eCHT REU 2024

Spectra, Spectral Sequences, and (Co)Fibers smashed with tmf

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Day 1

Lecture 1: Lot's of definitions!

Plan

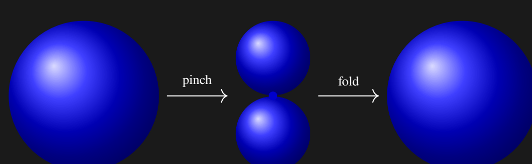
We are going to try and compute
$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).
- $E(n, p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G-action.

$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned} \qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be
$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$
- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.
$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$



The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{S^0, S^1, S^2, \ldots\right\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\times 2} \mathbb{Z} \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 2. \end{aligned}$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \overset{\times 2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \mathrm{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X)$, $g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, \quad g : S^m \to X.$$

What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$\begin{tikzcd} & X \wedge S^m \arrow[dr, "\mathbb{1} \times g"] & & \\ S^{n+m} = S^n \wedge S^m \arrow[ur, "f \wedge \mathbb{1}"] \arrow[rr, "f \wedge g"] \arrow[dr, "\mathbb{1} \wedge g"'] & & X \wedge X \arrow[r, "\mu"] & X \\ & S^n \wedge X \arrow[ur, "f \wedge \mathbb{1}"'] & & \end{tikzcd}$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A spectrum[a] X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n\in\mathbb{N}}$$

together with structure maps

$$\Sigma X_n \longrightarrow X_{n+1}.$$

[a] What we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as $X_n \to \Omega X_{n+1}$ and homeomorphisms.

Example 1

1. The sphere spectrum $\mathbb{S} = \left\{S^0, S^1, \dots\right\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.
2. Suspension spectrum $\Sigma^\infty X = \left\{X, \Sigma X, \Sigma^2 X, \dots\right\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.

$$\begin{tikzcd}
\mathsf{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left=60, "\Sigma^\infty"] & \mathsf{PreSp} \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & \mathsf{Sp} \arrow[l, shift left, "U"] \arrow[ll, bend left=60, "\Omega^\infty"]
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{PreSp} &\xrightarrow{\quad L \quad} \mathsf{Sp} \\
(X_n)_{n\in\mathbb{N}} &\longmapsto (LX_n)_{n\in\mathbb{N}}, \qquad\qquad LX_n := \operatorname{colim}\left(X_n \hookrightarrow \Omega X_{n+1} \hookrightarrow \Omega^2 X_{n+t} \hookrightarrow \cdots\right) \\
\mathsf{Sp} &\xrightarrow{\quad \Omega^\infty \quad} \mathsf{Top}_* \\
(X_n)_{n\in\mathbb{N}} &\longmapsto X_0.
\end{aligned}$$

"Why were spectra invented?" you may ask. One answer comes in the form of Brown's representability theorem. To understand this, we need a few definitions.

Definition 2

A generalized homology theory E is a functor

$$E : \mathsf{Spaces} \longrightarrow \mathsf{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.
- Exactness: Exact sequence in homology from a cofiber sequence.
- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \to E_*(X, B)$ is an isomorphism.

- Additivity: Coproducts in Spaces induce coproducts in homology.

For more details, see Wikipedia on generalized cohomology.

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized (co)homology theories and spectra. Given a spectrum $\mathcal{E}$, the homology is given by

$$\mathcal{E}_*(X) = \pi_*(\mathcal{E} \wedge X).$$

The cohomology associated to the spectrum $\mathcal{E}$ is given by

$$\mathcal{E}^*(X) = [X, \mathcal{E}].$$

Definition 3: Fiber Sequences

We'll come back to this! The key is that in spectra land, it goes back and forth in both directions.

FACT 1

Any fiber sequence $X \to Y \twoheadrightarrow Z$ gives rise to a long exact sequence in π_*,

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{k+1}Z \arrow[dll, out=-90, in=90] \\
\pi_k X \arrow[r] & \pi_k Y \arrow[r] & \pi_k Z \arrow[dll, out=-90, in=90] \\
\pi_{k-1}X \arrow[r] & \cdots &
\end{tikzcd}$$

Lecture 2: Algebra

Rings

First, let's talk about commutative *(order of multiplication doesn't matter)*, unital *(the ring has the element 1)* rings. Every time I write R as a ring, I mean this version of a ring.

Example 2

- $\mathbb{Z}$
- $\mathbb{Z}/n$ for $n \geq 2$
- $\mathbb{F}_p := \mathbb{Z}/p$ with p a prime. A special case of this is $\mathbb{F}_2 = (\{0, 1\}, +, \times)$.
- $\mathbb{Z}[x]$, $\mathbb{F}_2[x]$, $R[x]$, aka polynomial rings in one variable.
- $\frac{\mathbb{F}_2[x]}{(x^3+1)}$, a ring mod out by an ideal.
- $\mathbb{Z}[G]$ for G an abelian group, the group ring.
- $\mathbb{F}_4 := \dfrac{\mathbb{F}_2[x]}{(x^2+x+1)}$, <u>the</u> field with $4 (= 2^2)$ elements.
- $\mathbb{Z}[[x]] = \left\{\sum_0^\infty a_k x^k \mid \forall k, a_k \in \mathbb{Z}\right\}$, the power series ring
- $\mathbb{Z}((x))$, the Laurent series ring.

Modules

Definition 4: Module

module M over a commutative ring R is an abelian group M together with a scaling map

$$R \otimes M \to M$$

$$r \otimes m \mapsto r \cdot m.$$

Example 3

A vector space V over the ring $\mathbb{R}$ (or any field $\mathbb{F}$) is the same thing as an $\mathbb{R}$-module.

Example 4

If R is a ring, then an ideal $I \subseteq R$ is the same thing as a submodule of R.

Exact Sequences

Definition 5: Short Exact Sequence

A short exact sequence is

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

such that

$$\ker(\text{each map}) = \operatorname{Im}(\text{previous map}).$$

For specificity, we need

i. f is injective

ii. g is surjectve

iii. $\ker g = \operatorname{Im} f$.

Example 5

Let R be a field, say $\mathbb{F}_2$, let V be an R-vector space and let $W \leq V$ be a subspace. Then

$$0 \to W \to V \to V/W \to 0$$

is a short exact sequence.

Example 6

Let $R = \mathbb{Z}$. Consider the map

$$0 \to \mathbb{Z}/2 \xrightarrow{[1\ 0]} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \mathbb{Z}/2 \to 0.$$

What is the composition of these maps? Is this sequence exact?

Example 7

Let $R = \mathbb{Z}$. Then consider the sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{(-)\cdot 2} \mathbb{Z}/4 \xrightarrow{() \mod 2} \mathbb{Z}/2 \to 0.$$

Is this a short exact sequence? If so, how does it compare to the prior example?

Remark 1

$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4$ as groups. Prove it!

Oftentimes, we are interested in some module M, and we know that it fits into a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M', M'' are known. Determining M given M' and M'' is called an extension problem.

Definition 6: p-adics

Another ring of interest is the p-adic integers $\mathbb{Z}_p$ also denoted $\mathbb{Z}_p^\wedge$ where $\wedge$ means completed. Another way to write this is

$$\begin{aligned} \mathbb{Z}_p = \mathbb{Z}_p^\wedge &= \varprojlim \mathbb{Z}/p^k \\ &= \varprojlim(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0) \\ &= \left\{(a_1, a_2, \dots) \mid a_i \in \mathbb{Z}/p^i, \quad a_{i+1} \equiv a_i \mod p^i\right\}. \end{aligned}$$

Day 1 Exercises

Exercise 1

If $m, n > 1$ are integers, construct an exact sequence of abelian groups of the form

$$0 \longrightarrow \mathbb{Z}/m \longrightarrow \mathbb{Z}/mn \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Exercise 2

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that

$$\dim V = \dim V' + \dim V''$$

Exercise 3

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that $V \cong V' \oplus V''$. (*Bonus: Is this isomorphism canonical? In other words, does it require any choices?*)beginexercise Give an example of a short exact sequence of modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

such that M is NOT isomorphic to $M' \oplus M''$.

Exercise 4

(The Splitting Lemma) Suppose

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of modules. Prove that the following are equivalent.

1. $M \cong M' \oplus M''$, f is the standard inclusion, and g is the standard projection.
2. There exists a map $s : M'' \to M$ such that $g \circ s = id_{M''}$.
3. There exists a map $t : M \to M'$ such that $t \circ f = id_{M'}$.

Exercise 5

Generalizing 1., if

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

is an exact sequence of vector spaces, prove that

$$\dim V_n - \dim V_{n-1} + \cdots \pm \dim V_1 \mp \dim V_0 = 0$$

Exercise 6

What are all possible group maps from

1. $\mathbb{Z}$ to $\mathbb{F}_2$?
2. $\mathbb{F}_2$ to $\mathbb{F}_2$?
3. $\mathbb{Z}$ to $\mathbb{F}_4$?
4. $\mathbb{F}_4$ to $\mathbb{F}_2$?
5. $\mathbb{F}_2$ to $\mathbb{F}_4$?
6. $\mathbb{Z}_2$ to $\mathbb{F}_2$?

Day 2

The plan for the day is as follows.

1. Spectra
2. Algebra

The plan for tomorrow is to do the following.

1. An example of a spectral sequence. This will use some group cohomology.
2. Showing the `spectralsequences` package in LaTeX, and maybe some LaTeXpractice.

We aren't trying to give homework! This is your job! There is no need to work outside of working hours. Please try to get LaTeXinstalled into your computer by tomorrow so we can practice it! Feel free to ask Irina, Jack, and Scotty for help.

Lecture 1: Spectra

1. Σ, the reduced suspension, is a functor among topological spaces. π_* is another functor from topological spaces to groups. There is a map,

$$\begin{aligned} \pi_{r+n}(S^n) &\xrightarrow{\quad\Sigma\quad} \pi_{r+n+1}(S^{n+1}) \\ \left(S^{r+n} \xrightarrow{f} S^n\right) &\longmapsto \left(\underbrace{\Sigma S^{r+n}}_{S^{r+n+1}} \xrightarrow{\Sigma f} \underbrace{\Sigma S^n}_{S^{n+1}}\right) \\ f &\longmapsto \Sigma f \end{aligned}$$

   $\pi_{n+r}(S^n)$ only depends on r for n large enough because of the Freudenthal Suspension theorem. Here, we get the definition of the stable homotopy groups of spheres, which we see as

$$\pi_r^{st} := \varinjlim_{n\geq 0}\left(\cdots \xrightarrow{\Sigma} \pi_{n+r}(S^n) \xrightarrow{\Sigma} \pi_{n+1+r}(S^{n+1}) \xrightarrow{\Sigma} \cdots\right)$$

   More generally, for any space X, you get

$$\pi_{n+r}(\Sigma^n X) \xrightarrow{\cong} \pi_{n+1+r}(\Sigma^{n+1}X)$$

   as long as n is large enough.

   Theorem 2: Freudenthal Suspension Theorem

   Let X be $(n-1)$ connected ($\pi_k(X) = 0$ for $k = 0, \dots, n-1$). Note that we have maps

$$\Sigma : \pi_q X \longrightarrow \pi_{q+1}\Sigma X.$$

   Then, when $q < 2n-1$, Σ is a bijection and when $q = 2n-1$, it is a surjection.

   For us today, a "space" means a compactly generated, weakly Hausdorff topological space. This excludes spaces that are "bad." See here for more about compactly generated spaces and see here for weakly Hausdorff spaces.

2. A spectrum E is
   - A collection of spaces $\{E_n\}_{n=0}^{\infty}$
   - Structure maps $\alpha_n : \Sigma E_n \longrightarrow E_{n+1}$.

Example 8

- Given a space X, we can cook a suspension spectrum $\Sigma^\infty X$, with
  - Spaces: $\{(\Sigma^\infty X)_n\}_n$ where $(\Sigma^\infty X)_n := \Sigma^n X$
  - Structure maps: $\alpha_n : \Sigma(\Sigma^n X) \xrightarrow{\cong} \Sigma^{n+1} X$.
- $H_*(X;\mathbb{Z})$ comes from a spectrum called an Eilenberg-MacLane spectrum. The spectrum is denoted $H\mathbb{Z}$ and the data of $H\mathbb{Z}$ is given by
  - Spaces: $H\mathbb{Z}_n := K(\mathbb{Z}, n)$ the Eilenberg-MacLane space for $\mathbb{Z}$ in degree n. This means that if a space A is a $K(\mathbb{Z}, n)$, then $\pi_n(A) = \mathbb{Z}$ and $\pi_k(A) = 0$ for all $k \neq n$.[a]
  - Structure maps: $\Sigma K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1)$.

[a] $\mathbb{Z}$ is a $K(\mathbb{Z}, 0)$, S^1 is a $K(\mathbb{Z}, 1)$, and $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. S^0 is a $K(\mathbb{Z}/2, 0)$, and $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2, 1)$.

Definition 7: Eilenberg MacLane spaces

For any Abelian group G and $n \geq 0$ there exists a space X which is considered $K(G, n)$ such that

$$\pi_k(X) = \begin{cases} G & k = n \\ 0 & \text{else.} \end{cases}$$

See the construction here.

3. Homotopy Groups of a spectrum E

$$\pi_{n+r} E_n \xrightarrow{\Sigma} \pi_{n+1+r} E_{n+1} \xrightarrow{\alpha_n} \pi_{n+1+r} E_{n+1}.$$

On the left, the codimension (difference in the index) is r, and that matches with the group on the right! The codimension is also r there. The rth stable group should hopefully convince you at the very least that there is something to the idea that if we did this forever, the homotopy group would "stabilize" once n gets big enough!

Definition 8: Homotopy groups of a spectrum

Let E be a spectrum. The rth homotopy group of E is defined to be:

$$\pi_r E := \operatorname*{colim}_{n \to \infty} \pi_{n+r} E_n.$$

If we take $E = \Sigma^\infty X$, then $\pi_n E = \pi_n^{st} X$, the stable homotopy groups of X. This definition recovers the definition of stable homotopy groups! Great!

4. Maps between spectra.

Definition 9: Maps of spectra, version 1

Let E, F be two spectra. A map between spectra $f : E \to F$ is a collection of maps $f_n : E_n \to F_n$ such that these f_n are compatible with the structure maps (ε_n for E and φ_n for F), i.e. for each n:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"'] \arrow[dr, phantom, "\circlearrowleft"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r, "\varphi_n"] & F_{n+1} \end{tikzcd}$$

This definition seems good! Let's try another definition and then we can compare and contrast which will be better for our purposes. Which definition will have better theorems?

Here is another construction.

Definition 10: Maps between spectra, version 2

What if instead, we take maps $f_n : E_n \to F_{n-r}$ where we decrease degree by r. We'd still like this to be compatible with structure maps.

$$\begin{tikzcd} \Sigma^r E_{n-r} \arrow[r, "\varepsilon"] \arrow[d, "\Sigma f_{n-r}"'] \arrow[dr, phantom, "\circlearrowleft"] & E_n \arrow[d, "f_n"] \\ \Sigma^r F_{n-2r} \arrow[r, "\varphi"] & F_{n-r} \end{tikzcd}$$

Let's explore this a little bit.

Consider the map $f : \mathbb{S} \to \mathbb{S}$ which is a degree 2 map, so we have maps $S^{n+2} \to S^n$. Note that $S^2 \to S^0$ is nullhomotopic. $S^3 \to S^1$ is nullhomotopic. However, $S^4 \to S^2$ is not nullhomotopic and can be represented by η^2 (look up the Hopf map! There are several cool links if you ask one of us about them).

Here is yet another construction!

Definition 11: Maps between spectra, version 3

A map of spectra $f : E \to F$ of degree r is a homotopy class of functions of spectra $f : E \to F$ of degree r where the function is defined "in the limit." Find the maps between high enough E_N, and worry about the early ones later on. Scotty heard from his advisor that this philosophy is "cells now, maps later."

NOTATION: When we talk about maps from spectra to spectra of degree r, we denote the collection of homotopy classes of maps of degree r between the two spectra as

$$[E, F]_r.$$

5. Homology and cohomology. Let E, X be spectra.

Definition 12: Homology and Cohomology

The Eth cohomology of X in degree r is

$$E^r(X) := [X, E]_{-r},$$

maps of spectra $X \to E$ which lower degree by r.
The Eth homology of X in degree r is

$$E_r(X) = [\mathbb{S}, E \wedge X]_r = \pi_r(E \wedge X).$$

When I say coefficients of a spectrum E, what I really mean is $E_r(*) = \pi_r E = E^{-r}(*)$. This is usually written E_* to collect all r into one neat little package. By this we mean

$$E_* = \bigoplus_{r \in \mathbb{Z}} E_r(*).$$

6. Given a map of spectra $f : X \to Y$, define

$$(Y \cup_f CX)_n := Y_n \cup_{f_n} (I_+ \wedge X_n).$$

This gives us a long cofiber sequence of spectra

$$\cdots \to \Omega X \to \Omega Y \to \Omega \mathrm{Cof}\, f X \xrightarrow{f} Y \xrightarrow{i} \underbrace{Y \cup_f CX}_{\mathrm{Cof}\, f} \to \underbrace{(Y \cup_f CX) \cup_i CY}_{\Sigma X} \to \Sigma Y \to \cdots.$$

Here $\Omega(-) = \mathsf{Map}(\mathbb{S}^1, -)$, the loops. And in spectra, Ω is something like Σ^{-1}.

Here, we have been using a lot from the blue book: [Ada74, pg. 123, §3]. Here is a pdf link.

Definition 13: pullbacks

Let C be a category which contains the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd} .$$

A pullback of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $W \xrightarrow{p_1} X$
- A map $W \xrightarrow{p_2} Y$

such that

- The diagram $\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$ commutes [a]

- If someone hands you a commutative diagram $\begin{tikzcd} A \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{h}$ such that

$$\begin{tikzcd} A \arrow[dr, "\widetilde{h}"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\ & W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd} .$$

[b]

Let C be a category with a subcategory $\mathcal{D}$. We say $\mathcal{D}$ is closed under pullbacks by morphisms in C if for all arrows $X \xrightarrow{f} Z$ in $\mathcal{D}$ and for all $Y \xrightarrow{g} Z$ in C such that we can form the pullback $\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then the arrow $W \xrightarrow{p_1} Z$ is in $\mathcal{D}$.

[a]"$\circlearrowleft$" is a 'long'hand for commutes, and people usually suppress it from notation.
[b]As a shorthand, people usually write the pullback like this:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Definition 14: pushouts

Let C be a category which contains the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \\ X & \end{tikzcd}.$$

A pushout of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $X \xrightarrow{i_1} W$
- A map $Y \xrightarrow{i_2} W$

such that

- The diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \end{tikzcd}$ commutes
- If someone hands you a commutative diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\ell_2"] \\ X \arrow[r, "\ell_1"] & A \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{\ell}$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "\ell_2"] & \\ X \arrow[r, "i_1"] \arrow[drr, bend right, "\ell_1"] & W \arrow[dr, "\widetilde{\ell}"] & \\ & & A \end{tikzcd}.$$

[a]

[a] As a shorthand, people usually write the pushout like this:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

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