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Chapter 1

Week 1

Day 1

Plan

We are going to try and compute

$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).
- $E(n, p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G-action.

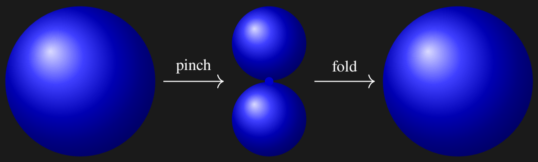
$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned} \qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be

$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.

$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$



The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{S^0, S^1, S^2, \dots\right\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\times 2} \mathbb{Z} \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 2. \end{aligned}$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i\in\mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \text{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X)$, $g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, g : S^m \to X.$$

What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$\begin{tikzcd}
 & X \wedge S^m \arrow[dr, "\mathbb{1}\times g"] & & \\
S^{n+m} = S^n \wedge S^m \arrow[ur, "f\wedge\mathbb{1}"] \arrow[rr, "f\wedge g"] \arrow[dr, "\mathbb{1}\wedge g"'] & & X \wedge X \arrow[r, "\mu"] & X \\
 & S^n \wedge X \arrow[ur, "f\wedge\mathbb{1}"'] & &
\end{tikzcd}$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

See [Bau08].

Biblio

Bibliography

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces (Tokyo 2005)*. Mathematical Sciences Publishers, February 2008.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

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