$$f_{\bar r}(\bar p) = \left(\frac{1}{2\pi m k_B T}\right)^{3/2} e^{-\frac{p^2}{2mk_BT}}\,\Theta\left[\epsilon_{max} - u(\bar r) - p^2/2m\right] A[T, \epsilon_{max} - u(\bar r)]$$

so $\int d^3p\, f_{\bar r}(\bar p) = 1$

Two-particle distribution $= f_{\bar r}(\bar p_1) f_{\bar r}(\bar p_2)$

$$f'_{\bar r}(\bar p_1, \bar p_2) = \left(\frac{1}{2\pi m k_B T}\right)^{3} e^{-\frac{p_1^2}{2mk_BT}} e^{-\frac{p_2^2}{2mk_BT}}\,\Theta\left[\ldots \frac{p_1^2}{2m}\right]\Theta\left[\ldots \frac{p_2^2}{2m}\right] A^2$$

relative momentum

$\mu = \frac{m_1 m_2}{m_1+m_2} = \frac{m}{2}$ $\quad \bar P_c = \bar p_1 + \bar p_2 \quad \bar p = \frac{\bar p_1 - \bar p_2}{2}$

$M = m_1 + m_2 = 2m$

Distribution of relative momentum

$$\tilde f_{\bar r}(\bar p) = \int d^3\bar P_c\, f'_{\bar r}(\bar p_1, \bar p_2)$$

$$= \left(\frac{1}{2\pi \mu k_B T}\right)^{3/2}\left(\frac{1}{2\pi M k_B T}\right)^{3/2}\int d^3\bar P_c\, e^{-\frac{P_c^2}{2Mk_BT}} e^{-\frac{p^2}{2\mu k_BT}}$$

$$\times\,\Theta\left[\epsilon_{max} - u(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} - \frac{\bar P_c\cdot\bar p}{2m}\right]\Theta\left[\epsilon_{max} - u(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} + \frac{\bar P_c\cdot\bar p}{2m}\right] A^2$$

$$= A^2\left(\frac{1}{2\pi\mu k_B T}\right)^{3/2} e^{-\frac{p^2}{2\mu k_BT}}\left(\frac{1}{2\pi M k_B T}\right)^{3/2} 2\pi\int_{-1}^{1}dx\int_0^\infty dP_c\, P_c^2\, e^{-\frac{P_c^2}{2Mk_BT}}\,\Theta\left[\epsilon_{max}(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} - \frac{P_c p x}{2m}\right]$$

$$\times\,\Theta\left[\epsilon_{max}(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} + \frac{P_c p x}{2m}\right]$$

$$= \left(\frac{1}{2\pi\mu k_B T}\right)^{3/2} e^{-\frac{p^2}{2\mu k_BT}}\left(\frac{1}{2\pi M k_B T}\right)^{3/2} 4\pi\underbrace{\int_0^1 dx\int_0^\infty dP_c\, P_c^2\, e^{-\frac{P_c^2}{2Mk_BT}}\,\Theta\left[\epsilon_{max}(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} - \frac{P_c p x}{2m}\right]}_{=\frac{\sqrt{\pi}}{4}(2Mk_BT)^{3/2}} A^2$$

if we let $\epsilon_{max}(\bar r) \to \infty$ $\quad$; $A[T,\infty] = 1$,

$$\Rightarrow \tilde f_{\bar r}(\bar p) = e^{-\frac{p^2}{2\mu k_BT}}\Big/(2\pi\mu k_B T)^{3/2} \quad \Rightarrow G\left[T, \infty, \frac{p^2}{2m}\right] = 1$$

in general

$$\tilde f_{\bar r}(\bar p) = \frac{1}{(2\pi\mu k_B T)^{3/2}} e^{-\frac{p^2}{2\mu k_BT}}\, G\left[T, \epsilon_{max}(\bar r), \frac{p^2}{2m}\right]$$

$\epsilon_{max} - u(\bar r)$

$$G\left[T, \epsilon_{max}(\bar r), \frac{p^2}{2m}\right] = \frac{4\pi}{(2\pi M k_B T)^{3/2}}\int_0^1 dx\int_0^\infty dP_c\, P_c^2\, e^{-\frac{P_c^2}{2Mk_BT}}\,\Theta\left[\epsilon_{max}(\bar r) - \frac{P_c^2}{8m} - \frac{p^2}{2m} - \frac{P_c p x}{2m}\right]A^2$$

$$\tilde{\epsilon} = \frac{p^2}{2\mu k_B T} = \frac{p^2}{m k_B T} \qquad p = \sqrt{2\mu k_B T \tilde{\epsilon}} = \sqrt{m k_B T \tilde{\epsilon}} \qquad \frac{dp}{d\tilde{\epsilon}} = \frac{1}{2}(2\mu k_B T)^{1/2}\tilde{\epsilon}^{-1/2}$$

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$$dp\,p^2 = \frac{1}{2}(2\mu k_B T)^{3/2}(\tilde{\epsilon})^{1/2}\,d\tilde{\epsilon} \qquad \int_0^{\infty} e^{-\tilde{\epsilon}}\,d\tilde{\epsilon} = 1$$

$$\tilde{f}_{\bar{r}}(\bar{p}) = \frac{1}{(2\pi\mu k_B T)^{3/2}} e^{-\tilde{\epsilon}} \left(\frac{1}{2\pi M k_B T}\right)^{3/2} 4\pi \int_0^1 dx \int_0^{\infty} dP_c\, P_c^2\, e^{-\frac{P_c^2}{2Mk_BT}}$$

$$\times\, \Theta\left[\epsilon_m - U(\bar{r}) - \frac{P_c^2}{8m} - \frac{\tilde{\epsilon} k_B T}{2} - \frac{P_c X \sqrt{m k_B T \tilde{\epsilon}}}{2m}\right] A^2$$

$$\cancel{\frac{\tilde{P}_c^2}{2} = \frac{P_c^2}{2Mk_BT}} \qquad \tilde{E} = \frac{P_c^2}{2Mk_BT} = \frac{P_c^2}{4mk_BT} \qquad P_c = \sqrt{4mk_BT\tilde{E}}$$

$$\frac{dP_c}{d\tilde{E}} = \frac{1}{2}(4mk_BT)^{1/2}(\tilde{E})^{-1/2}$$

$$dP_c\,P_c^2 = \frac{1}{2}(4mk_BT)^{3/2}(\tilde{E})^{1/2}\,d\tilde{E}$$

$$\tilde{f}_{\bar{r}}(\bar{p}) = \frac{e^{-\tilde{\epsilon}}}{(2\pi\mu k_B T)^{3/2}} \frac{4\pi}{(2\pi M k_B T)^{3/2}} \int_0^1 dx \int_0^{\infty} \frac{(4mk_BT)^{3/2}(\tilde{E})^{1/2}}{2}\, d\tilde{E}\, e^{-\tilde{E}}$$

$$\times\, \Theta\left[\epsilon_m - U(\bar{r}) - \frac{1}{2}\tilde{E}k_BT - \frac{\tilde{\epsilon}k_BT}{2} - \frac{X[4mk_BT\tilde{E}]^{1/2}[mk_BT\tilde{\epsilon}]^{1/2}}{2m}\right] A^2$$

$$= \frac{e^{-\tilde{\epsilon}}}{(2\pi\mu k_B T)^{3/2}} \frac{(4mk_BT)^{3/2}\cdot 4\pi}{2(2\pi\cdot 2mk_BT)^{3/2}} \int_0^1 dx \int_0^{\infty} d\tilde{E}\, e^{-\tilde{E}}\sqrt{\tilde{E}} \cdot \frac{2}{\sqrt{\pi}}\, \frac{\sqrt{\pi}}{2}$$

$$\frac{d\sqrt{ax}}{dx} = \sqrt{a}\frac{d}{dx}\sqrt{x} = \frac{1}{2}\frac{\sqrt{a}}{\sqrt{x}}$$

$$\times\, \Theta\Bigg[\underbrace{\frac{\epsilon_{max} - U(\bar{r})}{k_BT}}_{\equiv \eta(\bar{r})} - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - X\sqrt{\tilde{E}\tilde{\epsilon}}\Bigg] A^2$$

$$\boxed{\tilde{f}_{\bar{r}}(\bar{p}) = \frac{e^{-\tilde{\epsilon}}}{(2\pi\mu k_BT)^{3/2}} \int_0^1 dx\, \frac{2}{\sqrt{\pi}} \int_0^{\infty} d\tilde{E}\, e^{-\tilde{E}}\sqrt{\tilde{E}}\, \Theta\left[\eta(\bar{r}) - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - X\sqrt{\tilde{E}\tilde{\epsilon}}\right] A^2}$$

3

$$\int dp\,p^2 \int d\Omega_p \tilde{f}_{\bar{r}}(\bar{p}) = \int \hat{f}_{\bar{r}}(\hat{\epsilon})\, d\hat{\epsilon} = 1$$

$$\Rightarrow dp\,p^2\, 4\pi\, \tilde{f}_{\bar{r}}(\bar{p}) = \hat{f}_{\bar{r}}(\tilde{\epsilon})\, d\hat{\epsilon}$$

$$4\pi \frac{(2\mu k_B T)^{3/2}}{2} \sqrt{\tilde{\epsilon}}\, \tilde{f}_{\bar{r}}[\bar{p}(\epsilon)] = \hat{f}_r(\epsilon)$$

$$\tilde{\epsilon} = \frac{p^2}{2\mu k_B T} \quad p = \sqrt{2\mu k_B T \tilde{\epsilon}}$$

$$\frac{dp}{d\tilde{\epsilon}} = \frac{\sqrt{2\mu k_B T}}{2\sqrt{\tilde{\epsilon}}}$$

$$dp\,p^2 = \frac{(2\mu k_B T)^{3/2}}{2} \sqrt{\tilde{\epsilon}}\, d\tilde{\epsilon}$$

$$\hat{f}_r(\tilde{\epsilon}) = \frac{4\pi}{2} \frac{(2\mu k_B T)^{3/2}}{(2\pi\mu k_B T)^{3/2}} \underbrace{\sqrt{\tilde{\epsilon}}\, e^{-\tilde{\epsilon}} \frac{2}{\sqrt{\pi}}}\, \frac{\sqrt{\pi}}{2} A^2 \int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta(\bar{r}) - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]$$

$$\boxed{\hat{f}_r(\tilde{\epsilon}) = \frac{2\sqrt{\tilde{\epsilon}}}{\sqrt{\pi}} e^{-\tilde{\epsilon}} A^2 \int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta(\bar{r}) - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]}$$

$$\int_0^{2\eta} d\tilde{\epsilon}\, \hat{f}_r(\tilde{\epsilon}) = 1 \quad \text{set } A^2$$

$$f_r(\tilde{\epsilon}) = \frac{\frac{2}{\sqrt{\pi}} \sqrt{\tilde{\epsilon}}\, e^{-\tilde{\epsilon}} \int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta(\bar{r}) - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]}{\int_0^{2\eta} d\tilde{\epsilon} \frac{2}{\sqrt{\pi}} \sqrt{\tilde{\epsilon}}\, e^{-\tilde{\epsilon}} \int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]}$$

$$\boxed{\begin{aligned} \hat{f}_r(\tilde{\epsilon}) &\equiv \frac{2}{\sqrt{\pi}} \sqrt{\tilde{\epsilon}}\, e^{-\tilde{\epsilon}}\, \tilde{G}[\eta, \tilde{\epsilon}] \\ \tilde{G}[\eta, \tilde{\epsilon}] &= \frac{\int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]}{\int_0^{2\eta} d\tilde{\epsilon} \frac{2}{\sqrt{\pi}} \sqrt{\tilde{\epsilon}}\, e^{-\tilde{\epsilon}} \int_0^1 dx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tilde{E}\, e^{-\tilde{E}} \sqrt{\tilde{E}}\, \Theta\left[\eta - \frac{\tilde{E}}{2} - \frac{\tilde{\epsilon}}{2} - x\sqrt{\tilde{E}\tilde{\epsilon}}\right]} \end{aligned}}$$

$$\boxed{\int_0^{2\eta} d\tilde{\epsilon}\, \hat{f}_r(\tilde{\epsilon}) = 1}$$

$$\hat{f}_1(\tilde{\varepsilon}) = \frac{2}{\sqrt{\pi}}\sqrt{\tilde{\varepsilon}}\, e^{-\tilde{\varepsilon}}\, \tilde{G}[\eta, \tilde{\varepsilon}]$$

$$\boxed{\frac{\hbar^2 k^2}{2\mu} = \varepsilon \qquad \tilde{\varepsilon} = \frac{p^2}{2\mu k_B T} = \frac{\varepsilon}{k_B T}}$$

$$K = \frac{1}{h Q_T}\int_0^\infty d\varepsilon\, e^{-\varepsilon/k_BT} |S|^2 = \left(\frac{h^2}{2\pi k_B T_\mu}\right)^{3/2} \frac{1}{h}\int_0^\infty d\varepsilon\, e^{-\varepsilon/k_BT}$$

$$= \frac{h^2}{(2\pi k_B T_\mu)^{3/2}} \int_0^\infty d\varepsilon\, e^{-\varepsilon/k_BT} \frac{\Delta_2'^2 \gamma_1 \gamma_s(\varepsilon)}{\underbrace{\left[\Delta_1'\Delta_2' - \frac{\Omega_{12}^2}{2}\right]^2}_{①} + \underbrace{\left[\frac{\gamma_1 + \gamma_s(\varepsilon)}{2}\right]^2 \Delta_2'^2}_{②}}$$

Using $\gamma_s(\varepsilon) = 2 l_{opt} \gamma_1 \left[\frac{(2\mu\varepsilon)^{1/2}}{\hbar}\right]$

$$= \frac{\cancel{h^2}}{(2\pi k_B T_\mu)^{3/2}} \int_0^\infty d\varepsilon\, e^{-\varepsilon/k_BT} \frac{\Delta_2'^2\, 2 l_{opt}\gamma_1^2}{[①]^2 + [②]^2} (2\mu\varepsilon)^{1/2} \frac{2\pi}{\cancel{h}}$$

$$= \frac{2\pi h}{(2\pi k_B T_\mu)^{3/2}} \int_0^\infty d\varepsilon\, e^{-\varepsilon/k_BT} (2\mu\varepsilon)^{1/2} \frac{\Delta_2'^2\, 2 l_{opt}\gamma_1^2}{[①]^2 + [②]^2}.$$

Use $\varepsilon = k_BT\tilde{\varepsilon} \Rightarrow \frac{d\varepsilon}{d\tilde{\varepsilon}} = k_BT$

$$= \frac{2\pi (2\mu)^{1/2} h}{(2\pi k_B T_\mu)^{3/2}} \int_0^\infty d\tilde{\varepsilon}\, k_BT\, e^{-\tilde{\varepsilon}} \sqrt{k_BT\tilde{\varepsilon}}\, \frac{\Delta_2'^2\, 2 l_{opt}\gamma_1^2}{[①]^2 + [②]^2} \left\{\frac{\mu\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}\mu}\right\}$$

$$= \frac{(2\pi k_B T_\mu)^{3/2}\, h}{(2\pi k_B T_\mu)^{3/2}} \int_0^\infty d\tilde{\varepsilon}\, e^{-\tilde{\varepsilon}} \sqrt{\tilde{\varepsilon}}\, \frac{\Delta_2'^2 l_{opt}\gamma_1^2}{[①]^2 + [②]^2} \frac{2}{\sqrt{\pi}}\frac{1}{\mu}$$

$$= \frac{h}{\mu}\int_0^\infty \frac{2}{\sqrt{\pi}} d\tilde{\varepsilon}\, e^{-\tilde{\varepsilon}}\sqrt{\tilde{\varepsilon}}\, l_{opt} \frac{\Delta_2'^2\gamma_1^2}{[①]^2 + [②]^2}$$

$$= \frac{l_{opt} h}{\mu}\int_0^\infty \frac{2}{\sqrt{\pi}} d\tilde{\varepsilon}\, \hat{f}(\tilde{\varepsilon}, \eta \to \infty) \frac{\Delta_2'^2\gamma_1^2}{[①]^2 + [②]^2}$$

$$\boxed{= \frac{l_{opt} h}{\mu}\frac{2}{\sqrt{\pi}}\int_0^{2\eta} d\tilde{\varepsilon}\, \sqrt{\tilde{\varepsilon}}\, e^{-\tilde{\varepsilon}}\, \tilde{G}[\eta, \tilde{\varepsilon}] \frac{\Delta_2'^2\gamma_1^2}{[①]^2 + [②]^2}}$$

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