Contents

Chapter 1

Week 1

Day 1

Lecture 1: Lot's of definitions!

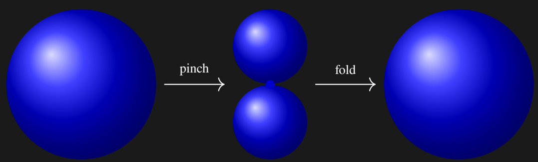
Plan
We are going to try and compute
$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).
- $E(n,p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G-action.

$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned} \qquad\qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be
$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$
- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.
$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$



The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{S^0, S^1, S^2, \dots\right\}.$$

  FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\times 2} \mathbb{Z} \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 2. \end{aligned}$$

  The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

  We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

  Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \mathrm{Maps}(S^n, X)_{/\text{homotopy}}.$$

  Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X)$, $g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, \quad g : S^m \to X.$$

  What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$\begin{tikzcd}
 & X \wedge S^m \arrow[dr, "\mathbb{1}\times g"] & & \\
S^{n+m} = S^n \wedge S^m \arrow[ur, "f\wedge\mathbb{1}"] \arrow[rr, "f\wedge g"] \arrow[dr, "\mathbb{1}\wedge g"'] & & X \wedge X \arrow[r, "\mu"] & X \\
 & S^n \wedge X \arrow[ur, "f\wedge\mathbb{1}"'] & &
\end{tikzcd}$$

  which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \mathrm{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A spectrum[a] X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \ldots\} = \{X_n\}_{n\in\mathbb{N}}$$

together with structure maps

$$\Sigma X_n \longrightarrow X_{n+1}.$$

[a]What we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as homeomorphisms.

Example 1

1. The sphere spectrum $\mathbb{S} = \left\{S^0, S^1, \ldots\right\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.
2. Suspension spectrum $\Sigma^\infty X = \left\{X, \Sigma X, \Sigma^2 X, \ldots\right\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.

$$\begin{tikzcd}
\mathsf{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left=60, "\Sigma^\infty"] & \mathsf{PreSp} \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & \mathsf{Sp} \arrow[l, shift left, "U"] \arrow[ll, bend left=60, "\Omega^\infty"]
\end{tikzcd}$$

"Why were spectra invented?" you may ask. The answer comes in the form of Brown's representability theorem. To understand this, we need a few definitions.

Definition 2

A generalized homology theory E E is a functor

$$E : \mathsf{Spaces} \longrightarrow \mathsf{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.
- Exactness: Exact sequence in homology from a cofiber sequence.
- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \longrightarrow E_*(X, B)$ is an isomorphism.
- Additivity: Coproducts in Spaces induce coproducts in homology.

For more details, see Wikipedia on generalized cohomology.

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized (co)homology theories and spectra. Given a spectrum $\mathcal{E}$, the homology is given by

$$\mathcal{E}_*(X) = \pi_*(\mathcal{E} \wedge X).$$

The cohomology associated to the spectrum $\mathcal{E}$ is given by

$$\mathcal{E}^*(X) = [X, \mathcal{E}].$$

Definition 3: Fiber Sequences

We'll come back to this! The key is that in spectra land, it goes back and forth in both directions.

FACT 1

Any fiber sequence $X \to Y \twoheadrightarrow Z$ gives rise to a long exact sequence in π_*,

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{k+1}Z \arrow[dll, out=-90, in=90] \\
\pi_k X \arrow[r] & \pi_k Y \arrow[r] & \pi_k Z \arrow[dll, out=-90, in=90] \\
\pi_{k-1} X \arrow[r] & \cdots &
\end{tikzcd}$$

Lecture 2: Algebra

Rings

First, let's talk about commutative *(order of multiplication doesn't matter)*, unital *(the ring has the element 1)* rings. Every time I write R as a ring, I mean this version of a ring.

Example 2

- $\mathbb{Z}$
- $\mathbb{Z}/n$ for $n \geq 2$
- $\mathbb{F}_p := \mathbb{Z}/p$ with p a prime. A special case of this is $\mathbb{F}_2 = (\{0, 1\}, +, \times)$.
- $\mathbb{Z}[x]$, $\mathbb{F}_2[x]$, $R[x]$, aka polynomial rings in one variable.
- $\frac{\mathbb{F}_2[x]}{(x^3+1)}$, a ring mod out by an ideal.
- $\mathbb{Z}[G]$ for G an abelian group, the group ring.
- $\mathbb{F}_4 := \dfrac{\mathbb{F}_2[x]}{(x^2 + x + 1)}$, <u>the</u> field with $4 (= 2^2)$ elements.
- $\mathbb{Z}[[x]] = \left\{\sum_0^\infty a_k x^k \mid \forall k, a_k \in \mathbb{Z}\right\}$, the power series ring
- $\mathbb{Z}((x))$, the laurent series ring.

Modules

Definition 4: Module

module M over a commutative ring R is an abelian group M together with a scaling map

$$R \otimes M \to M$$

$$r \otimes m \mapsto r \cdot m.$$

Example 3

A vector space V over the ring $\mathbb{R}$ (or any field $\mathbb{F}$) is the same thing as an $\mathbb{R}$-module.

Example 4

If R is a ring, then an ideal $I \subseteq R$ is the same thing as a submodule of R.

Exact Sequences

Definition 5: Short Exact Sequence

A short exact sequence is

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

such that

$$\ker(\text{each map}) = \operatorname{Im}(\text{previous map}).$$

For specificity, we need

i. f is injective

ii. g is surjectve

iii. $\ker g = \operatorname{Im} f$.

Example 5

Let R be a field, say $\mathbb{F}_2$, let V be an R-vector space and let $W \leq V$ be a subspace. Then

$$0 \to W \to V \to V/W \to 0$$

is a short exact sequence.

Example 6

Let $R = \mathbb{Z}$. Consider the map

$$0 \to \mathbb{Z}/2 \xrightarrow{[1\ 0]} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{Z}/2 \to 0.$$

What is the composition of these maps? Is this sequence exact?

Example 7

Let $R = \mathbb{Z}$. Then consider the sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{(-)\cdot 2} \mathbb{Z}/4 \xrightarrow{() \mod 2} \mathbb{Z}/2 \to 0.$$

Is this a short exact sequence? If so, how does it compare to the prior example?

Remark 1

$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4$ as groups. Prove it!

Oftentimes, we are interested in some module M, and we know that it fits into a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M', M'' are known. Determining M given M' and M'' is called an extension problem.

Definition 6: p-adics

Another ring of interest is the p-adic integers $\mathbb{Z}_p$ also denoted $\mathbb{Z}_p^\wedge$ where $\wedge$ means completed. Another way to write this is

$$\begin{aligned} \mathbb{Z}_p = \mathbb{Z}_p^\wedge &= \varprojlim \mathbb{Z}/p^k \\ &= \varprojlim(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0) \\ &= \left\{(a_1, a_2, \dots) \mid a_i \in \mathbb{Z}/p^i, \quad a_{i+1} \equiv a_i \mod p^i\right\}. \end{aligned}$$

Day 1 Exercises

Exercise 1

If $m, n > 1$ are integers, construct an exact sequence of abelian groups of the form

$$0 \longrightarrow \mathbb{Z}/m \longrightarrow \mathbb{Z}/mn \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Exercise 2

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that

$$\dim V = \dim V' + \dim V''$$

Exercise 3

If

$$0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$$

is a short exact sequence of vector spaces, prove that $V \cong V' \oplus V''$. (*Bonus: Is this isomorphism canonical? In other words, does it require any choices?*)beginexercise Give an example of a short exact sequence of modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

such that M is NOT isomorphic to $M' \oplus M''$.

Exercise 4

(The Splitting Lemma) Suppose

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of modules. Prove that the following are equivalent.

1. $M \cong M' \oplus M''$, f is the standard inclusion, and g is the standard projection.
2. There exists a map $s : M'' \longrightarrow M$ such that $g \circ s = id_{M''}$.
3. There exists a map $t : M \longrightarrow M'$ such that $t \circ f = id_{M'}$.

Exercise 5

Generalizing 1., if

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

is an exact sequence of vector spaces, prove that

$$\dim V_n - \dim V_{n-1} + \cdots \pm \dim V_1 \mp \dim V_0 = 0$$

Exercise 6

What are all possible group maps from

1. $\mathbb{Z}$ to $\mathbb{F}_2$?
2. $\mathbb{F}_2$ to $\mathbb{F}_2$?
3. $\mathbb{Z}$ to $\mathbb{F}_4$?
4. $\mathbb{F}_4$ to $\mathbb{F}_2$?
5. $\mathbb{F}_2$ to $\mathbb{F}_4$?
6. $\mathbb{Z}_2$ to $\mathbb{F}_2$?

Chapter 2

Appendix

Definition 7: pullbacks

Let C be a category which contains the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}.$$

A pullback of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $W \xrightarrow{p_1} X$
- A map $W \xrightarrow{p_2} Y$

such that

- The diagram $\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$ commutes [a]

- If someone hands you a commutative diagram $\begin{tikzcd} A \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{h}$ such that

$$\begin{tikzcd} A \arrow[dr, "\widetilde{h}"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\ & W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}.$$

[b]

Let C be a category with a subcategory $\mathcal{D}$. We say $\mathcal{D}$ is closed under pullbacks by morphisms in C if for all arrows $X \xrightarrow{f} Z$ in $\mathcal{D}$ and for all $Y \xrightarrow{g} Z$ in C such that we can form the pullback $\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, then the arrow $W \xrightarrow{p_1} Z$ is in $\mathcal{D}$.

[a]“$\circlearrowleft$” is a ‘long’hand for commutes, and people usually suppress it from notation.
[b]As a shorthand, people usually write the pullback like this:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Definition 8: pushouts

Let C be a category which contains the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \\ X & \end{tikzcd}.$$

A pushout of this diagram W is 3 pieces of information

- An object $W \in \mathrm{ob}(C)$
- A map $X \xrightarrow{i_1} W$
- A map $Y \xrightarrow{i_2} W$

such that

- The diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \end{tikzcd}$ commutes
- If someone hands you a commutative diagram $\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\ell_2"] \\ X \arrow[r, "\ell_1"] & A \end{tikzcd}$, then there is a *UNIQUE* map $\widetilde{\ell}$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "\ell_2"] & \\ X \arrow[r, "i_1"] \arrow[drr, bend right, "\ell_1"'] & W \arrow[dr, "\widetilde{\ell}"] & \\ & & A \end{tikzcd}.$$

[a]

[a]As a shorthand, people usually write the pushout like this:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & W \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Bibliography

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

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