A mixed formulation for the Stokes problem with varying density

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Abstract

We propose and analyse a mixed finite element method for the nonstandard pseudostress–velocity formulation of the Stokes problem with varying density ρ in $\mathbb{R}^d$, $d \in \{2,3\}$. Since the resulting variational formulation does not have the standard dual–mixed structure, we reformulate the continuous problem as an equivalent fixed–point problem. Then, we apply the classical Babuška–Brezzi theory to prove that the associated mapping $\mathbb{T}$ is well defined, and assuming that $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ is sufficiently small, we show that $\mathbb{T}$ is a contraction mapping, which implies that the variational formulation is well posed. Under the same hypothesis on ρ we prove stability of the continuous problem. Next, adapting the arguments of the continuous analysis to the discrete case, we establish suitable hypotheses on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. In addition, we derive a reliable and efficient residual–based a posteriori error estimator for the problem. Finally, several numerical results illustrating the performance of the method, confirming the theoretical rate of convergence and the theoretical properties of the estimator, and showing the behaviour of the associated adaptive algorithms, are provided.

1. Model problem

Let Ω be a domain in R^d, $d = 2, 3$, with Lipschitz–continuous boundary $\Gamma = \partial\Omega$, governed by the Stokes equations with varying density:

$$\boldsymbol{\sigma} \;=\; \nu(\rho\nabla\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega, \qquad -\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega,$$
$$\operatorname{div}(\rho\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega, \qquad \mathbf{u} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma, \qquad (p,1)_\Omega = 0.$$

Here, the unknowns are the pseudostress tensor $\boldsymbol{\sigma}$, the fluid velocity $\mathbf{u}$ and the pressure p. The given data are the extended force per unit mass $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the viscosity $\nu > 0$, which is assumed to be constant, and the density function $\rho \in H^1(\Omega) \cap W^{1,\infty}(\Omega)$, satisfying

$$\frac{\nabla\rho}{\rho} \in \mathbf{L}^\infty(\Omega) \quad \text{and} \quad 0 < \rho_0 < \rho(x) < \rho_1, \quad \text{a.e.} \quad \text{in} \quad \Omega,$$

where ρ_0 and ρ_1 are positive constants.

2. Continuous formulation

- Elimination of the pressure: $\quad p \;:=\; -\dfrac{1}{d}(\nu\mathbf{u}\cdot\nabla\rho + \operatorname{tr}\boldsymbol{\sigma}) \quad \text{in} \quad \Omega.$
- Introduction of new tensor: $\quad \boldsymbol{\sigma}_0 \;:=\; \boldsymbol{\sigma} + \dfrac{\nu}{d|\Omega|}(\mathbf{u}\cdot\nabla\rho, 1)_\Omega\mathbb{I} \quad \text{in} \quad \Omega.$
- Defining the spaces: $\quad \mathbb{H} := \mathbb{H}(\mathbf{div};\Omega), \quad \mathbb{H}_0 := \mathbb{H}_0(\mathbf{div};\Omega) \quad \text{and} \quad \mathbf{Q} := \mathbf{L}^2(\Omega).$

Then, we obtain the variational formulation: Find $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_0,\boldsymbol{\tau}) \;+\; b(\boldsymbol{\tau},\mathbf{u}) \;-\; c(\boldsymbol{\tau},\mathbf{u}) \;&=\; 0 && \forall\,\boldsymbol{\tau} \in \mathbb{H}_0, \\ b(\boldsymbol{\sigma}_0,\mathbf{v}) \;&=\; -(\mathbf{f},\mathbf{v})_\Omega && \forall\,\mathbf{v} \in \mathbf{Q}, \end{aligned} \tag{1}$$

where the bilinear forms $a(\cdot,\cdot) : \mathbb{H}\times\mathbb{H} \to \mathbf{R}$, $b(\cdot,\cdot) : \mathbb{H}\times\mathbf{Q} \to \mathbf{R}$ and $c(\cdot,\cdot) : \mathbb{H}\times\mathbf{Q} \to \mathbf{R}$, are defined by

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) := \nu^{-1}\left(\frac{1}{\rho}\boldsymbol{\sigma}^{\mathrm{D}},\boldsymbol{\tau}^{\mathrm{D}}\right)_\Omega, \quad b(\boldsymbol{\tau},\mathbf{v}) := (\mathbf{div}\boldsymbol{\tau},\mathbf{v})_\Omega, \quad c(\boldsymbol{\tau},\mathbf{v}) := \frac{1}{d}\left(\mathbf{v}\cdot\frac{\nabla\rho}{\rho}, \operatorname{tr}\boldsymbol{\tau}\right)_\Omega.$$

Theorem 2.1 *Assume that $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ is small enough. Then, there exists a unique $(\boldsymbol{\sigma}_0,\mathbf{u}) \in \mathbb{H}_0\times\mathbf{Q}$ solution of (1). Moreover, there exist constant $C_{\boldsymbol{\sigma}} > 0$ and $C_{\mathbf{u}} > 0$, such that*

$$\|\boldsymbol{\sigma}_0\|_{\mathbf{div};\Omega} \leqslant C_{\boldsymbol{\sigma}}\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \|\mathbf{u}\|_{0,\Omega} \leqslant C_{\mathbf{u}}\|\mathbf{f}\|_{0,\Omega}.$$

3. Discrete formulation

For each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k:

$$\mathbf{RT}_k(T) := [P_k(T)]^d \oplus P_k(T)\mathbf{x},$$

where $\mathbf{x} := (x_1,\cdots,x_d)^t$ is a generic vector of $\mathbb{R}^d$. Then, we choose:

$$\begin{aligned} \mathbf{H}_h &:= \{\tau \in \mathbf{H}(\operatorname{div};\Omega) : \quad \tau|_T \in \mathbf{RT}_k(T), \quad \forall T \in \mathcal{T}_h\}, \\ Q_h &:= \{v \in L^2(\Omega) : \quad v|_T \in P_k(T), \quad \forall T \in \mathcal{T}_h\}, \end{aligned}$$

and define

$$\mathbb{H}_h := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \mathbf{c}^t\boldsymbol{\tau} \in \mathbf{H}_h\ \forall\mathbf{c} \in \mathbf{R}^d\}, \quad \mathbb{H}_{h,0} := \mathbb{H}_h \cap \mathbb{H}_0(\mathbf{div};\Omega) \quad \text{and} \quad \mathbf{Q}_h := [Q_h]^d$$

Theorem 3.2 *Assume that $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ is small enough. Then, there exists a unique $(\boldsymbol{\sigma}_{h,0},\mathbf{u}_h) \in \mathbb{H}_{h,0}\times\mathbf{Q}_h$, such that*

$$\begin{aligned} a(\boldsymbol{\sigma}_{h,0},\boldsymbol{\tau}_h) \;+\; b(\boldsymbol{\tau}_h,\mathbf{u}_h) \;-\; c(\boldsymbol{\tau}_h,\mathbf{u}_h) \;&=\; 0 && \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}, \\ b(\boldsymbol{\sigma}_{h,0},\mathbf{v}_h) \;&=\; -(\mathbf{f},\mathbf{v}_h)_\Omega && \forall\,\mathbf{v}_h \in \mathbf{Q}_h. \end{aligned}$$

Moreover, let $(\boldsymbol{\sigma}_0,\mathbf{u}) \in \mathbb{H}_0\times\mathbf{Q}$ be the unique solution of the continuous problem (1) and assume that $\boldsymbol{\sigma}_0 \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\boldsymbol{\sigma}_0 \in \mathbf{H}^s(\Omega)$, and $\mathbf{u} \in \mathbf{H}^s(\Omega)$, for some $s \in (0, k+1]$. Then, there exists $C > 0$, independent of h, such that

$$\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div},\Omega} + \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \leq Ch^s\left\{\|\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega}\right\}.$$

4. A residual–based a posteriori error estimator

We define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$\begin{aligned} \theta_T^2 \;:=\; & \|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_{h,0}\|_{0,T}^2 \;+\; h_T^2\left\|\mathbf{rot}\left(\frac{\nu^{-1}}{\rho}\boldsymbol{\sigma}_{h,0}^{\mathrm{D}} - \frac{1}{2}\left(\mathbf{u}_h\cdot\frac{\nabla\rho}{\rho}\right)\mathbb{I}\right)\right\|_{0,T}^2 \\ & + h_T^2\left\|\nabla\mathbf{u}_h - \left(\frac{\nu^{-1}}{\rho}\boldsymbol{\sigma}_{h,0}^{\mathrm{D}} - \frac{1}{2}\left(\mathbf{u}_h\cdot\frac{\nabla\rho}{\rho}\right)\mathbb{I}\right)\right\|_{0,T}^2 \;+\; \sum_{e\in\mathcal{E}_h(T)} h_e\left\|\left[\left(\frac{\nu^{-1}}{\rho}\boldsymbol{\sigma}_{h,0}^{\mathrm{D}} - \frac{1}{2}\left(\mathbf{u}_h\cdot\frac{\nabla\rho}{\rho}\right)\mathbb{I}\right)\mathbf{t}\right]\right\|_{0,e}^2 \end{aligned}$$

As usual the expression $\boldsymbol{\theta} := \left\{\sum_{T\in\mathcal{T}_h}\theta_T^2\right\}^{1/2}$ is employed as the global residual error estimator.

Theorem 4.3 *Let $(\boldsymbol{\sigma}_0,\mathbf{u}) \in \mathbb{H}_0\times\mathbf{Q}$ and $(\boldsymbol{\sigma}_{h,0},\mathbf{u}_h) \in \mathbb{H}_{h,0}\times\mathbf{Q}_h$ be the unique solutions of the continuous and discrete formulations, respectively. Assume that $\|\nabla\rho/\rho\|_{0,\Omega}$ is small enough. Then, there exists positive constants $C_{\mathtt{eff}}$ and $C_{\mathtt{rel}}$, independent of h, such that*

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \textit{h.o.t.} \;\leq\; \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div},\Omega} \;+\; \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta},$$

where h.o.t. stands for one or several terms of higher order.

5. Numerical results

We consider the L–shaped domain given by $\Omega := (-1,1)^2 \setminus [0,1]^2$. Then, for all $(x_1,x_2) \in \Omega$, we choose the density

$$\rho(x_1,x_2) = (x_1 - 0.01)^2 + (x_2 - 0.01)^2$$

and the datum $\mathbf{f}$ so that the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \frac{1}{\rho(x_1,x_2)}\operatorname{curl}\left(x_1^2x_2^2(x_1^2-1)^2(x_1^2-1)^2\right) \quad \text{and} \quad p(x_1,x_2) = \frac{x_1 - 0.01}{\rho(x_1,x_2)} + p_0,$$

with $p_0 = 0.4153036413$.

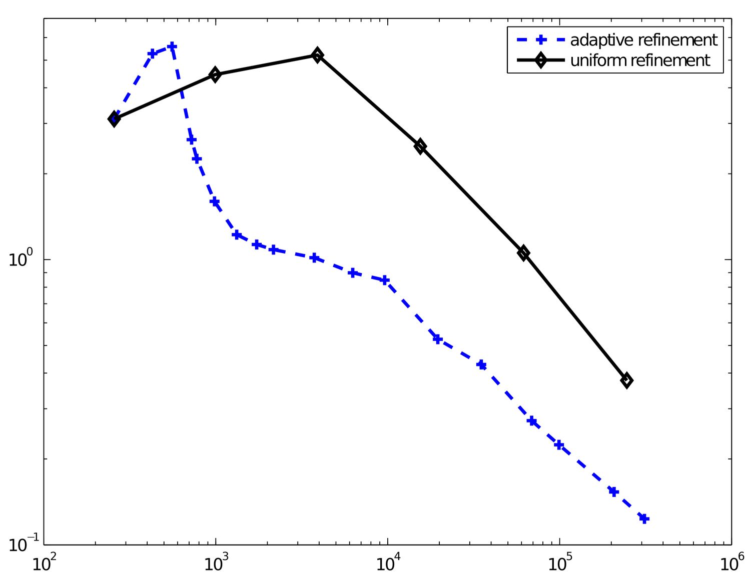

Figure 1: $e(\boldsymbol{\sigma}_0,\mathbf{u})$ *vs. degrees of freedom for uniform/adaptive schemes.*

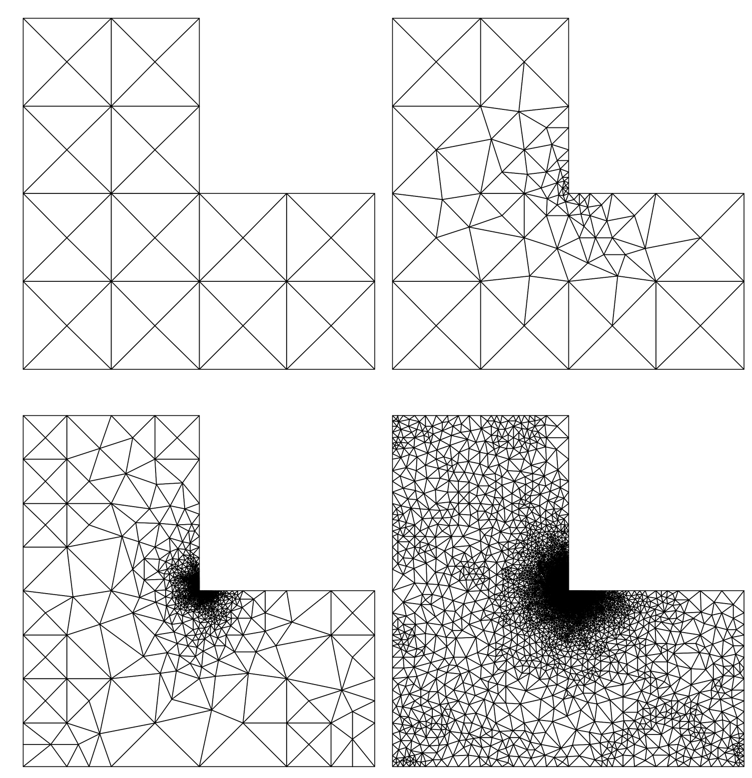

Figure 2: *Adapted meshes with 257, 777, 19569 and 311277 degrees of freedom*

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