Intro to Vector Calculus

Question 1:

Compute the following scalar line integrals:

- $\int_C (x+y)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t \\ 1-t \end{bmatrix}$ and $t \in [0,1]$
- $\int_C (x-y)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} 4t \\ 3t \end{bmatrix}$ and $t \in [0,2]$
- $\int_C (x^2+y^2+z^2)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ 8t \end{bmatrix}$ and $t \in [0,\pi/2]$
- $\int_C \sqrt{1+4y} \cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $t \in [0,2]$
- $\int_C xy^4 \cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} 2\sin t \\ 2\cos t \end{bmatrix}$ and $t \in [0,\pi]$
- $\int_C y^2 \cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} 3+2\cos t \\ 4+2\sin t \end{bmatrix}$ and $t \in [0,2\pi]$
- $\int_C \frac{ds}{x^2+y^2+z^2}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ and $t \in [0,2\pi]$

Solution:

For $\int_C (x+y)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} t \\ 1-t \end{bmatrix}$ and $t \in [0,1]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = \sqrt{2}$

$$\begin{aligned} \int_C (x+y)ds &= \int_{t=0}^{1} (x_C(t) + y_C(t)) \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\ &= \int_{t=0}^{1} (t + (1-t)) \cdot \sqrt{2} \cdot dt = \int_{t=0}^{1} \sqrt{2} \cdot dt = \sqrt{2} \cdot t\Big|_{t=0}^{1} = \sqrt{2} \end{aligned}$$

For $\int_C (x-y)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} 4t \\ 3t \end{bmatrix}$ and $t \in [0,2]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = 5$

$$\begin{aligned}
\int_C (x-y)ds &= \int_{t=0}^{2} (x_C(t) - y_C(t)) \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{2} (4t-3t)\cdot 5dt = \int_{t=0}^{2} 5tdt = \frac{5}{2}t^2\Big|_{t=0}^{2} = 10
\end{aligned}$$

For $\int_C (x^2+y^2+z^2)ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \\ z_C(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \cos(t) \\ 8t \end{bmatrix}$ and $t \in [0, \pi/2]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 8 \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t + 64} = \sqrt{65}$

$$\begin{aligned}
\int_C (x^2+y^2+z^2)ds &= \int_{t=0}^{\pi/2} (x_C(t)^2 + y_C(t)^2 + z_C(t)^2) \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2)\cdot\sqrt{65}\cdot dt \\
&= \sqrt{65}\int_{t=0}^{\pi/2} (1+64t^2)dt = \sqrt{65}\ (t + \frac{64}{3}t^3)\Big|_{t=0}^{\pi/2} \\
&= \sqrt{65}((\frac{\pi}{2} + \frac{8}{3}\pi^3) - 0) = \sqrt{65}(\frac{\pi}{2} + \frac{8}{3}\pi^3)
\end{aligned}$$

For $\int_C \sqrt{1+4y}\cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $t \in [0,2]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = \sqrt{1+4t^2}$

$$\begin{aligned}
\int_C \sqrt{1+4y}\cdot ds &= \int_{t=0}^{2} \sqrt{1+4y_C(t)} \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{2} \sqrt{1+4t^2}\cdot\sqrt{1+4t^2}\cdot dt = \int_{t=0}^{2} (1+4t^2)dt = (t + \frac{4}{3}t^3)\Big|_{t=0}^{2} \\
&= (2 + \frac{32}{3}) - 0 = \frac{38}{3}
\end{aligned}$$

For $\int_C xy^4 \cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} 2\sin t \\ 2\cos t \end{bmatrix}$ and $t \in [0, \pi]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 2\cos t \\ -2\sin t \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$

$$\begin{aligned}
\int_C xy^4 \cdot ds &= \int_{t=0}^{\pi} x_C(t) y_C(t)^4 \cdot \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{\pi} (2\sin t)(2\cos t)^4 \cdot 2dt = 64 \int_{t=0}^{\pi} \cos^4 t \sin t dt \\
&= 64(-\frac{1}{5}\cos^5 t\Big|_{t=0}^{\pi}) = 64((-\frac{1}{5}(-1)^5) - (-\frac{1}{5}(1)^5)) \\
&= 64(\frac{1}{5} + \frac{1}{5}) = \frac{128}{5}
\end{aligned}$$

For $\int_C y^2 \cdot ds$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} 3 + 2\cos t \\ 4 + 2\sin t \end{bmatrix}$ and $t \in [0, 2\pi]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = 2$

$$\begin{aligned}
\int_C y^2 \cdot ds &= \int_{t=0}^{2\pi} y_C(t)^2 \cdot \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{2\pi} (4 + 2\sin t)^2 \cdot 2dt = \int_{t=0}^{2\pi} (32 + 32\sin t + 8\sin^2 t)dt \\
&= \int_{t=0}^{2\pi} (32 + 32\sin t + 8\frac{1 - \cos(2t)}{2})dt \\
&= \int_{t=0}^{2\pi} (36 + 32\sin t - 4\cos(2t))dt = (36t - 32\cos t - 2\sin(2t))|_{t=0}^{2\pi} \\
&= (72\pi - 32 - 0) - (0 - 32 - 0) = 72\pi
\end{aligned}$$

For $\int_C \frac{ds}{x^2+y^2+z^2}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \\ z_C(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ and $t \in [0, 2\pi]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$ and $\left|\frac{d\mathbf{r}_C}{dt}\right| = \sqrt{2}$

$$\begin{aligned} \int_C \frac{ds}{x^2+y^2+z^2} &= \int_{t=0}^{2\pi} \frac{1}{x_C(t)^2+y_C(t)^2+z_C(t)^2} \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\ &= \int_{t=0}^{2\pi} \frac{\sqrt{2}\cdot dt}{\cos^2 t+\sin^2 t+t^2} = \int_{t=0}^{2\pi} \frac{\sqrt{2}\cdot dt}{1+t^2} = \sqrt{2}\,\mathrm{atan}(t)\Big|_{t=0}^{2\pi} \\ &= \sqrt{2}\cdot \mathrm{atan}(2\pi) \end{aligned}$$

Question 2:

Compute the following vector line integrals:

- $\int_C \begin{bmatrix} x^2+y \\ x^2-y \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $t \in [0, 1]$
- $\int_C \begin{bmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} R\cos(t) \\ R\sin(t) \end{bmatrix}$ and $t \in [0, 2\pi]$
- $\int_C \begin{bmatrix} 1/(x+y) \\ 1/(x+y) \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $t \in [0, \pi/2]$
- $\int_C \begin{bmatrix} zy \\ x \\ z^2x \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix}$ and $t \in [0, 1]$
- $\int_C \begin{bmatrix} x \\ y \\ -5z \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ t \end{bmatrix}$ and $t \in [0, 2\pi]$
- $\int_C \begin{bmatrix} -(5y+4)/2 \\ \sqrt{2x}-3 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t^2/2 \\ -2t \end{bmatrix}$ and $t \in [0, 3]$
- $\int_C \begin{bmatrix} y^2 \\ xy-x^2 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}(t) = \begin{bmatrix} t \\ 3t \end{bmatrix}$ and $t \in [0, 1]$
- $\int_C \begin{bmatrix} y^2 \\ xy-x^2 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}(t) = \begin{bmatrix} t^2/9 \\ t \end{bmatrix}$ and $t \in [0, 3]$

Solution:

For $\int_C \begin{bmatrix} x^2+y \\ x^2-y \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $t \in [0,1]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$

$$\begin{aligned}
\int_C \begin{bmatrix} x^2+y \\ x^2-y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{1} \begin{bmatrix} x_C(t)^2 + y_C(t) \\ x_C(t)^2 - y_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt \\
&= \int_{t=0}^{1} \begin{bmatrix} t^2+t^2 \\ t^2-t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2t \end{bmatrix} dt = \int_{t=0}^{1} \begin{bmatrix} 2t^2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2t \end{bmatrix} dt = \int_{t=0}^{1} 2t^2 dt = \frac{2}{3} t^3 \Big|_{t=0}^{1} = \frac{2}{3}
\end{aligned}$$

For $\int_C \begin{bmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} R\cos(t) \\ R\sin(t) \end{bmatrix}$ and $t \in [0, 2\pi]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix}$

$$\begin{aligned}
\int_C \begin{bmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{2\pi} \begin{bmatrix} -y_C(t)/(x_C(t)^2 + y_C(t)^2) \\ x_C(t)/(x_C(t)^2 + y_C(t)^2) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt \\
&= \int_{t=0}^{2\pi} \begin{bmatrix} -R\sin(t)/(R^2\cos^2(t) + R^2\sin^2(t)) \\ R\cos(t)/(R^2\cos^2(t) + R^2\sin^2(t)) \end{bmatrix} \cdot \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} dt \\
&= \int_{t=0}^{2\pi} \begin{bmatrix} -(1/R)\sin(t) \\ (1/R)\cos(t) \end{bmatrix} \cdot \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} dt \\
&= \int_{t=0}^{2\pi} (\sin^2(t) + \cos^2(t)) dt = \int_{t=0}^{2\pi} 1 \cdot dt = t|_{t=0}^{2\pi} = 2\pi
\end{aligned}$$

For $\int_C \begin{bmatrix} 1/(x+y) \\ 1/(x+y) \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $t \in [0, \pi/2]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$

$$\begin{aligned}
\int_C \begin{bmatrix} 1/(x+y) \\ 1/(x+y) \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{\pi/2} \begin{bmatrix} 1/(x_C(t)+y_C(t)) \\ 1/(x_C(t)+y_C(t)) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt \\
&= \int_{t=0}^{\pi/2} \begin{bmatrix} 1/(\cos(t)+\sin(t)) \\ 1/(\cos(t)+\sin(t)) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt \\
&= \int_{t=0}^{\pi/2} \frac{-\sin(t)+\cos(t)}{\cos(t)+\sin(t)} dt = \ln|\cos(t)+\sin(t)||_{t=0}^{\pi/2} \\
&= \ln|0+1| - \ln|1+0| = 0 - 0 = 0
\end{aligned}$$

For $\int_C \begin{bmatrix} zy \\ x \\ z^2x \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \\ z_C(t) \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix}$ and $t \in [0,1]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{aligned}
\int_C \begin{bmatrix} zy \\ x \\ z^2x \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{1} \begin{bmatrix} z_C(t)y_C(t) \\ x_C(t) \\ z_C(t)^2 x_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=0}^{1} \begin{bmatrix} (3t)(2t) \\ t \\ (3t)^2(t) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} dt \\
&= \int_{t=0}^{1} \begin{bmatrix} 6t^2 \\ t \\ 9t^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} dt = \int_{t=0}^{1} (27t^3 + 6t^2 + 2t) dt \\
&= (\frac{27}{4}t^4 + 2t^3 + t^2)\Big|_{t=0}^{1} = (\frac{27}{4} + 3) - 0 = \frac{39}{4}
\end{aligned}$$

For $\int_C \begin{bmatrix} x \\ y \\ -5z \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \\ z_C(t) \end{bmatrix} = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ t \end{bmatrix}$ and $t \in [0, 2\pi]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} -2\sin(t) \\ 2\cos(t) \\ 1 \end{bmatrix}$

$$\int_C \begin{bmatrix} x \\ y \\ -5z \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^{2\pi} \begin{bmatrix} x_C(t) \\ y_C(t) \\ -5z_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=0}^{2\pi} \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ -5t \end{bmatrix} \cdot \begin{bmatrix} -2\sin(t) \\ 2\cos(t) \\ 1 \end{bmatrix} dt$$
$$= \int_{t=0}^{2\pi} -5t dt = -\frac{5}{2}t^2 \Big|_{t=0}^{2\pi} = -10\pi^2$$

For $\int_C \begin{bmatrix} -(5y+4)/2 \\ \sqrt{2x} - 3 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} t^2/2 \\ -2t \end{bmatrix}$ and $t \in [0, 3]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} t \\ -2 \end{bmatrix}$

$$\int_C \begin{bmatrix} -(5y+4)/2 \\ \sqrt{2x} - 3 \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^{3} \begin{bmatrix} -(5y_C(t)+4)/2 \\ \sqrt{2x_C(t)} - 3 \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=0}^{3} \begin{bmatrix} -(-10t+4)/2 \\ \sqrt{t^2} - 3 \end{bmatrix} \cdot \begin{bmatrix} t \\ -2 \end{bmatrix} dt$$
$$= \int_{t=0}^{3} \begin{bmatrix} 5t - 2 \\ t - 3 \end{bmatrix} \cdot \begin{bmatrix} t \\ -2 \end{bmatrix} dt = \int_{t=0}^{3} (5t^2 - 4t + 6) dt$$
$$= (\frac{5}{3}t^3 - 2t^2 + 6t) \Big|_{t=0}^{3} = (45 - 18 + 18) - 0 = 45$$

For $\int_C \begin{bmatrix} y^2 \\ xy - x^2 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} x_C(t) \\ y_C(t) \end{bmatrix} = \begin{bmatrix} t \\ 3t \end{bmatrix}$ and $t \in [0, 1]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\int_C \begin{bmatrix} y^2 \\ xy - x^2 \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^{1} \begin{bmatrix} y_C(t)^2 \\ x_C(t)y_C(t) - x_C(t)^2 \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt$$
$$= \int_{t=0}^{1} \begin{bmatrix} (3t)^2 \\ (t)(3t) - (t)^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} dt = \int_{t=0}^{1} \begin{bmatrix} 9t^2 \\ 2t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} dt = \int_{t=0}^{1} 15t^2 dt$$
$$= 5t^3 \Big|_{t=0}^{1} = 5$$

For $\int_C \begin{bmatrix} y^2 \\ xy - x^2 \end{bmatrix} \cdot d\mathbf{r}$ where C has the parameterization $C : \mathbf{r}_C(t) = \begin{bmatrix} t^2/9 \\ t \end{bmatrix}$ and $t \in [0, 3]$,

The parameterization yields $\frac{d\mathbf{r}_C}{dt} = \begin{bmatrix} (2/9)t \\ 1 \end{bmatrix}$

$$\begin{aligned}
\int_C \begin{bmatrix} y^2 \\ xy - x^2 \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{3} \begin{bmatrix} y_C(t)^2 \\ x_C(t)y_C(t) - x_C(t)^2 \end{bmatrix} \cdot \frac{d\mathbf{r}_C(t)}{dt} dt \\
&= \int_{t=0}^{3} \begin{bmatrix} t^2 \\ (t^2/9)(t) - (t^2/9)^2 \end{bmatrix} \cdot \begin{bmatrix} (2/9)t \\ 1 \end{bmatrix} dt \\
&= \int_{t=0}^{3} ((2/9)t^3 + ((1/9)t^3 - (1/81)t^4))dt \\
&= \int_{t=0}^{3} ((1/3)t^3 - (1/81)t^4)dt \\
&= ((1/12)t^4 - (1/(5 \cdot 81))t^5)\Big|_{t=0}^{3} \\
&= (27/4 - 3/5) - 0 = \frac{135 - 12}{20} = \frac{123}{20}
\end{aligned}$$

Question 3:

For each of the following vector fields $\mathbf{F}(\mathbf{r})$, determine if the field is conservative, and if so, find a scalar field $f(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r})$.

- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} 2xy^3 \\ 3y^2x^2 \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} -y + e^x \sin(y) \\ (x+2)e^x \cos(y) \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} e^{2x} \sin(y) \\ e^{2x} \cos(y) \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} 6x + 5y \\ 5x + 4y \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} 2x\cos(y) - y\cos(x) \\ -x^2 \sin(y) - \sin(x) \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} 3x^2 - 4xy \\ 2x^2 - 4 \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} 3x^2 + 4xy \\ 2x^2 - 4 \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} e^{y^2 - y} + e^x \\ x(2y - 1)e^{y^2 - y} \end{bmatrix}$
- $\mathbf{F}(x, y) = \begin{bmatrix} F_x(x, y) \\ F_y(x, y) \end{bmatrix} = \begin{bmatrix} -(x^4 + y^2)^{-2} \cdot 4x^3 \\ -(x^4 + y^2)^{-2} \cdot 2y \end{bmatrix}$

Solution:

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} 2xy^3 \\ 3y^2x^2 \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 6xy^2 - 6xy^2 = 0$$

Therefore $\mathbf{F}(x,y)$ is conservative.
Staring with $f(0,0) = C$,

$$\begin{aligned} f(x,y) =& C + \int_{x_1=0}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\ =& C + \int_{x_1=0}^{x} 0dx_1 + \int_{y_1=0}^{y} 3y_1^2x^2dy_1 \\ =& C + x^2y_1^3\big|_{y_1=0}^{y} = C + x^2y^3 \end{aligned}$$

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} -y + e^x\sin(y) \\ (x+2)e^x\cos(y) \end{bmatrix}$,
The circulation density is:

$$\begin{aligned} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} =& (e^x\cos(y) + (x+2)e^x\cos(y)) - (-1 + e^x\cos(y)) \\ =& 1 + (1 + (x+2) - 1)e^x\cos(y) = 1 + (x+2)e^x\cos(y) \end{aligned}$$

Therefore $\mathbf{F}(x,y)$ is **not** conservative.

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} e^{2x}\sin(y) \\ e^{2x}\cos(y) \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2e^{2x}\cos(y) - e^{2x}\cos(y) = e^{2x}\cos(y)$$

Therefore $\mathbf{F}(x,y)$ is **not** conservative.

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} 6x+5y \\ 5x+4y \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 5 - 5 = 0$$

Therefore $\mathbf{F}(x,y)$ is conservative.
Starting with $f(0,0) = C$,

$$\begin{aligned} f(x,y) =& C + \int_{x_1=0}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\ =& C + \int_{x_1=0}^{x} 6x_1 dx_1 + \int_{y_1=0}^{y} (5x+4y_1)dy_1 \\ =& C + 3x_1^2\Big|_{x_1=0}^{x} + (5xy_1 + 2y_1^2)\Big|_{y_1=0}^{y} \\ =& C + 3x^2 + 5xy + 2y^2 \end{aligned}$$

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} 2x\cos(y) - y\cos(x) \\ -x^2\sin(y) - \sin(x) \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (-2x\sin(y) - \cos(x)) - (-2x\sin(y) - \cos(x)) = 0$$

Therefore $\mathbf{F}(x,y)$ is conservative.
Staring with $f(0,0) = C$,

$$\begin{aligned} f(x,y) =& C + \int_{x_1=0}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\ =& C + \int_{x_1=0}^{x} 2x_1 dx_1 + \int_{y_1=0}^{y} (-x^2\sin(y_1) - \sin(x))dy_1 \\ =& C + x_1^2\Big|_{x_1=0}^{x} + (x^2\cos(y_1) - y_1\sin(x))\Big|_{y_1=0}^{y} \\ =& C + x^2 + ((x^2\cos(y) - y\sin(x)) - x^2) \\ =& C + x^2\cos(y) - y\sin(x) \end{aligned}$$

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} 3x^2 - 4xy \\ 2x^2 - 4 \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 4x - (-4x) = 8x$$

Therefore $\mathbf{F}(x,y)$ is **not** conservative.

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} 3x^2 + 4xy \\ 2x^2 - 4 \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 4x - 4x = 0$$

Therefore $\mathbf{F}(x,y)$ is conservative.
Starting with $f(0,0) = C$,

$$\begin{aligned} f(x,y) =& C + \int_{x_1=0}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\ =& C + \int_{x_1=0}^{x} (3x_1^2)dx_1 + \int_{y_1=0}^{y} (2x^2-4)dy_1 \\ =& C + x_1^3\Big|_{x_1=0}^{x} + (2x^2-4)y_1\Big|_{y_1=0}^{y} \\ =& C + (x^3 - 0) + ((2x^2-4)y - 0) \\ =& C + x^3 + 2x^2y - 4y \end{aligned}$$

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} e^{y^2-y} + e^x \\ x(2y-1)e^{y^2-y} \end{bmatrix}$,
The circulation density is:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (2y-1)e^{y^2-y} - (2y-1)e^{y^2-y} = 0$$

Therefore $\mathbf{F}(x,y)$ is conservative.
Starting with $f(0,0) = C$,

$$\begin{aligned}
f(x,y) =& C + \int_{x_1=0}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\
=& C + \int_{x_1=0}^{x} (1+e^{x_1})dx_1 + \int_{y_1=0}^{y} x(2y_1-1)e^{y_1^2-y_1}dy_1 \\
=& C + (x_1+e^{x_1})\Big|_{x_1=0}^{x} + xe^{y_1^2-y_1}\Big|_{y_1=0}^{y} \\
=& C + ((x+e^x)-1) + (xe^{y^2-y} - x) \\
=& (C-1) + e^x + xe^{y^2-y}
\end{aligned}$$

For $\mathbf{F}(x,y) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \end{bmatrix} = \begin{bmatrix} -(x^4+y^2)^{-2} \cdot 4x^3 \\ -(x^4+y^2)^{-2} \cdot 2y \end{bmatrix}$,
The circulation density is:

$$\begin{aligned}
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} =& \frac{\partial}{\partial x}(-(x^4+y^2)^{-2} \cdot 2y) - \frac{\partial}{\partial y}(-(x^4+y^2)^{-2} \cdot 4x^3) \\
=& 2(x^4+y^2)^{-3} \cdot 4x^3 \cdot 2y - 2(x^4+y^2)^{-3} \cdot 2y \cdot 4x^3 = 0
\end{aligned}$$

Therefore $\mathbf{F}(x,y)$ is conservative.
There is a singularity at the origin, so we will instead start with $f(1,0) = C$

$$\begin{aligned}
f(x,y) =& C + \int_{x_1=1}^{x} F_x(x_1,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1)dy_1 \\
=& C + \int_{x_1=1}^{x} -4x_1^{-5}dx_1 + \int_{y_1=0}^{y} -(x^4+y_1^2)^{-2} \cdot 2y_1 dy_1 \\
=& C + x_1^{-4}\Big|_{x_1=1}^{x} + (x^4+y_1^2)^{-1}\Big|_{y_1=0}^{y} \\
=& C + (x^{-4}-1) + ((x^4+y^2)^{-1} - x^{-4}) \\
=& (C-1) + (x^4+y^2)^{-1}
\end{aligned}$$