Recall that $\mathbf{i}$ is a vector with a length of 1 that points in the direction of the positive x-axis, and that $\mathbf{j}$ is a vector with a length of 1 that points in the direction of the positive y-axis. An arbitrary vector $\mathbf{v} = \langle v_x, v_y \rangle$ is the sum of its horizontal component $v_x\mathbf{i}$ and its vertical component $v_y\mathbf{j}$:

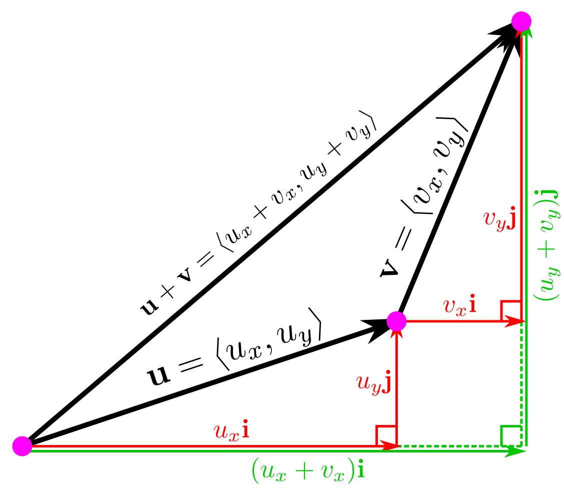
$$\mathbf{v} = \langle v_x, v_y \rangle = v_x\mathbf{i} + v_y\mathbf{j}$$

Component vector addition

A major benefit of vectors expressed using component form is that the addition and scalar multiplication of vectors is a trivial matter. Consider two vectors $\mathbf{u} = \langle u_x, u_y \rangle$ and $\mathbf{v} = \langle v_x, v_y \rangle$. Adding vectors $\mathbf{u}$ and $\mathbf{v}$, as seen in the image on the right, involves adding the horizontal components of $\mathbf{u}$ and $\mathbf{v}$ to get the horizontal component of $\mathbf{u} + \mathbf{v}$; and adding the vertical components of $\mathbf{u}$ and $\mathbf{v}$ to get the vertical component of $\mathbf{u} + \mathbf{v}$. The horizontal component of $\mathbf{u} + \mathbf{v}$ is $u_x\mathbf{i} + v_x\mathbf{i} = (u_x + v_x)\mathbf{i}$. The vertical component of $\mathbf{u} + \mathbf{v}$ is $u_y\mathbf{j} + v_y\mathbf{j} = (u_y + v_y)\mathbf{j}$. Therefore:

$$\mathbf{u} + \mathbf{v} = \langle u_x + v_x, u_y + v_y \rangle$$

Vectors are added by simply adding together their corresponding horizontal and vertical components.



Examples:

- $\langle 11, 9 \rangle + \langle -5, -7 \rangle = \langle 11 - 5, 9 - 7 \rangle = \langle 6, 2 \rangle$
- $\langle 34, -24 \rangle + \langle 23, 41 \rangle = \langle 34 + 23, -24 + 41 \rangle = \langle 57, 17 \rangle$
- $\langle 2, -24 \rangle + \langle 18, 21 \rangle = \langle 2 + 18, -24 + 21 \rangle = \langle 20, -3 \rangle$

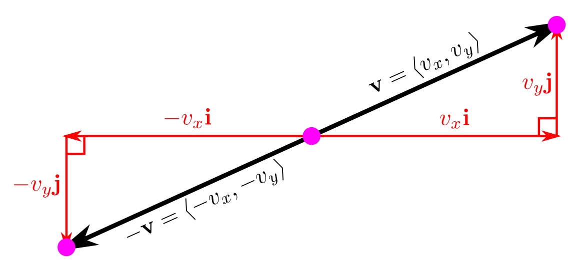
Component vector negatives

The negative of a vector $\mathbf{v} = \langle v_x, v_y \rangle$ is determined by simply reversing the direction of $\mathbf{v}$. In terms of components, the sign of each component is flipped:

$$-\mathbf{v} = \langle -v_x, -v_y \rangle$$

When a vector $\mathbf{v}$ is subtracted from a vector $\mathbf{u}$, what is happening is that the negative of $\mathbf{v}$ is being added to $\mathbf{u}$:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$



$$\langle u_x, u_y \rangle - \langle v_x, v_y \rangle = \langle u_x, u_y \rangle + \langle -v_x, -v_y \rangle = \langle u_x - v_x, u_y - v_y \rangle$$

Component vector scalar multiplication

Consider an arbitrary vector $\mathbf{v} = \langle v_x, v_y \rangle$. For an arbitrary natural number $n \in \{1, 2, 3, ...\}$, adding n copies of $\mathbf{v}$ together gives $n\mathbf{v}$:

$$\begin{aligned} n\mathbf{v} =& \underbrace{\mathbf{v} + \mathbf{v} + \cdots + \mathbf{v}}_{n} = \underbrace{\langle v_x, v_y \rangle + \langle v_x, v_y \rangle + \cdots + \langle v_x, v_y \rangle}_{n} = \left\langle \underbrace{v_x + v_x + \cdots + v_x}_{n}, \underbrace{v_y + v_y + \cdots + v_y}_{n} \right\rangle \\ =& \langle nv_x, nv_y \rangle \end{aligned}$$

Therefore:

$$n\mathbf{v} = \langle nv_x, nv_y \rangle$$

Generalizing the natural number n to a real number $c \in \mathbb{R}$ gives the general formula scalar multiplication:

$$c\mathbf{v} = \langle cv_x, cv_y \rangle$$

In essence, multiplication by a scalar c simply involves the multiplication of each component by c.

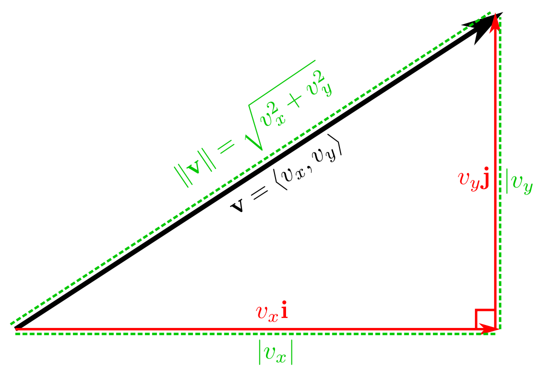
Examples:

- $3\langle -1, 5 \rangle = \langle 3 \cdot (-1), 3 \cdot 5 \rangle = \langle -3, 15 \rangle$
- $-2\langle 4, 7 \rangle = \langle (-2) \cdot 4, (-2) \cdot 7 \rangle = \langle -8, -14 \rangle$
- $5\langle 0.2, -1.2 \rangle = \langle 5 \cdot (0.2), 5 \cdot (-1.2) \rangle = \langle 1, -6 \rangle$

Component vector magnitude

Consider an arbitrary vector $\mathbf{v} = \langle v_x, v_y \rangle$. Through use of the Pythagorean theorem, the length (or magnitude) of $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, satisfies $\|\mathbf{v}\|^2 = |v_x|^2 + |v_y|^2$. Therefore:

$$\|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2}$$



Examples:

- $\|\langle -3, 4 \rangle\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$
- $\|\langle 12, -5 \rangle\| = \sqrt{12^2 + (-5)^2} = \sqrt{144 + 25} = \sqrt{169} = 13$
- $\|\langle -3, -1 \rangle\| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10} \approx 3.16228$

Putting it all together

- $6\langle 2, -3 \rangle - \langle -5, 7 \rangle = \langle 12, -18 \rangle + \langle 5, -7 \rangle = \langle 17, -25 \rangle$
- $\|\langle 3, 2 \rangle + 2\langle 1.5, -0.5 \rangle\| = \|\langle 3, 2 \rangle + \langle 3, -1 \rangle\| = \|\langle 6, 1 \rangle\| = \sqrt{6^2 + 1^2} = \sqrt{37} \approx 6.08276$