

### Problem 1.

Derive Equation 6.3. [Hint: What fraction of the original sample decays between  $t$  and  $t + dt$ ? What, is the (initial) probability,  $p(t)dt$ , of any given particle decaying between  $t$  and  $t + dt$ ? The average lifetime is  $\int_0^\infty tp(t) dt$ .]

Solution

$$\tau = \frac{1}{\Gamma}, \text{ Equation (6.3)}$$

Q. What fraction of the original sample decays between  $t$  and  $t + dt$ ?

$$\text{A. } dN = -\Gamma N dt, \text{ Equation (6.1)}$$

Q. What is the initial probability,  $p(t)dt$ ?

$$\text{A. } N(t) = N(0)e^{-\Gamma t}, \text{ Equation (6.2)}$$

Using probability density function, (from wiki),

$$1 = C \int_0^\infty p(t) dt = C \int_0^\infty N(0)e^{-\Gamma t} dt, \text{ where } C \text{ is normalizing coefficient.}$$

$$= C \cdot \frac{N_0}{\Gamma}$$

$$C = \frac{\Gamma}{N_0}$$

Q. The average lifetime?

$$\text{A. } \tau = \langle t \rangle = \int_0^\infty tp(t) dt = \int_0^\infty t \frac{\Gamma}{N_0} N(0)e^{-\Gamma t} dt = \Gamma \int_0^\infty te^{-\Gamma t} dt$$

$$\Gamma \int_0^\infty te^{-\Gamma t} dt = (\text{integration by parts}) = \frac{1}{\Gamma} = \tau$$

reference [wiki - Exponential decay, Probability Density function, integration by parts]

### Problem 2.

Nuclear physicist traditionally work with ‘half-life’ ( $t_{1/2}$ ) instead of mean life ( $\tau$ );  $t_{1/2}$  is the time it takes for half the members of a large sample to decay. For exponential decay(Equation 6.2), derive the formula for  $t_{1/2}$  (as a multiple of  $\tau$ ).

Solution

$$dN = -\Gamma N dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

Where  $\Gamma$  is the decay rate,  $-\Gamma N dt$  is the Number of decayed particles.

The mean lifetime is the reciprocal of the decay rate.

$$\tau = \frac{1}{\Gamma}$$

$t_{1/2}$  is the time when  $N(t) = \frac{N(0)}{2}$

So,  $N(t_{1/2}) = N(0)e^{-\Gamma t_{1/2}} = N(0)/2$

Then,  $e^{-\Gamma t_{1/2}} = \frac{1}{2}$

Taking  $\ln$  for both sides,  $-\Gamma t_{1/2} = \ln \frac{1}{2}$

$$t_{1/2} = \frac{\log 2}{\Gamma} = \tau \ln 2$$