eCHT REU 2024
Spectra, Spectral Sequences, and (Co)Fibers smashed with tmf

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Note: These are LiveTEXed notes from the eCHT REU written by Scotty Tilton based on lectures by Irina Bobkova and Jack Carlisle. Scotty may have added a little commentary to add to the exposition, and it's more than likely that Scotty added some of his own errors to the lecture. Please email me at `stilton@ucsd.edu` if (and when) you find errors. Thanks!

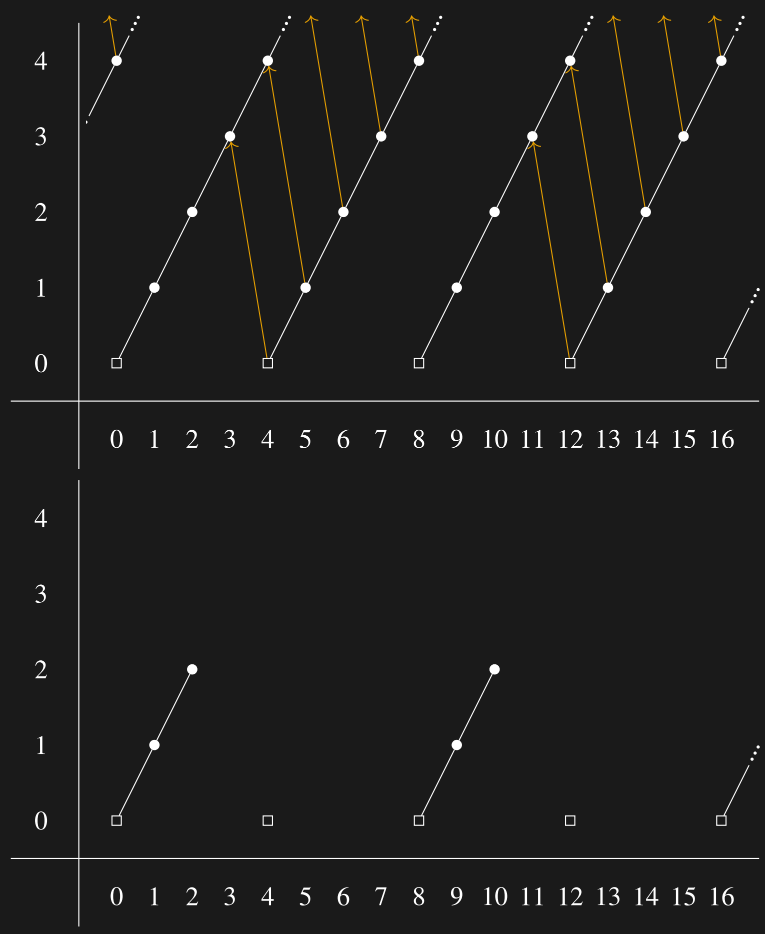
Contents

Day 4

Lecture 1

Yesterday we computed a homotopy fixed point spectral sequence. Suppose that $C_2 \curvearrowright \mathcal{E}$ where $\mathcal{E}$ is a spectrum.

$$E_2^{s,t} = H^s(C_2; \pi_t \mathcal{E}) \Rightarrow \pi_{t-s} \mathcal{E}^{hC_2}.$$



The final E_∞ page is $\pi_* E^{hC_2}$. This gives $\pi_* E = \mathbb{Z}[u^{\pm 1}]$ where $|u| = 2$. (If this were a polynomial ring where x is a variable, then u would have the same weight the x^2 "part.")

Question 1

How do we compute $\pi_*(E^{hC_2} \wedge V(0))$.

There is a spectral sequence

$$\underbrace{E_2^{s,t} = H^s(C_2, \pi_t(E \wedge V(0)))}_{\text{Question 1}} \Rightarrow \underbrace{\pi_{t-s} E^{hC_2} \wedge V(0)}_{\text{Question 2}}$$

For Question 1, we have

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to V(0) \qquad \text{(fiber sequence)}$$

$$E \wedge \mathbb{S} \xrightarrow{2} E \wedge \mathbb{S} \to E \wedge V(0) \qquad \text{(fiber sequence)}$$

This induces a long exact sequence in homotopy groups.

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{k+1}(E \wedge V(0)) \arrow[dll, bend left] \\
\pi_k(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_k(E \wedge \mathbb{S}) \arrow[r] & \pi_k(E \wedge \mathbb{S}) \arrow[dll, bend left] \\
\pi_{k-1}(E \wedge \mathbb{S}) \arrow[r, "2"] & \cdots .
\end{tikzcd}$$

After following the LES (using our spectral sequence E_∞ page on the the previous page), we get

$$\pi_t(E \wedge V(0)) = \begin{cases} \mathbb{Z}/2 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}.$$

Therefore, our $E_2^{s,t}$ group we were looking for is coming from

$$H^s(C_2, \pi_t(E \wedge V(0))).$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_3(E \wedge V(0)) \arrow[dll, bend left] \\
\pi_2(E \wedge \mathbb{S}) \arrow[r, "u \mapsto 2u"] & \pi_2(E \wedge \mathbb{S}) \arrow[r] & \pi_2(E \wedge \mathbb{S}) \arrow[dll, bend left] \\
\pi_1(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_1(E \wedge \mathbb{S}) \arrow[r] & \pi_1(E \wedge \mathbb{S}) \arrow[dll, bend left] \\
\pi_0(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_0(E \wedge \mathbb{S})
\end{tikzcd}$$

Definition 1: pullbacks

Let C be a category which contains the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd} .$$

A **pullback** of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $W \xrightarrow{p_1} X$
- A map $W \xrightarrow{p_2} Y$

such that

- The diagram $\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$ commutes [a]

- If someone hands you a commutative diagram $\begin{tikzcd} A \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{h}$ such that

$$\begin{tikzcd} A \arrow[dr, "\widetilde{h}"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\ & W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd} .$$

[b]

Let C be a category with a subcategory $\mathcal{D}$. We say $\mathcal{D}$ is **closed under pullbacks** by morphisms in C if for all arrows $X \xrightarrow{f} Z$ in $\mathcal{D}$ and for all $Y \xrightarrow{g} Z$ in C such that we can form the pullback $\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then the arrow $W \xrightarrow{p_1} Z$ is in $\mathcal{D}$.

[a] "$\circlearrowleft$" is a 'long'hand for commutes, and people usually suppress it from notation.
[b] As a shorthand, people usually write the pullback like this:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Definition 2: pushouts

Let C be a category which contains the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \\ X & \end{tikzcd}.$$

A **pushout** of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $X \xrightarrow{i_1} W$
- A map $Y \xrightarrow{i_2} W$

such that

- The diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \end{tikzcd}$ commutes

- If someone hands you a commutative diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\ell_2"] \\ X \arrow[r, "\ell_1"] & A \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{\ell}$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "\ell_2"] & \\ X \arrow[r, "i_1"] \arrow[drr, bend right, "\ell_1"] & W \arrow[dr, "\widetilde{\ell}"] & \\ & & A \end{tikzcd}.$$

[a]

[a] As a shorthand, people usually write the pushout like this:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] \arrow[ur, phantom, "\ulcorner"] & W \end{tikzcd}$$

Definition 3: (Homotopy) Fixed points

Let $G \curvearrowright X$ where X is a topological space. The **fixed points** of X are

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