Quadratic equations

A quadratic equation has the form:

$$ax^2 + bx + c = 0$$

where a, b, and c are fixed constants where $a \neq 0$. $ax^2 + bx + c$ is referred to as a "degree 2 polynomial", or a "quadratic polynomial". When solving quadratic equations, there will be multiple solutions. To assist with denoting multiple solutions, the following notation will be used:

- The notation $x = \pm v$ means that $x = v$ OR $x = -v$. There are two solutions for x, which are v and $-v$.
- The notation $x = v_1, v_2$ means that $x = v_1$ OR $x = v_2$. There are two solutions for x, which are v_1 and v_2.
- Generalizing, the notation $x = v_1, v_2, v_3$ means that $x = v_1$ OR $x = v_2$ OR $x = v_3$. There are three solutions for x, which are v_1,v_2, and v_3.
- etc.

Solving simple equations:

Equations are easy to solve when x appears only once in the equation. For example, consider the equation:

$$\frac{1}{5\sqrt{x+2}-7} = \frac{1}{8}$$

The expression $\frac{1}{5\sqrt{x+2}-7}$ is derived from x by the following sequence of single variable functions:

- The function $f_1(a) = a + 2$ is applied to x give $x + 2$. The inverse of this function is $f_1^{-1}(b) = b - 2$.
- The function $f_2(a) = \sqrt{a}$ is applied to $x+2$ to give $\sqrt{x+2}$. The inverse of this function is $f_2^{-1}(b) = b^2$, with b restricted to $[0, +\infty)$.
- The function $f_3(a) = 5a$ is applied to $\sqrt{x+2}$ to give $5\sqrt{x+2}$. The inverse of this function is $f_3^{-1}(b) = \frac{b}{5}$.
- The function $f_4(a) = a - 7$ is applied to $5\sqrt{x+2}$ to give $5\sqrt{x+2} - 7$. The inverse of this function is $f_4^{-1}(b) = b + 7$.
- The function $f_5(a) = \frac{1}{a}$ is applied to $5\sqrt{x+2} - 7$ to give $\frac{1}{5\sqrt{x+2}-7}$. The inverse of this function is $f_5^{-1}(b) = \frac{1}{b}$.

The equation $\frac{1}{5\sqrt{x+2}-7} = \frac{1}{8}$ is simply

$$f_5(f_4(f_3(f_2(f_1(x))))) = \frac{1}{8}$$

and finding x simply involves "undoing" each of the functions that have been applied to x. Starting with

$$\frac{1}{5\sqrt{x+2}-7} = \frac{1}{8}$$

applying $f_5^{-1}(b) = \frac{1}{b}$ to both sides gives

$$5\sqrt{x+2} - 7 = 8$$

applying $f_4^{-1}(b) = b + 7$ to both sides gives

$$5\sqrt{x+2} = 15$$

applying $f_3^{-1}(b) = \frac{b}{5}$ to both sides gives

$$\sqrt{x+2} = 3$$

applying $f_2^{-1}(b) = b^2$ to both sides gives

$$x + 2 = 9$$

applying $f_1^{-1}(b) = b - 2$ to both sides gives

$$x = 7$$

hence the equation has been solved for $x = 7$.

Another example:
With equations such as

$$2x + 5 = 9x - 16$$

x appears more than once in the equation, but the equation can be manipulated by subtracting $9x$ from both sides to give the equation:

$$-7x + 5 = -16$$

where x appears only once. This equation can now be solved as follows:

$$-7x + 5 = -16 \iff -7x = -21 \iff x = 3$$

Quadratic equations: special cases

With the quadratic equation

$$ax^2 + bx + c = 0$$

x appears twice. Solving the quadratic equation is not a straightforwards process.

case $b = 0$

When $b = 0$, the equation becomes

$$ax^2 + c = 0$$

x appears only once, so

$$ax^2 + c = 0 \iff ax^2 = -c \iff x^2 = -c/a \iff x = \pm\sqrt{-c/a}$$

(when inverting the function $f(u) = u^2$, there are two possible outcomes $f^{-1}(v) = \pm\sqrt{v}$)
Therefore:

$$x = \pm\sqrt{-c/a}$$

It is also important to note that if $-c/a < 0$, then no solutions exist.

Example 1
Consider the equation:

$$2x^2 - 18 = 0$$

$$2x^2 - 18 = 0 \iff 2x^2 = 18 \iff x^2 = 9 \iff x = \pm 3$$

$$x = \pm 3$$

Example 2
Consider the equation:

$$3x^2 = 0$$

$$3x^2 = 0 \iff x^2 = 0 \iff x = 0$$

$$x = 0$$

Example 3
Consider the equation:

$$7x^2 + 28 = 0$$

$$7x^2 + 28 = 0 \iff 7x^2 = -28 \iff x^2 = -4$$

Since $x^2 < 0$, there are **no solutions**.

Example 4
Consider the equation:

$$-2x^2 + 32 = 0$$

$$-2x^2 + 32 = 0 \iff -2x^2 = -32 \iff x^2 = 16 \iff x = \pm 4$$

$$x = \pm 4$$

case $c = 0$

When $c = 0$, the equation becomes

$$ax^2 + bx = 0$$

x appears twice, however x can be factored from the two terms to get

$$x(ax + b) = 0$$

It now seems like both sides can be divided by x to get a linear equation wherein x appears only once. However, one must also consider the case where $x = 0$. $x = 0$ is a solution, and when $x \neq 0$, both sides can be divided by x to get:

$$ax + b = 0 \iff ax = -b \iff x = -b/a$$

Therefore the solutions are:

$$x = 0, -b/a$$

Example 1
Consider the equation:

$$5x^2 - 10x = 0$$

$$5x^2 - 10x = 0 \iff x(5x - 10) = 0 \iff x = 0 \vee 5x - 10 = 0$$

(recall that $\vee$ = "OR")
The second alternative yields:

$$5x - 10 = 0 \iff 5x = 10 \iff x = 2$$

Therefore:

$$x = 0, 2$$

Example 2

Consider the equation:

$$-2x^2 - 6x = 0$$

$$-2x^2 - 6x = 0 \iff x(-2x - 6) = 0 \iff x = 0 \vee -2x - 6 = 0$$

The second alternative yields:

$$-2x - 6 = 0 \iff -2x = 6 \iff x = -3$$

Therefore:

$$x = 0, -3$$

Example 3
Consider the equation:

$$6x^2 = 0$$

$$6x^2 = 0 \iff x(6x) = 0 \iff x = 0 \vee 6x = 0$$

The second alternative yields:

$$6x = 0 \iff x = 0$$

Therefore:

$$x = 0$$

Quadratic equations: the general case

Now will be described how to solve the quadratic equation $ax^2 + bx + c = 0$ in the general case where it may not always be the case that $b = 0$ or $c = 0$.

When factorization is obvious

It is also important to note that if a quadratic equation $ax^2 + bx + c = 0$ can be factored via **mathematical intuition** to have the form $(a_1x + b_1)(a_2x + b_2) = 0$, then there are two cases: $a_1x + b_1 = 0$ or $a_2x + b_2 = 0$. The first case yields:

$$a_1x + b_1 = 0 \iff a_1x = -b_1 \iff x = -b_1/a_1$$

The second case yields:

$$a_2x + b_2 = 0 \iff a_2x = -b_2 \iff x = -b_2/a_2$$

The easiest quadratic polynomial to factorize via mathematical intuition has $a = 1$: $x^2 + bx + c$. A factorization of this polynomial will have the form $(x + r_1)(x + r_2)$ where constants r_1 and r_2 are what is sought.

$$x^2 + bx + c = (x + r_1)(x + r_2) \iff x^2 + bx + c = x^2 + (r_1 + r_2)x + r_1r_2$$

Given coefficients b and c, quantities r_1 and r_2 must be chosen so that the sum is b and that the product is c. Below are hints as to how one can find r_1 and r_2 intuitively.

- If c is positive and b is positive, then r_1 and r_2 must both be positive, and they must add to b.
- If c is positive and b is negative, then r_1 and r_2 must both be negative, and their absolute values must add to $|b|$.
- If c is negative and b is positive, then r_1 is positive and r_2 is negative, and the difference in their absolute values is $|b|$ with r_1 having the greater absolute value.
- If c is negative and b is negative, then r_1 is positive and r_2 is negative, and the difference in their absolute values is $|b|$ with r_2 having the greater absolute value.

Example 1 Consider the equation:

$$x^2 + 8x + 7 = 0$$

$r_1 = 1$ and $r_2 = 7$ are two numbers whose product is 7 and whose sum is 8. Therefore:

$$x^2 + 8x + 7 = 0 \iff (x + 1)(x + 7) = 0 \iff x + 1 = 0 \vee x + 7 = 0$$

The first alternative gives:

$$x + 1 = 0 \iff x = -1$$

The second alternative gives:

$$x + 7 = 0 \iff x = -7$$

Therefore:

$$x = -1, -7$$

Example 2 Consider the equation:

$$x^2 - 8x + 7 = 0$$

$r_1 = -1$ and $r_2 = -7$ are two numbers whose product is 7 and whose sum is -8. Therefore:

$$x^2 - 8x + 7 = 0 \iff (x - 1)(x - 7) = 0 \iff x - 1 = 0 \vee x - 7 = 0$$

The first alternative gives:

$$x - 1 = 0 \iff x = 1$$

The second alternative gives:

$$x - 7 = 0 \iff x = 7$$

Therefore:

$$x = 1, 7$$

Example 3 Consider the equation:

$$x^2 + 6x - 7 = 0$$

$r_1 = 7$ and $r_2 = -1$ are two numbers whose product is -7 and whose sum is 6. Therefore:

$$x^2 + 6x - 7 = 0 \iff (x + 7)(x - 1) = 0 \iff x + 7 = 0 \vee x - 1 = 0$$

The first alternative gives:

$$x + 7 = 0 \iff x = -7$$

The second alternative gives:

$$x - 1 = 0 \iff x = 1$$

Therefore:

$$x = -7, 1$$

Example 4 Consider the equation:

$$x^2 - 6x - 7 = 0$$

$r_1 = 1$ and $r_2 = -7$ are two numbers whose product is -7 and whose sum is -6. Therefore:

$$x^2 - 6x - 7 = 0 \iff (x + 1)(x - 7) = 0 \iff x + 1 = 0 \vee x - 7 = 0$$

The first alternative gives:

$$x + 1 = 0 \iff x = -1$$

The second alternative gives:

$$x - 7 = 0 \iff x = 7$$

Therefore:

$$x = -1, 7$$

Most polynomials do not have an obvious factorization. In addition, if the quadratic equation has no solutions, then there is no factorization. Hence a straightforwards general approach is needed.

Completing the square

Given a quadratic polynomial ax^2+bx+c where a, b, and c are **known** coefficients, "completing the square" refers to rewriting the polynomial to have the form $a(x+p)^2+q$ where p and q are chosen such that

$$ax^2+bx+c = a(x+p)^2+q$$

The advantage of the form $a(x+p)^2+q$ is that x **only appears once**.

Firstly,

$$ax^2+bx+c = a(x^2+\frac{b}{a}x)+c$$

The subexpression $x^2+\frac{b}{a}x$ will now be manipulated to match $(x+p)^2$ as much as possible, matching the higher degree terms before the lower degree terms. Expanding $(x+p)^2$ gives $(x+p)^2 = x^2+2px+p^2$. When comparing this expression to $x^2+\frac{b}{a}x$, we see that the coefficients of x^2 are both 1, and the coefficients of x are $2p$ and $\frac{b}{a}$ respectively. The coefficients of x must equal each other, so $2p=\frac{b}{a}$ which is equivalent to $p=\frac{b}{2a}$. Now we know that $p=\frac{b}{2a}$. Since the coefficients of x match, the coefficient of x can be expressed as either $\frac{b}{a}$ or $2p$:

$$x^2+\frac{b}{a}x = x^2+2px$$

Now for x^2+2px to equal $x^2+2px+p^2$, we need to add and subtract p^2 to "complete the square":

$$x^2+2px = (x^2+2px+p^2)-p^2 = (x+p)^2-p^2$$

In total,

$$\begin{aligned} ax^2+bx+c =& a(x^2+\frac{b}{a}x)+c = a(x^2+2px)+c = a((x+p)^2-p^2)+c \\ =& a(x+p)^2+(c-ap^2) = a(x+p)^2+q \end{aligned}$$

where $p=\frac{b}{2a}$ and $q=c-ap^2$.

Examples:

- $x^2+6x+10 = (x^2+6x)+10 = ((x^2+2(3x)+3^2)-9)+10 = (x+3)^2+1$
- $x^2-8x = (x^2+2(-4x)+(-4)^2)-16 = (x-4)^2-16$
- $x^2-4x-7 = (x^2-4x)-7 = ((x^2+2(-2x)+(-2)^2)-4)-7 = (x-2)^2-11$
- $x^2+5x+7 = (x^2+5x)+7 = ((x^2+2(\frac{5}{2}x)+(\frac{5}{2})^2)-\frac{25}{4})-7 = (x+\frac{5}{2})^2-\frac{53}{4}$
- $$\begin{aligned} 2x^2-8x+7 =& 2(x^2-4x)+7 = 2((x^2+2(-2x)+(-2)^2)-4)+7 \\ =& 2(x-2)^2-8+7 = 2(x-2)^2-1 \end{aligned}$$
- $$\begin{aligned} 3x^2+2x-5 =& 3(x^2+\frac{2}{3}x)-5 = 3((x^2+2(\frac{1}{3}x)+(\frac{1}{3})^2)-\frac{1}{9})-5 \\ =& 3(x+\frac{1}{3})^2-\frac{1}{3}-5 = 3(x+\frac{1}{3})^2-\frac{16}{3} \end{aligned}$$

Completing the square and the quadratic formula

The most general approach to solving the quadratic equation is to "complete the square". In the equation $ax^2 + bx + c = 0$, x appears twice, however in the equation

$$a(x + p)^2 + q = 0$$

where p and q are as of yet undetermined constants, x appears once. Now will be described how to manipulate and rewrite the quadratic polynomial $ax^2 + bx + c$ to have the form $a(x + p)^2 + q$ via "completing the square". The expression $(x + p)^2$ expands to equal $x^2 + 2px + p^2$. Using this expansion as a "goal",

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + 2\frac{b}{2a}x) + c = a((x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2) - (\frac{b}{2a})^2) + c$$

By adding and subtracting $(\frac{b}{2a})^2$ the square of the binomial has been "completed".

$$= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$$

The quadratic polynomial now has the form $a(x + p)^2 + q$ where $p = \frac{b}{2a}$ and $q = c - \frac{b^2}{4a}$. The quadratic equation $ax^2 + bx + c = 0$ has become

$$a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a}) = 0$$

With x now appearing once, "unwrapping" x gives:

$$a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a}) = 0 \iff a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a}$$
$$\iff (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \iff x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is referred to as the **quadratic formula** and can be used to solve any quadratic equation with the form $ax^2 + bx + c = 0$.

The expression

$$\Delta = b^2 - 4ac$$

inside the square root is known as the "discriminant" and its sign indicates the number of solutions that the quadratic equation has:

- If $\Delta > 0$, then there are **two solutions**: $x = \frac{-b\pm\sqrt{\Delta}}{2a}$
- If $\Delta = 0$, then there is only **one solution**: $x = -\frac{b}{2a}$
- If $\Delta < 0$, then there is **no solutions**.

Examples:

- Consider the equation:

$$x^2 + 14x + 40 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 196 - 160 = 36$$

Since $\Delta > 0$, there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-14 \pm 6}{2} = -4, -10$$

- Consider the equation:

$$2x^2 - 20x + 48 = 0$$

to simplify the equation, divide both sides by the common factor of 2 to get:

$$x^2 - 10x + 24 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 100 - 96 = 4$$

Since $\Delta > 0$, there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{10 \pm 2}{2} = 6, 4$$

- Consider the equation:

$$2x^2 - 5x - 3 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 25 + 24 = 49$$

Since $\Delta > 0$, there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{5 \pm 7}{4} = 3, -\frac{1}{2}$$

- Consider the equation:

$$3x^2 + 12x + 12 = 0$$

to simplify the equation, divide both sides by the common factor of 3 to get:

$$x^2 + 4x + 4 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 16 - 16 = 0$$

Since $\Delta = 0$, there is only one solution:

$$x = -\frac{b}{2a} = -\frac{4}{2} = -2$$

- Consider the equation:

$$9x^2 + 6x + 1 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 36 - 36 = 0$$

Since $\Delta = 0$, there is only one solution:

$$x = -\frac{b}{2a} = -\frac{6}{18} = -\frac{1}{3}$$

- Consider the equation:

$$-3x^2 + 30x - 78 = 0$$

to simplify the equation, divide both sides by the common factor of -3 to get:

$$x^2 - 10x + 26 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 100 - 104 = -4$$

Since $\Delta < 0$, there is **no solutions**.

- Consider the equation:
$$-4x^2 + 4x - 5$$
The discriminant is:
$$\Delta = b^2 - 4ac = 16 - 80 = -64$$
Since $\Delta < 0$, there is **no solutions**.

- Consider the equation:
$$-7x^2 + 21x + 70 = 0$$
to simplify the equation, divide both sides by the common factor of -7 to get:
$$x^2 - 3x - 10 = 0$$
The discriminant is:
$$\Delta = b^2 - 4ac = 9 + 40 = 49$$
Since $\Delta > 0$, there are two solutions:
$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{3 \pm 7}{2} = 5, -2$$

- Consider the equation:
$$-6x^2 - x + 1 = 0$$
The discriminant is:
$$\Delta = b^2 - 4ac = 1 + 24 = 25$$
Since $\Delta > 0$, there are two solutions:
$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 5}{-12} = -\frac{1}{2}, \frac{1}{3}$$

The equations of circles in general form

Using Cartesian Coordinates, the **standard form** for the equation of a circle centered on point (a, b) with radius R is:
$$(x - a)^2 + (y - b)^2 = R^2$$

Fully expanded, equations with the **general form** of:
$$x^2 + y^2 + Ax + By + C = 0$$
where A, B, and C are arbitrary coefficients, represent a circle provided that there exists a point that satisfies the above equation.

Converting from the standard form to the general form gives:
$$(x-a)^2 + (y-b)^2 = R^2 \iff (x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = R^2$$
$$\iff x^2 + y^2 + (-2a)x + (-2b)y + (a^2 + b^2 - R^2) = 0$$

Converting from the general form to the standard form by completing the square gives:
$$x^2 + y^2 + Ax + By + C = 0 \iff (x^2 + Ax + (A/2)^2) + (y^2 + By + (B/2)^2) = (A/2)^2 + (B/2)^2 - C$$
$$\iff (x + A/2)^2 + (y + B/2)^2 = (A^2 + B^2 - 4C)/4 \iff \left(x - \frac{-A}{2}\right)^2 + \left(y - \frac{-B}{2}\right)^2 = \left(\frac{\sqrt{A^2 + B^2 - 4C}}{2}\right)^2$$

Examples:

- The circle $(x+6)^2+(y-2)^2=4$ is equivalent to:

$$(x+6)^2+(y-2)^2=4 \iff (x^2+12x+36)+(y^2-4y+4)=4$$
$$\iff x^2+y^2+12x-4y+36=0$$

- The circle $x^2+y^2+18x-12y-12=0$ is equivalent to:

$$x^2+y^2+18x-12y-12=0 \iff (x+2(9x)+81)+(y^2+2(-6y)+36)=12+81+36$$
$$\iff (x+9)^2+(y-6)^2=129$$

- The circle $x^2+y^2-14x+8y+5=0$ is equivalent to:

$$x^2+y^2-14x+8y+5=0 \iff (x^2+2(-7x)+49)+(y+2(4y)+16)=49+16-5$$
$$\iff (x-7)^2+(y+4)^2=60$$

- The circle $x^2+y^2-2x-4y+10=0$ is equivalent to:

$$x^2+y^2-2x-4y+10=0 \iff (x^2+2(-x)+1)+(y^2+2(-2y)+4)=1+4-10$$
$$\iff (x-1)^2+(y-2)^2=-5$$

  Since the right hand side is negative, no points satisfy this equation, and no circle actually exists.

- The circle $x^2+y^2-6x+4y+13=0$ is equivalent to:

$$x^2+y^2-6x+4y+13=0 \iff (x^2+2(-3x)+9)+(y^2+2(2y)+4)=9+4-13$$
$$\iff (x-3)^2+(y+2)^2=0$$

  Since the right hand side is 0, the only point that satisfies this equation is the point $(3,-2)$, so this "circle" has a radius of 0.