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Chapter 1

Week 1

Day 1

Lecture 1: Lot's of definitions!

Plan

We are going to try and compute

$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E -theory and this is a spectrum (think a space).
- $E(n, p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G -action.

$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned}$$

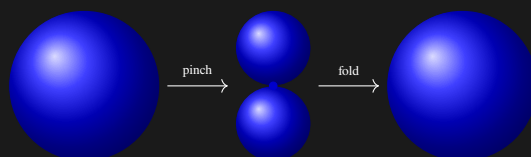
$$E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be

$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n , and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.

$$S^n \xrightarrow{2} S^n \vee S^n \rightarrow S^n.$$



The thing to take away is that for a degree m -map between n -spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \{S^0, S^1, S^2, \dots\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences: In spectra, fiber and cofiber sequences are the same! This is an analog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 2. \end{array}$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \rightarrow \underbrace{V(0)}_{\text{cofiber}(2)} \rightarrow \Sigma \mathbb{S} \xrightarrow{\Sigma 2} \Sigma \mathbb{S} \rightarrow \Sigma V(0) \rightarrow \dots$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ , aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n . In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.
- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \text{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X), g \in \pi_m(X)$, meaning that we have

$$f : S^n \rightarrow X, \quad g : S^m \rightarrow X.$$

What is $f \cdot g$ if we're talking about π_* having a “ring structure.” Then we have

$$\begin{array}{ccccc} & & X \wedge S^m & & \\ & \nearrow f \wedge \mathbb{1} & & \searrow \mathbb{1} \times g & \\ S^{n+m} = S^n \wedge S^m & \xrightarrow{f \wedge g} & X \wedge X & \xrightarrow{\mu} & X \\ & \searrow \mathbb{1} \wedge g & & \nearrow f \wedge \mathbb{1} & \\ & & S^n \wedge X & & \end{array}$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0 . Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A spectrum^a X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n \in \mathbb{N}}$$

together with structure maps

$$\Sigma X_n \rightarrow X_{n+1}.$$

^aWhat we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as homeomorphisms.

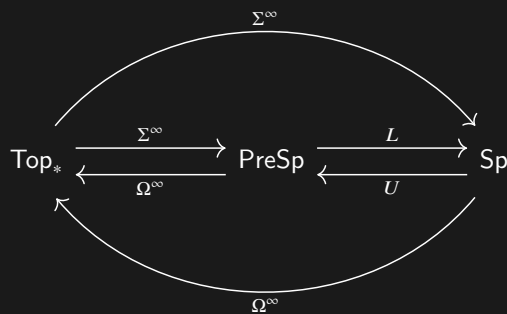
Example 1

1. The sphere spectrum $\mathbb{S} = \{S^0, S^1, \dots\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.

2. Suspension spectrum $\Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \dots\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma \Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.



“Why were spectra invented?” you may ask. The answer comes in the form of Brown’s representability theorem. To understand this, we need a few definitions.

Definition 2

A generalized homology theory E is a functor

$$E : \text{Spaces} \rightarrow \text{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.
- Exactness: Exact sequence in homology from a cofiber sequence.
- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \rightarrow E_*(X, B)$ is an isomorphism.
- Additivity: Coproducts in Spaces induce coproducts in homology.

For more details, see [Wikipedia on generalized cohomology](#).

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized (co)homology theories and spectra. Given a spectrum \mathcal{E} , the homology is given by

$$\mathcal{E}_*(X) = \pi_*(\mathcal{E} \wedge X).$$

The cohomology associated to the spectrum \mathcal{E} is given by

$$\mathcal{E}^*(X) = [X, \mathcal{E}].$$

Definition 3: Fiber Sequences

We'll come back to this! The key is that in spectra land, it goes back and forth in both directions.

FACT 1

Any fiber sequence $X \rightarrow Y \rightarrow Z$ gives rise to a long exact sequence in π_* ,

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & \pi_{k+1}Z \\ & & & \swarrow & & & \uparrow \\ \pi_k X & \longrightarrow & \pi_k Y & \longrightarrow & \pi_k Z & & \\ & & \swarrow & & \uparrow & & \\ \pi_{k-1} X & \longrightarrow & \cdots & & & & \end{array}$$

Lecture 2: Algebra

Rings

First, let's talk about commutative (*order of multiplication doesn't matter*), unital (*the ring has the element 1*) rings. Every time I write R as a ring, I mean this version of a ring.

Example 2

- \mathbb{Z}
- \mathbb{Z}/n for $n \geq 2$
- $\mathbb{F}_p := \mathbb{Z}/p$ with p a prime. A special case of this is $\mathbb{F}_2 = (\{0, 1\}, +, \times)$.
- $\mathbb{Z}[x]$, $\mathbb{F}_2[x]$, $R[x]$, aka polynomial rings in one variable.
- $\frac{\mathbb{F}_2[x]}{(x^3+1)}$, a ring mod out by an ideal.
- $\mathbb{Z}[G]$ for G an abelian group, the group ring.
- $\mathbb{F}_4 := \frac{\mathbb{F}_2[x]}{(x^2+x+1)}$, the field with $4(=2^2)$ elements.
- $\mathbb{Z}[[x]] = \{\sum_0^\infty a_k x^k \mid \forall k, a_k \in \mathbb{Z}\}$, the power series ring
- $\mathbb{Z}((x))$, the laurent series ring.

Modules

Definition 4: Module

module M over a commutative ring R is an abelian group M together with a scaling map

$$R \otimes M \rightarrow M$$

$$r \otimes m \mapsto r \cdot m.$$

Example 3

A vector space V over the ring \mathbb{R} (or any field \mathbb{F}) is the same thing as an \mathbb{R} -module.

Example 4

If R is a ring, then an ideal $I \subseteq R$ is the same thing as a submodule of R .

Exact Sequences

Definition 5: Short Exact Sequence

A short exact sequence is

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0,$$

such that

$$\ker(\text{each map}) = \text{Im}(\text{previous map}).$$

For specificity, we need

- i. f is injective
- ii. g is surjective
- iii. $\ker g = \operatorname{Im} f$.

Example 5

Let R be a field, say \mathbb{F}_2 , let V be an R -vector space and let $W \leq V$ be a subspace. Then

$$0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$$

is a short exact sequence.

Example 6

Let $R = \mathbb{Z}$. Consider the map

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{Z}/2 \rightarrow 0.$$

What is the composition of these maps? Is this sequence exact?

Example 7

Let $R = \mathbb{Z}$. Then consider the sequence

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{(-) \cdot 2} \mathbb{Z}/4 \xrightarrow{(\) \bmod 2} \mathbb{Z}/2 \rightarrow 0.$$

Is this a short exact sequence? If so, how does it compare to the prior example?

Remark 1

$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4$ as groups. Prove it!

Oftentimes, we are interested in some module M , and we know that it fits into a short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

where M', M'' are known. Determining M given M' and M'' is called an extension problem.

Definition 6: p -adics

Another ring of interest is the p -adic integers \mathbb{Z}_p also denoted \mathbb{Z}_p^\wedge where \wedge means completed. Another way to write this is

$$\begin{aligned} \mathbb{Z}_p &= \mathbb{Z}_p^\wedge = \varprojlim \mathbb{Z}/p^k \\ &= \varprojlim (\cdots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0) \\ &= \{(a_1, a_2, \dots) \mid a_i \in \mathbb{Z}/p^i, \quad a_{i+1} \equiv a_i \pmod{p^i}\}. \end{aligned}$$

Day 1 Exercises

Exercise 1

If $m, n > 1$ are integers, construct an exact sequence of abelian groups of the form

$$0 \longrightarrow \mathbb{Z}/m \longrightarrow \mathbb{Z}/mn \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Exercise 2

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that

$$\dim V = \dim V' + \dim V''$$

Exercise 3

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that $V \cong V' \oplus V''$. (Bonus: Is this isomorphism canonical? In other words, does it require any choices?)

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

such that M is NOT isomorphic to $M' \oplus M''$.

Exercise 4

(The Splitting Lemma) Suppose

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of modules. Prove that the following are equivalent.

1. $M \cong M' \oplus M''$, f is the standard inclusion, and g is the standard projection.
2. There exists a map $s : M'' \rightarrow M$ such that $g \circ s = id_{M''}$.
3. There exists a map $t : M \rightarrow M'$ such that $t \circ f = id_{M'}$.

Exercise 5

Generalizing 1., if

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

is an exact sequence of vector spaces, prove that

$$\dim V_n - \dim V_{n-1} + \cdots \pm \dim V_1 \mp \dim V_0 = 0$$

Exercise 6

What are all possible group maps from

1. \mathbb{Z} to \mathbb{F}_2 ?
2. \mathbb{F}_2 to \mathbb{F}_2 ?
3. \mathbb{Z} to \mathbb{F}_4 ?
4. \mathbb{F}_4 to \mathbb{F}_2 ?
5. \mathbb{F}_2 to \mathbb{F}_4 ?
6. \mathbb{Z}_2 to \mathbb{F}_2 ?

Chapter 2

Appendix

Definition 7: pullbacks

Let \mathcal{C} be a category which contains the diagram

$$\begin{array}{ccc} & Y & \\ & \downarrow g & \\ X & \xrightarrow{f} & Z \end{array}$$

A pullback of this diagram W is 3 pieces of information

- An object $W \in \text{ob}(\mathcal{C})$
- A map $W \xrightarrow{p_1} X$
- A map $W \xrightarrow{p_2} Y$

such that

- The diagram $\begin{array}{ccccc} W & \xrightarrow{p_2} & Y \\ p_1 \downarrow & \circlearrowleft & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$ commutes^a

- If someone hands you a commutative diagram $\begin{array}{ccccc} A & \xrightarrow{h_2} & Y \\ h_1 \downarrow & \circlearrowleft & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$, then there is a *UNIQUE* map \tilde{h} such that

$$\begin{array}{ccccc} A & & & & Y \\ & \searrow \tilde{h} & & \searrow h_2 & \\ & W & \xrightarrow{p_2} & Y & \\ & p_1 \downarrow & \circlearrowleft & \downarrow g & \\ & X & \xrightarrow{f} & Z & \end{array}$$

^b

Let \mathcal{C} be a category with a subcategory \mathcal{D} . We say \mathcal{D} is closed under pullbacks by morphisms in \mathcal{C} if for all

arrows $X \xrightarrow{f} Z$ in \mathcal{D} and for all $Y \xrightarrow{g} Z$ in \mathcal{C} such that we can form the pullback

$$\begin{array}{ccc} W & \xrightarrow{p_1} & Y \\ p_2 \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

$W \xrightarrow{p_1} Z$ is in \mathcal{D} .

^a“ \cup ” is a ‘long’ hand for commutes, and people usually suppress it from notation.

^bAs a shorthand, people usually write the pullback like this:

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Definition 8: pushouts

Let \mathcal{C} be a category which contains the diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \\ X & & \end{array}.$$

A pushout of this diagram W is 3 pieces of information

- An object $W \in \text{ob}(\mathcal{C})$
- A map $X \xrightarrow{i_1} W$
- A map $Y \xrightarrow{i_2} W$

such that

- The diagram $\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \downarrow i_2 \\ X & \xrightarrow{i_1} & W \end{array}$ commutes

- If someone hands you a commutative diagram $\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \downarrow \ell_2 \\ X & \xrightarrow{\ell_1} & A \end{array}$, then there is a *UNIQUE* map $\tilde{\ell}$ such that

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \downarrow i_2 \\ X & \xrightarrow{i_1} & W \end{array} \quad \begin{array}{c} \searrow \ell_2 \\ \downarrow \tilde{\ell} \\ \searrow \ell_1 \end{array} \quad \begin{array}{c} \\ \\ A \end{array}.$$

^a

^aAs a shorthand, people usually write the pushout like this:

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & \lrcorner & \downarrow i_2 \\ X & \xrightarrow{i_1} & W \end{array}$$

Bibliography

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

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