Switched Optimal Control and Dwell Time Constraints: A Preliminary Study

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***Abstract*— Most modern control systems are switched, meaning they have continuous as well as discrete decision variables. Switched systems often have constraints called dwell-time constraints, (e.g., cycling constraints in a heat pump), on the switching rate. This paper introduces an embedding-based-method to solve optimal control problems that have both discreet and continuous decision variables. unlike existing methods, the developed technique can heuristically incorporate dwell-time constraints while also preserving other state and control constrains of the problem. Simulations are run for a switched optimal control problem with and without the auxiliary cost to showcase the utility of the developed method.**

I. Introduction

The field of switched optimal control has grown exponentially in recent years due to the advances in computing technology. While optimal control methods for continuous systems have been well-explored, a significant portion of modern controlled systems have continuous as well as binary (or integer) decision variables. For example, a refrigeration system with multiple fixed speed compressors and multiple cooling racks where the compressors and the valves represent binary on/off decision variables, yet the dynamics of the system are continuous. Another example is the automatic transmission in a vehicle, which has a discrete number of gears but continuous dynamics for the torque from the engine and load from the wheels. Due to the presence of discrete decision variables, switched optimal control problems (SOCPs) cannot be solved using traditional gradient-based methods developed for continuous optimal control problems (OCPs). Direct transcription of OCPs with continuous and discrete decision variables often results in mixed-integer nonlinear programs that are computationally intensive to solve.

A variety of methods for solving SOCPs have been developed over the past several decades, which can be roughly classified as follows. In the first category are methods that fix a mode sequence [1], [2], or use predefined switching surfaces [3], [4] and optimize the timing of switching using smooth optimization techniques such as gradient descent. The method developed in this paper aims to optimize the sequence itself.

Another category of methods to solve SOCPs optimize both the mode sequence and the switching time instances [3], [5], [6]. In [7] the author develops a theoretical framework for a general OCP of variable structure systems (VSS), which can be interpreted as a sub-class of switched systems. The framework is based on a specific approximation method and results in a differential inclusion. In [8] a method is developed based on approximations of the differential constraints that are assumed to be given in the form of controlled differential inclusions. However, these techniques do not incorporate dwell-time constraints, and as such are not applicable to the problem at hand.

Others analyze OCPs for switched systems with dwell-time using a dynamic programming method, but have very restrictive assumptions such as having stable state matrices for all of the subsystems and for all of the time instances [9], or consider only discrete time autonomous systems [10]. While mode insertion methods such as [11] can incorporate dwell-time constraints, the dwell-time is implemented by filtering the optimal controller after-the-fact, resulting in loss of satisfaction guarantees for other state and control constraints of the problem. The embedding method in [12] also makes changes to the optimal mode sequence after-the-fact, which results in the loss of satisfaction guarantees for the boundary constraints of the problem.

The goal in this paper is to develop a method to optimize continuous controllers, mode sequences, and mode switching time, while guaranteeing satisfaction of boundary constraints and dwell-time constraints.

The paper extends the embedding approach introduced by [12], which involves solving the SOCP as a continuous OCP, to heuristically incorporate dwell-time constraints and to preserve boundary constraints by avoiding after-the-fact modification of the optimal mode sequence. The switched system is embedded in a continuous system by implementing the switched control signals as continuous variables, as explained in Section II. Section III details the primary contributions of this paper, which is adding a switching cost to the cost function in the problem formulation that results in a bang-bang type solution. Section IV simulates this method with a simple nonlinear example and discusses how the magnitude of the added switching cost heuristically controls the switching rate of the binary/integer control variables in numerical solutions. Section V develops a framework to address the discrepancy between the theoretical solution and the numerical solution found in Sections III and IV, respectively, and Section VI concludes with the way the developed technique improves the current embedding method and future work that is being done.

II. Embedded Optimal Control Formulation

For ease of exposition, this paper focuses on a SOCP with two subsystems, but the developed method can be readily extended to switched systems with a higher number of subsystems. The system state is represented by $x : \mathbb{R}_{\geq 0} \to$

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$\mathbb{R}^n$ with dynamics[1]

$$\dot{x}(t) = f_{v(t)}(t, x(t), u(t)) \tag{1}$$

$$x(t_0) = x_0 \text{ for almost all } t \in [t_0, t_f]$$

where $v : \mathbb{R}_{\geq 0} \to \{0, 1\}$ is the mode sequence, $u : \mathbb{R}_{\geq 0} \to \Omega \subset \mathbb{R}^m$ is the control input constrained to the compact set Ω, and $f_{v(t)} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $\forall t \in [t_0, t_f]$. The control functions $t \mapsto v(t)$ and $t \mapsto u(t)$ must be selected such that the following constraints are satisfied: $(t_0, x(t_0)) \in \mathcal{T}_0 \times \mathcal{B}_0$ and $(t_f, x(t_f)) \in \mathcal{T}_f \times \mathcal{B}_f$, where the endpoint constraint set $\mathcal{B} \triangleq \mathcal{T}_0 \times \mathcal{B}_0 \times \mathcal{T}_f \times \mathcal{B}_f$ is contained in a compact set in $\mathcal{B} \subseteq \mathbb{R}^{2n+2}$. The switched cost functional is defined as

$$J(t_0, x_0, u(\cdot), v(\cdot)) = \int_{t_0}^{t_f} L_{v(t)}(t, x(t), u(t))\mathrm{d}t + K(t_0, x_0, t_f, x_f) \tag{2}$$

where $L_{v(t)} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$ $\forall t \in [t_0, t_f]$, $K \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. The SOCP is formulated as

$$\begin{aligned} \min_{u,v} \quad & J(t_0, x_0, u(\cdot), v(\cdot)) \quad \text{subject to:} \\ & \text{(i) } x(\cdot) \text{ satisfies(1)}, \\ & \text{(ii) } (t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B} \\ & \text{(iii) } v(t) \in \{0, 1\}, u(t) \in \Omega, \ \forall t \in [t_0, t_f] \\ & \text{(iv) } \forall t_1, t_2 \in [t_0, t_f] \text{with} v(t_1^-) \neq v(t_1^+) \text{ and} \\ & \qquad v(t_2^-) \neq v(t_2^+), |t_1 - t_2| \geq T > 0 \end{aligned}$$

where (iv) encodes the dwell-time constraint. In order to solve this problem using conventional gradient-based techniques that stem from dynamic programming or Pontryagin's minimum principle, all the decision variables in the optimization problem need to be continuous. Therefore the SOCP is embedded in a larger domain by replacing the mode sequence with a continuous function $\bar{v} : \mathbb{R}_{\geq 0} \to [0, 1]$, and letting $u_i : \mathbb{R}_{\geq 0} \to \Omega \, \forall i \in \{0, 1\}$ be the control input in vector field f_i. The dynamics of the embedded system are now defined as[2]

$$\dot{x}(t) = [1 - \bar{v}(t)] f_0(t, x(t), u_0(t)) + \bar{v}(t) f_1(t, x(t), u_1(t)), \quad x(t_0) = x_0 \,, \forall t \in [t_0, t_f], \tag{3}$$

and the cost functional is defined as

$$\begin{aligned} J(t_0, x_0, u_0(\cdot), u_1(\cdot), \bar{v}(\cdot)) = \\ \int_{t_0}^{t_f} \Big([1 - \bar{v}(t)] L_0(t, x(t), u_0(t)) \\ + \bar{v}(t) L_1(t, x(t), u_1(t)) \Big) \mathrm{d}t + K(t_0, x_0, t_f, x_f) \end{aligned} \tag{4}$$

[1]$\mathbb{R}_{\geq a}$ represents all positive real numbers greater than or equal to a.

[2]For the Extension to a switched systems with higher number of subsystems see [12].

The embedded optimal control problem (EOCP) is formulated as

$$\begin{aligned} \min_{u_0, u_1, \bar{v}} \quad & J(t_0, x_0, u_0(\cdot), u_1(\cdot), \bar{v}(\cdot)) \quad \text{subject to:} \\ & \text{(i) } x(\cdot) \text{ satisfies (3)}, \\ & \text{(ii) } (t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B} \\ & \text{(iii) } \bar{v}(t) \in [0, 1], \ u_0(t), u_1(t) \in \Omega, \ \forall t \in [t_0, t_f] \end{aligned}$$

By allowing $\bar{v}(\cdot)$ to be continuous, all the decision variables are now continuous which allows the classical necessary and sufficient conditions of optimal control theory to be applied [13]. However, the solution of the EOCP is not always equivalent to the solution of the SOCP since the mode sequence is allowed to be continuous. Furthermore, the dwell-time constraint (iv) of the SOCP cannot be easily incorporated in the ECOP. To ensure that the EOCP produces bang-bang solutions, the embedding process is modified in Section III.

III. Modified Formulation

When the solution to the EOCP is a regular solution, otherwise known as a bang-bang type solution, the range of the embedded mode sequence $\bar{v}$ is restricted to $\{0, 1\}$, so it can be identified with the switched mode sequence v. In fact, regular solutions of the EOCP are also the solutions of the SOCP as shown in [12]. However, whenever the EOCP has a singular solution, meaning $\bar{v}(t) \in (0, 1)$ over a time interval of nonzero measure, the embedded mode sequence cannot be identified directly with any switched mode sequence. Using the Chattering Lemma [14], a sub-optimal bang-bang solution can be determined within an arbitrary ϵ-distance from the singular solution. However, such a solution may not satisfy the boundary constraints and the path constraints of the SOCP. In this paper, a method to ensure that the EOCP produces bang-bang solutions that meet all the boundary and path constraints, while heuristically accounting for dwell-time constraints of the SCOP, is developed

In the modified EOCP (MEOCP), a concave auxiliary cost function $L_{\bar{v}} : [0, 1] \to \mathbb{R}$ is added such that $L_{\bar{v}}(\bar{v}(t)) = 0$ whenever $\bar{v}(t)$ is equal to one of the modes in the switched mode sequence, and $L_{\bar{v}}(\bar{v}(t)) > 0$ whenever $\bar{v}(t)$ is not. For example, an inverted parabola of the form $L_{\bar{v}}(\bar{v}(\cdot)) = 4\beta(\bar{v}(\cdot) - \bar{v}(\cdot)^2)$ with β being a positive constant could be used, which outputs 0 at $\bar{v}(\cdot) \in \{0, 1\}$ and reaches a maximum value of β when $\bar{v}(\cdot) = 0.5$. Since the minima of the auxiliary cost function will be the modes of the SOCP, the effect of the auxiliary cost is that in minimizing the cost functional to find the optimal solution, the embedded mode sequence is pushed towards the modes of the SOCP and away from modes which are not in the SOCP. For the two-switched system, in the following, Pontryagin's minimum principle is used to prove that the solutions of the MEOCP will be of a bang-bang type, which can be directly implemented on the switched system, allowing the SOCP to be solved as a continuous OCP using conventional techniques.[3]

Let $V := [u_0, u_1, \bar{v}]^T$ denote the augmented control input. The Hamiltonian for the MEOCP is defined as

$$H(x, \lambda, V, t) = \langle \lambda, [1 - \bar{v}] f_0(t, x, u_0) + \bar{v} f_1(t, x, u_1) \rangle + [1 - \bar{v}] L_0(t, x, u_0) + \bar{v} L_1(t, x, u_1) + L_{\bar{v}}(\bar{v}). \quad (5)$$

According to Pontryagin's minimum principle, when evaluated along the optimal state trajectory, $x^*(\cdot)$, and the optimal costate trajectory, $\lambda^*(\cdot)$, the optimal augmented control $V^*(\cdot) = [u_0^*(\cdot), u_1^*(\cdot), \bar{v}^*(\cdot)]^T$ minimizes the Hamiltonian among all other controllers. That is, $\forall t$, $V^*(t) \in \mathbb{R}^3$ minimizes the function $V \mapsto H(x^*(t), \lambda^*(t), V, t)$.

As a result, it can be concluded that the function $\bar{v} \mapsto H(x^*(t), \lambda^*(t), [u_0^*(t), u_1^*(t), \bar{v}]^T, t)$ is minimized by the optimal mode sequence $\bar{v}^*(t)$, $\forall t$. Since $L_{\bar{v}}$ is a concave function and since the function $\bar{v} \mapsto \langle \lambda^*(t), [1 - \bar{v}] f_0(t, x^*(t), u_0^*(t)) + \bar{v} f_1(t, x^*(t), u_1^*(t)) \rangle + [1 - \bar{v}] L_0(t, x^*(t), u_0^*(t) + \bar{v} L_1(t, x^*(t), u_1^*(t))$ is affine $\forall t$, the function $\bar{v} \mapsto H(x^*(t), \lambda^*(t), [u_0^*(t), u_1^*(t), \bar{v}]^T, t)$ is a sum of a concave function and an affine function, and as a result, is concave. Since concave functions over a compact set have their minima on the boundary of the compact set (see [15, Theorem 3], the function $\bar{v} \mapsto H(x^*(t), \lambda^*(t), [u_0^*(t), u_1^*(t), \bar{v}]^T, t)$ is minimized at the boundary of the embedded mode sequence.

In the two-switched system, the Hamiltonian is minimized when the embedded mode sequence is either 0 or 1, which can be identified with the modes of the SOCP. Thus, the solution found for the MEOCP can be implemented in the original switched system. In summary the auxiliary cost ensures that solutions of the MEOCP are of a bang-bang type, and as a result, can be implemented on the original switched system. However, bang-bang solutions of the MEOCP are not necessarily solutions of the SOCP. Determination of optimality of bang-bang solutions of the MEOCP with respect to the SOCP is an open question that requires further research.

IV. Simulation Example

An example is used to showcase the developed method, featuring a valve which empties into a tank which empties into a second tank and then out. The flow rate out of each tank is modeled as the square root of the water level, which is the state of the system, and the objective is to maintain a given water level in the second tank, with the control input being the switched flow rate of the input valve. The valve can either be a high flow rate of 2 or a low flow rate of 1. The dynamics of this switched system are defined as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, & v(t) = 0 \\ \begin{bmatrix} 2 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, & v(t) = 1 \end{cases} \quad (6)$$

[3]Future work is examining the addition of an auxiliary cost function for a switched system with three subsystems where the Hamiltonian is equal to zero when evaluated on the vertices of a polygon.

where $x_1(\cdot)$ is the water level in the first tank, and $x_2(\cdot)$ is the water level in the second tank, and $v(\cdot) \in \{0, 1\}$ is the switched control signal. Fig. 1 shows a diagram of the simulated system. The cost functional on this system is defined as

$$J(t, x(\cdot)) = \int_{t_0}^{t_f} \alpha (x_2(t) - 3)^2 \mathrm{d}t. \quad (7)$$

where in this example $t_0 = 0$, $t_f = 20$, $\alpha = 2$, and $x(0) = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$. The goal is to achieve a water level of 3 in the second tank, so the SOCP is formulated as

$$\begin{aligned} \min_{v} \quad & J(t_0, x_0) \quad \text{subject to:} \\ & \text{(i) } x(\cdot) \text{ satisfies(6)}, \\ & \text{(ii) } (t_0, x(t_0), t_f, x(t_f)) \in \\ & \qquad \{0\} \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} \times \{20\} \times \begin{bmatrix} [0, 4] \\ 3 \end{bmatrix} \\ & \text{(iii) } v(t) \in \{0, 1\}, \ \forall t \in [t_0, t_f] \end{aligned}$$

For the water level in the second tank to be maintained at 3, we need $\dot{x}_2(t) = 0$, which means $\sqrt{x_1(t)} = \sqrt{x_2(t)} \implies x_1(t) = x_2(t) = 3$. This means the water level in the first tank must also be a constant 3, so $\dot{x}_1(t) = 0$, which means the flow rate of the valve must be $\sqrt{x_1(t)} = \sqrt{3}$ and between the high and the low valve state. In the SOCP, this flow rate is impossible to achieve without the solution flipping infinitely fast between a high and a low valve state, which is characteristic of a singular solution. Following the embedding method, the constraints on the control are modified so that the switched control signal is a continuous variable where $\bar{v}(\cdot) \in [0, 1]$. The state of the embedded system is found using the above method and simplifying, and is defined by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 + \bar{v}(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, x(t_0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \forall t \in [t_0, t_f], \quad (8)$$

with the cost functional remaining the same as the SOCP formulation. This EOCP is solved as is to show that the solution is singular and cannot be readily implemented in the switched system. To solve the OCP, GPOPS [16] was used, with the differentiation method set to finite-difference and the maximum mesh iterations set to 5. As can be seen in Fig. 2, the control signal settles to a value of 1.73 as predicted above, and both of the tank levels settle to 3, as shown in Fig. 3. To eliminate the singular solution, a cost function is added to the control signal of the form $4\beta(\bar{v}(\cdot) - \bar{v}^2(\cdot))$, which is 0 at both extremes of the control and a maximum of β between. The auxiliary cost function was derived using the embedding process above by setting $L_0(t, x(t), \bar{v}(t)) = 2\beta\bar{v}(t) + \alpha(x_2(t) - 3)^2$ and $L_1(t, x(t), \bar{v}(t)) = 2\beta(1 - \bar{v}(t)) + \alpha(x_2(t) - 3)^2$, which after simplifying, makes the cost functional

$$J(t, x(t), v(t)) = \int_0^{20} (\alpha (x_2(t) - 3)^2 + 4\beta(v(t) - v^2(t)) \mathrm{d}t. \quad (9)$$

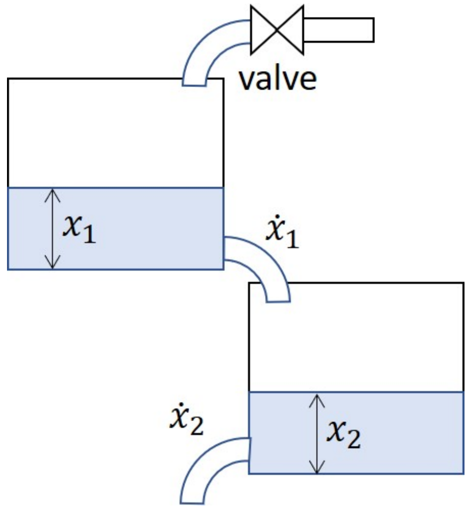

Fig. 1. The two-tank system model used in the simulation.

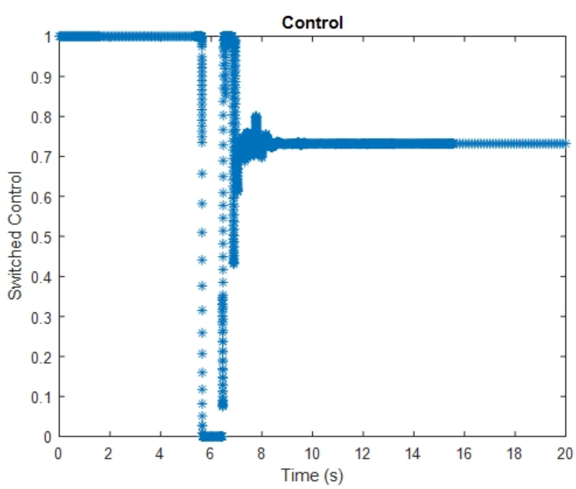

Fig. 2. The optimal control signal for the unmodified EOCP.

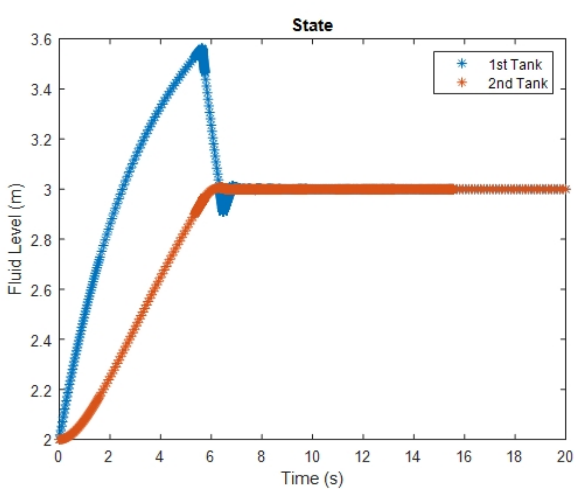

Fig. 3. The states of the system subject to the optimal control signal for the unmodified EOCP, with a final cost of 4.7312.

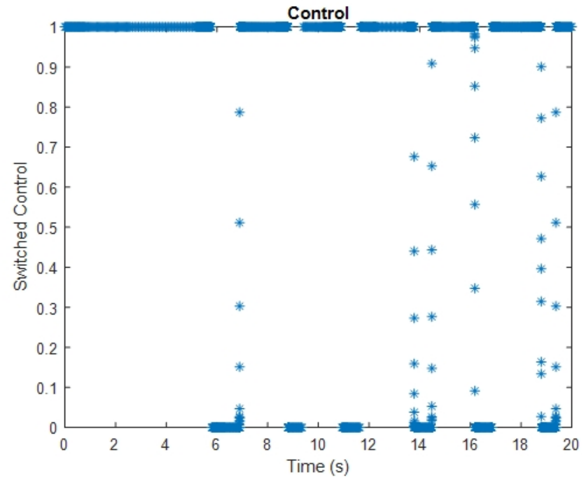

Fig. 4. The optimal control signal for the EOCP with $\beta = 0.01$.

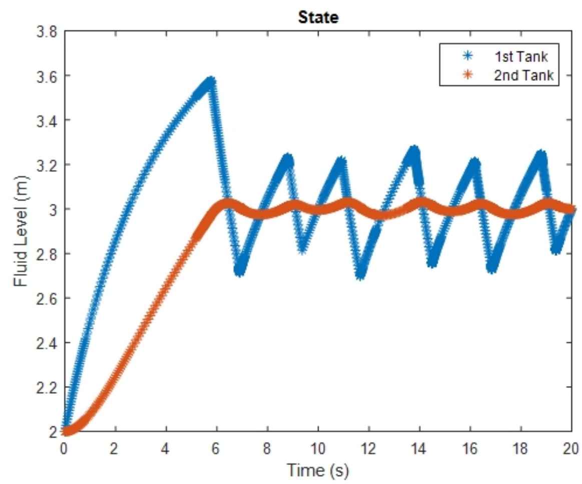

Fig. 5. The states of the system subject to the optimal control signal for the EOCP with $\beta = 0.01$, with a final cost of 4.7355.

The numerical results indicate that the magnitude of the added switching cost can be modified in accordance with the desired dwell-time constraint, with a higher magnitude resulting in a longer dwell-time, and vice versa. Figs. 4 and 5 show the solution to the modified EOCP when $\beta = 0.01$, which results in a fairly small auxiliary cost relative to the state cost, a linear switching rate, and a lower dwell-time. Fig. 6 and 7 show the solution when $\beta = 0.2$, in which case, the switching rate is lower and the dwell-time is higher. In each solution run with a different value of β, the control is a bang-bang type, which can be directly implemented in the original switched system. Changing β changes the frequency of the switching, thus a dwell-time constraint can be implemented heuristically by tuning β. A greater value of β results in a lower switching rate, so the water level in the second tank is not kept as close to the desired level and the optimal cost is higher. Figs. 4 and 6 indicate that increasing the constant β in the auxiliary cost function

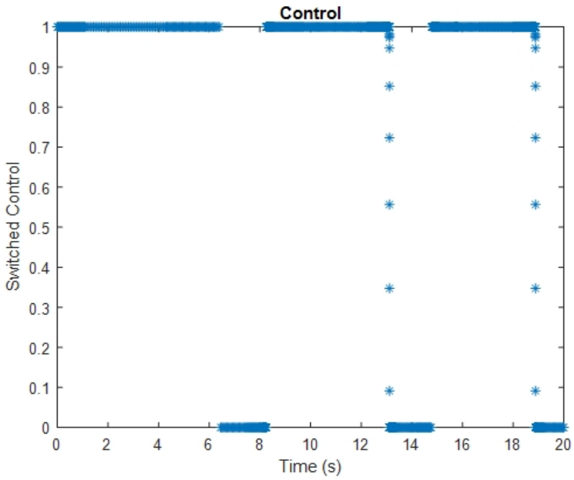

Fig. 6. The optimal control signal for the EOCP with $\beta = 0.2$

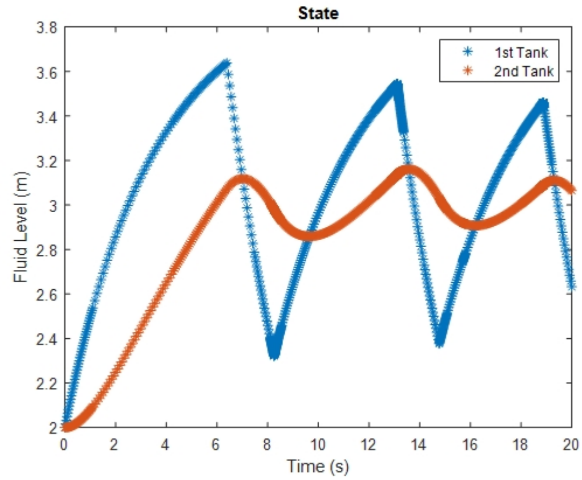

Fig. 7. The states of the system subject to the optimal control signal for the EOCP with $\beta = 0.2$, with a final cost of 4.8032.

increases the time between two consecutive switches of the solution generated by the EOCP solver.

Since the auxiliary cost function evaluates to zero when the embedded mode sequence takes the values 0 or 1, the contribution of this function to the total cost should be zero when the solution of the EOCP is bang-bang. Furthermore, the costate dynamics are also independent of the auxiliary cost function. *It is therefore concluded that there should not be any correlation between β and the dwell-time of the bang-bang solution and that Pontryagin's minimum principle fails to explain the numerical results*. In fact, a heuristic examination of the two-tank MEOCP reveals that the optimal solution is a sliding mode solution that keeps the water level in the second tank at exactly the required height using infinite frequency switching. The authors hypothesize that the correlation observed in Figs. 4 and 6 is a result of the approximations introduced by the Lagrange interpolation-based quadrature methods utilized by the numerical solver. For a brief discussion on the validity of the said hypothesis using a simpler numerical example, see the Appendix.

V. Augmenting the Modified System

Since the theoretical results predict that the contribution due to the auxiliary cost of an optimal trajectory should be zero, but the example in Section IV shows otherwise, an investigation of the discrepancy is needed. One possible reason for the discrepancy is that the theoretical result is based on the input $v(t)$ taking discrete values, which contradicts the continuity assumptions implicit in the numerical implementation of the previous section. In order to develop a theoretical framework that takes into account input values that vary continuously between 0 and 1 , which would take into account the effect of the auxiliary cost function, an augmented system is constructed as such that The dynamics of the augmented system are now defined as

$$\dot{x}(t) = [1 - x_{n+1}(t)] f_0(t, x(t), u_0(t)) + x_{n+1}(t) f_1(t, x(t), u_1(t)),$$

$$\dot{x}_{n+1}(t) = \gamma w(t) \tag{10}$$

and the cost functional is defined as

$$\begin{aligned} J(t_0, x_0, u_0(\cdot), u_1(\cdot), w(\cdot)) = \\ \int_{t_0}^{t_f} \Big([1 - x_{n+1}(t)] L_0(t, x(t), u_0(t)) \\ + x_{n+1}(t) L_1(t, x(t), u_1(t))) + L_{\bar{v}}(x_{n+1}(t)) \Big) \mathrm{d}t \\ + K(t_0, x_0, t_f, x_f). \end{aligned} \tag{11}$$

The augmented MECOP (AMEOCP) is formulated as

$$\min_{u_0, u_1, w} \quad J(t_0, x_0, u_0(\cdot), u_1(\cdot), w(\cdot)) \quad \text{subject to:}$$

(i) $\Big(x(\cdot), x_{n+1}(\cdot) \Big)$ satisfies (10),
(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B}$
(iii) $w(t) \in [-1, 1],\ u_0(t), u_1(t) \in \Omega,\ \forall t \in [t_0, t_f]$
(iv) $x_{n+1}(t) \in [0, 1],\ u_0(t), u_1(t) \in \Omega,$
$\forall t \in [t_0, t_f]$

Due to the constraint (iv), the AMEOCP is a path-constrained OCP, and as such, is difficult to analyze. In this paper, it is analyzed using Maurer's work in [17], where Pontryagin's minimum principle is developed for path-constrained OCPs in the Mayer form with control appearing linearly in the dynamics. To facilitate the application of the results in [17], the AMEOCP is also converted from Bolza form to Mayer form by augmenting the dynamics with the cost. For the two-tank system, the augmentation consists of adding the input as a continuous state x_3 and the Lagrange cost as x_4, where $\dot{x}_4 = (\alpha(x_2(t) - 3)^2 + 4\beta(x_3(t) - x_3^2(t))$, that is,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 1 + x_3(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \\ \gamma w(t) \\ (\alpha(x_2(t) - 3)^2 + 4\beta(x_3(t) - x_3^2(t)) \end{bmatrix}, \forall t \in [t_0, t_f], \tag{12}$$

following the procedure in [17], the dynamics are split into the control-affine form

$$\dot{\mathcal{X}}(t) = g(X(t)) + [\mathbf{0}^n \ \gamma \ 0]^\intercal w(t) \tag{13}$$

where $g(X(\cdot)) \in \mathbb{R}^4$, $\mathcal{X}(\cdot) \in \mathbb{R}^2 \times [0,1] \times \mathbb{R}^+$, and $X(\cdot) \in \mathbb{R}^2 \times [0,1]$

$$g(X(t)) = \begin{bmatrix} 1 + x_3(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \\ 0 \\ (\alpha(x_2(t)-3)^2 + 4\beta(x_3(t) - x_3^2(t)) \end{bmatrix}, \forall t \in [t_0, t_f]. \tag{14}$$

The cost function is simply $J(\mathcal{X}(t_f)) = x_4(t_f)$, which is the final value of the augmented cost. Other requirements of the formulation are that the state is constrained, and that the constraint is expressed as $S(X) \leq \alpha$, for a scalar valued function $S : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ and a constant $\alpha \geq 0$. In this example, the constraint $|x_3(\cdot)| \leq 1$ is expressed as $x_3^2(\cdot) - x_3(\cdot) \leq 0$ with $S(X) = x_3^2 - x_3$ and $\alpha = 0$

Define the Hamiltonian as:

$$H(\mathcal{X}(\cdot), \mathcal{P}(\cdot), w(\cdot), \eta(\cdot)) = \\ \mathcal{P}^T(\cdot)g(X(\cdot)) + \gamma p_3(\cdot)w(\cdot) + \eta(\cdot)S(X(\cdot)) \tag{15}$$

where $\mathcal{P}(\cdot) \in \mathbb{R}^4$ are the costates and $\eta(\cdot) \in \mathbb{R}$. The necessary conditions of Pontryagin's Minimum Principle are:

(i) There exists a function $\eta^*(\cdot)$ such that the optimal costates in this problem satisfy:

$$\begin{aligned} &\dot{p}^{*T}(t) = -\mathcal{P}^{*T}(t)g_x(X^*(t)) \\ &\dot{p}_3^*(t) = -\mathcal{P}^{*T}(t)g_{x_3}(X^*(t)) - \eta^*(t)S_{x_3}(X^*(t)) \\ &\dot{p}_4^*(t) = 0 \ \forall t \in [t_0, t_f] \\ &p^*(t_f) = \mathbf{0}^n \\ &p_3^*(t_f) = \eta_0^* \\ &p_4^*(t_f) = 0 \end{aligned}$$

where the subscript on a function represents the partial derivative according to that variable.

(ii) The function $\eta^*(\cdot)$ satisfies $\eta^*(t)S(X^*(t)) \equiv 0 \ \forall t \in [t_0, t_f]$

(iii) $H(\mathcal{X}^*(t), \mathcal{P}^*(t), w^*(t), \eta^*(t)) \leq H(\mathcal{X}^*(t), \mathcal{P}^*(t), w(t), \eta^*(t)) \ \forall \ w(t) \in [-1,1]$ and $t \in [t_0, t_f]$ where $\mathcal{X}^*$, $\mathcal{P}^*$, w^*, and η^* denote the optimal state, costate, and control trajectories.

The coefficient of $w(\cdot)$ in (15) is called the switching function, $\phi(\cdot) = \gamma p_3(\cdot) \in \mathbb{R}$, the switching function determines the optimal control of the system. The optimal solution to the problem in (12) is comprised of three types of arcs dependent on whether or not the system is on the boundary of the augmented state constraint [17]. When the system is on a boundary arc, that is, when $x_3(\cdot) \in \{0,1\}$, the optimal control is $w(\cdot) = 0$. The switching function is defined by:

$$\begin{aligned} &\phi(t) = \mathcal{P}^T(t)\varphi(t)_0 = 0, \\ &\dot{\phi}(t) = \mathcal{P}^T(t)\varphi(t)_1 - \eta(t)b(t) = 0 \ \forall t \in [t_0, t_f] \end{aligned}$$

where $b(\cdot) = \gamma S_{x_3}(X(\cdot))$. The number of derivatives of $\phi(t)$ that are needed is dependent on how many time derivatives of $S(\cdot)$ are needed before the control appears explicitly. Since the state constraint uses the augmented state $x_3(\cdot)$ that is explicitly controlled by $w(\cdot)$ in its dynamics, only a single time derivative is needed.

When $x_3(\cdot)$ is in the interior of its range, the system can be in one of the other two different arcs. The first is an interior singular arc, which means $\phi(\cdot) \equiv 0$ and thus $p_3(\cdot) \equiv 0$. This is assumed to never occur, since it implies that the Hamiltonian is equal to 0 on a nonzero time horizon only when every other costate is also equal to 0 on that time horizon, thus the Hamiltonian becomes meaningless for finding the optimal control. In the example used in this paper, the costates are never equal to 0 on a nonzero time horizon, so an interior singular arc never occurs. The other possible arc is an interior nonsingular arc, and according to Maurer the switching function is equal to:

$$\phi(\cdot)^{(i)} = xP^T(\cdot)\varphi(\cdot)_{(i)}, \ i \in 0,1$$

where the equation $\varphi(\cdot) \in \mathbb{R}^4$ is defined recursively as:

$$\varphi_0(\cdot) = [\mathbf{0}^n \ \gamma \ 0]^T, \ \varphi_1(\cdot) = \dot{\varphi}_0(\cdot) - g_{\mathcal{X}}(\cdot)\varphi_0(\cdot)$$

and the superscript is the time derivative and the subscript is the partial derivatives according to $\mathcal{X}$ if not specified otherwise. In this problem, $\phi(\cdot) = \gamma p_3(\cdot)$ and $\dot{\phi}(\cdot) = -\gamma \mathcal{P}^T g_{x_3}(X(\cdot))$. Since H is linear with respect to the control w, the optimal controller is given by $w(\cdot) = -\mathrm{sgn}(\phi(\cdot))$, which means the optimal control is dependent on the sign of the switching function which depends on the sign of the costate $p_3(\cdot)$.

Essentially, the terms containing $\eta(\cdot)$ are slack terms, such that when the system hits the boundary, the function $\eta(\cdot)$ starts changing from 0 until the next switching time when $\eta(\cdot)$ reaches 0 again. $\phi(\cdot)$ is the same as in the boundary arc and $\dot{\phi}(\cdot) = -\gamma\mathcal{P}^T(\cdot)g_{x_3}(X(\cdot)) - \gamma S_{x_3}(X(\cdot))\eta(\cdot) = 0$. Solving for $\eta(\cdot)$ gives:

$$\eta(t) = \frac{-\mathcal{P}^T(t)g_{x_3}(X(t))}{S_{x_3}(X(t))} \tag{16}$$

where $t \in \{[t_0, t_f] \mid x_3(t) \in \{0,1\}\}$. The constant γ disappears, meaning the weight put on the control does not matter, but the weight on the added cost function does matter since it appears in the derivative of $L_v(x_3(\cdot))$ with respect to x_3 in $g_{x_3}(X(\cdot))$.

The system then follows a duality, changing between a boundary arc with $w(\cdot) = 0$ but $\eta(\cdot) \neq 0$, and the interior arc representing the switching times where the augmented state is in the interior of its range and the control is at a boundary. Thus the augmented state $x_3(\cdot)$ is at a constant 0 or 1 most of the time, and whenever $\eta(\cdot) = 0$ the augmented state switches at a rate $\pm\gamma$ to the other side. Augmenting the state allows for the same optimal solution as the modified problem by increasing γ to an arbitrarily high number to reduce the time taken whenever the augmented state is switching; thus the majority of the solution is in a boundary arc which can be taken directly as a solution to the original switched problem.

The difference now is that the weight on the added cost explicitly affects the dwell-time of the system, which can be calculated using the $\eta(\cdot)$ function. A minimum dwell-time constraint can then be met by adjusting the added cost coefficient such that the switching frequency of the system is below the threshold set by the constraint. Unfortunately, since η depends on the costate trajectory, selection of the added cost coefficient to maintain a prescribed dwell-time requires trail and error. Further research is needed to formally quantify the relationship between the added cost and the resulting minimum dwell-time.

VI. Conclusion

This paper introduces a way to solve SOCPs with dwell-time constraints via embedding, by imposing a cost on the embedded mode sequence to encourage solutions of a bang-bang type, which can be directly identified with modes switched system. A dwell-time constraint can be implemented heuristically using this method by changing the magnitude of the constant introduced in the cost on the embedded mode sequences. Unlike existing methods, in spite of the heuristics involved in tuning the dwell-time, the solutions produced arc is guaranteed to meet other state and control constraints of the original problem. The embedding facilitates the use of fast gradient-based numerical methods to solve the SOCP and can be modified to include a dwell-time constraint directly. As a result, it can be utilized in a real time solver. The development of a method for implementing embedding in a switched receding horizon format is a part of future research.

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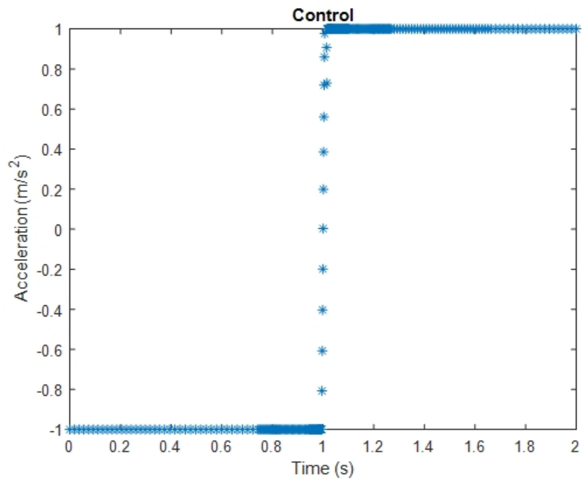

Fig. 8. The optimal control signal found for the double integrator.

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Appendix

Analysis of the Numerical Results

According to [16], GPOPS uses Legendre–Gauss–Radau points to determine where to do orthogonal collocation. Collocation inherently is an interpolation technique on continuous functions, so when calculating the cost the solver interpolates between the high and the low embedded mode sequence values in the collocation points on either side of the mode sequence switch. As a result, for a few collocation points, the embedded mode sequence is no longer at 0 or 1, and it incurs a cost dependent on β. When GPOPS is used in a multi-phase mode such that a mesh division occurs right at a switching point, GPOPS collocates each section of the trajectory separately so no interpolation occurs over the switching point and the cost no longer depends on β. Such a mesh division is possible only when the time of the switching point is known and the mesh division is set exactly a priori; otherwise the cost is dependent on β.

The hypothesis can be tested by the following example using a time-minimization problem with a known solution, as shown in Fig. 8 and 9. In this example, a double integrator is used where the states are defined as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}x_2(t)\\ u(t)\end{bmatrix}, x(t_0) = \begin{bmatrix}1\\ 0\end{bmatrix}, x(t_f) = \begin{bmatrix}0\\ 0\end{bmatrix} \quad (17)$$

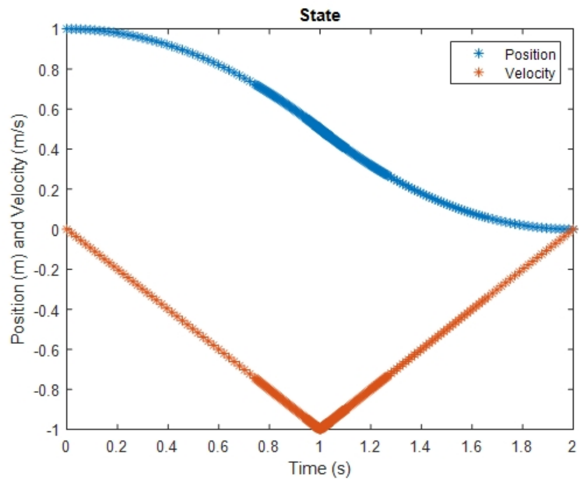

Fig. 9. The optimal states for the double integrator.

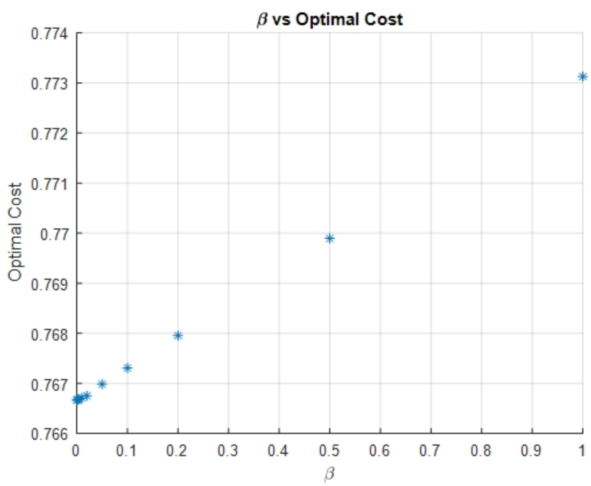

Fig. 10. The relationship of β and the optimal cost in the numerical solver.

where $t \in [0, 2]$ and the control signal $u(\cdot) \in [-1, 1]$. The cost function for this system is defined as

$$J(t, x(\cdot), u(\cdot)) = \int_{t_0}^{t_f} x_1^2(t) + \beta(1 - u^2(t))\mathrm{d}t \qquad (18)$$

This problem can be solved analytically for $\beta = 0$, with the optimal control being $u(t) = -1 \ \forall t \in [0, 1)$ and $u(t) = 1 \ \forall t \in [1, 2]$, and the minimal cost being 23/30. There should be no cost dependence on the β term because the optimal control of -1 or 1 results in the auxiliary cost being zero. However, when simulated the β term has a direct relation to the cost without affecting the solution found. The numerical solver matches the analytical solution with an optimal cost of 0.7666, but increases linearly with an increase in the β term as shown in Fig. 10. Changing this β term does not have an effect on the optimal mode sequence found for this problem, because this problem is setup for a bang-bang solution already, so all of the plots for other β values are identical and not shown.