Contents

Chapter 1

Week 1

Day 1

Lecture 1: Lot's of definitions!

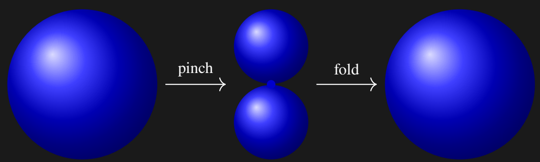
Plan
We are going to try and compute
$$\pi_*(E^{hC_6} \wedge V(0)).$$

Let's define a few things.

- C_6 is a cyclic group of order 6.
- E^{hC_6} is a Morava E-theory and this is a spectrum (think a space).
- $E(n,p)$ has n the chromatic height and p a prime.
- $G \curvearrowright$ on sets, spaces, or spectra.
- Let S be a space with a G-action.

$$\begin{aligned} S^G &= \{s \in S \mid g \cdot s = s \ \forall g \in G\} \\ &= \{G\text{-fixed points of } S\} \end{aligned} \qquad E^{hC_6} := \{\text{homotopy } G \text{ fixed points}\}$$

- $X \wedge Y$ is the smash product of X, Y and is defined to be
$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$
- $V(0) := \mathbb{S}/2$ the Moore space. Take a sphere S^n, and consider the degree map $S^n \xrightarrow{m} S^n$. Here is an instance of this map.
$$S^n \xrightarrow{2} S^n \vee S^n \to S^n.$$



The thing to take away is that for a degree m-map between n-spheres, you can create this map as a composition

$$S^n \xrightarrow{\text{pinch}} \bigvee_1^m S^n \xrightarrow{\text{fold}} S^n$$

to get a degree m map. More details about this can be found in [Hat02, §2.2]

- The sphere spectrum is a topological object which can be written as

$$\mathbb{S} := \left\{S^0, S^1, S^2, \ldots\right\}.$$

FACT: We can define a degree m map on the sphere spectrum.

- Fiber/cofiber sequences:. In spectra, fiber and cofiber sequences are the same! This is an anolog of a short exact sequence for groups. Here's an example. Consider the map

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\times 2} \mathbb{Z} \\ 0 &\longmapsto 0 \\ 1 &\longmapsto 2. \end{aligned}$$

The kernel of this map is 0! The cokernel of this map is $\mathbb{Z}/2$. This gives a short exact sequence of groups

$$0 \to \mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

We can do an analog with spectra to get

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \underbrace{V(0)}_{\text{cofiber}(2)} \to \Sigma\mathbb{S} \xrightarrow{\Sigma 2} \Sigma\mathbb{S} \to \Sigma V(0) \to \cdots.$$

Note: there is a way to understand fibers and cofibers as pushout and pullback diagrams.

- For spaces Σ, aka reduced suspension, exists for all $n \in \mathbb{N}$; you can suspend a space however many times you want, Σ^n. In spectra-land, you can *negatively*-suspend a space, aka desuspend the space, i.e. you can do Σ^n for all $n \in \mathbb{Z}$.

- $\pi_* = \bigoplus_{i \in \mathbb{Z}} \pi_i$. Here

$$\pi_n(X) := \text{Maps}(S^n, X)_{/\text{homotopy}}.$$

Sometimes we write this as $[S^n, X]$ so we have to type less!

- Let X be a space, and let $f \in \pi_n(X)$, $g \in \pi_m(X)$, meaning that we have

$$f : S^n \to X, g : S^m \to X.$$

What is $f \cdot g$ if we're talking about π_* having a "ring structure." Then we have

$$\begin{tikzcd} & X \wedge S^m \arrow[dr, "\mathbb{1} \times g"] & & \\ S^{n+m} = S^n \wedge S^m \arrow[ur, "f \wedge \mathbb{1}"] \arrow[rr, "f \wedge g"] \arrow[dr, "\mathbb{1} \wedge g"'] & & X \wedge X \arrow[r, "\mu"] & X \\ & S^n \wedge X \arrow[ur, "f \wedge \mathbb{1}"'] & & \end{tikzcd}$$

which gives us a map $\pi_{n+m}(X \wedge X)$. If we have a map $X \wedge X \xrightarrow{\mu} X$, then we're good; this is an honest to goodness ring! An instance of this is S^0. Try it out! For us $V(0) = \text{Cofiber}(2)$ is not a ring.

Spectra

Definition 1: Spectrum

A spectrum[a] X is a collection of pointed spaces

$$\{X_0, X_1, X_2, \ldots\} = \{X_n\}_{n \in \mathbb{N}}$$

together with structure maps

$$\Sigma X_n \longrightarrow X_{n+1}.$$

[a] What we describe here is sometimes referred to as a prespectrum. Some people require a spectrum to have the structure maps as homeomorphisms.

Example 1

1. The sphere spectrum $\mathbb{S} = \left\{S^0, S^1, \ldots\right\}$ and homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$.
2. Suspension spectrum $\Sigma^\infty X = \left\{X, \Sigma X, \Sigma^2 X, \ldots\right\}$ with structure maps

$$\Sigma(\Sigma^\infty X)_n = \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X = (\Sigma^\infty X)_{n+1}.$$

3. For some (non-suspension) spectra, we can describe the spaces, but for the majority of spectra, we cannot.

$$\begin{tikzcd}
\mathsf{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left=50, "\Sigma^\infty"] & \mathsf{PreSp} \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & \mathsf{Sp} \arrow[l, shift left, "U"] \arrow[ll, bend left=50, "\Omega^\infty"]
\end{tikzcd}$$

"Why were spectra invented?" you may ask. The answer comes in the form of Brown's representability theorem. To understand this, we need a few definitions.

Definition 2

A generalized homology theory E E is a functor

$$E : \mathsf{Spaces} \longrightarrow \mathsf{GradedAbGrps}$$

with the properties

- Homotopy: Homotopic spaces have the same homology.
- Exactness: Exact sequence in homology from a cofiber sequence.
- Excision: If $X = A \cup B$, then $E_*(A, A \cap B) \to E_*(X, B)$ is an isomorphism.
- Additivity: Coproducts in Spaces induce coproducts in homology.

For more details, see Wikipedia on generalized cohomology.

Theorem 1: Brown's representability Theorem

There is an isomorphism between generalized (co)homology theories and spectra. Given a spectrum $\mathcal{E}$, the homology is given by

$$\mathcal{E}_*(X) = \pi_*(\mathcal{E} \wedge X).$$

The cohomology associated to the spectrum $\mathcal{E}$ is given by

$$\mathcal{E}^*(X) = [X, \mathcal{E}].$$

Definition 3: Fiber Sequences

We'll come back to this! The key is that in spectra land, it goes back and forth in both directions.

FACT 1

Any fiber sequence $X \to Y \twoheadrightarrow Z$ gives rise to a long exact sequence in π_*,

$$\begin{tikzcd}
& \cdots \arrow[r] & \pi_{k+1}Z \arrow[dll, out=-90, in=90] \\
\pi_k X \arrow[r] & \pi_k Y \arrow[r] & \pi_k Z \arrow[dll, out=-90, in=90] \\
\pi_{k-1}X \arrow[r] & \cdots &
\end{tikzcd}$$

Biblio

Bibliography

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

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