eCHT REU 2024

Spectra, Spectral Sequences, and (Co)Fibers smashed with tmf

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Note: These are LiveTEXed notes from the eCHT REU written by Scotty Tilton based on lectures by Irina Bobkova and Jack Carlisle. Scotty may have added a little commentary to add to the exposition, and it's more than likely that Scotty added some of his own errors to the lecture. Please email me at `stilton@ucsd.edu` if (and when) you find errors. Thanks!

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Lecture 1 (Irina Bobkova): Lot's of definitions!

Plan
We are going to try and compute

$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).
- $E(n,p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G-action.

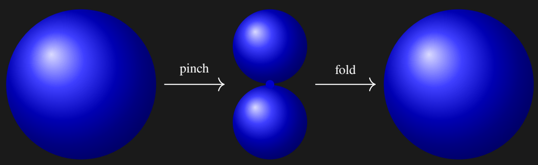
$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned} \qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be

$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.

$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$



The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{S^0, S^1, S^2, \ldots\right\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\times 2} \mathbb{Z} \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 2. \end{aligned}$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \overset{\times 2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \text{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X)$, $g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, \quad g : S^m \to X.$$

What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$\begin{tikzcd} & X \wedge S^m \arrow[dr, "\mathbb{1}\times g"] & & \\ S^{n+m} = S^n \wedge S^m \arrow[ur, "f\wedge\mathbb{1}"] \arrow[rr, "f\wedge g"] \arrow[dr, "\mathbb{1}\wedge g"'] & & X \wedge X \arrow[r, "\mu"] & X \\ & S^n \wedge X \arrow[ur, "f\wedge\mathbb{1}"'] & & \end{tikzcd}$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A **spectrum**[a] X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n\in\mathbb{N}}$$

together with structure maps

$$\Sigma X_n \longrightarrow X_{n+1}.$$

[a] What we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as $X_n \to \Omega X_{n+1}$ and homeomorphisms.

Example 1

1. The sphere spectrum $\mathbb{S} = \left\{S^0, S^1, \dots\right\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.
2. Suspension spectrum $\Sigma^\infty X = \left\{X, \Sigma X, \Sigma^2 X, \dots\right\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.

$$\begin{tikzcd}
\mathsf{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left=50, "\Sigma^\infty"] & \mathsf{PreSp} \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & \mathsf{Sp} \arrow[l, shift left, "U"] \arrow[ll, bend left=50, "\Omega^\infty"]
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{PreSp} &\xrightarrow{\quad L \quad} \mathsf{Sp} \\
(X_n)_{n\in\mathbb{N}} &\longmapsto (LX_n)_{n\in\mathbb{N}}, \qquad LX_n := \operatorname{colim}\left(X_n \hookrightarrow \Omega X_{n+1} \hookrightarrow \Omega^2 X_{n+t} \hookrightarrow \cdots\right) \\
\mathsf{Sp} &\xrightarrow{\quad \Omega^\infty \quad} \mathsf{Top}_* \\
(X_n)_{n\in\mathbb{N}} &\longmapsto X_0.
\end{aligned}$$

"Why were spectra invented?" you may ask. One answer comes in the form of Brown's representability theorem. To understand this, we need a few definitions.

Definition 2

A **generalized homology** theory E is a functor

$$E : \mathsf{Spaces} \longrightarrow \mathsf{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.
- Exactness: Exact sequence in homology from a cofiber sequence.

- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \longrightarrow E_*(X, B)$ is an isomorphism.
- Additivity: Coproducts in Spaces induce coproducts in homology.

For more details, see Wikipedia on generalized cohomology.

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized (co)homology theories and spectra. Given a spectrum $\mathcal{E}$, the homology is given by

$$\mathcal{E}_*(X) = \pi_*(\mathcal{E} \wedge X).$$

The cohomology associated to the spectrum $\mathcal{E}$ is given by

$$\mathcal{E}^*(X) = [X, \mathcal{E}].$$

Definition 3: Fiber Sequences

We'll come back to this! The key is that in spectra land, it goes back and forth in both directions.

FACT 1

Any fiber sequence $X \longrightarrow Y \twoheadrightarrow Z$ gives rise to a long exact sequence in π_*,

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{k+1}Z \arrow[dll, out=-90, in=90] \\
\pi_k X \arrow[r] & \pi_k Y \arrow[r] & \pi_k Z \arrow[dll, out=-90, in=90] \\
\pi_{k-1}X \arrow[r] & \cdots &
\end{tikzcd}$$

Lecture 2 (Jack Carlisle): Algebra

Rings

First, let's talk about commutative *(order of multiplication doesn't matter)*, unital *(the ring has the element 1)* rings. Every time I write R as a ring, I mean this version of a ring.

Example 2

- $\mathbb{Z}$
- $\mathbb{Z}/n$ for $n \geq 2$
- $\mathbb{F}_p := \mathbb{Z}/p$ with p a prime. A special case of this is $\mathbb{F}_2 = (\{0, 1\}, +, \times)$.
- $\mathbb{Z}[x]$, $\mathbb{F}_2[x]$, $R[x]$, aka polynomial rings in one variable.
- $\frac{\mathbb{F}_2[x]}{(x^3+1)}$, a ring mod out by an ideal.
- $\mathbb{Z}[G]$ for G an abelian group, the group ring.
- $\mathbb{F}_4 := \dfrac{\mathbb{F}_2[x]}{(x^2+x+1)}$, the field with $4(= 2^2)$ elements.
- $\mathbb{Z}[[x]] = \left\{\sum_0^\infty a_k x^k \mid \forall k, a_k \in \mathbb{Z}\right\}$, the power series ring
- $\mathbb{Z}((x))$, the Laurent series ring.

Modules

Definition 4: Module

module M over a commutative ring R is an abelian group M together with a scaling map

$$R \otimes M \to M$$

$$r \otimes m \mapsto r \cdot m.$$

Example 3

A vector space V over the ring $\mathbb{R}$ (or any field $\mathbb{F}$) is the same thing as an $\mathbb{R}$-module.

Example 4

If R is a ring, then an ideal $I \subseteq R$ is the same thing as a submodule of R.

Exact Sequences

Definition 5: Short Exact Sequence

A **short exact sequence** is

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

such that

$$\ker(\text{each map}) = \operatorname{Im}(\text{previous map}).$$

For specificity, we need

i. f is injective

ii. g is surjectve

iii. $\ker g = \operatorname{Im} f$.

Example 5

Let R be a field, say $\mathbb{F}_2$, let V be an R-vector space and let $W \leq V$ be a subspace. Then

$$0 \to W \to V \to V/W \to 0$$

is a short exact sequence.

Example 6

Let $R = \mathbb{Z}$. Consider the map

$$0 \to \mathbb{Z}/2 \xrightarrow{[1\ 0]} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{Z}/2 \to 0.$$

What is the composition of these maps? Is this sequence exact?

Example 7

Let $R = \mathbb{Z}$. Then consider the sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{(-)\cdot 2} \mathbb{Z}/4 \xrightarrow{() \mod 2} \mathbb{Z}/2 \to 0.$$

Is this a short exact sequence? If so, how does it compare to the prior example?

Remark 1

$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4$ as groups. Prove it!

Oftentimes, we are interested in some module M, and we know that it fits into a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M', M'' are known. Determining M given M' and M'' is called an extension problem.

Definition 6: p-adics

Another ring of interest is the **p-adic integers** $\mathbb{Z}_p$ also denoted $\mathbb{Z}_p^\wedge$ where $\wedge$ means completed. Another way to write this is

$$\begin{aligned}
\mathbb{Z}_p = \mathbb{Z}_p^\wedge &= \varprojlim \mathbb{Z}/p^k \\
&= \varprojlim(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0) \\
&= \left\{(a_1, a_2, \ldots) \mid a_i \in \mathbb{Z}/p^i, \quad a_{i+1} \equiv a_i \mod p^i\right\}.
\end{aligned}$$

Day 1 Exercises

Exercise 1

If $m, n > 1$ are integers, construct an exact sequence of abelian groups of the form

$$0 \longrightarrow \mathbb{Z}/m \longrightarrow \mathbb{Z}/mn \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Exercise 2

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that

$$\dim V = \dim V' + \dim V''$$

Exercise 3

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that $V \cong V' \oplus V''$. (*Bonus: Is this isomorphism canonical? In other words, does it require any choices?*)

Exercise 4

Give an example of a short exact sequence of modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

such that M is NOT isomorphic to $M' \oplus M''$.

Exercise 5

(The Splitting Lemma) Suppose

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of modules. Prove that the following are equivalent.

1. $M \cong M' \oplus M''$, f is the standard inclusion, and g is the standard projection.
2. There exists a map $s : M'' \to M$ such that $g \circ s = id_{M''}$.
3. There exists a map $t : M \to M'$ such that $t \circ f = id_{M'}$.

Exercise 6

Generalizing 1., if

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

is an exact sequence of vector spaces, prove that

$$\dim V_n - \dim V_{n-1} + \cdots \pm \dim V_1 \mp \dim V_0 = 0$$

Exercise 7

What are all possible group maps from

1. $\mathbb{Z}$ to $\mathbb{F}_2$?
2. $\mathbb{F}_2$ to $\mathbb{F}_2$?
3. $\mathbb{Z}$ to $\mathbb{F}_4$?
4. $\mathbb{F}_4$ to $\mathbb{F}_2$?
5. $\mathbb{F}_2$ to $\mathbb{F}_4$?
6. $\mathbb{Z}_2$ to $\mathbb{F}_2$?

Day 2

The plan for the day is as follows.

1. Spectra
2. Algebra

The plan for tomorrow is to do the following.

1. An example of a spectral sequence. This will use some group cohomology.
2. Showing the `spectralsequences` package in LaTeX, and maybe some LaTeXpractice.

We aren't trying to give homework! This is your job! There is no need to work outside of working hours. Please try to get LaTeXinstalled into your computer by tomorrow so we can practice it! Feel free to ask Irina, Jack, and Scotty for help.

Lecture 1 (Irina Bobkova): Spectra

1. Σ, the reduced suspension, is a functor among topological spaces. π_* is another functor from topological spaces to groups. There is a map,

$$\begin{array}{ccc} \pi_{r+n}(S^n) & \xrightarrow{\quad\Sigma\quad} & \pi_{r+n+1}(S^{n+1}) \\ \left(S^{r+n} \xrightarrow{f} S^n\right) & \longmapsto & \left(\underbrace{\Sigma S^{r+n}}_{S^{r+n+1}} \xrightarrow{\Sigma f} \underbrace{\Sigma S^n}_{S^{n+1}}\right) \\ f & \longmapsto & \Sigma f \end{array}$$

$\pi_{n+r}(S^n)$ only depends on r for n large enough because of the Freudenthal Suspension theorem. Here, we get the definition of the stable homotopy groups of spheres, which we see as

$$\pi_r^{st} := \varinjlim_{n\geq 0} \left(\cdots \xrightarrow{\Sigma} \pi_{n+r}(S^n) \xrightarrow{\Sigma} \pi_{n+1+r}(S^{n+1}) \xrightarrow{\Sigma} \cdots\right)$$

More generally, for any space X, you get

$$\pi_{n+r}(\Sigma^n X) \xrightarrow{\cong} \pi_{n+1+r}(\Sigma^{n+1} X)$$

as long as n is large enough.

Theorem 2: Freudenthal Suspension Theorem

Let X be $(n-1)$ connected ($\pi_k(X) = 0$ for $k = 0, \ldots, n-1$). Note that we have maps

$$\Sigma : \pi_q X \longrightarrow \pi_{q+1}\Sigma X.$$

Then, when $q < 2n-1$, Σ is a bijection and when $q = 2n-1$, it is a surjection.

For us today, a "space" means a compactly generated, weakly Hausdorff topological space. This excludes spaces that are "bad." See here for more about compactly generated spaces and see here for weakly Hausdorff spaces.

2. A spectrum E is

   - A collection of spaces $\{E_n\}_{n=0}^{\infty}$
   - Structure maps $\alpha_n : \Sigma E_n \longrightarrow E_{n+1}$.

Example 8

- Given a space X, we can cook a suspension spectrum $\Sigma^\infty X$, with
  - Spaces: $\{(\Sigma^\infty X)_n\}_n$ where $(\Sigma^\infty X)_n := \Sigma^n X$
  - Structure maps: $\alpha_n : \Sigma(\Sigma^n X) \xrightarrow{\cong} \Sigma^{n+1} X$.
- $H_*(X;\mathbb{Z})$ comes from a spectrum called an Eilenberg-MacLane spectrum. The spectrum is denoted $H\mathbb{Z}$ and the data of $H\mathbb{Z}$ is given by
  - Spaces: $H\mathbb{Z}_n := K(\mathbb{Z}, n)$ the Eilenberg-MacLane space for $\mathbb{Z}$ in degree n. This means that if a space A is a $K(\mathbb{Z}, n)$, then $\pi_n(A) = \mathbb{Z}$ and $\pi_k(A) = 0$ for all $k \neq n$.[a]
  - Structure maps: $\Sigma K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1)$.

[a] $\mathbb{Z}$ is a $K(\mathbb{Z}, 0)$, S^1 is a $K(\mathbb{Z}, 1)$, and $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. S^0 is a $K(\mathbb{Z}/2, 0)$, and $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2, 1)$.

Definition 7: Eilenberg MacLane spaces

For any Abelian group G and $n \geq 0$ there exists a space X which is considered $K(G, n)$, or an **Eilenberg MacLane space**, such that

$$\pi_k(X) = \begin{cases} G & k = n \\ 0 & \text{else.} \end{cases}$$

See the construction here.

3. Homotopy Groups of a spectrum E

$$\pi_{n+r} E_n \xrightarrow{\Sigma} \pi_{n+1+r} E_{n+1} \xrightarrow{\alpha_n} \pi_{n+1+r} E_{n+1}.$$

On the left, the codimension (difference in the index) is r, and that matches with the group on the right! The codimension is also r there. The rth stable group should hopefully convince you at the very least that there is something to the idea that if we did this forever, the homotopy group would "stabilize" once n gets big enough!

Definition 8: Homotopy groups of a spectrum

Let E be a spectrum. The rth homotopy group of E is defined to be:

$$\pi_r E := \operatorname*{colim}_{n \to \infty} \pi_{n+r} E_n.$$

If we take $E = \Sigma^\infty X$, then $\pi_n E = \pi_n^{st} X$, the stable homotopy groups of X. This definition recovers the definition of stable homotopy groups! Great!

4. Maps between spectra.

Definition 9: Maps of spectra, version 1

Let E, F be two spectra. A **map between spectra** $f : E \to F$ is a collection of maps $f_n : E_n \to F_n$ such that these f_n are compatible with the structure maps (ε_n for E and φ_n for F), i.e. for each n:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"'] \arrow[dr, phantom, "\circlearrowleft"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r, "\varphi_n"] & F_{n+1} \end{tikzcd}$$

This definition seems good! Let's try another definition and then we can compare and contrast which will be better for our purposes. Which definition will have better theorems?

Here is another construction.

Definition 10: Maps between spectra, version 2

What if instead, we take maps $f_n : E_n \to F_{n-r}$ where we decrease degree by r. We'd still like this to be compatible with structure maps.

$$\begin{tikzcd} \Sigma^r E_{n-r} \arrow[r, "\varepsilon"] \arrow[d, "\Sigma f_{n-r}"'] \arrow[dr, phantom, "\circlearrowleft"] & E_n \arrow[d, "f_n"] \\ \Sigma^r F_{n-2r} \arrow[r, "\varphi"] & F_{n-r} \end{tikzcd}$$

Let's explore this a little bit.

Consider the map $f : \mathbb{S} \to \mathbb{S}$ which is a degree 2 map, so we have maps $S^{n+2} \to S^n$. Note that $S^2 \to S^0$ is nullhomotopic. $S^3 \to S^1$ is nullhomotopic. However, $S^4 \to S^2$ is not nullhomotopic and can be represented by η^2 (look up the Hopf map! There are several cool links if you ask one of us about them).

Here is yet another construction!

Definition 11: Maps between spectra, version 3

A map of spectra $f : E \to F$ of degree r is a homotopy class of functions of spectra $f : E \to F$ of degree r where the function is defined "in the limit." Find the maps between high enough E_N, and worry about the early ones later on. Scotty heard from his advisor that this philosophy is "cells now, maps later."

NOTATION: When we talk about maps from spectra to spectra of degree r, we denote the collection of homotopy classes of maps of degree r between the two spectra as

$$[E, F]_r.$$

5. Homology and cohomology. Let E, X be spectra.

Definition 12: Homology and Cohomology

The E-cohomology of X in degree r is

$$E^r(X) := [X, E]_{-r},$$

maps of spectra $X \to E$ which lower degree by r.
The E-homology of X in degree r is

$$E_r(X) = [\mathbb{S}, E \wedge X]_r = \pi_r(E \wedge X).$$

When I say coefficients of a spectrum E, what I really mean is $E_r(*) = \pi_r E = E^{-r}(*)$. This is usually written E_* to collect all r into one neat little package. By this we mean

$$E_* = \bigoplus_{r \in \mathbb{Z}} E_r(*).$$

6. Given a map of spectra $f : X \to Y$, define

$$(Y \cup_f CX)_n := Y_n \cup_{f_n} (I_+ \wedge X_n).$$

This gives us a long cofiber sequence of spectra

$$\cdots \to \Omega X \to \Omega Y \to \Omega \mathrm{Cof}\, f X \xrightarrow{f} Y \xrightarrow{i} \underbrace{Y \cup_f CX}_{\mathrm{Cof}\, f} \to \underbrace{(Y \cup_f CX) \cup_i CY}_{\Sigma X} \to \Sigma Y \to \cdots .$$

Here $\Omega(-) = \mathsf{Map}(\mathbb{S}^1, -)$, the loops. And in spectra, Ω is something like Σ^{-1}.

Here, we have been using a lot from the blue book: [Ada74, pg. 123, §3]. Here is a pdf link.

Lecture 2 (Jack Carlisle): Homological Algebra and Group Cohomology

Homological Algebra

Definition 13: Chain complex

A **chain complex** is

$$\cdots \xrightarrow{d_{n+2}} C_{n+1}C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

such that

- For all i, $d_{i-1}d_i = 0$. Another way people write this is $d \circ d = 0$ or $d^2 = 0$.

Example 9

$$0 \to \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0.$$

Notice that composing any two differentials, we get 0! This means that we have a chain complex.

Example 10

An exact sequence of R-modules

$$\cdots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \xrightarrow{f_{n-2}} \cdots$$

is an example of an exact sequence. By exactness, we get that

$$f_{n+1}(m) \in \text{Im }(f_{n+1}) = \ker(f_n) \implies f_n f_{n+1} m = 0$$

so the composition of any two is zero.
Here recall that exactness means that for all n,

$$\ker f_n = \text{Im } f_{n+1}.$$

Definition 14: Homology of a chain complex

The **homology of a chain complex** $C = (C_\bullet, d_\bullet)$ is defined to be

$$H_n(C) := \frac{\ker d_n}{\text{Im } d_{n+1}}, \qquad H_*(C) := \bigoplus_{n \in \mathbb{Z}} H_n(C).$$

Example 11: Example above continued

Let's calculate the homology from 1,

$$\underbrace{0}_{4} \to \underbrace{\mathbb{Z}}_{3} \xrightarrow{\times 5} \underbrace{\mathbb{Z}}_{2} \xrightarrow{\times 0} \underbrace{\mathbb{Z}}_{1} \xrightarrow{\times 2} \underbrace{\mathbb{Z}}_{0} \to \underbrace{0}_{-1}.$$

From this, we get

$$H_0(C) = \frac{\ker(\mathbb{Z} \to 0)}{\text{Im } (\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z})} = \mathbb{Z}/2$$

$$H_1(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z})}{\text{Im } (\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z})} = 0$$

$$H_2(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z})}{\text{Im } (\mathbb{Z} \xrightarrow{\times 5} \mathbb{Z})} = \mathbb{Z}/5$$

$$H_3(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\times 5} \mathbb{Z})}{\text{Im } (0 \to \mathbb{Z})} = 0$$

This means that

$$H_n(C) := \begin{cases} \mathbb{Z}/2 & n = 0 \\ \mathbb{Z}/5 & n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Definition 15: Map of Chain complexes

A **map of chain complexes**

$$f : C \to D$$

is

- A collection of maps $f_i : C_i \to D_i$

such that

- $\begin{array}{ccc} C_i & \xrightarrow{d_i^C} & C_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ D_i & \xrightarrow{d_i^D} & D_{i-1} \end{array}$, or in equation form, we have $d_i^D \circ f_i = f_{i-1} \circ d_i^C$.

Lemma 1

If $f : C \to D$ is a chain map, then f induces a map

$$H_*(f) : H_*(C) \to H_*(D).$$

Proof. The proof should come from this diagram! Take a look at how to make it work.

$$\begin{tikzcd}
& C_n \arrow[r, "f_n"] & D_n & \\
& \ker(d_n^C) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[dr] & \ker(d_n^D) \arrow[u, phantom, "\cup"] \arrow[d, two heads] & \\
H_n(C) \arrow[r, phantom, "="] & \dfrac{\ker d_n^c}{\text{Im } d_{n+1}^c} \arrow[r] & \dfrac{\ker d_n^D}{\text{Im } d_{n+1}^D} \arrow[r, phantom, "="] & H_n(D)
\end{tikzcd}$$

Our condition implies that $f(\ker(d_n^C)) \subset \ker d_n^D$. Also it means that $f(\text{Im }(d_{n+1}^C)) \subset \text{Im }(d_{n+1}^D)$. This completes our proof, so long as you work through the details! $\square$

Definition 16: Chain homotopy

Suppose $f, g : C \longrightarrow D$ are chain maps. A **chain homotopy** h from f to g is a collection of maps $h_i : C_i \longrightarrow D_{i+1}$ such that

$$f_i - g_i = dh_i + h_{i-1}d.$$

FACT 2

If $f, g : C \longrightarrow D$ are homotopic chain maps, then

$$H_*(f) = H_*(g)$$

as maps between $H_*(C) \longrightarrow H_*(D)$.
This will appear in your problem set!

Hey, Jack! What about cohomology?

Remark 2

Sometimes we will work instead of with chain complexes, we'll work with **cochain complexes**. These are the same as chain complexes except the differentials *INCREASE* degree. Let's compare

$$\text{Chain Complex} \quad \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

$$\text{Cochain Complex} \quad \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

If $C = (C^\bullet, d^\bullet)$ is a cochain complex, then the cohomology of C is

$$H^n(C) := \frac{\ker d^n}{\text{Im } d^{n-1}}.$$

Group Cohomology

Group cohomology is a "machine" which eats a finite Abelian group G an spits out a cohomology group $H^*(G;\mathbb{Z})$. More generally, we can start with a finite group G and a coefficient group M to get the group cohomology G with coefficients in M (M ends up being a $\mathbb{Z}[G]$-module), written as $H^*(G;M)$.

But Jack! How do we calculate this?

Definition 17: Calculation of Group Cohomology

Step 1: Find a free resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. In other words, we'd like an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

with F_i a free $\mathbb{Z}[G]$-module. In other words, we want $F_i \cong \bigoplus_? \mathbb{Z}[G]$.[a]

Step 2: Apply the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(-,M)$. Now, take the *just the free part* of our resolution

$$\begin{gathered}\cdots \to F_2 \to F_1 \to F_0 \\ \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbb{Z}[G]}(-,M)} \\ \cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_2,M) \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_1,M) \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_0,M).\end{gathered}$$

This second sequence is a cochain complex of Abelian groups ($\mathbb{Z}[G]$-modules).

Step 3: Take cohomology groups of this cochain complex![b]

$$H^n(G;M) := \text{The } n\text{th cohomology group of the cochain complex above.}$$

[a] Recall that $\mathbb{Z}[G] = \left\{\sum_1^n a_i g_i \mid a_i \in \mathbb{Z}, g_i \in G\right\}$. Another way to say this is all finite sums of elements of G where multiplication works the way you'd want it. By this, I specifically mean $(5g+2g') \times (6h) = 30(gh) + 12(g'h)$.
For a specific example, $\mathbb{Z}[C_2] = \mathbb{Z}[x]/(x^2-1=0)$.
For the general definition, $\mathbb{Z}[G] = \mathbb{Z}[x_g \mid g \in G]/x_g x_h = x_{gh}, x_e = 1$.
Note that $\mathbb{Z}$ is a $\mathbb{Z}[G]$-module by G acting trivially on $\mathbb{Z}$.

[b] This is also recognizable as the Ext functor!

Example 12: Example calculation of group cohomology

Let $G = C_2$ and suppose $M = \mathbb{F}_2$ with trivial C_2-action (C_2 does nothing). Let's calculate $H^*(C_2;\mathbb{F}_2)$.

Step 1: Let's find our free resolution. Lucky for us, Jack found one already, and let's see what it is and verify this is in fact a good resolution.
For notation, let $\mathbb{Z}[C_2] = \mathbb{Z}[y]/(y^2-1)$. The following sequence is a free resolution:

$$\cdots \xrightarrow{(-)\cdot(y-1)} \underbrace{\mathbb{Z}[C_2]}_{F_2} \xrightarrow{(-)\cdot(y+1)} \underbrace{\mathbb{Z}[C_2]}_{F_1} \xrightarrow{(-)\cdot(y-1)} \underbrace{\mathbb{Z}[C_2]}_{F_0} \xrightarrow{y\mapsto 1} \mathbb{Z} \to 0.$$

Prove that this is an exact sequence!

Step 2: Apply $\mathrm{Hom}_{\mathbb{Z}[C_2]}(-,\mathbb{F}_2)$. We'll drop the subscript to declutter.

$$\cdots \leftarrow \mathrm{Hom}(\mathbb{Z}[C_2],\mathbb{F}_2) \leftarrow \mathrm{Hom}(\mathbb{Z}[C_2],\mathbb{F}_2) \leftarrow \mathrm{Hom}(\mathbb{Z}[C_2],\mathbb{F}_2)$$

which corresponds to

$$\cdots \xleftarrow{(-)\cdot 0} \mathbb{F}_2 \xleftarrow{(-)\cdot 0} \mathbb{F}_2 \xleftarrow{(-)\cdot 0} \mathbb{F}_2 \xleftarrow{(-)\cdot 0} \mathbb{F}_2 \xleftarrow{(-)\cdot 0} \mathbb{F}_2 \xleftarrow{(-)\cdot 0} \mathbb{F}_2.$$

Step 3: Let's take cohomology groups!

$$H^0(C_2;\mathbb{F}_2) = \frac{\ker(\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2)}{\mathrm{Im}\ (0 \to \mathbb{F}_2)} = \mathbb{F}_2$$
$$H^1(C_2;\mathbb{F}_2) = \frac{\ker(\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2)}{\mathrm{Im}\ (\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2)} = \mathbb{F}_2$$
$$H^2(C_2;\mathbb{F}_2) = \frac{\ker(\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2)}{\mathrm{Im}\ (\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2)} = \mathbb{F}_2$$

Finally, this gives us

$$H^n(C_2;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

Example 13: $H^*(C_2;\mathbb{Z})$ **where** $\mathbb{Z}$ **has a trivial** C_2 **action (**C_2 **does nothing)**

Step 1: Take this free resolution:

$$\xrightarrow{(-)\cdot(y-1)} \underbrace{\mathbb{Z}[C_2]}_{F_2} \xrightarrow{(-)\cdot(y+1)} \underbrace{\mathbb{Z}[C_2]}_{F_1} \xrightarrow{(-)\cdot(y-1)} \underbrace{\mathbb{Z}[C_2]}_{F_0} \xrightarrow{y\mapsto 1} \mathbb{Z} \to 0.$$

Step 2: Apply $\mathrm{Hom}_{\mathbb{Z}[C_2]}(-,\mathbb{Z})$ to the complex to get a cochain complex

$$\cdots \xleftarrow{(-)\cdot 2} \mathbb{Z} \xleftarrow{(-)\cdot 0} \mathbb{Z} \xleftarrow{(-)\cdot 2} \mathbb{Z} \xleftarrow{(-)\cdot 0} \mathbb{Z}$$

Step 3: Take group cohomology!

$$H^0(C_2;\mathbb{Z}) = \frac{\ker(\mathbb{Z} \xrightarrow{0} \mathbb{Z})}{\mathrm{Im}\ (0 \to \mathbb{Z})} = \mathbb{Z}$$
$$H^1(C_2;\mathbb{Z}) = \frac{\ker(\mathbb{Z} \xrightarrow{2} \mathbb{Z})}{\mathrm{Im}\ (\mathbb{Z} \xrightarrow{0} \mathbb{Z})} = 0$$
$$H^2(C_2;\mathbb{Z}) = \frac{\ker(\mathbb{Z} \xrightarrow{0} \mathbb{Z})}{\mathrm{Im}\ (\mathbb{Z} \xrightarrow{2} \mathbb{Z})} = \mathbb{Z}/2$$
$$\vdots$$

Which ends up resulting in

$$H^n(C_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n > 0 \text{ and even} \\ 0 & \text{else.} \end{cases}$$

As a fun little surprise, look up $H^*(\mathbb{RP}^\infty;\mathbb{Z})$. Do you notice anything?

Exercises

Exercise 8

Calculate the homology of the following chain complex:

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0.$$

where the right-most copy of $\mathbb{Z}$ is in degree 0.

Exercise 9

Calculate the homology of the following chain complex:

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0.$$

where the right-most copy of $\mathbb{Z}$ is in degree 0.

Exercise 10

Prove that if $f, g : C \to D$ are chain homotopic, then $H_*(f) = H_*(g)$.

Exercise 11

Let $\mathbb{Z}^\sigma$ denote the $\mathbb{Z}[C_2]$-module $\mathbb{Z}$, where the non-trivial element of C_2 acts by $n \mapsto -n$. Calculate $H^*(C_2, \mathbb{Z}^\sigma)$, the group cohomology of C_2 with coefficients in $\mathbb{Z}^\sigma$.

Exercise 12

Calculate the cohomology of C_p with p a prime with $\mathbb{F}_2$ coefficients with trivial action.
A good reference for this stuff is Brown's *Cohomology of Groups*.

Day 3

Spectral Sequences

I apologize, I was unable to take as detailed notes today. I'll add some good references, and then I'll try and give a TL;DR of spectral sequences.

References

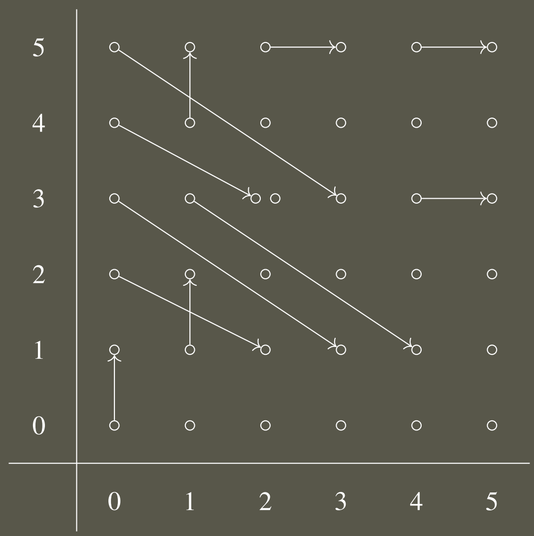
- Ravi Vakil's Puzzling Through Spectral Sequences
- Hatcher's Chapter 5 on Spectral Sequences
- ⋆ Haynes-Miller's Algebraic Topology II course notes. Specifically, check out the spectral sequences chapter.
- McCleary's User's guide to spectral sequences.

Exercises

These are exercises based on the package `spectralsequences` by Hood Chatham. See the documentation for an incredible review of what this package can do.

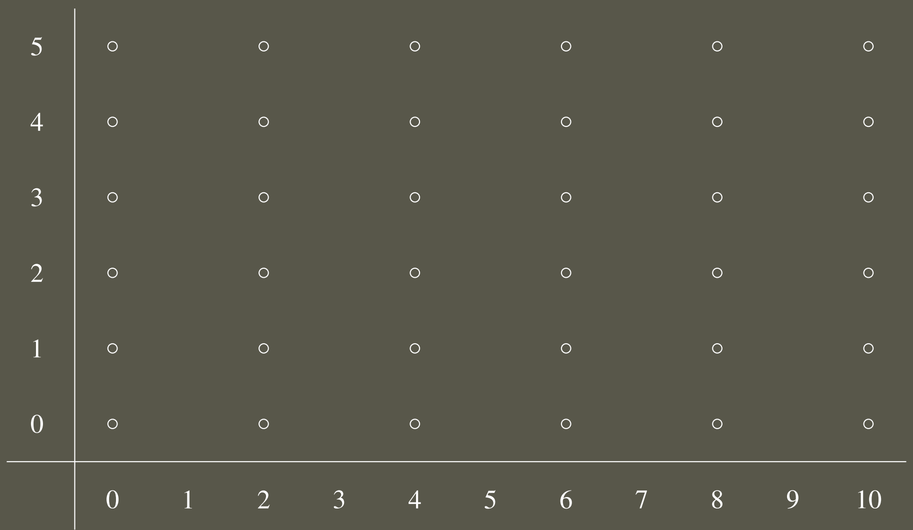
Exercise 13: (Harder, try last! Test out your skills with the package, not a real spectral sequence)

Make the following diagram on your own. (Hint: I'm on page 0, using cohomological Serre grading, and I used a for loop to make my life easier.) What is the E_4 page of this spectral sequence? Why does Scotty like the E_4 page?



Exercise 14: (Start here!)

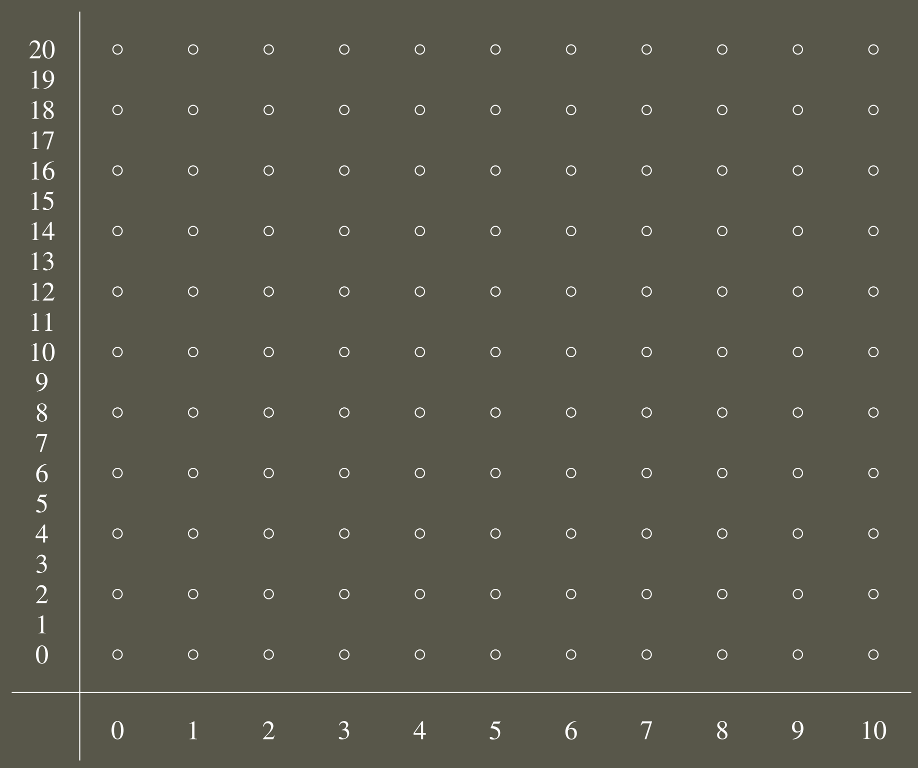
Make the following diagram on your own! (Hint: I'm on page 0, using Adams grading, and I used some loops plus math to make my life easier)



Question: What differentials have a chance to be supported here? Remember, we're using Adams grading!

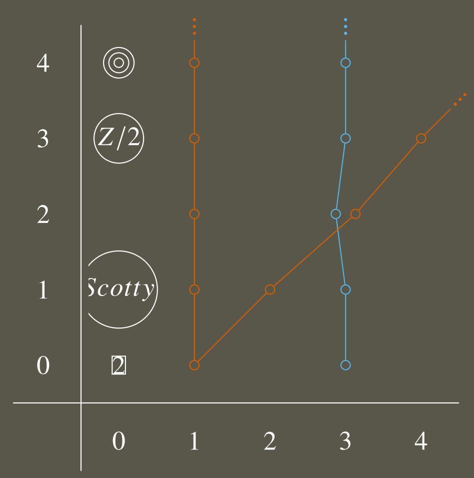
Exercise 15

What differentials could be supported on this spectral sequence using the Adams grading? Note that the scaling is different between the two axes!



Exercise 16

Create this spectral sequence page 0 with Adams grading!



Exercise 17

Create a spectral sequence with Adams grading using differentials going from 0 to 3 to make some dots in a pattern that is your first initial.

Solution TEX

Exercise 13

```
\begin{sseqdata}[name = Exercise 1, cohomological Serre grading]
    \foreach \x in {0,...,5}{
        \foreach \y in {0,...,5}{
            \class(\x,\y)
        }
    }
    \d0(1,4)
    \d0(1,1)
    \d0(0,0)
    \d1(2,5)
    \d1(4,5)
    \d1(4,3)
    \d2(0,2)
    \d2(0,4)
    \d3(0,3)
    \d3(1,3)
    \d3(0,5)
    \class(2,3)
\end{sseqdata}
\begin{center}
    \printpage[name = Exercise 1,page = 0]
\end{center}
```

Exercise 14

```
\begin{sseqdata}[name = ex2,Adams grading]
    \foreach \x in {0,...,5}{
        \foreach \y in {0,...,5}{
            \class({2*\x},\y)
        }
    }
\end{sseqdata}
\begin{center}
    \printpage[name = ex2,page = 0]
\end{center}
```

Exercise 15

```
\begin{sseqdata}[name = ex3,Adams grading, yscale = .4]
    \foreach \x in {0,...,10}{
        \foreach \y in {0,...,10}{
            \class (\x,{2*\y})
        }
    }
\end{sseqdata}
\begin{center}
    \printpage[name = ex3,page =0]
\end{center}
```

Exercise 16

```
\DeclareSseqGroup\tower{}{
    \class (0,0)
    \foreach \i in {1,...,5}{
        \class (0,\i)
        \structline(0,{\i-1},-1)(0,\i,-1)
    }
}
\DeclareSseqGroup\diagonal{}{
    \tower(0,0)
    \foreach\i in {1,...,5}{
        \class(\i,\i)
        \structline(\i-1,\i-1,-1)(\i,\i,-1)
    }
}
\begin{sseqdata}[name = ex4,Adams grading]
    \tower[wonglightblue](3,0) % I created this color
    \diagonal[wongred](1,0) % I created this color
    \class["2",rectangle] (0,0)
    \class["Scotty"] (0,1)
    \class["Z/2"] (0,3)
    \class[circlen =3] (0,4)
\end{sseqdata}
\begin{center}
    \printpage[name = ex4,x range = {0}{4},y range = {0}{4}]
\end{center}
```

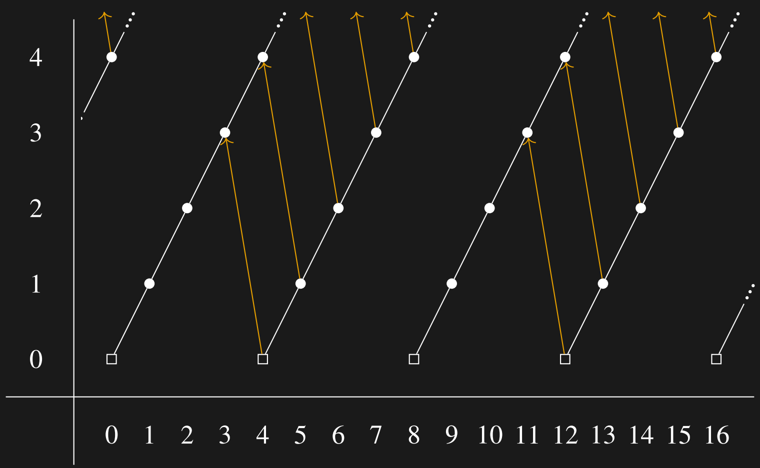
Day 4

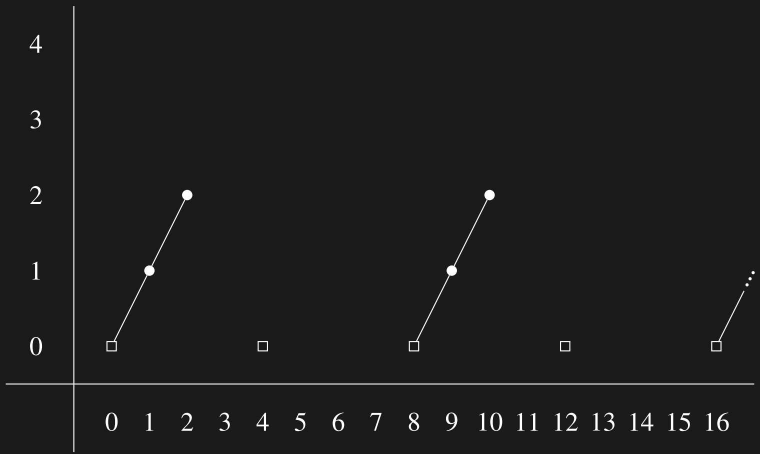
Lecture 1

Disclaimer: today had a lot of spectral sequences, and I'm not sure how coherent the writing is below here. I tried to get the spectral sequences correct, but I may have missed some important language helping us make these.

Yesterday we computed a homotopy fixed point spectral sequence. Suppose that $C_2 \curvearrowright \mathcal{E}$ where $\mathcal{E}$ is a spectrum.

$$E_2^{s,t} = H^s(C_2; \pi_t \mathcal{E}) \Rightarrow \pi_{t-s}\mathcal{E}^{hC_2}.$$





The final E_∞ page is $\pi_* E^{hC_2}$. This gives $\pi_* E = \mathbb{Z}[u^{\pm 1}]$ where $|u| = 2$. (If this were a polynomial ring where x is a variable, then u would have the same weight the x^2 "part.")

Question 1

How do we compute $\pi_*(E^{hC_2} \wedge V(0))$.

There is a spectral sequence

$$\underbrace{E_2^{s,t} = H^s(C_2, \pi_t(E \wedge V(0)))}_{\text{Question 1}} \Rightarrow \underbrace{\pi_{t-s}E^{hC_2} \wedge V(0)}_{\text{Question 2}}$$

For Question 1, we have

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to V(0) \qquad \text{(fiber sequence)}$$

$$E \wedge \mathbb{S} \xrightarrow{2} E \wedge \mathbb{S} \to E \wedge V(0) \qquad \text{(fiber sequence)}$$

This induces a long exact sequence in homotopy groups.

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{k+1}(E \wedge V(0)) \arrow[dll, out=-30, in=150] \\
\pi_k(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_k(E \wedge \mathbb{S}) \arrow[r] & \pi_k(E \wedge \mathbb{S}) \arrow[dll, out=-30, in=150] \\
\pi_{k-1}(E \wedge \mathbb{S}) \arrow[r, "2"] & \cdots .
\end{tikzcd}$$

After following the LES (using our spectral sequence E_∞ page on the the previous page), we get

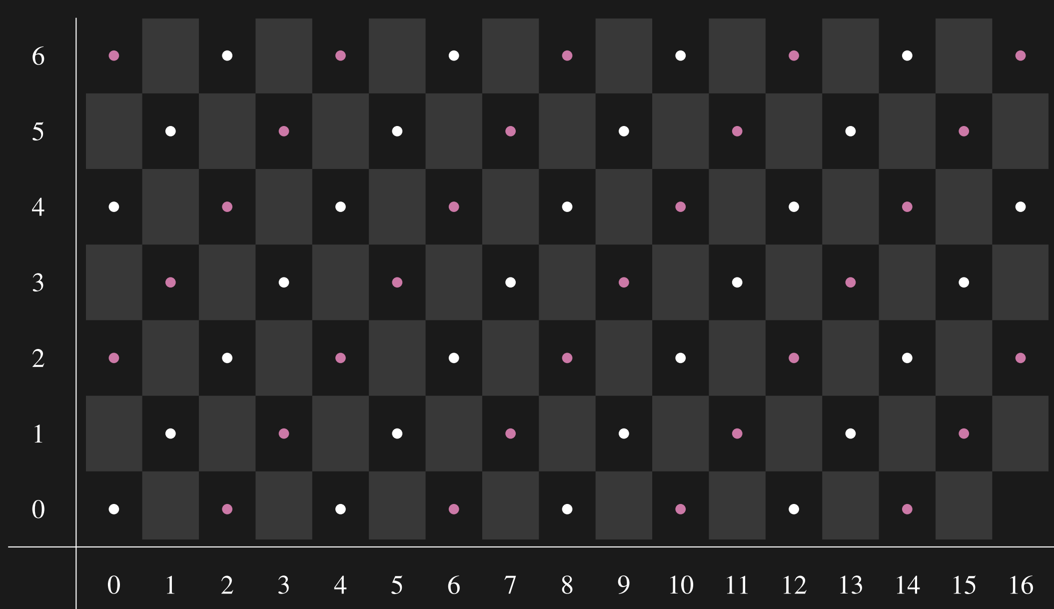
$$\pi_t(E \wedge V(0)) = \begin{cases} \mathbb{Z}/2 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}.$$

Therefore, our $E_2^{s,t}$ group we were looking for is coming from

$$H^s(C_2, \pi_t(E \wedge V(0))).$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_3(E \wedge V(0)) \arrow[dll, out=-30, in=150] \\
\pi_2(E \wedge \mathbb{S}) \arrow[r, "2", "u \mapsto 2u"'] & \pi_2(E \wedge \mathbb{S}) \arrow[r] & \pi_2(E \wedge \mathbb{S}) \arrow[dll, out=-30, in=150] \\
\pi_1(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_1(E \wedge \mathbb{S}) \arrow[r] & \pi_1(E \wedge \mathbb{S}) \arrow[dll, out=-30, in=150] \\
\pi_0(E \wedge \mathbb{S}) \arrow[r, "2"] & \pi_0(E \wedge \mathbb{S})
\end{tikzcd}$$

Here, we need to find $H^s(C_2; \pi_2(E \wedge V(0)))$ and put this into an Adams grading. This means the x-axis doesn't represent t but rather represents $t - s$. When you do this computation, we get the following picture for the E_2 page.



The pink dot represents the module $\mathbb{Z}/2\{u\}$ and the white dot represents the module $\mathbb{Z}/2\{1\}$. The map $E \xrightarrow{\times 2} E \to E \wedge V(0)$ as maps of spectra induces a long exact sequence in π_* which in this case is a short exact sequence!

$$0 \to \pi_{2t}E \to \pi_{2t}E \to \pi_{2t}E \wedge V(0) \to 0$$

$$0 \to \pi_*E \to \pi_*E \to \pi_*E \wedge V(0) \to 0.$$

FACT 3

A short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence in group cohomology,

$$\cdots \to H^{s-1}(G; M'') \to H^s(G; M') \to H^s(G; M) \to H^s(G; M'') \to H^{s+1}(G; M') \to \cdots.$$

Now, since we have the ses in modules

$$0 \to \pi_*(E) \to \pi_*(E) \to \pi_*(E \wedge V(0)) \to 0,$$

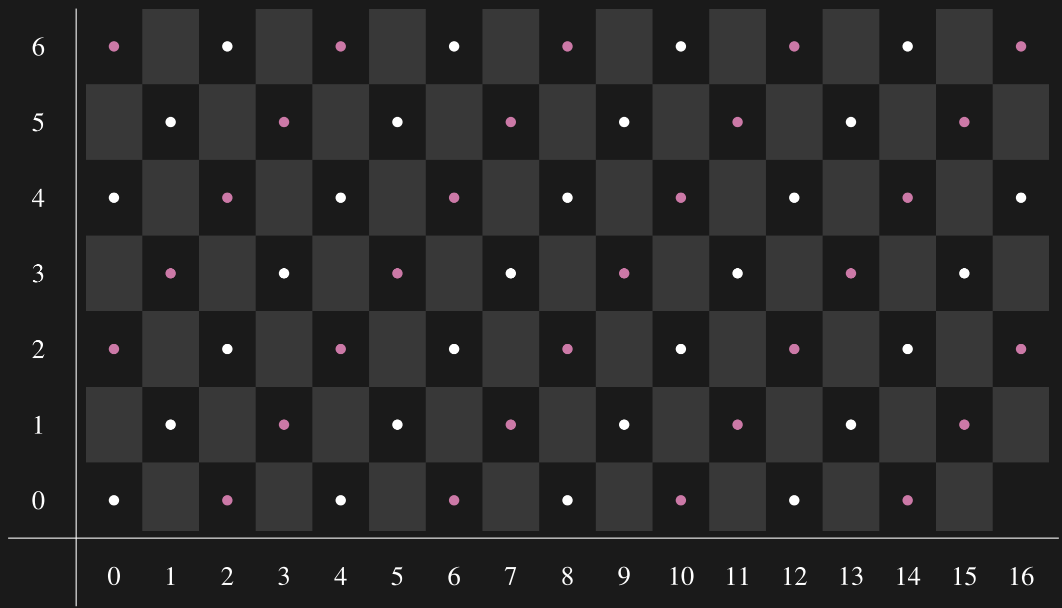
we get the LES in group cohomology

$$\cdots \to H^s(G; \pi_*E) \to \underbrace{H^s(G; \pi_*E)}_{E_2^{s,*}(E)} \to \underbrace{H^s(G, \pi_*E \wedge V(0))}_{E_2^{s,*}(E \wedge V(0))} \to \underbrace{H^{s+1}(G; \pi_*E)}_{E_2^{s+1,*}(E)} \to \cdots.$$

Since the elements are inside of our spectral sequences, we actually get maps between spectral sequences! This now tells us information about our differentials between the different spectral sequences!

Lecture 2

Again, we'll use this spectral sequence



and the maps

$$\cdots \to H^s(G;\pi_*E) \to \underbrace{H^s(G;\pi_*E)}_{E_2^{s,*}(E)} \to \underbrace{H^s(G,\pi_*E\wedge V(0))}_{E_2^{s,*}(E\wedge V(0))} \to \underbrace{H^{s+1}(G;\pi_*E)}_{E_2^{s+1,*}(E)} \to \cdots.$$

From this les in cohomology, we can see that the spectral sequences associated to each piece converge at E_∞ to the long exact sequence

$$\cdots \to \pi_{t-s}E^{hG} \to \pi_{t-s}E^{hG} \to \pi_{t-s}E^{hG}\wedge V(0) \to \pi_{t-s-1}E^{hG} \to \cdots.$$

The question which started out at the beginning of the lecture was (effectively) this:

Question 2

What is a morphism of spectral sequences? If we have a LES in cohomology which induces a map for spectral sequences, how can we say this generally?

Remark 3: Method 1

Let's look at the LES sequence,

$$\begin{tikzcd}
\overset{0}{\pi_6 E^{hC_2}} \arrow[r, "2"] & \overset{0}{\pi_6 E^{hC_2}} \arrow[r] & \pi_6 E^{hC_2} \wedge V(0) \arrow[dll, out=-90, in=90] \\
\overset{0}{\pi_5 E^{hC_2}} \arrow[r, "2"] & \overset{0}{\pi_5 E^{hC_2}} \arrow[r] & \pi_5 E^{hC_2} \wedge V(0) \arrow[dll, out=-90, in=90] \\
\overset{\mathbb{Z}}{\pi_4 E^{hC_2}} \arrow[r, "2"] & \overset{\mathbb{Z}}{\pi_4 E^{hC_2}} \arrow[r] & \pi_4 E^{hC_2} \wedge V(0) \arrow[dll, out=-90, in=90] \\
\pi_3 E^{hC_2} \arrow[r] & \cdots &
\end{tikzcd}.$$

This means that $\pi_6(E^{hC_2} \wedge V(0)) = 0$ and $\pi_5(E^{hC_2} \wedge V(0)) = 0$.

FACT 4

If we're given an exact sequence

$$A \xrightarrow{f} B \to C \to D \xrightarrow{g} E,$$

where these are $\mathbb{Z}/2$-modules (i.e. $\mathbb{Z}/2$ vector spaces), then $C \cong \operatorname{coker} f \oplus \ker g$.

Theorem 3

Let G be a finite group. If E^{hG} satisfies the property that there is a number P, called the period, such that for any n

$$\pi_{n+P}(E^{hG}) = \pi_n(E^{hG}),$$

then if M is a finite complex[a] $\pi_*(E^{hF} \wedge M)$ satisfies the same periodic property with period $p \le P$ and this period is a factor of this number.

[a] suspension spectrum of a finite CW complex is an instance of this

This should convince us that our result is going to be something less than or equal to 8-periodic since E^{hC_2} is 8-periodic.

Remark 4: Method 2 for calculating spectral sequences

The second method goes as follows.

$$\begin{tikzcd}
E_3^{s,t}(E^{hC_2}) \arrow[r, "f"] \arrow[d, "d_3"'] \arrow[dr, phantom, "\circlearrowleft"] & E_3^{s,t}(E^{hC_2} \wedge V(0)) \arrow[d, "d_3"] \\
E_3^{s+3,t+2}(E^{hC_2}) \arrow[r, "f"'] & E_3^{s+3,t+2}(E^{hC_2} \wedge V(0))
\end{tikzcd}$$

This is commutative, so in other words, if x is in the top left corner, we require

$$d_3(f(x)) = f(d_3(x)).$$

This allows us to import information from one spectral sequence to the other and vice versa!

After Hours

Definition 18: Morphism of spectral sequences

A morphism of spectral sequences

$$(E_\bullet^{*,*}, d_\bullet) \xrightarrow{f} (\widetilde{E}_\bullet^{*,*}, \widetilde{d}_\bullet)$$

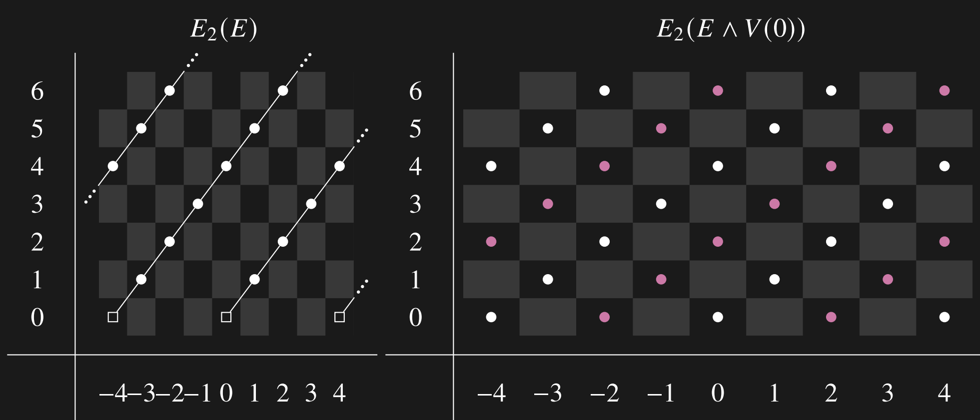
is a collection of maps

- $E_k^{s,t} \xrightarrow{f_k^{s,t}} \widetilde{E}_k^{s,t}$

such that

- $$\begin{tikzcd} E_k^{s,t} \arrow[r, "f_k^{s,t}"] \arrow[d, "d_k"'] & \widetilde{E}_k^{s,t} \arrow[d, "\widetilde{d}_k"] \\ E_k^{s+k,t-k+1} \arrow[r, "f_k^{s+k,t-k+1}"] & \widetilde{E}_k^{s+k,t-k+1} \end{tikzcd}$$, and
- $f_{k+1}^{s,t} = H_*(f_k^{s,t})$.

Now, let's do something concrete.



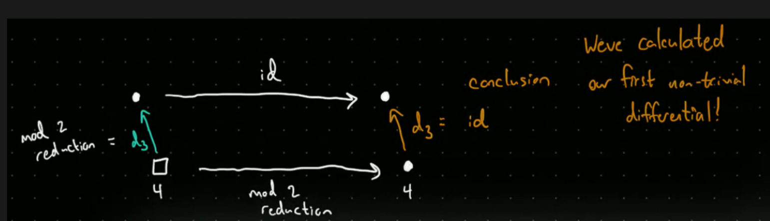
Here, we'd like to see what is going on where $(t-s, s) = (3, 3)$, and this gives a map

$$\square \xrightarrow{f} \bullet$$
$$\mathbb{Z} \xrightarrow{?} \mathbb{Z}/2$$

Now, we can look at the $E_2(E)$ corresponding page, and we see an exact sequence

$$\begin{array}{ccccccc} H^0(C_2;\mathbb{Z}) & \xrightarrow{2} & H^0(C_2;\mathbb{Z}) & \longrightarrow & H^0(C_2;\mathbb{Z}) & \xrightarrow{0} & 0 \\ \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2. & & \end{array}$$

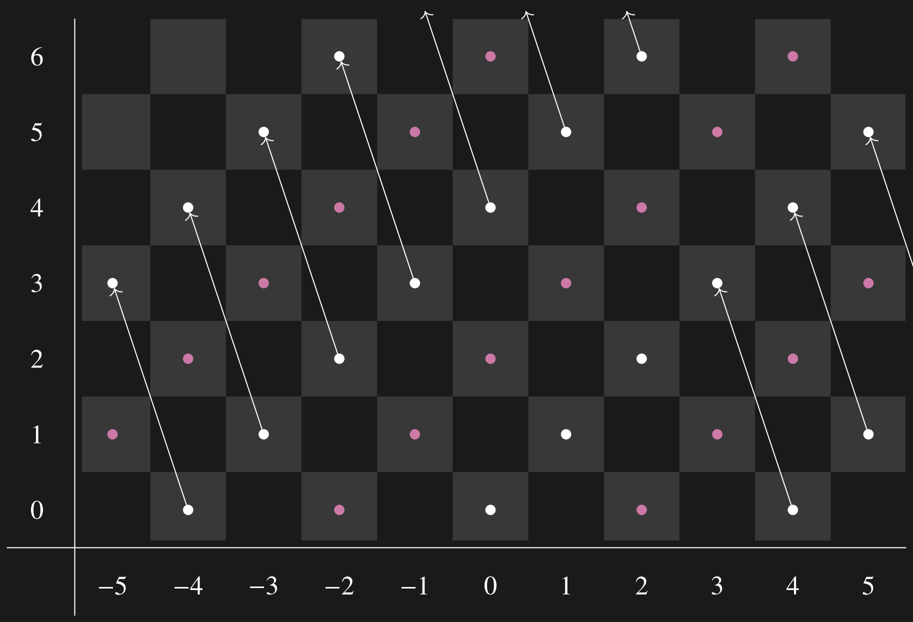
So here, when we map using d_3, we get from the LES in cohomlogy, we get $d_3 : \mathbb{Z}/2 \to \mathbb{Z}/2$ is the identity.

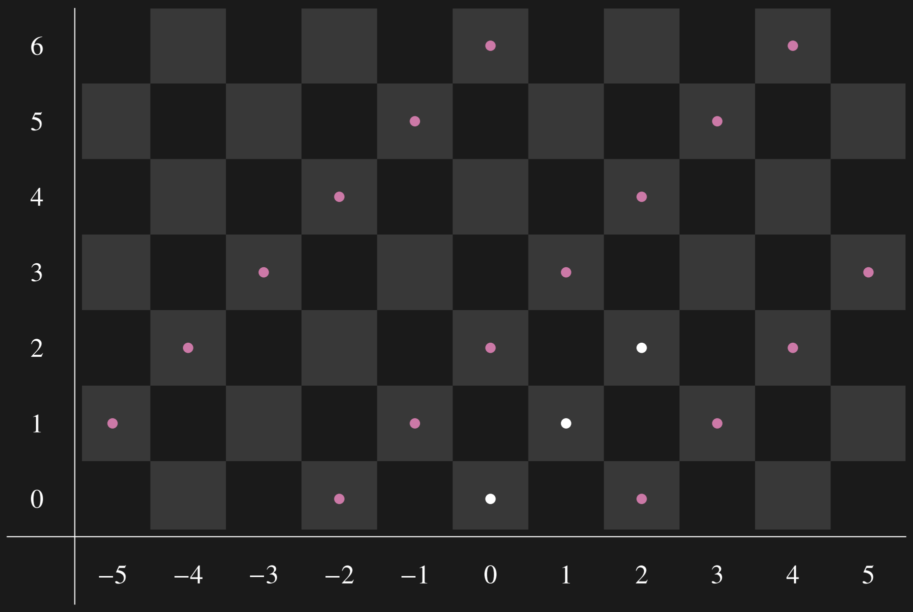


Proposition 1

We have a tower of nontrivial differentials as shown in the image above.

Proof. Either do direct calculation or use the $\pi_*\mathbb{S}$-module structure to figure it out. $\square$





We still don't know a lot about the pink diagonal which starts at $(-2, 0)$.

We've exhausted the data of $E \longrightarrow E \wedge V(0)$.

Question 3

How can we proceed?

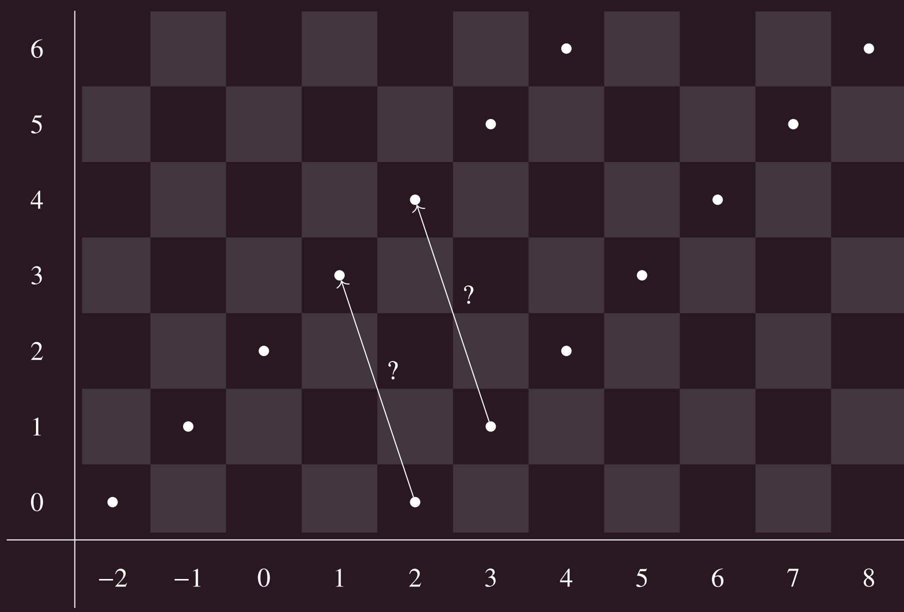
Options.

1. We can use knowledge of the final answer (which we can find from the long exact sequence in $\pi_*(0)$ associated to the cofiber sequence $E \xrightarrow{2} E \to E \wedge V(0)$.) This is a good bet!
2. We did all of this work using $E \to E \wedge V(0)$ to get a lot of information. Why don't we also try using $E \wedge V(0) \to \Sigma E$? This is because of extending the cofiber sequence![1] Note that $\pi_t \Sigma E = \pi_{t\pm 1} E$. This is a good fact to have!

□

Remark 5

Suppose there exists a spectral sequence with E_3 page like



If we know that $\pi_2(E) = 0$, then we can deduce that this d_3 differential must exist! Because if it didn't exist, then we wouldn't get that $\pi_2 = 0$.

[1]The Puppe sequence is a sequence of spaces that you arrive at from taking consecutive cofibers. It's slick!

Question 4: Why is $\pi_*\mathbb{S}$ a ring?

Concrete answer: How do we multiply? Suppose $\alpha \in \pi_k\mathbb{S}$ and $\beta \in \pi_\ell\mathbb{S}$. We can make a new element $\alpha\beta \in \pi_{k+\ell}\mathbb{S}$.
We can represent

$$\alpha = [S^{k+i} \xrightarrow{a} S^i], \quad \beta = [S^{\ell+j} \xrightarrow{b} S^j].$$

We can define

$$\alpha\beta := \left[S^{k+\ell+i+j} \cong S^{k+i} \wedge S^{\ell+j} \xrightarrow{a\wedge b} S^i \wedge S^j \cong S^{i+j}\right] \in \pi_{k+\ell}\mathbb{S}.$$

Furthermore, $\alpha\beta = (-1)^{|\alpha|+|\beta|}\beta\alpha$. This is called "graded commutative."

Formal Answer: $\mathbb{S}$ is a **ring spectrum**. A ring spectrum R is a spectrum with some extra data. This extra data is

- the unit map: $\mathbb{S} \xrightarrow{\eta} R$
- the multiplication map: $R \wedge R \xrightarrow{\mu} R$

such that several diagrams commute.

Claim 1

If R is a ring spectrum, then $\pi_*(R)$ is a ring.

Proof. $\pi_k R$ = homotopy classes of maps of spectra from $\mathbb{S}^k := \Sigma^\infty S^k$. Now, if we have $\alpha = [S^k \xrightarrow{[} a]R] \in \pi_k R$, $\beta = [S^\ell \xrightarrow{b} R] \in \pi_\ell R$, then we can define

$$\alpha\beta = \left[S^{k+\ell} \cong S^k \wedge S^\ell \xrightarrow{a\wedge b} R \wedge R \xrightarrow{\mu} R\right] \in \pi_{k+\ell}R.$$

□

Question 5: What is η from all the spectral sequences we've been looking at?

η is an element $\eta \in \pi_1\mathbb{S}$. First, note that $S^3 \cong$ unit sphere $\subset \mathbb{C}^2$. Then we have a map

$$\begin{aligned} S^3 &\longrightarrow \mathbb{CP}^1 \cong S^2 \\ (z, w) &\longmapsto [z : w] \end{aligned}$$

Day 5

Lecture 1

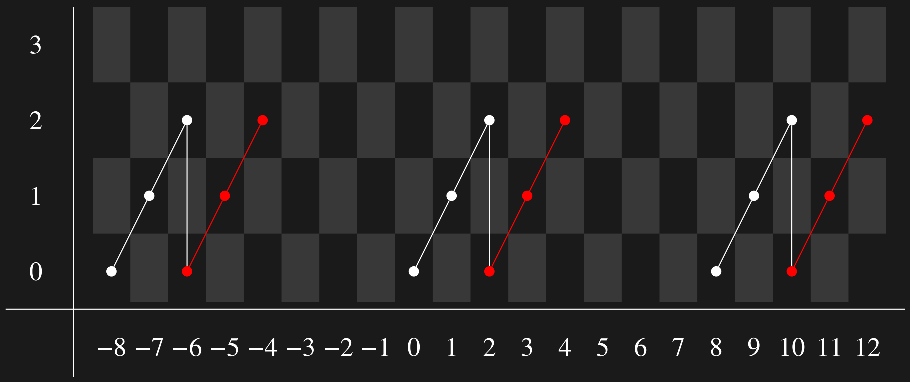
Today we started on this Miro board where we went over our calculation of the HFPSS for $E^{hC_2} \wedge V(0)$ using method 1.

The fiber sequence $E^{hC_2} \xrightarrow{2} E^{hC_2} \to E^{hC_2} \wedge V(0)$ induces a long exact sequence in homotopy

$$
\begin{aligned}
\cdots \to \underbrace{\pi_8(E^{hC_2})}_{\mathbb{Z}} &\xrightarrow{2} \underbrace{\pi_8(E^{hC_2})}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_8(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/2} \\
\to \underbrace{\pi_7(E^{hC_2})}_{0} &\xrightarrow{2} \underbrace{\pi_7(E^{hC_2})}_{0} \longrightarrow \underbrace{\pi_7(E^{hC_2} \wedge V(0))}_{0} \\
\to \underbrace{\pi_6(E^{hC_2})}_{0} &\xrightarrow{2} \underbrace{\pi_6(E^{hC_2})}_{0} \longrightarrow \underbrace{\pi_6(E^{hC_2} \wedge V(0))}_{0} \\
\to \underbrace{\pi_5(E^{hC_2})}_{0} &\xrightarrow{2} \underbrace{\pi_5(E^{hC_2})}_{0} \longrightarrow \underbrace{\pi_5(E^{hC_2} \wedge V(0))}_{0} \\
\to \underbrace{\pi_4(E^{hC_2})}_{\mathbb{Z}} &\xrightarrow{2} \underbrace{\pi_4(E^{hC_2})}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_4(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/2} \\
\to \underbrace{\pi_3(E^{hC_2})}_{0} &\xrightarrow{2} \underbrace{\pi_3(E^{hC_2})}_{0} \longrightarrow \underbrace{\pi_3(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/2} \\
\to \underbrace{\pi_2(E^{hC_2})}_{\mathbb{Z}/2} &\xrightarrow{2} \underbrace{\pi_2(E^{hC_2})}_{\mathbb{Z}/2} \longrightarrow \underbrace{\pi_2(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/4} \\
\to \underbrace{\pi_1(E^{hC_2})}_{\mathbb{Z}/2} &\xrightarrow{2} \underbrace{\pi_1(E^{hC_2})}_{\mathbb{Z}/2} \longrightarrow \underbrace{\pi_1(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/2} \\
\to \underbrace{\pi_0(E^{hC_2})}_{\mathbb{Z}} &\xrightarrow{2} \underbrace{\pi_0(E^{hC_2})}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_0(E^{hC_2} \wedge V(0))}_{\mathbb{Z}/2} \to 0
\end{aligned}
$$

In the long exact sequnce above, we figured out all the homotopy groups of $E^{hC_2} \wedge V(0)$. Now, we can look at the spectral sequence to figure out way more of the differentials!

Now, we'll draw the E_∞ page of our spectral sequence for $E^{hC_2} \wedge V(0)$.



Theorem 4

The homotopy fixed point spectral sequences for E^{hG} has a vanishing line $E_\infty^{s,t} = 0$ for $s > s_0$ where s_0 depends on E^{hG} for G a finite group.

The homotopy fixed point spectral sequence for $E^{hG} \wedge M$ for a finite complex M has a vanishing line for $s > s_0 + \varepsilon$. ε depends on M.

Note that E^{hC_2} has a vanishing line at 2.

Tasks for the afternoon, plan for next week

Afternoon

1. Can we prove that $E_4 = E_\infty$ in the homotopy fixed point spectral sequence for $E^{hC_2} \wedge V(0)$.
   a. Using nothing.
   b. Using the vanishing line theorem.
2. Figure out the answer to the question: How many group extensions are there of $\mathbb{F}_2$-modules of rank 3. Some examples include
$$\cdots \to (\mathbb{Z}/2)^3 \to \cdots$$
$$\cdots \to \mathbb{Z}/4 \oplus \mathbb{Z}/2 \to \cdots$$
$$\cdots \to \mathbb{Z}/8 \to \cdots$$
3. Each question has 2 parts. First part is finding the group cohomology. The second part is working out the spectral sequence calculation. This needs to be written in detail!
   a. (Known, public) Find $H^*(C_2; \pi_* E_2) \Rightarrow \pi_* E_2^{hC_2}$. Here, use that
$$\pi_* E_2 = \mathbb{Z}[[u_1]][u^{\pm 1}], \qquad |u| = 2, |u_1| = 0.$$
   The actions of C_2 on these elements are
$$C_2 \cdot u = -u, \qquad C_2 \cdot u_1 = u_1.$$
   b. (Known but not public) $H^*(C_6, \pi_* E_2) \Rightarrow \pi_* E_2^{hC_6}$
   c. (Not for sure known, but some) $H^*(C_6, \pi_* E_2 \wedge V(0)) \Rightarrow \pi_*(E_2^{hC_2} \wedge V(0))$
   d. (Plan for REU!) $H^*(C_6, \pi_*(E_2 \wedge Y)) \Rightarrow \pi_*(E_2^{hC_6} \wedge Y)$
4. Come up with things that we'd like to discuss in more detail.
5. Type up a couple of lemmas. Type up the computation we did today. The goal of this one is to work with the `spectralsequences` package and to get more comfortable with the theorems and lemmas we used. The more we have locked into our brains, the easier it will be to do our harder calculations as this REU moves forward!

Ideas for Lectures for next week

1. (Irina) Chromatic Homotopy Theory
2. Student lectures:
    - Spectral Sequences
    - ??
    - ??

After Lunch

For the last two days we've been looking at a spectrum[2] $E = E_1$ with $\pi_* E = \mathbb{Z}[u^{\pm 1}]$, aka

$$\pi_k(E) = \begin{cases} \mathbb{Z} & k = 4\ell \\ \mathbb{Z}^\sigma & k = 4\ell + 2 \\ 0 & \text{else} \end{cases} = \begin{cases} \mathbb{Z}\left\{u^{2\ell}\right\} & k = 4\ell \\ \mathbb{Z}\left\{u^{2\ell+1}\right\} & k = 4\ell + 2 \\ 0 & \end{cases}.$$

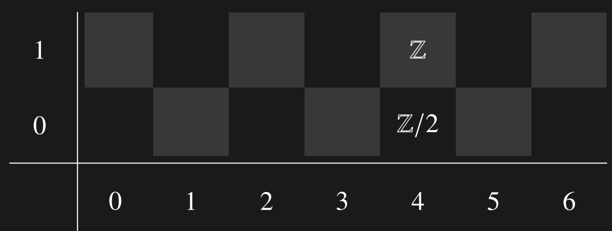
Now, we are using a new spectrum[3] called E_2 with

$$\pi_*(E_2) = \mathbb{Z}[[u_1]][u^{\pm 1}], \qquad |u_1| = 0, |u| = 2.$$

Alternatively, we can write this as

$$\pi_k(E_2) = \begin{cases} \mathbb{Z}[[u_1]]\left\{u^{2\ell}\right\} & k = 4\ell \\ \mathbb{Z}^\sigma[[u_1]] \text{ or } \mathbb{Z}[[u_1]]\left\{u^{2\ell+1}\right\} & k = 4\ell + 2 \\ 0 & \text{otherwise} \end{cases}.$$

If our E_∞ page is



We can read off $\pi_4 E$ as $\mathbb{Z}/2 \oplus \mathbb{Z}$.

[2] height 1 Morava E-theory
[3] height 2 Morava E-theory

Chapter 2

Appendix

Appendix A

Definition 19: (Homotopy) Fixed points

Let $G \curvearrowright X$ where X is a topological space. The **fixed points** of X are, all equivalently,

$$\begin{aligned} X^G &:= \{x \in X \mid gx = x \; \forall g \in G\} \\ &:= G\mathsf{Maps}(*, X) \\ &:= \mathsf{Maps}(*, X)^G = \left\{f : * \to X \mid gf(g^{-1}*) = f(*) \; \forall g \in G\right\}. \end{aligned}$$

The **homotopy fixed points** of the topological space X are, all equivalently,

$$\begin{aligned} X^{hG} &:= G\mathsf{Maps}(EG, X) \\ &:= \mathsf{Maps}(EG, X)^G = \left\{f : EG \to X \mid \; gf(g^{-1}e) = f(e) \forall g \in G\right\} \\ &= \mathsf{Maps}(BG, X). \end{aligned}$$

Here BG is the **classifying space** for G and EG is the total space of the classifying space. Key properties of EG are that $EG \simeq *$, i.e. EG is contractible, and EG has a G action which is free on EG. BG is defined to be the quotient space

$$BG := EG/G = \frac{EG}{\sim}, \quad e \sim e' \iff \exists g \in G \text{ such that } e = e' \cdot g.$$

Here are some instances of BG and EG.

G	EG	BG
$\{e\}$	$*$	$*$
C_2	$S^\infty \subset \mathbb{R}^{\oplus\infty}$	$\mathbb{RP}^\infty$
$\mathbb{Z}$	$\mathbb{R}$	S^1
S^1	$S^\infty \subset \mathbb{C}^{\oplus\infty}$	$\mathbb{CP}^\infty$
$\mathsf{SU}(2)$	$S^\infty \subset \mathbb{H}^{\oplus\infty}$	$\mathbb{HP}^\infty$
$\mathsf{O}(k)$	$V_k(\infty) \subset (\mathbb{R}^k)^{\oplus\infty}$	$\mathsf{Gr}_k(\mathbb{R}^\infty)$
$\mathsf{U}(k)$	$V_k(\infty) \subset (\mathbb{C}^k)^{\oplus\infty}$	$\mathsf{Gr}_k(\mathbb{C}^\infty)$

Some letters I wrote down are O the orthogonal group, U, the unitary group and SU the special unitary group, V_k the Stiefel manifold, Gr the Grassmannian, and $\mathbb{H}$ the quaternions which make a 4-dimensional real vector space.

Homotopy groups

FACT 5: Homotopy groups of E^{hC_2}

$$\pi_0(E^{hC_2}) = \mathbb{Z} \qquad \pi_5(E^{hC_2}) = 0$$
$$\pi_1(E^{hC_2}) = \mathbb{Z}/2 \qquad \pi_6(E^{hC_2}) = 0$$
$$\pi_2(E^{hC_2}) = \mathbb{Z}/2 \qquad \pi_7(E^{hC_2}) = 0$$
$$\pi_3(E^{hC_2}) = 0 \qquad \pi_8(E^{hC_2}) = \mathbb{Z}$$
$$\pi_4(E^{hC_2}) = \mathbb{Z} \qquad \text{This is 8 periodic}$$

FACT 6: Homotopy groups of $E^{hC_2} \wedge V(0)$

$$\pi_0(E^{hC_2} \wedge V(0)) = \mathbb{Z}/2 \qquad \pi_5(E^{hC_2} \wedge V(0)) = 0$$
$$\pi_1(E^{hC_2} \wedge V(0)) = \mathbb{Z}/2 \qquad \pi_6(E^{hC_2} \wedge V(0)) = 0$$
$$\pi_2(E^{hC_2} \wedge V(0)) = \mathbb{Z}/4 \qquad \pi_7(E^{hC_2} \wedge V(0)) = 0$$
$$\pi_3(E^{hC_2} \wedge V(0)) = \mathbb{Z}/2 \qquad \pi_8(E^{hC_2} \wedge V(0)) = \mathbb{Z}/2$$
$$\pi_4(E^{hC_2} \wedge V(0)) = \mathbb{Z}/2 \qquad \text{This is} \leq 8 \text{ periodic}$$

FACT 7: A couple of important maps in stable homotopy

Here, when I have a finite dimensional vector space V, when I write S^V I mean V^+, the one-point compactification of V.

$$\begin{array}{lll}
2 \in \pi_0 & \mathbb{R}^2 \setminus \{0\} \supset \mathbb{S}^1 \xrightarrow[2]{(x,y)\mapsto xy^{-1}} S^{\mathbb{R}} = \mathbb{S}^1 & C_2 = \Sigma^{-1}\Sigma^\infty \mathbb{RP}^2 \\
\eta \in \pi_1 & \mathbb{C}^2 \setminus \{0\} \supset \mathbb{S}^3 \xrightarrow[\eta]{(z_1,z_2)\mapsto z_1 z_2^{-1}} S^{\mathbb{C}} = \mathbb{S}^2 & C_\eta = \Sigma^{-2}\Sigma^\infty \mathbb{CP}^2 \\
\nu \in \pi_3 & \mathbb{H}^2 \setminus \{0\} \supset \mathbb{S}^7 \xrightarrow[\nu]{(w_1,w_2)\mapsto w_1 w_2^{-1}} S^{\mathbb{H}} = \mathbb{S}^4 & C_\nu = \Sigma^{-4}\Sigma^\infty \mathbb{HP}^2 \\
\sigma \in \pi_7 & \mathbb{O}^2 \setminus \{0\} \supset \mathbb{S}^{15} \xrightarrow[\sigma]{(\omega_1,\omega_2)\mapsto \omega_1 \omega_2^{-1}} S^{\mathbb{O}} = \mathbb{S}^8 & C_\sigma = \Sigma^{-8}\Sigma^\infty \mathbb{OP}^2
\end{array}.$$

Alternatively, note that $S^1 \cong \mathbb{RP}^1, S^2 \cong \mathbb{CP}^1, S^4 \cong \mathbb{HP}^1, S^8 \cong \mathbb{OP}^1$ and these maps are all $(a,b) \mapsto [a:b]$ in homogeneous coordinates.

Definition 20: pullbacks

Let C be a category which contains the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}.$$

A **pullback** of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $W \xrightarrow{p_1} X$
- A map $W \xrightarrow{p_2} Y$

such that

- The diagram $\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$ commutes [a]
- If someone hands you a commutative diagram $\begin{tikzcd} A \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{h}$ such that

$$\begin{tikzcd} A \arrow[dr, "\widetilde{h}"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\ & W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}.$$

[b]

Let C be a category with a subcategory $\mathcal{D}$. We say $\mathcal{D}$ is **closed under pullbacks** by morphisms in C if for all arrows $X \xrightarrow{f} Z$ in $\mathcal{D}$ and for all $Y \xrightarrow{g} Z$ in C such that we can form the pullback $\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then the arrow $W \xrightarrow{p_1} Z$ is in $\mathcal{D}$.

[a] "$\circlearrowleft$" is a 'long'hand for commutes, and people usually suppress it from notation.
[b] As a shorthand, people usually write the pullback like this:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Definition 21: pushouts

Let C be a category which contains the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \\ X & \end{tikzcd}.$$

A **pushout** of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $X \xrightarrow{i_1} W$
- A map $Y \xrightarrow{i_2} W$

such that

- The diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \end{tikzcd}$ commutes
- If someone hands you a commutative diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\ell_2"] \\ X \arrow[r, "\ell_1"] & A \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{\ell}$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "\ell_2"] & \\ X \arrow[r, "i_1"] \arrow[drr, bend right, "\ell_1"'] & W \arrow[dr, "\widetilde{\ell}"] & \\ & & A \end{tikzcd}.$$

[a]

[a] As a shorthand, people usually write the pushout like this:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

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