the nonuniform acceptance, thus solving azimuthal bias in detectors.

## 3 Multiparticle Cumulants

Up to now we were only handling statistically independent particles. But we want to isolate genuine multiparticle correlations, from multiparticle correlations which are just a trivial superposition of few-particle correlations. That's why we use cumulants. This was first propossed by Borghini et. al. [10] at the turn of the Millenium. The work was very influential and changed the way anisotropic flow analysis is performed in high-energy physics. Cumulants will provide less biased estimators true harmonics $v_n$. For instance, the simplest multivariate cumulant can be defined as:

$$\langle X_1X_2\rangle_c = \langle X_1X_2\rangle - \langle X_1\rangle\langle X_2\rangle\,. \tag{15}$$

One important fundamental property of cumulants is described in theorem 1 from the paper of Kubo [1], and we interpret it as follows:

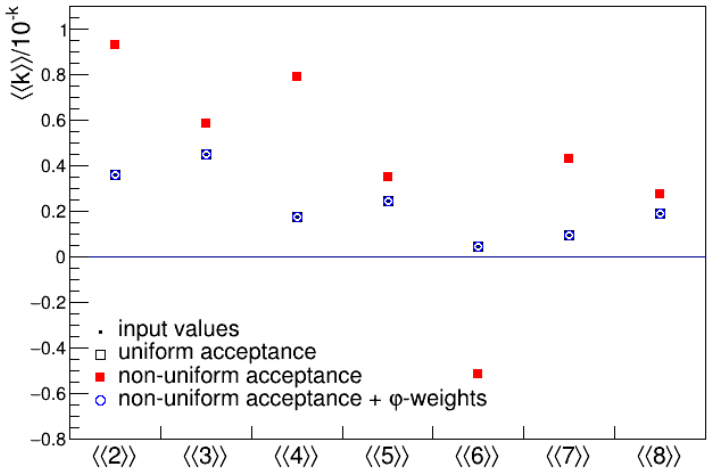


Figure 7: Results showing the correlations using harmonics of Eq. (13). As we see here, the use of weights (non-uniform acceptance + weights, blue dots) corrects the azimuthal bias of detectors (non-uniform acceptance, red dots). For a better view, k-correlation values were divided by $10^{-k}$.

1. non-vanishing multi-particle cumulant is equivalent to the existence of genuine multi-body interaction;
2. if there is no genuine multi-body interaction, then the corresponding multi-particle cumulant is identically zero;
3. however, if multi-particle cumulant is identically zero, that does NOT mean that there is no genuine multi-body interaction => cumulant can also be trivially zero, due to underlying symmetries.

In the past decade physicists took many multivariate estimators as cumulants, but sadly most of them were found to be incorrect. Last year, the first strict mathematical formalism was published by Bilandzic et. al. [12]. In this section we will summarize them.

It is clear that we need first a mathematical notation to refer our cumulants, and study their properties using it. In each event we have a set of N stochastic variables $X_1, ..., X_N$ and we would like to find the corresponding multivariate p.d.f. $f(X_1, ..., X_N)$, because it gives information of all statistical properties. Experimentally in our colliders, it is very difficult to determine these p.d.f's. Instead of a long computational calculation, we use multivariate moments $\mu$ and cumulants $\kappa$. With them we save time. In this paper we will take in consideration the following notation:

$$\mu_{\nu_1,...,\nu_N} \equiv \mu(X_1^{\nu_1}, ..., X_N^{\nu_N}) \equiv \left\langle X_1^{\nu_1}, ..., X_N^{\nu_N} \right\rangle \tag{16}$$

$$\kappa_{\nu_1,...,\nu_N} \equiv \kappa(X_1^{\nu_1}, ..., X_N^{\nu_N}) \equiv \left\langle X_1^{\nu_1}, ..., X_N^{\nu_N} \right\rangle_c \tag{17}$$

The subindice $\nu_i$, where i is at least 1 and at most N, in the first term, determines the order of variable $X_i$. For example $\nu_1$ is the order of the variable $X_1$. When we define the moments, we don't specify the type; they are not necessarily the azimuthal moments in a non-uniform acceptance, like in our generic framework. In the most general way the statistical moment is defined the following way:

$$\mu_{\nu_1,...,\nu_N} \equiv \int X_1^{\nu_1} \cdots X_1^{\nu_N} f(X_1,...,X_N) dX_1 \cdots dX_N \tag{18}$$

Of course, we would need the p.d.f. if we just use this definition. But in this paper we calculate correlations from our toy Monte Carlo simulations, so we can avoid this integrals and just use Eq. (16).

### 3.1 Definition

The definition of multivariate moments $\mu_{\nu_1,...,\nu_N}$ in Eq. (16) can be rewritten very compactly in terms of the moment generating function, $M(\xi_1,...,\xi_N)$, wich is defined as,

$$M(\xi_1,...,\xi_N) = \left\langle e^{\sum_{j=1}^{N} \xi_j X_j} \right\rangle \tag{19}$$

We can do a Taylor expansion of the exponential function in in auxiliary variables $\xi_1,...,\xi_N$ about zero, replace all averages of $X_1,...,X_N$ with $\mu_{\nu_1,...,\nu_N}$ [see Eq. (16)], and obtain all multivariable moments $\mu_{\nu_1,...,\nu_N}$ as coefficients of different orders in auxiliary variables $\xi_1,...,\xi_N$:

$$M(\xi_1,...,\xi_N) = \sum_{\nu_1,...,\nu_N} \left( \prod_j \frac{\xi_j^{v_j}}{\nu_j!} \right) \mu_{\nu_1,...,\nu_N} \tag{20}$$

where all indices $\nu_1,...,\nu_N$ run from zero. The multivariate moments can therefore be obtained directly from their generating function with the following standard expression:

$$\mu_{\nu_1,...,\nu_N} = \frac{\partial^{\nu_1}}{\partial \xi_N^{\nu_1}} \cdots \frac{\partial^{\nu_N}}{\partial \xi_N^{\nu_N}} M(\xi_1,...,\xi_N) \Bigg|_{\xi_1=\xi_2=\cdots=\xi_N=0} \tag{21}$$

The generating function for cumulants, $K_(\xi_1,...,\xi_N)$, is defined in terms of the moment generating function:

$$K(\xi_1,...,\xi_N) \equiv \ln M(\xi_1,...,\xi_N) = \ln \left\langle e^{\sum_{j=1}^{N} \xi_j X_j} \right\rangle \tag{22}$$

By definition, multivariate cumulants $\kappa_{\nu_1,...,\nu_N}$ are coefficients in the formal Taylor expansion of their generating function $K(\xi_1,...,\xi_N)$ about zero:

$$K_{\nu_1,...,\nu_N} = \sum_{\nu_1,...,\nu_N} \left( \prod_j \frac{\xi_j^{v_j}}{\nu_j!} \right) \kappa_{\nu_1,...,\nu_N} \tag{23}$$

In the sum $\sum_{\nu_1,...,\nu_N}$, the term $\nu_1 = \cdots = \nu_N = 0$ was excluded from summation. Analogously to moments, cumulants can be obtained directly from their generating function:

$$\kappa_{\nu_1,...,\nu_N} = \frac{\partial^{\nu_1}}{\partial \xi_N^{\nu_1}} \cdots \frac{\partial^{\nu_N}}{\partial \xi_N^{\nu_N}} K(\xi_1,...,\xi_N) \Bigg|_{\xi_1=\xi_2=\cdots=\xi_N=0} \tag{24}$$

The details of underlying multivariate p.d.f. f $(X_1,...,X_N)$ can be estimated equivalently either with moments or with cumulants, because from relation of Eq. (17), it can be shown that all cumulants $\kappa_{\nu_1,...,\nu_N}$ can be uniquely expressed in terms of moments $\mu_{\nu_1,...,\nu_N}$, and vice versa. We are easily measurable moments with the average $\left\langle X_1^{\nu_1} \cdots X_N^{\nu_N} \right\rangle$, while cumulants $\left\langle X_1^{\nu_1} \cdots X_N^{\nu_N} \right\rangle_c$ are obtained with multiparticle calculations, like in our toy monte carlo section. In practice, one first measures moments, then in the next step calculates cumulants from them, and finally from cumulants draws the physics conclusions and constraints on the many-body problem in question.

After clarifying notation and definitions, we can finally introduce the general properties. Here we take them as sufficient requirements; when a multivariate matematical expression satisfy all of them then it is a cumulant. It is not clear if there are more properties, that is a subject of new studies. In any case, with the properties of this paper, physicists can discard many wrongly consider multivariate cumulants of flow amplitudes. The detailed proof of all properties can be found in Appendix A 3 from the paper of Bilandzic et. al. [12].

## 3.2 Statistical independence.

A multivariate cumulant $\kappa_{\nu_1,...,\nu_N} \equiv \langle X_1^{\nu_1},...,X_1^{\nu_N}\rangle_c$ is identically zero if its stochastic variables $X_1,...,X_N$ can be divided into two groups which are statistically independent. This implies that the cumulant is zero if at least one of its variables is statistically independent from the rest. When some group of variables are correlated, then their cumulant can be nonzero.

## 3.3 Reduction.

If in a set of variables $X_1,...,X_N$ two or more are identical, the resulting cumulant has lower number of variables; we can group them and add their indices:

$$\kappa_{\nu_1,...,\nu_N} = \left\langle X_1^{\nu_1}\cdots X_N^{\nu_N}\right\rangle_c = \left\langle X_1^{\tilde{\nu}_1}\cdots X_M^{\tilde{\nu}_M}\right\rangle_c = \kappa_{\tilde{\nu}_1,...,\tilde{\nu}_M}, \tag{25}$$

where $\tilde{\nu}_i$ is the sum of indices of all variables which are equal to $X_i$. Clearly $M \leq N$.

For example, if we have the set $X_1, X_2, X_3, X_4$ and $X_1$ is equal to $X_2$, then we can convert the cumulant $\kappa_{\nu_1,\nu_2,\nu_3,...,\nu_N}$ and conclude:

$$\kappa_{\nu_1,\nu_2,\nu_3,\nu_4} = \left\langle X_1^{\nu_1}X_2^{\nu_2}X_3^{\nu_3}X_4^{\nu_4}\right\rangle_c = \left\langle X_1^{\nu_1+\nu_2}X_3^{\nu_3}X_4^{\nu_4}\right\rangle_c = \kappa_{\nu_1+\nu_2,\nu_3,\nu_4} \tag{26}$$

The four variate cumulant became a three variate cumulant. Similarly, if we choose to equalize all variables in the set $X_1,...,X_N$ to $X$, then the final expression is a univariate cumulant of $X$ of order $\nu_1 + \cdots \nu_N$:

$$\kappa_{\nu_1,...,\nu_N} = \left\langle X_1^{\nu_1}\cdots X_N^{\nu_N}\right\rangle_c = \left\langle X^{\nu_1+\cdots\nu_N}\right\rangle_c = \kappa_{\nu_1+\cdots+\nu_N} \tag{27}$$

We can further assume that the orders of all variables are equal to 1, $\nu_1 = \nu_2 = \cdots = \nu_N = 1$, then the original multivariate cumulant of N random variables reduces to the Nth-order univariate cumulant, i.e., $\kappa_{1,...,1} = \kappa_N$.

## 3.4 Semi-invariance.

We will see what happens if we sum constants to each variable in the set $X_1,...,X_N$. If the sum of indices in the multivariate cumulant $\kappa_{\nu_1,...,\nu_N}$ is equal or bigger than 2, then it follows that adding constants $c_i$, where $1 \leq i \leq N$, won't change the original cumulant:

$$\kappa([X_1 + c_1]^{\nu_1},...,[X_N + c_N]^{\nu_N}) = \kappa(X_1^{\nu_1},...,X_N^{\nu_N}) = \kappa_{\nu_1,...,\nu_N}. \tag{28}$$

In the case that one variable is equal to 1 and the rest to zero, then adding a constant to that variable has the same effect that adding the constant to the orignal cumulant:

$$\kappa(1, ..., 1, X_i + c_i, 1, ..., 1) = c_i + \kappa(1, ..., 1, X_i, 1, ..., 1). \tag{29}$$

For the univariate case, this property will made all cumulants shift-invariant, i.e., for any constant $c$,

$$\kappa[(X + c)^\nu] = \kappa(X^\nu), \tag{30}$$

where $\nu$ is equal or bigger than 2. Last case is the first-order cumulant; this will add the constant to the cumulant:

$$\kappa(X + c) = c + \kappa(X). \tag{31}$$

### 3.5 Homogeneity.

A mathematical expression will be a multivariate cumulant of flow amplitudes, if and only if multiplying constants $c_i$ to each variable $X_i$ will result in the original cumulant multiplied by the product of all constants, where $1 \leq i \leq N$. This is called the homogeneity of cumulants. For instance, if we have a set of variables $X_1, ..., X_N$ and a cumulant $\kappa_{\nu_1,...,\nu_N}$, then the following expression must be true:

$$\kappa[(c_1X_1)^{\nu_1}, ..., (c_NX_N)^{\nu_N}] = c_1^{\nu_1} \cdots c_N^{\nu_N} \kappa\Big(X_1^{\nu_1}, ..., X_N^{\nu_N}\Big) = c_1^{\nu_1} \cdots c_N^{\nu_N} \kappa_{\nu_1,...,\nu_N}. \tag{32}$$

For the univariate case, this requierement reduces to

$$\kappa([cX]^\nu) = c^\nu \kappa(X^\nu) \tag{33}$$

### 3.6 Multilinearity.

A mathematical expression is a multivariate cumulant of flow amplitudes, if and only if the following is true. When we have a set of variables $Z_1, ..., Z_N$ and one of the variables is linear (of order 1), let's say $X_1$, if we express it as a sum of other M statistical variables $\sum_{i=1}^{M} X_i$, then the resulting cumulant is the sum of cumulants with each variable $X_i$ as first variable:

$$\kappa\left[\Big(\sum_{i=1}^{M} X_i\Big), Z_2^{\nu_2}, ..., Z_N^{\nu_N}\right] = \sum_{i=1}^{M} \kappa\big[X_i, Z_2^{\nu_2}, ..., Z_N^{\nu_N}\big] \tag{34}$$

We can see here that the rest of variables can be linear or not ($\nu_i > 1$). For the univariate case we have:

$$\kappa\left(\sum_{i=1}^{M} X_i\right) = \sum_{i=1}^{M} \kappa(X_i), \tag{35}$$

wich is a linear univariate estimator.

### 3.7 Examples of cumulants

From the definition of Eq. (16), (17) and Taylor series, we can express the first cumulants in terms of the moments, although, in this paper, we will focus more in the verification of cumulants.

$$\begin{aligned}
\kappa_1 &= \mu_1, \\
\kappa_2 &= \mu_2 - \mu_1^2, \\
\kappa_{1,1} &= \mu_{1,1} - \mu_{0,1}\mu_{1,0}, \\
\kappa_{2,1} &= 2\mu_{0,1}\mu_{1,0}^2 - 2\mu_{1,1}\mu_{1,0} - \mu_{0,1}\mu_{2,0} + \mu_{2,1}, \\
\kappa_{1,2} &= 2\mu_{1,0}\mu_{0,1}^2 - 2\mu_{1,1}\mu_{0,1} - \mu_{0,2}\mu_{1,0} + \mu_{1,2}, \\
\kappa_{1,1,1} &= 2\mu_{0,0,1}\mu_{0,1,0}\mu_{1,0,0} - \mu_{0,1,1}\mu_{1,0,0} - \mu_{0,1,0}\mu_{1,0,1} - \mu_{0,0,1}\mu_{1,1,0} + \mu_{1,1,1}.
\end{aligned} \tag{36}$$

Having well defined properties, we can continue to illustrate examples of true multivariate cumulants of flow amplitudes. With these examples we will improve our understanding and will be able to recognize a true cumulant. This will help us in the next section, where a new and original multivariate cumulant is proposed and demonstrated. On this paper we will only handle square of flow amplitudes as stochastic variables.

Let's begin with the easiest two particle cumulant of Eq. (15). The stochastic variables are chosen to be $X_1 \equiv v_m^2$ and $X_2 \equiv v_n^2$.

- **Statistical independence** Like we already did in the beginning of the section, by definition this cumulant is zero if the two variables are statistically independent.

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle = 0 \tag{37}$$

- **Reduction** If we have that $X_1 = X_2 = X = v^2$ then it follows:

$$\begin{aligned}
\langle X_1 X_2 \rangle_c &= \langle XX \rangle - \langle X \rangle \langle X \rangle = \left\langle X^2 \right\rangle - \langle X \rangle^2 \\
&= \left\langle v^4 \right\rangle - \left\langle v^2 \right\rangle^2
\end{aligned} \tag{38}$$

  wich is an univariate cumulant $\kappa_2$ (Appendix B of [12]).

- **Semi-invariance** Adding two constants $c_1$ and $c_2$ will have the following effect:

$$\begin{aligned}
\langle (X_1 + c_1)(X_2 + c_2) \rangle_c &= \left\langle (v_m^2 + c_1)(v_n^2 + c_2) \right\rangle - \left\langle (v_m^2 + c_1) \right\rangle \left\langle (v_n^2 + c_2) \right\rangle \\
&= \left\langle (v_m^2 v_n^2 + v_m^2 c_2 + c_1 v_n^2 + c_1 c_2) \right\rangle \\
&\quad - \left\langle (v_m^2 + c_1) \right\rangle \left\langle (v_n^2 + c_2) \right\rangle \\
&= \left\langle v_m^2 v_n^2 \right\rangle + c_2 \left\langle v_m^2 \right\rangle + c_1 \left\langle v_n^2 \right\rangle + c_1 c_2 \\
&\quad - (\left\langle v_m^2 \right\rangle + c_1)(\left\langle v_n^2 \right\rangle + c_2) \\
&= \left\langle v_m^2 v_n^2 \right\rangle + c_2 \left\langle v_m^2 \right\rangle + c_1 \left\langle v_n^2 \right\rangle + c_1 c_2 \\
&\quad - \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle - c_1 \left\langle v_n^2 \right\rangle - c_2 \left\langle v_m^2 \right\rangle - c_1 c_2 \\
&= \left\langle v_m^2 v_n^2 \right\rangle - \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle = \langle X_1 X_2 \rangle_c \,,
\end{aligned} \tag{39}$$

  which does not change anything, as expected for a true cumulant.

- **Homogeneity** Now we multiply the variables $v_m^2$ and $v_n^2$ with the constants $c_1$ and $c_2$, respectively:

$$\begin{aligned}
\langle (c_1 X_1)(c_2 X_2) \rangle_c &= \left\langle (c_1 v_m^2)(c_2 v_n^2) \right\rangle - \left\langle c_1 v_m^2 \right\rangle \left\langle c_2 v_n^2 \right\rangle \\
&= c_1 c_2 \left( \left\langle v_m^2 v_n^2 \right\rangle - \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle \right) \\
&= c_1 c_2 \langle X_1 X_2 \rangle_c
\end{aligned} \tag{40}$$

  and see that the homogeneity is satisfied.

- **Multilinearity** Let's take the first variable $X_1 = v_m^2$ as linear. When it is the sum of other two variables $v_m^2 = z_1^2 + z_2^2$, where $Z_1 = z_1^2$ and $Z_2 = z_2^2$, then the expression becomes:

$$\begin{aligned}
\langle (Z_1 + Z_2)X_2 \rangle_c &= \left\langle (z_1^2 + z_2^2)v_n^2 \right\rangle_c \equiv \left\langle (z_1^2 + z_2^2)v_n^2 \right\rangle - \left\langle (z_1^2 + z_2^2) \right\rangle \left\langle v_n^2 \right\rangle \\
&= \left\langle z_1^2 v_n^2 \right\rangle + \left\langle z_2^2 v_n^2 \right\rangle - \left( \left\langle z_1^2 \right\rangle \left\langle v_n^2 \right\rangle + \left\langle z_2^2 \right\rangle \left\langle v_n^2 \right\rangle \right) \\
&= \left\langle z_1^2 v_n^2 \right\rangle - \left\langle z_1^2 \right\rangle \left\langle v_n^2 \right\rangle + \left\langle z_2^2 v_n^2 \right\rangle - \left\langle z_2^2 \right\rangle \left\langle v_n^2 \right\rangle \\
&= \langle Z_1 X_2 \rangle_c + \langle Z_2 X_2 \rangle_c
\end{aligned} \tag{41}$$

Symmetric cumulants of flow amplitudes are defined by $SC(k,l) \equiv \left\langle v_k^2 v_l^2 \right\rangle - \left\langle v_k^2 \right\rangle \left\langle v_l^2 \right\rangle$, with the angular brackets denoting an average over all events. In recent studies it was found that different SC(k,l) observables provide new and independent constraints for both the initial conditions and the QGP properties. For instance, It was showed that the different $SC(k,l)$ observables have different sensitivities to the initial conditions of a heavy-ion collision and properties of the created system, and can therefore help in separating the effects of $\eta/s$ in the final state anisotropies from the contributions originating in the initial state. Furthermore, it was demonstrated that the SC observables are more sensitive to the temperature dependence $\eta/s(T)$ than the individual flow amplitudes, which are sensitive only to the average values $\langle \eta/s \rangle$ [11]. Asymmetric cumulants (ACs), on the other hand, only differ with SCs that the even exponents are not equal. Let's continue with an AC, because it has the same type of our new cumulant from the next section:

$$AC_{2,1}(m,n) = \left\langle v_m^4 v_n^2 \right\rangle - \left\langle v_m^4 \right\rangle \left\langle v_n^2 \right\rangle - 2 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle + 2 \left\langle v_m^2 \right\rangle^2 \left\langle v_n^2 \right\rangle \tag{42}$$

Here the subindices $2,1$ mean that the flow amplitudes $v_m^2$ and $v_n^2$ are elevated to the maximal exponents 2 and 1, respectively. We are proceeding now to prove that it is a cumulant.

- **Statistical independence** If the fluctuations $v_n^2$ and $v_m^2$ are completely uncorrelated, then Eq. (42) becomes

$$AC_{2,1}(m,n) = \left\langle v_m^4 \right\rangle \left\langle v_n^2 \right\rangle - \left\langle v_m^4 \right\rangle \left\langle v_n^2 \right\rangle - 2 \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle + 2 \left\langle v_m^2 \right\rangle^2 \left\langle v_n^2 \right\rangle = 0, \tag{43}$$

  as expected in absence of genuine correlations between the two variables.

- **Reduction** Again, if we set the two flow amplitudes equal to the same quantity $v_m^2 = v_n^2 = v^2$ then we have

$$\begin{aligned}
AC_{2,1}(m,n) &= \left\langle v^6 \right\rangle - \left\langle v^4 \right\rangle \left\langle v^2 \right\rangle - 2 \left\langle v^4 \right\rangle \left\langle v^2 \right\rangle + 2 \left\langle v^2 \right\rangle^2 \left\langle v^2 \right\rangle \\
&= \left\langle v^6 \right\rangle - 3 \left\langle v^4 \right\rangle \left\langle v^2 \right\rangle + 2 \left\langle v^2 \right\rangle^3 .
\end{aligned} \tag{44}$$

  This is a valid univariate cumulant with $v^2$ as fundamental variable, $\kappa_3$ from the paper of Bilandzic et. al. [12].

- **Semi-invariance** We will add now two constants $c_1$ and $c_2$ to the variables $X_1$ and

$X_2$, respectively. Let's see if this property is fulfilled.

$$\begin{aligned}
\big\langle (v_m^2+c_m)^2(v_n^2+c_n)\big\rangle_c &= \big\langle (v_m^2+c_m)^2(v_n^2+c_n)\big\rangle - \big\langle (v_m^2+c_m)^2\big\rangle\big\langle (v_n^2+c_n)\big\rangle \\
&\quad -2\big\langle (v_m^2+c_m)(v_n^2+c_n)\big\rangle\big\langle v_m^2+c_m\big\rangle \\
&\quad +2\big\langle v_m^2+c_m\big\rangle^2+\big\langle v_n^2+c_n\big\rangle \\
&= \big\langle (v_m^4+2c_m v_m^2+c_m^2)(v_n^2+c_n)\big\rangle \\
&\quad -\big\langle v_m^4+2c_m v_m^2+c_m^2\big\rangle\big\langle v_n^2+c_n\big\rangle \\
&\quad -2\big\langle v_m^2 v_n^2+c_m v_n^2+c_n v_m^2+c_m c_n\big\rangle\big\langle v_m^2+c_m\big\rangle \\
&\quad +2\big(\big\langle v_m^2\big\rangle^2+2c_m\big\langle v_m^2\big\rangle+c_m\big)\big\langle v_n^2+c_m\big\rangle \\
&= AC_{2,1}(m,n) \\
&\quad +2c_m\big(\big\langle v_m^2 v_n^2\big\rangle-2\big\langle v_m^2\big\rangle\big\langle v_n^2\big\rangle-\big\langle v_m^2 v_n^2\big\rangle+ \\
&\quad 2\big\langle v_m^2\big\rangle\big\langle v_n^2\big\rangle\big) \\
&\quad +c_n\big(\big\langle v_m^4\big\rangle-\big\langle v_m^4\big\rangle 2\big\langle v_m^2\big\rangle^2-2\big\langle v_m^2\big\rangle^2\big) \\
&\quad +c_m^2\big(3\big\langle v_n^2\big\rangle-3\big\langle v_n^2\big\rangle\big)+2c_m c_n\big(3\big\langle v_m^2\big\rangle-3\big\langle v_m^2\big\rangle\big) \\
&\quad +c_m^2 c_n(3-3) \\
&= AC_{2,1}(m,n) \qquad (45)
\end{aligned}$$

As expected, the cumulant won't change when adding constants.

- **Homogeneity** Let's multiply the constants $c_m$ and $c_n$ to the variables $v_m^2$ and $v_n^2$, respectively:

$$\begin{aligned}
\big\langle (c_m v_m^2)^2(c_n v_n^2)\big\rangle_c &= \big\langle c_m^2 v_m^4 c_n v_n^2\big\rangle-\big\langle c_m^2 v_m^4\big\rangle\big\langle c_n v_n^2\big\rangle-2\big\langle c_m v_m^2 c_n v_n^2\big\rangle\big\langle c_m v_m^2\big\rangle \\
&\quad +2\big\langle c_m v_m^2\big\rangle^2\big\langle c_n v_n^2\big\rangle \\
&= c_m^2 c_n AC_{2,1}(m,n). \qquad (46)
\end{aligned}$$

As we see here, homogeneity is valid for our expression $AC_{2,1}(m,n)$.

- **Multilinearity** We consider now the second variable linear, so we have three variables, $v_m^2$, $v_n^2$ and $v_k^2$. Let's see if our expression $AC_{2,1}(m,n)$ is linear:

$$\begin{aligned}
AC_{2,1}(m,n+k) &= \big\langle v_m^4(v_n^2+v_k^2)\big\rangle-\big\langle v_m^4\big\rangle\big\langle v_n^2+v_k^2\big\rangle-2\big\langle v_m^2(v_n^2+v_k^2)\big\rangle\big\langle v_m^2\big\rangle \\
&\quad +2\big\langle v_m^2\big\rangle\big\langle v_n^2+v_k^2\big\rangle \\
&= AC_{2,1}(m,n)+AC_{2,1}(m,k). \qquad (47)
\end{aligned}$$

And yes, it is, so our expression shows multilinearity.

We have just desmostrated that the expression $AC_{2,1}(m,n)$ given by Eq. (42) is a valid multivariate cumulant with $v_m^2$ and $v_n^2$ being the fundamental stochastic variables. After all this long examples we have now a clear understanding of cumulants and can proceed to analyse our new cumulant.

# 4 Proposal and demonstration of a new cumulant

In this section we have taken the asymmetric cumulant of flow amplitudes $AC_{3,1}(m,n)$ from the paper of Bilandzic et. al. (Eq. 38 in [12]), and we will show that this a proper cumulant that meets all the requirements from section 3. By definition, it can probe the genuine correlations between the different moments, 3 and 1, of different flow harmonics $v_m^2$ and $v_n^2$, respectively. Therefore, it has access to new and independent information to constrain different stages in the heavy-ion evolution. The multivariate estimator has following form:

$$\begin{aligned} AC_{3,1}(m,n) &= \left\langle v_m^6 v_n^2\right\rangle - \left\langle v_m^6\right\rangle\left\langle v_n^2\right\rangle - 3\left\langle v_m^2 v_n^2\right\rangle\left\langle v_m^4\right\rangle - 3\left\langle v_m^4 v_n^2\right\rangle\left\langle v_m^2\right\rangle \\ &\quad +6\left\langle v_m^4\right\rangle\left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle + 6\left\langle v_m^2 v_n^2\right\rangle\left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3\left\langle v_n^2\right\rangle \\ &= \left\langle X_1^3 X_2\right\rangle_c \end{aligned} \tag{48}$$

We now demonstrate with the detailed calculus that this observable satisfies all fundamental mathematical properties of cumulants. The first stochastic variable is $X_1 = v_m^2$ and the second $X_2 = v_n^2$.

## 4.1 Statistical independence

Let's begin with the easiest one. Statistical independence means that the mean of a products is equal to the product of the mean of its factors. We see in the following expression that when the variables are independent to each other then the asymmetric cumulant becomes zero:

$$\begin{aligned} AC_{3,1}(m,n) &= \left\langle v_m^6\right\rangle\left\langle v_n^2\right\rangle - \left\langle v_m^6\right\rangle\left\langle v_n^2\right\rangle - 3\left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle\left\langle v_m^4\right\rangle - 3\left\langle v_m^4\right\rangle\left\langle v_n^2\right\rangle\left\langle v_m^2\right\rangle \\ &\quad +6\left\langle v_m^4\right\rangle\left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle + 6\left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle\left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3\left\langle v_n^2\right\rangle \\ &= 0. \end{aligned} \tag{49}$$

## 4.2 Reduction

Now we go to the second property. To prove the reduction we need to equalize the two variables, $X_1 = X_2 = v^2$. Then Eq. (48) becomes:

$$\begin{aligned} AC_{3,1}(m,n) &= \left\langle v^6 v^2\right\rangle - \left\langle v^6\right\rangle\left\langle v^2\right\rangle - 3\left\langle v^2 v^2\right\rangle\left\langle v^4\right\rangle - 3\left\langle v^4 v^2\right\rangle\left\langle v^2\right\rangle \\ &\quad +6\left\langle v^4\right\rangle\left\langle v^2\right\rangle\left\langle v^2\right\rangle + 6\left\langle v^2 v^2\right\rangle\left\langle v^2\right\rangle^2 - 6\left\langle v^2\right\rangle^3\left\langle v^2\right\rangle \\ &= \left\langle v^8\right\rangle - \left\langle v^6\right\rangle\left\langle v^2\right\rangle - 3\left\langle v^4\right\rangle^2 - 3\left\langle v^6\right\rangle\left\langle v^2\right\rangle \\ &\quad +6\left\langle v^4\right\rangle\left\langle v^2\right\rangle^2 + 6\left\langle v^4\right\rangle\left\langle v^2\right\rangle^2 - 6\left\langle v^2\right\rangle^3\left\langle v^2\right\rangle \\ &= \left\langle v^8\right\rangle - 4\left\langle v^6\right\rangle\left\langle v^2\right\rangle - 3\left\langle v^4\right\rangle^2 + 12\left\langle v^4\right\rangle\left\langle v^2\right\rangle^2 - 6\left\langle v^2\right\rangle^4 \\ &= \kappa_4. \end{aligned} \tag{50}$$

$\kappa_4$ is a univariate cumulant as mentioned in Appendix B from the paper of Bilandzic et. al. [12]. Thus we have reduced our two-variate cumulant to the valid univariate cumulant and proved this property.

### 4.3 Semi-invariance

In Eq. (48) we add two constants $c_1$ and $c_2$ to the squared flow amplitudes $v_m^2$ and $v_n^2$, respectively. If the cumulant dones not change, then the semi-invariance is fulfilled.

$$
\begin{aligned}
AC_{3,1}(m,n) \;=\;& \left\langle (v_m^2+c_1)^3(v_n^2+c_2)\right\rangle - \left\langle (v_m^2+c_1)^3\right\rangle \left\langle v_n^2+c_2\right\rangle \\
&+3\left\langle (v_m^2+c_1)(v_n^2+c_2)\right\rangle \left\langle (v_m^2+c_1)^2\right\rangle \\
&-3\left\langle (v_m^2+c_1)^2(v_n^2+c_2)\right\rangle \left\langle v_m^2+c_1\right\rangle \\
&+6\left\langle (v_m^2+c_1)^2\right\rangle \left\langle v_m^2+c_1\right\rangle \left\langle v_n^2+c_2\right\rangle \\
&+6\left\langle (v_m^2+c_1)(v_n^2+c_2)\right\rangle \left\langle v_m^2+c_1\right\rangle^2 - 6\left\langle v_m^2+c_1\right\rangle^3 \left\langle v_n^2+c_2\right\rangle
\end{aligned}
\tag{51}
$$

Expressions inside the parentheses on each term can be expanded.

$$
\begin{aligned}
AC_{3,1}(m,n) \;=\;& \left\langle (v_m^6+3v_m^4c_1+3v_m^2c_1^2+c_1^3)(v_n^2+c_2)\right\rangle \\
&-\left\langle (v_m^6+3v_m^4c_1+3v_m^2c_1^2+c_1^3)\right\rangle \left\langle (v_n^2+c_2)\right\rangle \\
&-3\left\langle (v_m^2v_n^2+v_m^2c_2+c_1v_n^2+c_1c_2)\right\rangle \left\langle (v_m^4+2v_m^2c_1+c_1^2)\right\rangle \\
&-3\left\langle (v_m^4+2v_m^2c_1+c_1^2)(v_n^2+c_2)\right\rangle \left\langle v_m^2+c_1\right\rangle \\
&+6\left\langle (v_m^4+2v_m^2c_1+c_1^2)\right\rangle \left\langle v_m^2+c_1\right\rangle \left\langle v_n^2+c_2\right\rangle \\
&+6\left\langle (v_m^2v_n^2+v_m^2c_2+c_1v_n^2+c_1c_2)\right\rangle \left\langle v_m^2+c_1\right\rangle^2 \\
&-6\left\langle v_m^2+c_1\right\rangle^3 \left\langle v_n^2+c_2\right\rangle
\end{aligned}
\tag{52}
$$

If we expand further, we obtain the next long expression. Here by definition we can express the average of a sum as the sum of averages, and the average of a variable multiplied with a constant as the product constant and average. For a better view all terms appear on the position of their ancestor product from Eq. (52).

$$
\begin{aligned}
AC_{3,1}(m,n) \;=\;& \left\langle v_m^6 v_n^2 \right\rangle + 3c_1 \left\langle v_m^4 v_n^2 \right\rangle + 3c_1^2 \left\langle v_m^2 v_n^2 \right\rangle + c_1^3 \left\langle v_n^2 \right\rangle \\
& + c_2 \left\langle v_m^6 \right\rangle + 3c_1 c_2 \left\langle v_m^4 \right\rangle + 3c_1^2 c_2 \left\langle v_m^2 \right\rangle + c_1^3 c_2 \\
& - \left\langle v_m^6 \right\rangle \left\langle v_n^2 \right\rangle - 3c_1 \left\langle v_m^4 \right\rangle \left\langle v_n^2 \right\rangle - 3c_1^2 \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle - c_1^3 \left\langle v_n^2 \right\rangle \\
& - c_2 \left\langle v_m^6 \right\rangle - 3c_1 c_2 \left\langle v_m^4 \right\rangle - 3c_1^2 c_2 \left\langle v_m^2 \right\rangle - c_1^3 c_2 \\
& - 3 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^4 \right\rangle - 3c_2 \left\langle v_m^2 \right\rangle \left\langle v_m^4 \right\rangle - 3c_1 \left\langle v_n^2 \right\rangle \left\langle v_m^4 \right\rangle - 3c_1 c_2 \left\langle v_m^4 \right\rangle \\
& - 6c_1 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle - 6c_1 c_2 \left\langle v_m^2 \right\rangle^2 - 6c_1^2 \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle - 6c_1^2 c_2 \left\langle v_m^2 \right\rangle \\
& - 3c_1^2 \left\langle v_m^2 v_n^2 \right\rangle - 3c_2 c_1^2 \left\langle v_m^2 \right\rangle - 3c_1^3 \left\langle v_n^2 \right\rangle - 3c_1^3 c_2 \\
& - 3 \left\langle v_m^4 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle - 6c_1 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle - 3c_1^2 \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle - 3c_2 \left\langle v_m^4 \right\rangle \left\langle v_m^2 \right\rangle \\
& - 6c_1 c_2 \left\langle v_m^2 \right\rangle^2 - 3c_1^2 c_2 \left\langle v_m^2 \right\rangle - 3c_1 \left\langle v_m^4 v_n^2 \right\rangle - 6c_1^2 \left\langle v_m^2 v_n^2 \right\rangle \\
& - 3c_1^3 \left\langle v_n^2 \right\rangle - 3c_1 c_2 \left\langle v_m^4 \right\rangle - 6c_1 c_2 \left\langle v_m^2 \right\rangle - 3c_1^3 c_2 \\
& + 6 \left\langle v_m^4 \right\rangle \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle + 12c_1 \left\langle v_m^2 \right\rangle^2 \left\langle v_n^2 \right\rangle + 6c_1 \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle + 6c_1 \left\langle v_m^4 \right\rangle \left\langle v_n^2 \right\rangle \\
& + 12c_1^2 \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle + 6c_1^3 \left\langle v_n^2 \right\rangle + 6c_2 \left\langle v_m^4 \right\rangle \left\langle v_m^2 \right\rangle + 12c_1 c_2 \left\langle v_m^2 \right\rangle^2 \\
& + 6c_1 c_2 \left\langle v_m^2 \right\rangle + 6c_1 c_2 \left\langle v_m^4 \right\rangle + 12c_1^2 c_2 \left\langle v_m^2 \right\rangle + 6c_1^3 c_2 \\
& + 6 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle^2 + 6c_2 \left\langle v_m^2 \right\rangle \left\langle v_m^2 \right\rangle^2 + 6c_1 \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle^2 + 6c_1 c_2 \left\langle v_m^2 \right\rangle^2 \\
& + 12c_1 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle + 12c_1 c_2 \left\langle v_m^2 \right\rangle \left\langle v_m^2 \right\rangle + 12c_1^2 \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle + 12c_1^2 c_2 \left\langle v_m^2 \right\rangle \\
& + 6c_1^2 \left\langle v_m^2 v_n^2 \right\rangle + 6c_1^2 c_2 \left\langle v_m^2 \right\rangle + 6c_1^3 \left\langle v_n^2 \right\rangle + 6c_1^3 c_2 \\
& - 6 \left\langle v_m^2 \right\rangle^3 \left\langle v_n^2 \right\rangle - 18 \left\langle v_m^2 \right\rangle^2 \left\langle v_n^2 \right\rangle - 18c_1^2 \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle - 6c_1^3 \left\langle v_n^2 \right\rangle \\
& - 6c_2 \left\langle v_m^2 \right\rangle^3 - 18c_2 \left\langle v_m^2 \right\rangle^2 - 18c_1^2 c_2 \left\langle v_m^2 \right\rangle - 6c_1^3 c_2
\end{aligned} \tag{53}
$$

We can separate the terms that does not have our introduced constants $c_1$ and $c_2$. As consequence we see that all this terms form the original expression $AC_{3,1}(m,n)$ from Eq. (48). With a long and exhausting calculation by hand, we can check that the rest terms sum zero. This means that $AC_{3,1}(m,n)$ hasn't changed after adding constants to its variables, the semi-invariance is proven.

### 4.4 Homogeneity

We will multiply each squared flow amplitude $v_m^2$ and $v_n^2$ by the constants $c_1$ and $c_2$, respectively. As we see in the following expression, the original estimator $AC_{3,1}(m,n)$ only changes by a new factor $c_1^3 c_2$, thus proving the homogeneity.

$$
\begin{aligned}
AC_{3,1}(m,n) \;=\;& \left\langle (c_1 v_m^2)^3 (c_2 v_n^2) \right\rangle - \left\langle (c_1 v_m^2)^3 \right\rangle \left\langle c_2 v_n^2 \right\rangle \\
& - 3 \left\langle (c_1 v_m^2)(c_2 v_n^2) \right\rangle \left\langle (c_1 v_m^2)^2 \right\rangle - 3 \left\langle (c_1 v_m^2)^2 (c_2 v_n^2) \right\rangle \left\langle (c_1 v_m^2) \right\rangle \\
& + 6 \left\langle (c_1 v_m^2)^2 \right\rangle \left\langle (c_1 v_m^2) \right\rangle \left\langle (c_2 v_n^2) \right\rangle + 6 \left\langle (c_1 v_m^2)(c_2 v_n^2) \right\rangle \left\langle (c_1 v_m^2) \right\rangle^2 \\
& - 6 \left\langle (c_1 v_m^2) \right\rangle^3 \left\langle (c_2 v_n^2) \right\rangle \\
=\;& c_1^3 c_2 \left\langle v_m^6 v_n^2 \right\rangle - c_1^3 c_2 \left\langle v_m^6 \right\rangle \left\langle v_n^2 \right\rangle - 3c_1^3 c_2 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^4 \right\rangle - 3c_1^3 c_2 \left\langle v_m^4 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle \\
& + 6c_1^3 c_2 \left\langle v_m^4 \right\rangle \left\langle v_m^2 \right\rangle \left\langle v_n^2 \right\rangle + 6c_1^3 c_2 \left\langle v_m^2 v_n^2 \right\rangle \left\langle v_m^2 \right\rangle^2 - 6c_1^3 c_2 \left\langle v_m^2 \right\rangle^3 \left\langle v_n^2 \right\rangle \\
=\;& c_1^3 c_2 AC_{3,1}(m,n).
\end{aligned} \tag{54}
$$

### 4.5 Multilinearity

In our asymmetric estimator $AC_{3,1}(m,n)$, the second stochastic variable is linear. We can express it as the sum of two squared flow amplitudes $v_n^2 + v_k^2$. The following form is taken:

$$\begin{aligned} AC_{3,1}(m,n+k) &= \left\langle v_m^6(v_n^2+v_k^2)\right\rangle - \left\langle v_m^6\right\rangle \left\langle (v_n^2+v_k^2)\right\rangle - 3\left\langle v_m^2(v_n^2+v_k^2)\right\rangle \left\langle v_m^4\right\rangle \\ &\quad -3\left\langle v_m^4(v_n^2+v_k^2)\right\rangle \left\langle v_m^2\right\rangle + 6\left\langle v_m^4\right\rangle \left\langle v_m^2\right\rangle \left\langle (v_n^2+v_k^2)\right\rangle \\ &\quad +6\left\langle v_m^2(v_n^2+v_k^2)\right\rangle \left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3 \left\langle (v_n^2+v_k^2)\right\rangle \end{aligned} \tag{55}$$

We can proceed to solve the averages and express everything in terms of correlations.

$$\begin{aligned} AC_{3,1}(m,n+k) &= \left\langle v_m^6 v_n^2\right\rangle + \left\langle v_m^6 v_k^2\right\rangle - \left\langle v_m^6\right\rangle \left\langle v_n^2\right\rangle - \left\langle v_m^6\right\rangle \left\langle v_k^2\right\rangle - 3\left\langle v_m^2 v_n^2\right\rangle \left\langle v_m^4\right\rangle \\ &\quad -3\left\langle v_m^2 v_k^2\right\rangle \left\langle v_m^4\right\rangle - 3\left\langle v_m^4 v_n^2\right\rangle \left\langle v_m^2\right\rangle - 3\left\langle v_m^4 v_k^2\right\rangle \left\langle v_m^2\right\rangle \\ &\quad +6\left\langle v_m^4\right\rangle \left\langle v_m^2\right\rangle \left\langle v_n^2\right\rangle + 6\left\langle v_m^4\right\rangle \left\langle v_m^2\right\rangle \left\langle v_k^2\right\rangle + 6\left\langle v_m^2 v_n^2\right\rangle \left\langle v_m^2\right\rangle^2 \\ &\quad +6\left\langle v_m^2 v_k^2\right\rangle \left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3 \left\langle v_n^2\right\rangle - 6\left\langle v_m^2\right\rangle^3 \left\langle v_k^2\right\rangle \end{aligned} \tag{56}$$

If we separate the terms with $v_n^2$ from the ones with $v_k^2$, then we get the sum of two estimators, $AC_{3,1}(m,n)$ and $AC_{3,1}(m,k)$. This proves the multilinearity.

$$\begin{aligned} AC_{3,1}(m,n+k) &= \left\langle v_m^6 v_n^2\right\rangle - \left\langle v_m^6\right\rangle \left\langle v_n^2\right\rangle - 3\left\langle v_m^2 v_n^2\right\rangle \left\langle v_m^4\right\rangle - 3\left\langle v_m^4 v_n^2\right\rangle \left\langle v_m^2\right\rangle \\ &\quad +6\left\langle v_m^4\right\rangle \left\langle v_m^2\right\rangle \left\langle v_n^2\right\rangle + 6\left\langle v_m^2 v_n^2\right\rangle \left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3 \left\langle v_n^2\right\rangle \\ &\quad +\left\langle v_m^6 v_k^2\right\rangle - \left\langle v_m^6\right\rangle \left\langle v_k^2\right\rangle - 3\left\langle v_m^2 v_k^2\right\rangle \left\langle v_m^4\right\rangle - 3\left\langle v_m^4 v_k^2\right\rangle \left\langle v_m^2\right\rangle \\ &\quad +6\left\langle v_m^4\right\rangle \left\langle v_m^2\right\rangle \left\langle v_k^2\right\rangle + 6\left\langle v_m^2 v_k^2\right\rangle \left\langle v_m^2\right\rangle^2 - 6\left\langle v_m^2\right\rangle^3 \left\langle v_k^2\right\rangle \\ &= AC_{3,1}(m,n) + AC_{3,1}(m,k) \end{aligned} \tag{57}$$

After this demonstrations we can conclude that our estimator $AC_{3,1}(m,n)$ meets all the requirements for a cumulant. In the following section we will review its characteristic with a toy Monte Carlo simulation using our generic framework from chapter 2.

## 5 Toy Monte Carlo studies

Theoretically, the cumulant's value must be zero, when its variables are statistically independent (no genuine multi-body interaction). In our generic framework, We can see this if we take the values of flow amplitudes in Eq. (12) to solve Eq. (48). We will proceed to demostrate this in another toy monte carlo simulation.

To illustrate this behavior, we first will introduce previous studies and then we will compare them with our own results. With the use of the realistic Monte Carlo (MC) event generator HIJING, Bilandzic et. al. have demostrated the robustness of the proposed observables against few-particle nonflow correlations [13, 14, 12]. HIJING (for Heavy-Ion Jet INteraction Generator) is a combination of models describing jet and nuclear-related mechanisms, like jet production and fragmentation or nuclear shadowing to cite only a few. It is very useful for us, because it does not include any collective effects such as anisotropic flow, and because we want to study the sensitivity of a flow estimator against nonflow.

Figures (10 and 9) show the centrality dependence of our two-harmonic between different moments of $v_2^2$ and $v_4^2$, $AC_{3,1}(2,4)$ and $AC_{2,1}(2,4)$, respectively. This results were obtained in the paper [12]. Data from Pb-Pb collisions was simulated at a center-of-mass energy of $\sqrt{s_{NN}} = 2.76\,\text{TeV}$. Two kinetic criteria have been applied as well: $0.2 < \text{pT} < 5.0$ GeV/c and $|\eta| < 0.8$. The different correlators involved in the expression of $AC_{2,1}(2,4)$

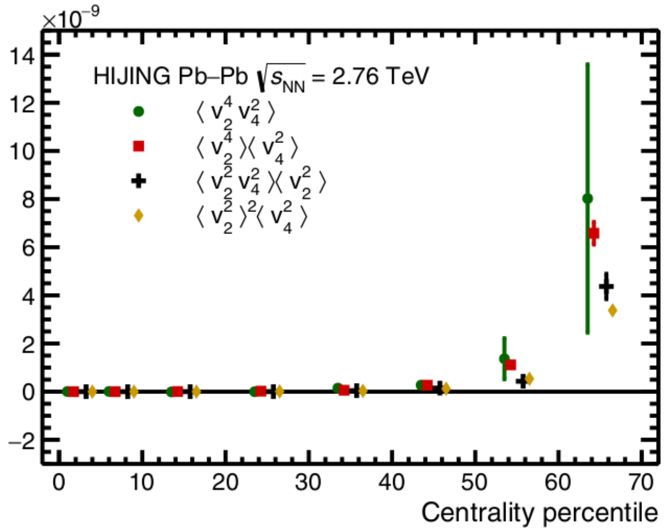


Figure 8: Centrality dependence of the correlators that are used in the calculus of our cumulants.

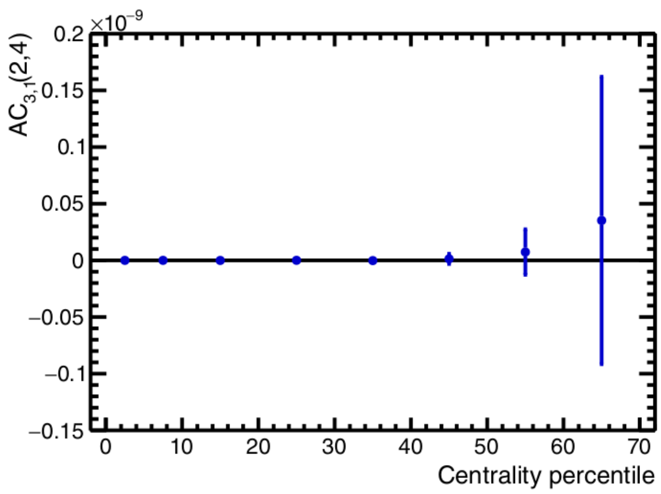


Figure 9: Centrality dependence of the correlator $AC_{3,1}(2,4)$.

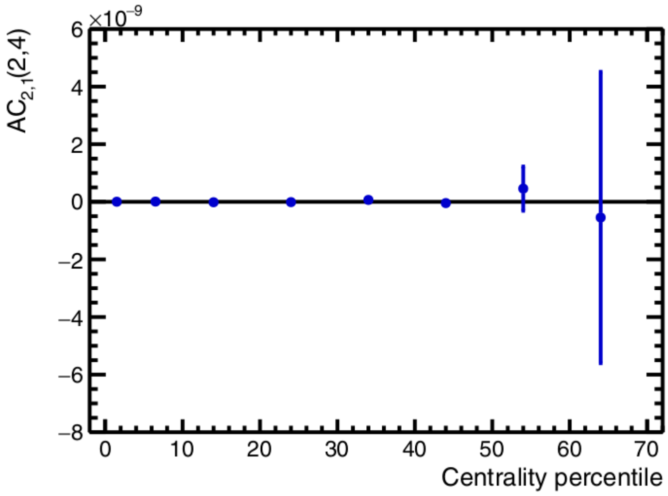


Figure 10: Centrality dependence of the correlator $AC_{2,1}(2,4)$.

are shown in Fig.(8). These two quantities are in agreement with zero for the full centrality range, meaning they are robust against few-particle nonflow correlations.

After introducing these studies we will now find the values of our cumulants $AC_{3,1}(2,4)$ and $AC_{2,1}(2,4)$. Experimentally, we can use estimators to calculate our ACs from the multiparticle correlations. We just need the Eq. (9) to eliminate any symmetry plane contribution, by setting zero to the sum of harmonics. These estimators have the following form (Eq. C6 and C7 from [12]):

$$\begin{aligned} AC_{3,1}(m,n) \;=\; & \left\langle\left\langle e^{i(m\varphi_1+m\varphi_2+m\varphi_3+n\varphi_4-m\varphi_5-m\varphi_6-m\varphi_7-n\varphi_8)}\right\rangle\right\rangle \\ & -\left\langle\left\langle e^{i(m\varphi_1+m\varphi_2+m\varphi_3-m\varphi_4-m\varphi_5-m\varphi_6)}\right\rangle\right\rangle\left\langle\left\langle e^{i(n\varphi_1-n\varphi_2)}\right\rangle\right\rangle \\ & -3\left\langle\left\langle e^{i(m\varphi_1+m\varphi_2-m\varphi_3-m\varphi_4)}\right\rangle\right\rangle\left\langle\left\langle e^{i(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)}\right\rangle\right\rangle \\ & -3\left\langle\left\langle e^{i(m\varphi_1+m\varphi_2+n\varphi_3-m\varphi_4-m\varphi_5-n\varphi_6)}\right\rangle\right\rangle\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle \\ & +6\left\langle\left\langle e^{i(m\varphi_1+m\varphi_2-m\varphi_3-m\varphi_4)}\right\rangle\right\rangle\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle\left\langle\left\langle e^{i(n\varphi_1-n\varphi_2)}\right\rangle\right\rangle \\ & +6\left\langle\left\langle e^{i(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)}\right\rangle\right\rangle\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle^2 \\ & -6\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle^3\left\langle\left\langle e^{i(n\varphi_1-n\varphi_2)}\right\rangle\right\rangle . \end{aligned} \tag{58}$$

$$\begin{aligned} AC_{2,1}(m,n) \;=\; & \left\langle\left\langle e^{i(m\varphi_1+m\varphi_2+n\varphi_3-m\varphi_4-m\varphi_5-n\varphi_6)}\right\rangle\right\rangle \\ & -\left\langle\left\langle e^{i(m\varphi_1+m\varphi_2-m\varphi_3-m\varphi_4)}\right\rangle\right\rangle\left\langle\left\langle e^{i(n\varphi_1-n\varphi_2)}\right\rangle\right\rangle \\ & -2\left\langle\left\langle e^{i(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)}\right\rangle\right\rangle\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle \\ & +2\left\langle\left\langle e^{i(m\varphi_1-m\varphi_2)}\right\rangle\right\rangle^2\left\langle\left\langle e^{i(n\varphi_1-n\varphi_2)}\right\rangle\right\rangle \end{aligned} \tag{59}$$

Double angle parenthesis means that the code first gets the means in an event and then calculates the mean in all events. After this, we can recognize the correlators of Eq. (11).

| type of distribution | average of $AC_{3,1}(2,4)$ | uncertainty |
|---|---|---|
| uniform acceptance | $5.761\,44 \times 10^{-10}$ | $2.570\,64 \times 10^{-9}$ |
| nouniform acceptance | $-3.606\,15 \times 10^{-8}$ | $1.6916 \times 10^{-8}$ |
| nonuniform acceptance + $\varphi-$weights | $-8.077\,76 \times 10^{-10}$ | $1.6097 \times 10^{-9}$ |

Table 1: Results of the first Monte Carlo simulation for our proposed cumulant $AC_{3,1}(2,4)$. We used azimuthal distribution from section 2; the application of weights corrects any biases. Only with a detector bias, the value of the cumulant $AC_{3,1}(2,4)$ cannot be identically zero. This means, in this case, that there is a multi-body interaction.

| type of distribution | average of $AC_{2,1}(2,4)$ | uncertainty |
|---|---|---|
| uniform acceptance | $-1.813\,16 \times 10^{-10}$ | $4.2044 \times 10^{-9}$ |
| nouniform acceptance | $-6.731\,24 \times 10^{-7}$ | $1.583\,83 \times 10^{-7}$ |
| nonuniform acceptance + $\varphi-$weights | $5.475\,84 \times 10^{-10}$ | $5.004\,23 \times 10^{-9}$ |

Table 2: Results of the second Monte Carlo simulation. Like previously, we used here the azimuthal distributions from section 2. The theoretical value for the cumulant $AC_{2,1}(2,4)$ is near zero. Only with a detector bias, the value of the cumulant $AC_{3,1}(2,4)$ cannot be identically zero. This means, in this case, that there is a multi-body interaction.

Like in previous studies we have selected $m = 2$ and $n = 4$. The cumulants become:

$$
\begin{aligned}
AC_{3,1}(2,4) \quad = \quad & \left\langle \langle 8 \rangle_{2,2,2,4,-2,-2,-2,-4} \right\rangle \\
& - \left\langle \langle 6 \rangle_{2,2,2,-2,-2,-2} \right\rangle \left\langle \langle 2 \rangle_{4,-4} \right\rangle \\
& -3 \left\langle \langle 4 \rangle_{2,2,-2,-2} \right\rangle \left\langle \langle 4 \rangle_{2,4,-2,-4} \right\rangle \\
& -3 \left\langle \langle 6 \rangle_{2,2,4,-2,-2,4} \right\rangle \left\langle \langle 2 \rangle_{2,-2} \right\rangle \\
& +6 \left\langle \langle 4 \rangle_{2,2,-2,-2} \right\rangle \left\langle \langle 2 \rangle_{2,-2} \right\rangle \left\langle \langle 2 \rangle_{4,-4} \right\rangle \\
& +6 \left\langle \langle 4 \rangle_{2,4,-2,-4} \right\rangle \left\langle \langle 2 \rangle_{2,-2} \right\rangle^2 \\
& -6 \left\langle \langle 2 \rangle_{2,-2} \right\rangle^3 \left\langle \langle 2 \rangle_{4,-4} \right\rangle
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
AC_{2,1}(2,4) \quad = \quad & \left\langle \langle 6 \rangle_{2,2,4,-2,-2,-4} \right\rangle \\
& - \left\langle \langle 4 \rangle_{2,2,-2,-2} \right\rangle \left\langle \langle 2 \rangle_{4,-4} \right\rangle \\
& -2 \left\langle \langle 4 \rangle_{2,4,-2,-4} \right\rangle \left\langle \langle 2 \rangle_{2,-2} \right\rangle \\
& +2 \left\langle \langle 2 \rangle_{2,-2} \right\rangle^2 \left\langle \langle 2 \rangle_{4,-4} \right\rangle
\end{aligned}
\tag{61}
$$

Figures (11) show the distribution used to obtain the azimuthal angles. Like in section 2 we have here a bias in angles between 30° and 60°, they are 50% less probable to appear. We used the weights from figure (12) to correct this bias. The tables (1) and (2) shows the results of simulation for the estimators $AC_{3,1}(2,4)$ and $AC_{2,1}(2,4)$, respectively.

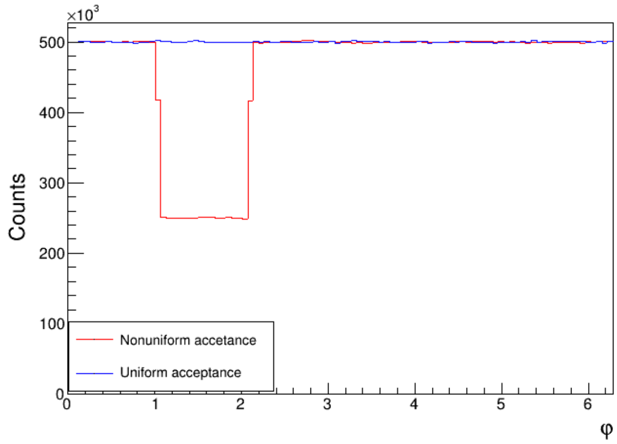


Figure 11: Azimuthal distributions for the monte carlo simulation. The red line represents a detector that has bias in angles between 30° and 60°.

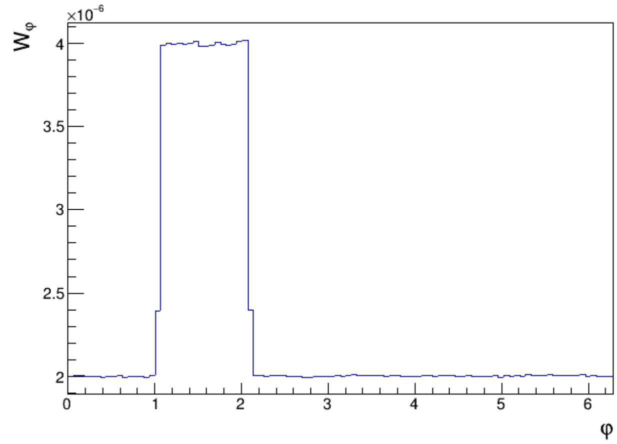


Figure 12: Weights obtained inverting the red histogram from Fig. (11). This corrects the azimuthal bias of the detector.

For the uniform acceptance and nonuniform acceptance + weights, their values are in order of $10^{-10}$. The respective uncertainties were calculated using the bootstrap technique described in Appendix B. They are in the order of $10^{-9}$ and $10^{-7}$, respectively. The theoretical value is zero, and it is located inside the accepted interval. This was totally expected due to our estimators being a cumulant. In the case of the nonuniform acceptance, we see that the accepted intervals does not include zero. This means, in this case, that there is a multi-body interaction.

If we compare with preovious studies, for $AC_{2,1}(2,4)$ From the figures (10) (9), we can see that at least 55% of events have the values in our simulations from table (1) and (2), respectively. Thus we have experimentally demonstrated that our estimators can be classified as cumulants.

# Appendix A Demonstrations

## A.1 $\langle \cos [n(\varphi - \Psi_n)] \rangle = v_n$

If we replace and calculate the following integral, we obtain the flow harmonics $v_n$:

$$\begin{aligned}\langle \cos [n(\varphi - \psi_n)] \rangle = & \int_0^{2\pi} \cos [n(\varphi - \psi_n)] \cdot f(\varphi) d\varphi \\ = & \frac{1}{2\pi} \left[ \int_0^{2\pi} \cos [n(\varphi - \psi_n)] \, d\varphi \right] \\ & + \frac{1}{\pi} \left[ \sum_{m=1}^{\infty} v_m \int_0^{2\pi} \cos [n(\varphi - \psi_n)] \cdot \cos [m(\varphi - \psi_m)] \, d\varphi \right] \\ = & \frac{1}{2\pi} \left[ \int_0^{2\pi} \cos [n(\varphi - \psi_n)] \, d\varphi + 2 \sum_{m=1}^{\infty} v_m \pi \cos(\psi_n - \psi_m) \delta_{mn} \right] \\ = & \frac{1}{2\pi} \left[ \int_0^{2\pi} \cos [n(\varphi - \psi_n)] \, d\varphi + 2\pi v_n \right] = v_n \end{aligned} \tag{62}$$

The integral of the first term is zero:

$$\int_0^{2\pi} \cos [n(\varphi - \psi_n)] = \int_0^{2\pi} (\cos(nx) \cos(n\psi_n) - \sin(nx) \sin(n\psi_n)) \, dx = 0 \tag{63}$$

## A.2 $\langle \cos [n(\varphi_1 - \varphi_2)] \rangle = v_n^2$

With the p.d.f from Eq. (2) and definition from Eq. (4) we obtain directly the result:

$$\begin{aligned}\langle \cos [n(\varphi_1 - \varphi_2)] \rangle = & \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\varphi_1 d\varphi_2 \cos(n(\varphi_1 - \varphi_2)) \Bigg[ 1 \\ & + \sum_{p=1}^{\infty} v_p \cos(p(\varphi_1 - \psi_p)) \\ & + 2 \sum_{m=1}^{\infty} v_m \cos(m(\varphi_1 - \psi_m)) \\ & + 4 \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} v_p v_m \cos(m(\varphi_1 - \psi_m)) \cos(p(\varphi_2 - \psi_p)) \Bigg] \end{aligned} \tag{64}$$

The first term is trivially similar to Eq. (63)

$$\int_0^{2\pi} \int_0^{2\pi} d\varphi_1 d\varphi_2 \cos(n(\varphi_1 - \varphi_2)) = 0 \tag{65}$$

The second term is zero. We use the orthogonality relation of trigonometric functions:

$$\begin{aligned} & 2 \sum_m^{\infty} v_m \int_0^{2\pi} \int_0^{2\pi} d\varphi_1 d\varphi_2 \cos(m(\varphi_1 - \psi_m)) \cos(n(\varphi_1 - \varphi_2)) \\ & = 2 \sum_m^{\infty} v_m \int_0^{2\pi} d\varphi_2 \int_0^{2\pi} d\varphi_1 \cos(m(\varphi_1 - \psi_m)) \cos(n(\varphi_1 - \varphi_2)) \\ & = 2 \sum_m^{\infty} v_m \pi \int_0^{2\pi} d\varphi_2 \cos(m(\varphi_2 - \psi_m)) \delta_{mn} = 2 v_n \pi \int_0^{2\pi} d\varphi_2 \cos(\varphi_2 - \psi_n) = 0 \end{aligned} \tag{66}$$

```
Double_t bootstr(Double_t data[10000], Int_t N, Double_t mean0){
Double_t sum=0;
Int_t p=0;
Int_t n=10000/N; //number of entries per subsample
for (Int_t j=0;j<N;j++){
        Double_t sumj=0, meanv2j=0;
        for (Int_t s=0;s<n;s++){
                p=j*n+s;
                sumj=sumj+data[p];
        }
        meanv2j=sumj/n;
        sum=sum+(meanv2j-mean0)*(meanv2j-mean0);
}
Double_t var=sum/(N-1);
Double_t uncertainty=sqrt(var/N);
return uncertainty;
}
```

Figure 13: Function to get the bootstrap uncertainty. Data was stored on array data. There were N subsamples and the variable mean0 is the mean of the original sample.

The third term is also zero, and for the last term we also use the orthogonality relations:

$$\begin{aligned}
&4\sum_{p=1}^{\infty}\sum_{m=1}^{\infty} v_p v_m \int_0^{2\pi} d\varphi_2 \cos(p(\varphi_2-\psi_p)) \int_0^{2\pi} d\varphi_1 \cos(m(\varphi_1-\psi_m))\cos(n(\varphi_1-\varphi_2)) \\
&= 4\sum_{p=1}^{\infty} v_p \int_0^{2\pi} d\varphi_2 \cos(p(\varphi_2-\psi_p)) \sum_{m=1}^{\infty} v_m \pi \cos(n\varphi_2 - m\varphi_m)\delta_{mn} \\
&= 4\pi v_n \sum_{p=1}^{\infty} v_p \int_0^{2\pi} d\varphi_2 \cos(p(\varphi_2-\psi_p)\cos(n(\varphi_2-\psi_n))) = 4\pi^2 \sum_{p=1}^{\infty} \cos(\psi_p-\psi_n)\delta_{mn} \\
&= 4\pi^2 v_n^2.
\end{aligned} \tag{67}$$

This give us the expected result $v_n^2$.

### A.3 Two particle correlation

In terms of the Q vector this can be expressed as:

$$\begin{aligned}
\sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{in(\varphi_i-\varphi_j)} &= \sum_{i,j=1}^{M} e^{in(\varphi_i-\varphi_j)} - \sum_{i=j}^{M} e^{in(\varphi_i-\varphi_j)} = \sum_{i,j=1}^{M} e^{in(\varphi_i-\varphi_j)} - M \\
&= \left(\sum_{i=1}^{M} e^{in\varphi_i}\right)\left(\sum_{j=1}^{M} e^{in\varphi_j}\right) - M = Q_n Q_n^* - M \\
&= |Q_n|^2 - M
\end{aligned} \tag{68}$$

## Appendix B Boostrap technique

The frequent problem after measuring an experiment, is finding the uncertainty. This becomes even harder for compound observables. For instance, if the directly measured observables are $x$ and $y$, but we are looking for the quantity $z \equiv x^2 + e^{-y} + \cos xy$, it is very difficult to perform the standard error propagation and obtain the statistical uncertainty for desired observable $z$, from the statistical uncertainties of $\sigma_x$ , $\sigma_y$ and their covariance $\text{Cov}(x, y)$, which can be obtained directly, because only $x$ and $y$ are measured like that. This sort of problem in general can be resolved with the bootstrap technique. In ALICE, physicists have endorsed the simplified version of bootstrap technique to find statistical uncertainties. The following steps must be done:

1. Divide the initial sample in 10 subsamples, so that each subsample contains roughly the same statistics;

2. For each subsample, perform an independent measurement and get for the observable of interest the results $\mu_0, \mu_1, ..., \mu_9$ , respectively;

3. Compute an unbiased variance from these 10 subsamples; i.e.

$$\text{Var} = \frac{1}{10-1}\sum_{i}^{10}(\mu_i - \mu)^2, \tag{69}$$

   where the result for the starting large sample of the observable of interest is denoted by $\mu$ in the above formula.

4. The final result for observable of interest and its statistical uncertainty is:

$$\mu \pm \sqrt{\frac{\text{Var}}{10}}. \tag{70}$$

A function to get the uncertainty using this technique is shown in Fig. (13).

# Appendix C Code to calculate the correlations

The main idea is shown in Fig. (14). In each event (in total nEvents), we uniformly extract the value of symmetry plane between the range 0 and $2\pi$. Then, in line 569, we sample the angles (in total nAngles) from a distribution fvarphi, also Eq. (14). Between lines 570 and 577, we calculate the detector's bias. Finally after line 585 we start to find the correlations and save their values in TProfile histograms. These will give us the averages and uncertainties.

```
//==========================================================================================
//NONUNIFORM + Weights Filling
for (int t=0;t<nEvents;t++){ //events
        fvarphi->SetParameter(0,gRandom->Uniform(TMath::TwoPi())); // sample randomly reaction plane

//Filling angles and weights
        for(int r=0;r<nParticles;r++){ //particles
                 // Azimuthal angles:
                angle=fvarphi->GetRandom();
                while (angle > TMath::TwoPi()/6 && angle < TMath::TwoPi()/3){
                        if (gRandom -> Uniform(0,1)<0.5){
                                break;
                        }
                        else {
                                angle=fvarphi->GetRandom();
                        }
                }
                angles[r]=angle;
                //Find the corresponding weights
                weights[r] = w->GetBinContent(w->FindBin(angles[r]));
        }

        //To do for each event:

        bUseWeights = kTRUE;

         // c) Calculate Q-vectors for available angles and weights;
        CalculateQvectors();

         // d) Calculate n-particle correlations from Q-vectors (using standalone formulas), and with th
         //  2-p correlations:
         TComplex two = Two(-2,2)/Two(0,0).Re();
         Double_t wTwo = Two(0,0).Re(); // Weight is 'number of combinations' by default
         nonuniformw[0][0]->Fill(0.5,two.Re(),wTwo); // <<cos(h1*phi1+h2*phi2)>>
```

Figure 14: Code to sample angles and calculate correlators from them.

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