Contents

Chapter 1

Week 1

Day 1

Lecture 1: Lot's of definitions!

> **Plan**
> We are going to try and compute
> $$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.

- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).

- $E(n, p)$ has n the chromatic height and p a prime.

- $G \curvearrowright$ on sets, spaces, or spectra.

- Let S be a space with a G-action.

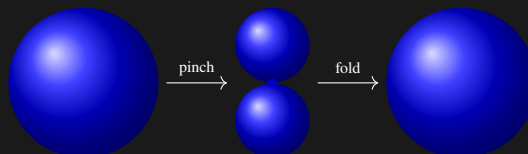$$S^G = \{s \in S \mid g \cdot s = s \ \forall g \in G\} \qquad\qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$
$$= \{G\text{-fixed points of } S\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be
$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.
$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$

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The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_{1}^{m} S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{ S^0, S^1, S^2, \ldots \right\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$
$$0 \longmapsto 0$$
$$1 \longmapsto 2.$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \text{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X), g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, g : S^m \to X.$$

What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$
\begin{array}{ccccccc}
& & X \wedge S^m & & & & \\
& \overset{f \wedge \mathbb{1}}{\nearrow} & & \overset{\mathbb{1} \times g}{\searrow} & & & \\
S^{n+m} = S^n \wedge S^m & \xrightarrow{\quad f \wedge g \quad} & & & X \wedge X & \xrightarrow{\mu} & X \\
& \underset{\mathbb{1} \wedge g}{\searrow} & & \underset{f \wedge \mathbb{1}}{\nearrow} & & & \\
& & S^n \wedge X & & & &
\end{array}
$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A spectrum[a] X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \ldots\} = \{X_n\}_{n \in \mathbb{N}}$$

together with structure maps

$$\Sigma X_n \to X_{n+1}.$$

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[a]What we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as homeomorphisms.
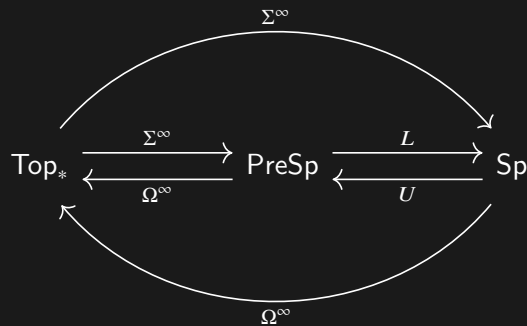
Example 1

1. The sphere spectrum $\mathbb{S} = \{S^0, S^1, \ldots\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.

2. Suspension spectrum $\Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \ldots\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma \Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.

"Why were spectra invented?" you may ask. The answer comes in the form of Brown's representability theorem. To understand this, we need a few definitions.

Definition 2

A generalized homology theory E E is a functor

$$E : \mathsf{Spaces} \to \mathsf{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.

- Exactness: Exact sequence in homology from a cofiber sequence.

- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \to E_*(X, B)$ is an isomorphism.

- Additivity: Coproducts in Spaces induce coproducts in homology.

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized homology theories and spectra. Given a spectrum $\mathcal{E}$, the homology is given by
$$\mathcal{E}_*(X) = \pi_a st(\mathcal{E} \wedge X).$$

See [Bau08].

Biblio

Bibliography

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces (Tokyo 2005)*. Mathematical Sciences Publishers, February 2008.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

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