Lines and Planes

Define the points $P(5,0,4)$; $Q(1,1,1)$; $R(7,3,1)$; and $S(-5,-1,1)$.

Question 1:

part 1a:

Compute the displacement of Q relative P: $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \begin{bmatrix} 1-5 \\ 1-0 \\ 1-4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$$

part 1b:

Compute the magnitude $\left|\overrightarrow{PQ}\right|$.

$$\left|\overrightarrow{PQ}\right| = \left|\begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}\right| = \sqrt{16+1+9} = \sqrt{26}$$

part 1c:

Compute a unit vector that shares the same direction of $\overrightarrow{PQ}$.

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{1}{\sqrt{26}} \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{26} \\ 1/\sqrt{26} \\ -3/\sqrt{26} \end{bmatrix}$$

part 1d:

Compute parametric and implicit equations of a line L_{PQ} that contains P and Q.

Parametric equation:

Starting at P with direction vector $\overrightarrow{PQ}$, one possible parameterization of L_{PQ} is:

$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$ which is equivalent to $\begin{cases} x(t) = & 5-4t \\ y(t) = & t \\ z(t) = & 4-3t \end{cases}$

Implicit equations:

The parameterization gives $\begin{cases} t = & (x-5)/(-4) \\ t = & y \\ t = & (z-4)/(-3) \end{cases}$

Since all t's must be equal, the implicit equations are: $\frac{x-5}{-4} = y = \frac{z-4}{-3}$

Question 2:

part 2a:

Compute the angle $\angle RPQ$ using the dot product.

$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 7-5 \\ 3-0 \\ 1-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$ so $\overrightarrow{PQ} \cdot \overrightarrow{PR} = -8 + 3 + 9 = 4$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \left|\overrightarrow{PQ}\right| \left|\overrightarrow{PR}\right| \cos(\angle RPQ)$ gives

$$\angle RPQ = \arccos\left(\frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\left|\overrightarrow{PQ}\right| \left|\overrightarrow{PR}\right|}\right) = \arccos\left(\frac{4}{\sqrt{16+1+9} \cdot \sqrt{4+9+9}}\right) = \arccos\left(\frac{4}{\sqrt{26} \cdot \sqrt{22}}\right)$$

part 2b:

Compute the area of triangle ΔPQR using the cross product.

$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$ so $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} (1)(-3) - (-3)(3) \\ (-3)(2) - (-4)(-3) \\ (-4)(3) - (1)(2) \end{bmatrix} = \begin{bmatrix} -3+9 \\ -6-12 \\ -12-2 \end{bmatrix} = \begin{bmatrix} 6 \\ -18 \\ -14 \end{bmatrix}$

The area is:

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{36+324+196} = \frac{\sqrt{556}}{2}$$

part 2c:

Compute an implicit equation of a plane M_{PQR} that contains P, Q and R.

The cross product $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 6 \\ -18 \\ -14 \end{bmatrix}$ is a normal vector to M_{PQR}.

Using $P(5,0,4)$ as the point on the plane, the implicit equation is:

$$\mathbf{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{n} \cdot \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} \iff 6x - 18y - 14z = 30 + 0 - 56 \iff 6x - 18y - 14z = -26$$
$$\iff 3x - 9y - 7z = -13$$

Therefore M_{PQR} has the implicit equation: $3x - 9y - 7z = -13$

part 2d:

Find the shortest distance between R and the line L_{PQ} that contains P and Q.

The shortest distance is:

$$d = \left|\mathbf{perp}_{\overrightarrow{PQ}}(\overrightarrow{PR})\right| = \frac{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}{\left|\overrightarrow{PQ}\right|} = \frac{\sqrt{36+324+196}}{\sqrt{16+1+9}} = \frac{\sqrt{556}}{\sqrt{26}}$$

part 2e:

Find the intersection between the line L_{PQ} that contains P and Q and the line L_{RS} that contains R and S, if this intersection exists.

From question 1d, line L_{PQ} has the parameterization $\begin{cases} x(t) = & 5-4t \\ y(t) = & t \\ z(t) = & 4-3t \end{cases}$

Via similar steps, line L_{RS} has the parameterization $\begin{cases} x(t) = & 7-12t \\ y(t) = & 3-4t \\ z(t) = & 1 \end{cases}$

Let t_1 be the parameter value for L_{PQ}, and t_2 be the parameter value for L_{RS} such that the same point is generated by both lines. This gives the 3 equations:

$$\begin{cases} 5-4t_1 = & 7-12t_2 \\ t_1 = & 3-4t_2 \\ 4-3t_1 = & 1 \end{cases}$$

Solving the top equation gives $5-4t_1 = 7-12t_2 \iff 12t_2 = 2+4t_1 \iff t_2 = 1/6+(1/3)t_1$. Eliminating t_2 in the second equation gives $t_1 = 3+(-2/3-(4/3)t_1) \iff (7/3)t_1 = 7/3 \iff t_1 = 1$, which substituting into $t_2 = 1/6+(1/3)t_1$ gives $t_2 = 1/6+1/3 = 1/2$.

Lastly, substituting into the bottom equation yields $1 = 1$. This means that there **is** a solution and that the lines **intersect**. If a contradiction, like $1 = 2$, was attained, then there would be no solution and no intersection.

Using $t_1 = 1$ and $t_2 = 1/2$, L_{PQ} and L_{RS} intersect at $(1,1,1)$.

part 2f:

Find the volume of the parallelepiped bounded by $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$. Explain your result.

The volume is:

$$\left|\overrightarrow{PS}\cdot(\overrightarrow{PQ}\times\overrightarrow{PR})\right| = \left|\begin{bmatrix}-10\\-1\\-3\end{bmatrix}\cdot\begin{bmatrix}6\\-18\\-14\end{bmatrix}\right| = |-60+18+42| = 0$$

The 0 volume means that the parallelepiped is flat and that P, Q, R, and S all lie in the same plane.

Question 3:

part 3a:

Given the line $L_1 : \begin{cases} x(t) = & 1+t \\ y(t) = & 2+3t \\ z(t) = & 1+t \end{cases}$ and the plane $M_1 : 2x-y+7z = 9$, find the intersection between L_1 and M_1.

Finding the value of parameter t in line L_1 that will generate a point in plane M_1 requires that $x(t) = 1+t$; $y(t) = 2+3t$; and $z(t) = 1+t$ satisfy $2x(t)-y(t)+7z(t) = 9$.

$$2(1+t)-(2+3t)+7(1+t) = 9 \iff (2-3+7)t+(2-2+7) = 9 \iff 6t = 2 \iff t = 1/3$$

$t = 1/3$ generates the intersection point $(4/3, 3, 4/3)$

part 3b:

Given the plane $M_2 : x + y + z = 3$, find the intersection between M_1 and M_2.

The set of points that satisfy the equations $\begin{cases} 2x - y + 7z = & 9 \\ x + y + z = & 3 \end{cases}$ form the intersection.

The top equation gives $z = 9/7 - (2/7)x + (1/7)y$, which when substituted into the bottom equation gives $x + y + (9/7 - (2/7)x + (1/7)y) = 3 \iff (5/7)x + (8/7)y = 12/7 \iff y = 3/2 - (5/8)x$. Substituting the expression for y into $z = 9/7 - (2/7)x + (1/7)y$ gives $z = 9/7 - (2/7)x + ((3/14) - (5/56)x) = 21/14 - (21/56)x = 3/2 - (3/8)x$.

Letting $x = t$ yields the parameterization of the intersection $\begin{cases} x(t) = & t \\ y(t) = & 3/2 - (5/8)t \\ z(t) = & 3/2 - (3/8)t \end{cases}$

part 3c:

Given the point $T(3, 3, 3)$, find the closest distance between T and M_1.

The normal vector to M_1 is $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$

The closest distance is:

$$d = \frac{\left| \mathbf{n} \cdot \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - 9 \right|}{|\mathbf{n}|} = \frac{|6 - 3 + 21 - 9|}{\sqrt{4 + 1 + 49}} = \frac{15}{\sqrt{54}}$$

part 3d:

The plane $M_3 : -6x + 3y - 21z = -33$ is parallel to M_1. Find the separation between M_1 and M_3.

The equation of M_1 is $2x - y + 7z = 9$, and the equation of M_3 is equivalent to $2x - y + 7z = 11$. The left hand side of both equations are now equal with the same normal vector $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$.

The perpendicular separation is:

$$d = \frac{|11 - 9|}{|\mathbf{n}|} = \frac{2}{\sqrt{4 + 1 + 49}} = \frac{2}{\sqrt{54}}$$

Question 4:

part a)

Given vector $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, find x and y such that $\mathbf{u}||\mathbf{v}$.

$\mathbf{u}||\mathbf{v}$ if and only if the ratios between corresponding components are all equal: $x/2 = y/5 = 1/(-7)$. This gives $x = -2/7$ and $y = -5/7$.

part b)

Given vector $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ z \end{bmatrix}$, find z such that $\mathbf{u} \perp \mathbf{v}$.

$\mathbf{u} \perp \mathbf{v}$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$ which yields $-2 + 5 - 3z = 0 \iff z = 1$