A Note for Abstract Algebra

Meterial List

- Lieback
- Rotman - Group
- Rings Notes
- Midterm

Additional material: e-book from wechat group contribute by Fanlue. It might be useful when having no problem sheets and solutions.

Chapters and Contents

1. Lieback
    - Chapter 19
    - Chapter 20
    - Chapter 25
    - Chapter 26
2. Rotman - Groups
    - Subgroup
    - Lagrange
    - Cyclic Groups
    - Normal Groups
    - Isomorphism
    - Correspondence
    - Quotient Groups
    - Direct product
    - Conjugates
    - G_set
    - Symmetric Groups
    - Corollay Burnside
    - Burnside Lemma
    - Burnside Colouring Flag
    - Burnside 4×4
    - Burnside Bracelet
3. Ring Notes
    - Rings 1 - 8
    - Rings 2.3 - 2.12
    - Foctorisation
    - Ishort
    - Division in quadratic integers

Note that: *groups actions* and ***Burnside's Lemma*** (**Section factorisation in integral domains**) are not examinable!!!

Groups

This part is the most important and abundant part. Problems mainly have no solutions.

1. ***Permutation* and *cycles***
2. ***Transportation***
3. ***Disjoint = Commute:*** if $\alpha(x) \neq x$, then all $\beta(x) = x$
4. ***Cancellation Law:*** if either $\alpha\beta = \alpha\gamma$ or $\beta\alpha = \gamma\alpha$ $\beta = \gamma$
5. ***Regular permutation:*** $\alpha = 1$ or α is a product of disjoint cycles.
6. ***Even/odd permutation:*** Even/Odd numbers of r-cycles, where r is an even number.
7. ***Group:***
   - **Closure Axiom:** for any $\alpha, \beta \in G,\ \text{then}\ \alpha\beta \in G$
   - **Associative Axiom:** for $\forall\alpha, \beta \in G, (a * b) * c = a * (b * c)$
   - **Identity Axiom:** for $\exists e \in G, \text{such that}\ a * e = e * a = a$, for all $a \in G$
   - **Inverse Axiom** for $\forall a \in G$ there exist $b \in G,\ s.t.,\ a * b = b * a = e$
8. ***Semi-group:*** Associative Axiom
9. $\mathbf{S_x}$ $\mathbf{S_n}$**:** ***Symmitric Group*** permutation groups $\mathbf{A_x}$ $\mathbf{A_n}$**:** ***Alternating Group***) **even** permutation groups
10. ***Comute=ablelian***
11. - [ ] ***Congruence Class*** $\mathbf{Z_m}$
12. - [ ] **Unit** in Ring
13. ***Four Group***
14. ***Multiplication table*** **NB:** $a_{row} * a_{colum}$
15. ***Homomorphism:*** $f(a * b) = f(a) \circ b$
16. ***Isomorphism*** $\cong$ ***:*** Homomorphism & Bijection
    Imbedded: S an be imebedded in $G \Leftrightarrow S \cong\ G'$, where $G' \leq G$
17. ***Subgroup:*** $S \leq G$ **Determine:**
    1. if $s, t \in S$, then $s^{-1} \in S$ and $st \in S \Rightarrow X \leq G$
    2. $1 \in S$ and if $s, t \in S$ then $st^{-1} \in S \Rightarrow S \leq G$
    3. G is a **finite** group and if $s, t \in G$ then $st \in S \Rightarrow S \leq G$
18. ***Cyclic (Sub)Group:*** $< a >$
    Subgroup generated by a***:*** $< A >$
19. ***Order: number of elements***
    Index: $[G : S]$***, number of (right) cosets of*** S ***in*** G
    Exponent: $x^n = 1$ ***for all*** $x \in G$
20. ***Proper:*** Subgroup $\neq$ Group
21. ***Trivial:*** Subgroup 1
22. ***Kernel:*** $kerf = \{a \in G : f(a) = 1\}$ ***Image:*** $Imf = \{h \in H : h = f(a)\}$ **NB:** For $G \to S$, $kerf \in G$ and $Imf \in H$
23. ***Word:*** A conception to build $< X >$, for a subset X of a group G, $w = x_1^{e1} x_2^{e2} x_3^{e3} x_4^{e4} \cdots x_n^{en}$, where $x_i \in X, e_n = \pm 1$, then $< X >$ is either 1 if $X = \emptyset$, or $< X >= \{$all *words* of $X\}$
24. ***General linear Group***$GL(n, k)$***:*** **N.B.** k **is a *field*.** $GL(n, k) = (A, \times)$, where $A = \{n{\times}n\ \textit{matrix: all entries(or say elements)}\ a \in k\}$
    Special Linear Group$SL(n, k)$***:*** matrices have determine 1.
    N.B. for $n = 1, GL(n, k)$ is **abelian** and $n \geq 2, GL(n, k)$ is ***NOT* Abelian**

25. ***(Right) Coset of S:*** $St = \{st : s \in S\}$, t is called representative.

Rings

1. ***Ring***: $(F, +, \cdot)$ is a ring if:
   - $(R, +)$ is a *Group*;
   - $(R, +)$ is a *Semigroup*;
   - $\forall a, b, c,\ a(b+c) = ab + ac$ and $(a+b)c = ac + bc$
2. ***Devision Ring***: $\forall a \neq 0, \exists b,\ s.t. ab = ba = 1$
3. ***Field***: Commutative division ring.
4. ***Subring:***
   - R is a *ring*, $S \subset R$
   - $0, 1 \in S$
   - for $\forall s, t \in S$, $s + t, st, s - t \in S$
5. ***Integral domain***:
   1. R is commutative
   2. $0 \neq 1$
   3. R has ***NO*** zero divisor
6. ***Examples:***

   Rings:
   - $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
   - $\mathbb{R}[x]$ (polynomials with real coeffcients)
   - $M_n(R)$ (Matrix ring, considering $GL(n, k)$) e.g. $M_2(R), M_3(R)...$
   - $\mathbb{Z}/m\mathbb{Z} = \{[0]_m, [1]_m, ...[m-1]_m\}$ *(Residue Class Rings)* **N.B.:** $m > 1$ **can be non-prime integer** e.g. $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, ...$

   Division Rings:
   - $\mathbb{C}, \mathbb{Q}, \mathbb{R}$

   ***NOT* Division Rings:**
   - $\mathbb{Z}$
   - $\mathbb{R}[x]$

   Subrings
   - $\mathbb{Z}$ of $\mathbb{Q}$
   - $\mathbb{Q}$ of $\mathbb{R}$
   - $\mathbb{R}$ of $\mathbb{C}$
   - $\mathbb{Z}[\sqrt{d}]$ of $\mathbb{C}$
   - $\mathbb{Q}[\sqrt{d}]$ of $\mathbb{C}$ **(Also a field)**
7. ***Left/Right Zero Divisor***: $a \neq 0$ is called left zero divisor if $\exists b \neq 0, s.t. ab = 0$

8. ***Unit*:** ab=ba=1 **no need for left/right**
   R^*: group of units.
9. ***A finit integral domain is a field*:**

   - [ ] *Proof*: Find inverse.

   $\Rightarrow$ The ring $\mathbb{Z}/m\mathbb{Z}$ is a field iff m is prime.
10.
11. ***Ring homorphism*:**
    - $\gamma(0) = 0, \gamma(1) = 1$
    - $\gamma(a+b) = \gamma(a) + \gamma(b)$
    - $\gamma(ab) = \gamma(a)\gamma(b)$