Problem 1.

Show that $\partial\phi/\partial x^{\mu}$ is a covariant four-vector (ϕ is a scalar function of x, y, z, and t).
Hint: First determine (from Equation 3.8) how covariant four-vectors transform; then use $\partial\phi$ / $\partial x^{\mu'}$ = ($\partial\phi$ / ∂x^{ν})(∂x^{ν} / $\partial x^{\mu'}$) transforms.

Solution

The four-vectors in x, y, z, t :
$x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Equation (3.7)

Equation (3.8)

$x^{0'} = \gamma(x^0$ - $\beta x^1)$
$x^{1'} = \gamma(x^1$ - $\beta x^0)$
$x^{2'} = x^2$
$x^{3'} = x^3$
$x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}$ $(\nu = 0, 1, 2, 3) = \Lambda^{\mu}_{\nu} x^{\nu}$

This is a contravariant tensor.

A contravariant tensor: axes changes $\rightarrow$ the components **inversly** change. And the vector does not change.

This is why it is called contravariant tensor.

A covariant : axes changes $\rightarrow$ the components also changes in the same way : vector $\rightarrow$ scalar.

A covariant vectors' components change the same as coordinates' change.

From Equation 3.8,

$x^{0'} = \gamma(x^0$ - $\beta x^1) \rightarrow x'_0 = \gamma(x^0$ + $\beta x^1)$

(The details are in Equation 3.1 and 3.3 , 3.3 is the inverse transform of 3.1. using the same inverse transform here.)

$x^{1'} = \gamma(x^1$ - $\beta x^0) \rightarrow x'_1 = \gamma(x^1$ + $\beta x^0)$

$x'_2 = x^2$

$x'_3 = x^3$

These are the covariant transform.

Q. What is the scalar function ϕ?

The quantity I (in Equation 3.13) and $r^2 = x^2 + y^2 + z^2$ can be scalar function. (not sure)

So, using, I = $(x^{0'})^2$ -$(x^{1'})^2$-$(x^{2'})^2$-$(x^{3'})^2$,

Since $I = x_\mu x^\mu$,

$\partial\phi$ / $\partial x^{\mu'}$ = x_μ, covariant tensor.

https://en.wikipedia.org/wiki/covariance_and_contravariance_of_vectors

Problem 2

7.2 Show that Equation 7.17 satisfies Equation 7.15

Solution

Equation 7.15 : $\{\gamma^\mu,\ \gamma^\nu\} = 2g^{\mu\nu}$

Equation 7.17 : $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

But, here, 1 denotes the 2 × 2 unit matrix, and 0 is the 2 × 2 matrix of zeros.

So, $\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & 0 & \sigma^i & 0 \\ 0 & 0 & 0 & \sigma^i \\ -\sigma^i & 0 & 0 & 0 \\ 0 & -\sigma^i & 0 & 0 \end{pmatrix}$

Anticommutator $\{A, B\}$ = AB + BA

So,

$\{\gamma^0, \gamma^i\} = \gamma^0\gamma^i + \gamma^i\gamma 0 =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sigma^i & 0 \\ 0 & 0 & 0 & \sigma^i \\ -\sigma^i & 0 & 0 & 0 \\ 0 & -\sigma^i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sigma^i & 0 \\ 0 & 0 & 0 & \sigma^i \\ -\sigma^i & 0 & 0 & 0 \\ 0 & -\sigma^i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \sigma^i & 0 \\ 0 & 0 & 0 & \sigma^i \\ \sigma^i & 0 & 0 & 0 \\ 0 & \sigma^i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\sigma^i & 0 \\ 0 & 0 & 0 & -\sigma^i \\ -\sigma^i & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, is this $2g^{0i}$?

Equation (3.14) $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

This is the minkowski metric, $g_{\mu\nu}$.
$g^{\mu\nu}$ is simply g^{-1}, and $g = g^{-1}$ (in a flat space time) (from wiki-minkowski metric)
However except g^{ii} or g^{00} every g is zero.