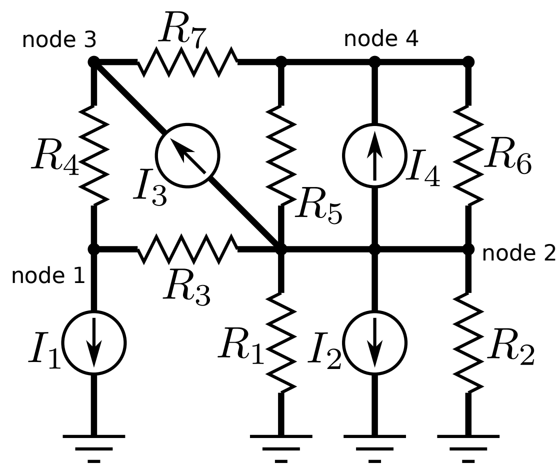
MATH2860 - 01 Project #1

Summer 2021

This project will involve the analysis of electric circuits such as the circuit depicted on the right. Essential terminology will be defined, and an analysis approach known has **node analysis** will be presented.

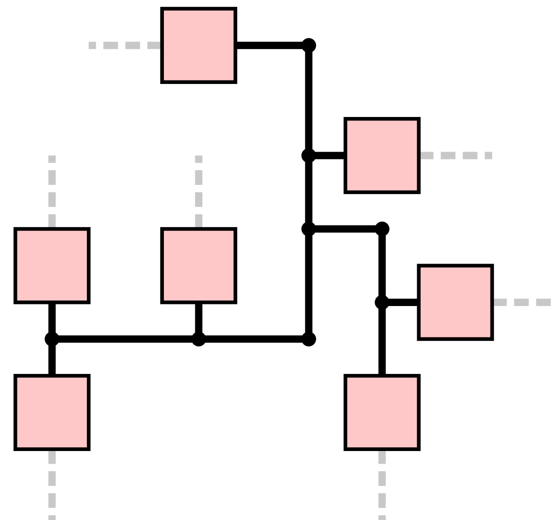


1 Terminology

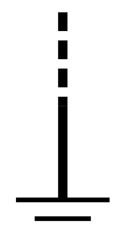
To describe simple electric circuits, a few basic terms need to be defined:

- **Voltage**, also referred to as "electric potential" is in essence "electrical pressure". When two conductors are at different voltages, electric charge flows from the conductor at a higher electric potential to the conductor at a lower electric potential. Voltage is measured using "volts" (V), and voltage variables often use the symbol V.
- **Current** is the rate at which electric charge passes a specified boundary, often a circuit component. The passage of negative charge backwards is indistinguishable from the passage of positive charge forwards. When current is measured in the opposite direction, its sign is reversed. Current is measured using "Amperes" (A). Amperes also measure the rate of charge accumulation at a specific point. Current variables often use the symbol I.

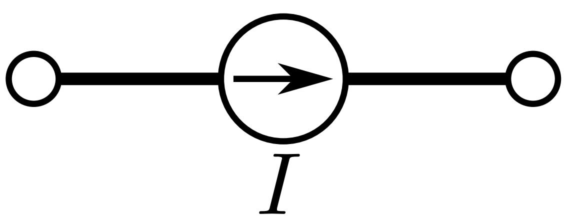
- In an electric circuit, a **node** is a continuous uninterrupted span of conducting material. Any two points in the circuit that can be connected by a path that does not pass through any circuit components or empty space are part of the same node, an example of which is depicted on the right. On the right, the branching wires that connect the visible components form a single node. The most important detail regarding nodes is that all points on the node have the same voltage. In circuit analysis, nodes are treated as single points connected by the various circuit components.



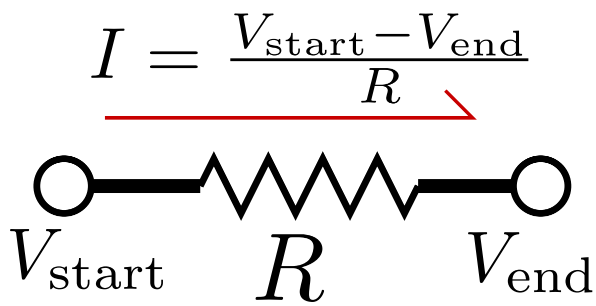
- The **ground node** is a node whose voltage is set to 0V. The voltage of all nodes is measured "relative" to the ground node. The ground node is not explicitly depicted in circuit diagrams, but an electrical connection to the ground node is denoted by the symbol on the right.



- A **circuit component** is an object that connects 2 or more nodes in an electric circuit. In this project, the only components that will be used are "current sources" and "resistors".
- A **current source** transfers a fixed current of I from a source node to a destination node.



- A **resistor** is a circuit component that connects two nodes, and the current that flows from the high voltage node to the low voltage node is proportional to the voltage difference between the two nodes.



The **resistance** R is the voltage required to propel each ampere of current through the resistor. The voltage difference required to propel a current of I is:

$$V_{\text{start}} - V_{\text{end}} = RI$$

Resistance is measured using "Ohms" (Ω). 1Ω is 1V per 1A.

The symbol G will often be used to denote the reciprocal of the resistance, referred to as the "conductivity": $G = 1/R$.

2 Setting up the linear system

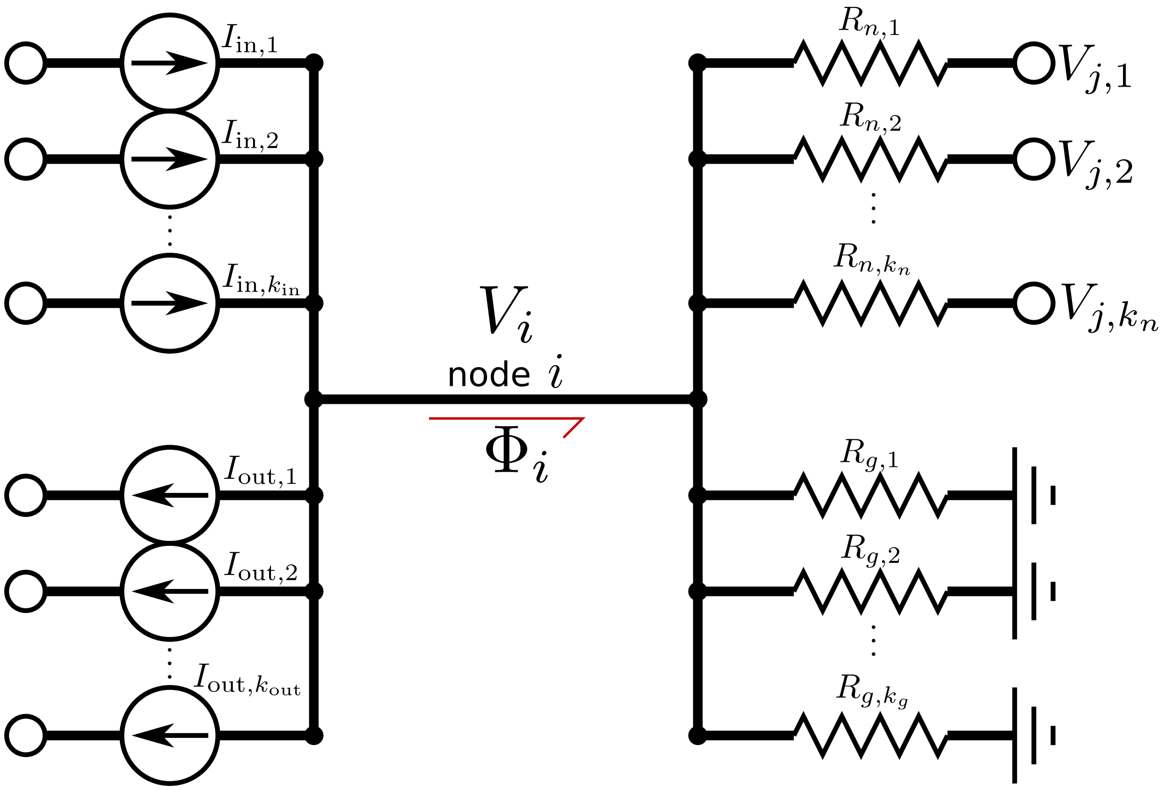
Given a circuit diagram with all current sources and resistances known, the goal of **node analysis** is to compute the voltage at each of the non-ground nodes. Let n denote the number of non-ground nodes, and index the non-ground nodes from 1 to n. Let V_1, V_2, ..., V_n denote the voltages at each of the non-ground nodes. All charge that enters a node must also leave it, and the total current that is leaving or entering a node must be 0. This current can be divided into two parts: the current that is entering the node as a result of the current sources, and the current that is draining from the node via the resistors. The rate at which current is pumped into node i by the current sources is equal to the rate at which current is draining through the resistors, and this common value will be denoted via Φ_i. The value of Φ_i can be computed directly from the circuit diagram by identifying the current sources that are connected to node i. The current sources that pump into node i add to Φ_i, while the current sources that drain from node i subtract from Φ_i. In the image below,

$$\Phi_i = (I_{\text{in},1} + I_{\text{in},2} + ... + I_{\text{in},k_{\text{in}}}) - (I_{\text{out},1} + I_{\text{out},2} + ... + I_{\text{out},k_{\text{out}}})$$

The rate at which current drains through the resistors is determined by the node voltages. Each connected resistor adds to the total current that is draining through the resistors. The current that drains through a resistor R_g that connects node i to the ground node is $\frac{V_i - 0}{R_g} = \frac{V_i}{R_g}$. The current that drains through a resistor R_n that connects node i to another node j is $\frac{V_i - V_j}{R_n}$. In the image below, the total current drain through the resistors is:

$$\begin{aligned}\Phi_i &= \left(\frac{V_i}{R_{g,1}} + \frac{V_i}{R_{g,2}} + ... + \frac{V_i}{R_{g,k_g}}\right) + \left(\frac{V_i - V_{j,1}}{R_{n,1}} + \frac{V_i - V_{j,2}}{R_{n,2}} + ... + \frac{V_i - V_{j,k_n}}{R_{n,k_n}}\right) \\ &= \left((\frac{1}{R_{g,1}} + \frac{1}{R_{g,2}} + ... + \frac{1}{R_{g,k_g}}) + (\frac{1}{R_{n,1}} + \frac{1}{R_{n,2}} + ... + \frac{1}{R_{n,k_n}})\right) V_i - \frac{V_{j,1}}{R_{n,1}} - \frac{V_{j,2}}{R_{n,2}} - ... - \frac{V_{j,k_n}}{R_{n,k_n}}\end{aligned}$$

By equating the two different expressions for Φ_i, a linear equation associated with node i is formed. The value of Φ_i computed from the current sources is the constant, and the expression for Φ_i above computed from the resistances gives the coefficients of the voltages V_1, V_2, ..., V_n.



Similar to how there is an unknown voltage V_i associated with each non-ground node i, there is an equation associated associated with each non-ground node i centered around the common value of Φ_i. In summary, the linear system in matrix vector form is:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix}$$

To derive the constant vector $\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix}$, each of the current sources need to be examined in sequence. Start by initializing $\Phi_i = 0$ for each non-ground node i. For each current source, do the following: Let I denote the current being pumped by the current source under consideration.

- If the starting node is not a ground-node, and has an index of i, then **subtract** I from Φ_i.
- If the ending node is not a ground-node, and has an index of i, then **add** I to Φ_i.

To derive the coefficient matrix, $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$, each of the resistors need to be examined in sequence. Start by initializing $a_{i,j} = 0$ for each pair of non-ground nodes i and j. For each resistor, do the following: Let R denote the resistance of the current resistor.

- If both terminals of the resistor are not the ground-node, and have indices of i and j, then **add** $1/R$ to both $a_{i,i}$ and $a_{j,j}$, and **subtract** $1/R$ from both $a_{i,j}$ and $a_{j,i}$.
- If one of the terminals is the ground node, let i denote the index of the non-ground node. **Add** $1/R$ to $a_{i,i}$.
- If both terminals are the ground node, then ignore the resistor.

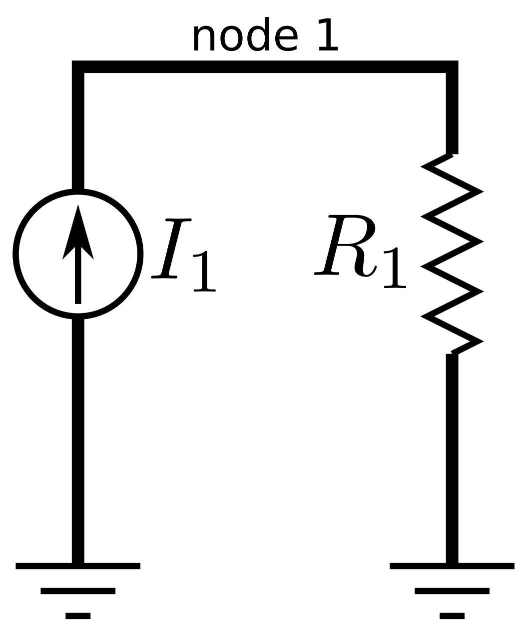
To better explain the process of computing the constant vector and coefficient matrix, the following examples will be given:

3 Example circuits

3.1 The simplest circuit

This is the simplest possible circuit.
Let $G_1 = 1/R_1$.
There is only 1 non-ground node, so the only unknown is V_1.
The single current source delivers a current of I_1 to node 1, so $\Phi_1 = I_1$.
The single resistor contributes $G_1 = 1/R_1$ to $a_{1,1}$, the only entry of A.
The matrix vector system is:

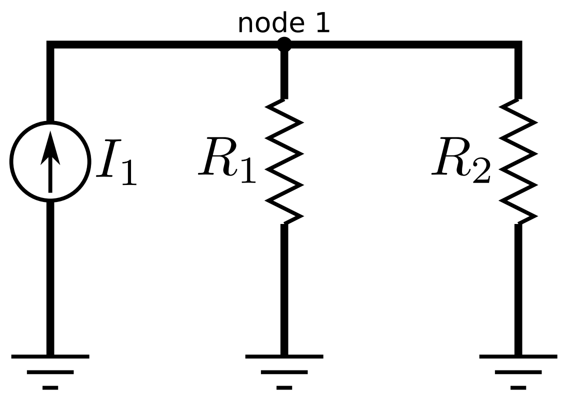
$$\begin{bmatrix} G_1 \end{bmatrix} \begin{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} I_1 \end{bmatrix}$$



3.2 A 1 node parallel circuit

Let $G_1 = 1/R_1$ and $G_2 = 1/R_2$.
There is only 1 non-ground node, so the only unknown is V_1.
The single current source delivers a current of $\Phi_1 = I_1$ to node 1.
Resistor R_1 contributes G_1 to $a_{1,1}$, and resistor R_2 contributes G_2 to $a_{1,1}$.
The matrix vector system is:

$$\begin{bmatrix} G_1 + G_2 \end{bmatrix} \begin{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} I_1 \end{bmatrix}$$



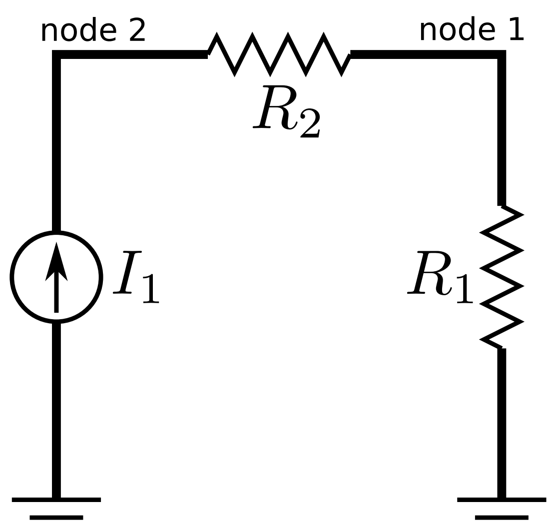
3.3 A 2 node series circuit

Let $G_1 = 1/R_1$ and $G_2 = 1/R_2$.
There are 2 non-ground nodes, so the unknowns are V_1 and V_2.
The single current source delivers a current of $\Phi_2 = I_1$ to node 2, while Φ_1 remains at 0.

- Resistor R_1 contributes G_1 to $a_{1,1}$.
- Resistor R_2 contributes G_2 to $a_{1,1}$ and $a_{2,2}$, and subtracts G_2 from $a_{1,2}$ and $a_{2,1}$.

The matrix vector system is:

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ I_1 \end{bmatrix}$$



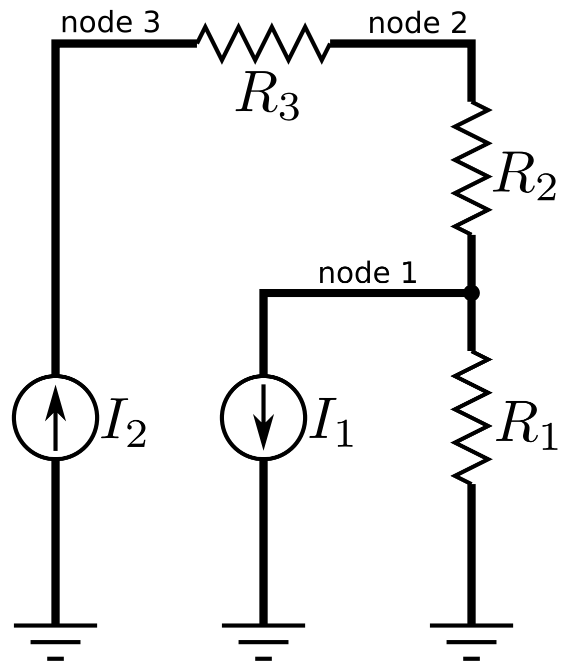
3.4 A 3 node circuit

Let $G_1 = 1/R_1$, $G_2 = 1/R_2$, and $G_3 = 1/R_3$.
There are 3 non-ground nodes, so the unknowns are V_1, V_2, and V_3.
The current sources deliver a current of $\Phi_1 = -I_1$ to node 1, and a current of $\Phi_3 = I_2$ to node 3, while Φ_2 remains at 0.

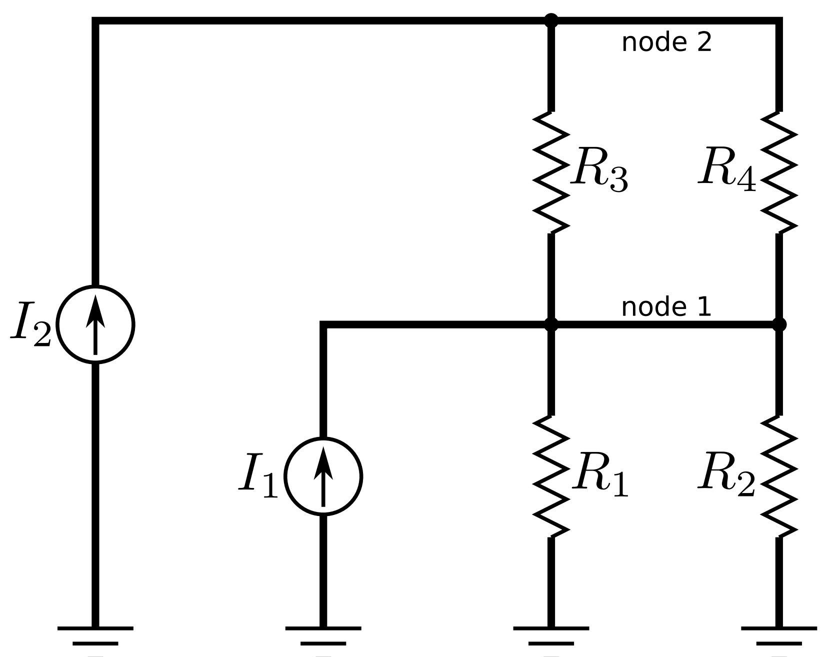
- The resistor that connects node 1 to the ground contributes G_1 to $a_{1,1}$.
- The resistor that links nodes 1 and 2 contributes G_2 to $a_{1,1}$ and $a_{2,2}$, and subtracts G_2 from $a_{1,2}$ and $a_{2,1}$.
- The resistor that links nodes 2 and 3 contributes G_3 to $a_{2,2}$ and $a_{3,3}$, and subtracts G_3 from $a_{2,3}$ and $a_{3,2}$.

The matrix vector system is:

$$\begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 & -G_3 \\ 0 & -G_3 & G_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -I_1 \\ 0 \\ I_2 \end{bmatrix}$$



3.5 A circuit with multiply linked nodes



Let $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$, and $G_4 = 1/R_4$.

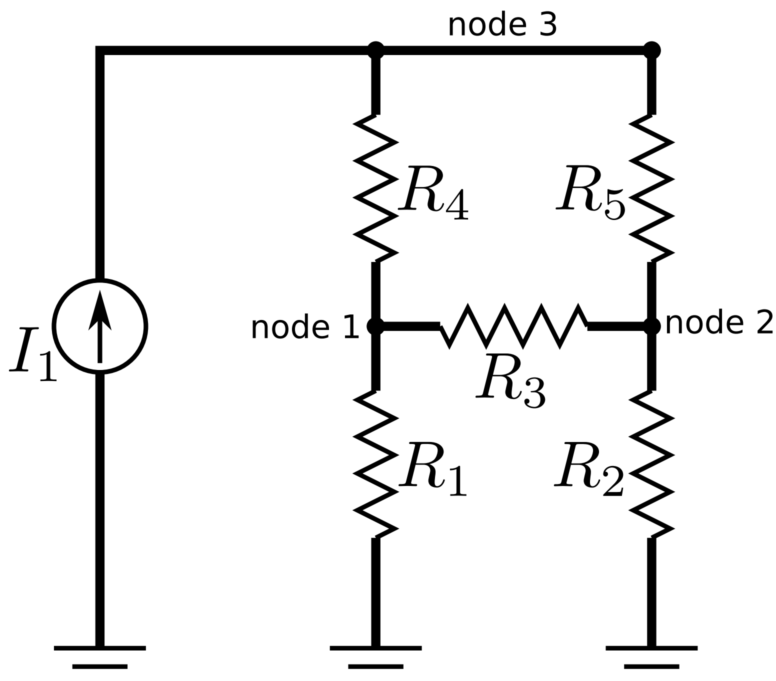
There are 4 non-ground nodes, so the unknowns are V_1, V_2, V_3, and V_4.
The current sources deliver a current of $\Phi_1 = I_1$ to node 1, and deliver a current of $\Phi_2 = I_2$ to node 2.

- The resistors that connect node 1 to the ground contribute $G_1 + G_2$ to $a_{1,1}$.
- The resistors that link nodes 1 and 2 contribute $G_3 + G_4$ to $a_{1,1}$ and $a_{2,2}$, and subtract $G_3 + G_4$ from $a_{1,2}$ and $a_{2,1}$.

The matrix vector system is:

$$\begin{bmatrix} G_1 + G_2 + G_3 + G_4 & -G_3 - G_4 \\ -G_3 - G_4 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

3.6 A circuit with a bridge



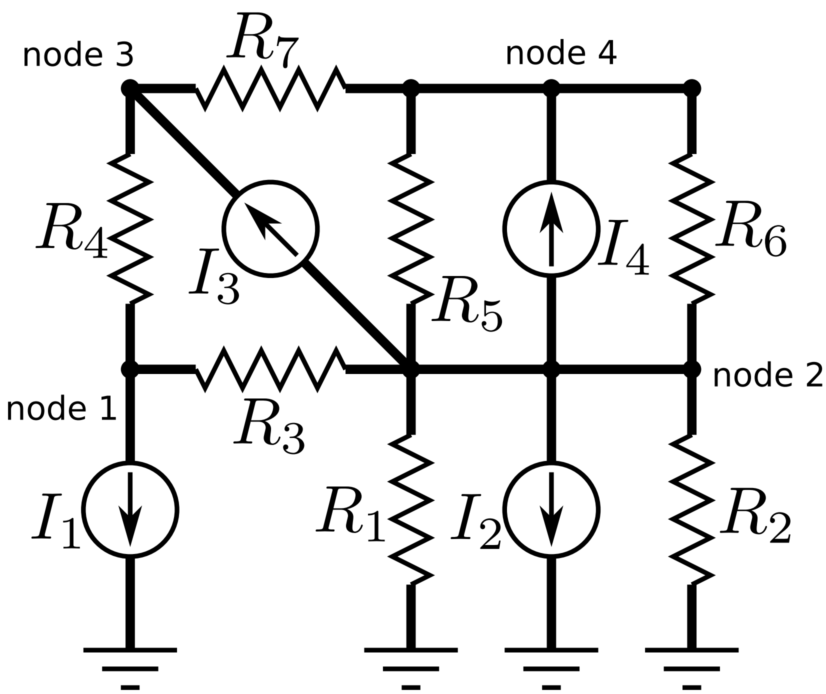
Let $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$, $G_4 = 1/R_4$, and $G_5 = 1/R_5$.
There are 3 non-ground nodes, so the unknowns are V_1, V_2, and V_3.
The current sources deliver a current of $\Phi_3 = I_1$ to node 3, while Φ_1 and Φ_2 remain at 0.

- The resistor that connects node 1 to the ground contributes G_1 to $a_{1,1}$.
- The resistor that connects node 2 to the ground contributes G_2 to $a_{2,2}$.
- The resistor that links nodes 1 and 2 contributes G_3 to $a_{1,1}$ and $a_{2,2}$, and subtracts G_3 from $a_{1,2}$ and $a_{2,1}$.
- The resistor that links nodes 1 and 3 contributes G_4 to $a_{1,1}$ and $a_{3,3}$, and subtracts G_4 from $a_{1,3}$ and $a_{3,1}$.
- The resistor that links nodes 2 and 3 contributes G_5 to $a_{2,2}$ and $a_{3,3}$, and subtracts G_5 from $a_{2,3}$ and $a_{3,2}$.

The matrix vector system is:

$$\begin{bmatrix} G_1+G_3+G_4 & -G_3 & -G_4 \\ -G_3 & G_2+G_3+G_5 & -G_5 \\ -G_4 & -G_5 & G_4+G_5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_1 \end{bmatrix}$$

3.7 A complex circuit



Let $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$, $G_4 = 1/R_4$, $G_5 = 1/R_5$, $G_6 = 1/R_6$, and $G_7 = 1/R_7$.

There are 4 non-ground nodes, so the unknowns are V_1, V_2, V_3, and V_4.

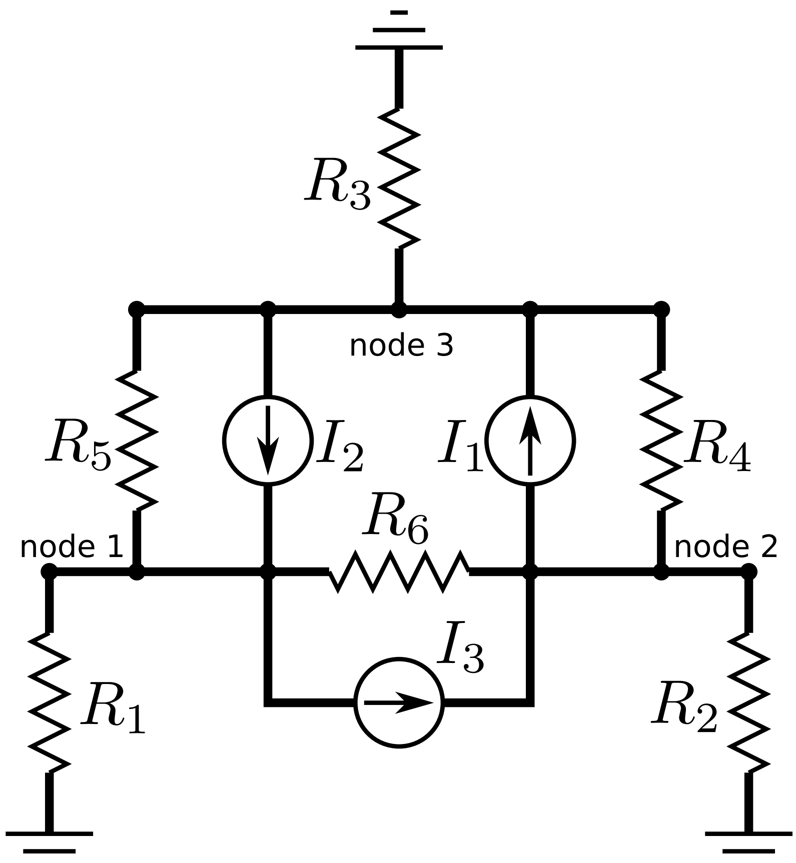
The current sources delivers a total current of $\Phi_1 = -I_1$ to node 1, a total current of $\Phi_2 = -I_2 - I_3 - I_4$ to node 2, a total current of $\Phi_3 = I_3$ to node 3, and a total current of $\Phi_4 = I_4$ to node 4.

- The resistors that connect node 2 to the ground contribute $G_1 + G_2$ to $a_{2,2}$.
- The resistor that connects nodes 1 and 2 contributes G_3 to $a_{1,1}$ and $a_{2,2}$, and subtracts G_3 from $a_{1,2}$ and $a_{2,1}$.
- The resistor that connects nodes 1 and 3 contributes G_4 to $a_{1,1}$ and $a_{3,3}$, and subtracts G_4 from $a_{1,3}$ and $a_{3,1}$.
- The resistors that connect nodes 2 and 4 contribute $G_5 + G_6$ to $a_{2,2}$ and $a_{4,4}$, and subtract $G_5 + G_6$ from $a_{2,4}$ and $a_{4,2}$.
- The resistor that connects nodes 3 and 4 contributes G_7 to $a_{3,3}$ and $a_{4,4}$, and subtracts G_7 from $a_{3,4}$ and $a_{4,3}$.

The matrix vector system is:

$$\begin{bmatrix} G_3+G_4 & -G_3 & -G_4 & 0 \\ -G_3 & G_1+G_2+G_3+G_5+G_6 & 0 & -G_5-G_6 \\ -G_4 & 0 & G_4+G_7 & -G_7 \\ 0 & -G_5-G_6 & -G_7 & G_5+G_6+G_7 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -I_1 \\ -I_2-I_3-I_4 \\ I_3 \\ I_4 \end{bmatrix}$$

3.8 A 2^{nd} complex circuit



Let $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$, $G_4 = 1/R_4$, $G_5 = 1/R_5$, and $G_6 = 1/R_6$

There are 3 non-ground nodes, so the unknowns are V_1, V_2, and V_3.

The current sources delivers a total current of $\Phi_1 = I_2 - I_3$ to node 1, a total current of $\Phi_2 = I_3 - I_1$ to node 2, a and total current of $\Phi_3 = I_1 - I_2$ to node 3.

- The resistor that connects node 1 to the ground contributes G_1 to $a_{1,1}$.
- The resistor that connects node 2 to the ground contributes G_2 to $a_{2,2}$.
- The resistor that connects node 3 to the ground contributes G_3 to $a_{3,3}$.
- The resistor that connects nodes 2 and 3 contributes G_4 to $a_{2,2}$ and $a_{3,3}$, and subtracts G_4 from $a_{2,3}$ and $a_{3,2}$.

- The resistor that connects nodes 1 and 3 contributes G_5 to $a_{1,1}$ and $a_{3,3}$, and subtracts G_5 from $a_{1,3}$ and $a_{3,1}$.
- The resistor that connects nodes 1 and 2 contributes G_6 to $a_{1,1}$ and $a_{2,2}$, and subtracts G_6 from $a_{1,2}$ and $a_{2,1}$.

The matrix vector system is:

$$\begin{bmatrix} G_1+G_5+G_6 & -G_6 & -G_5 \\ -G_6 & G_2+G_4+G_6 & -G_4 \\ -G_5 & -G_4 & G_3+G_4+G_5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_2-I_3 \\ I_3-I_1 \\ I_1-I_2 \end{bmatrix}$$

4 Assignment

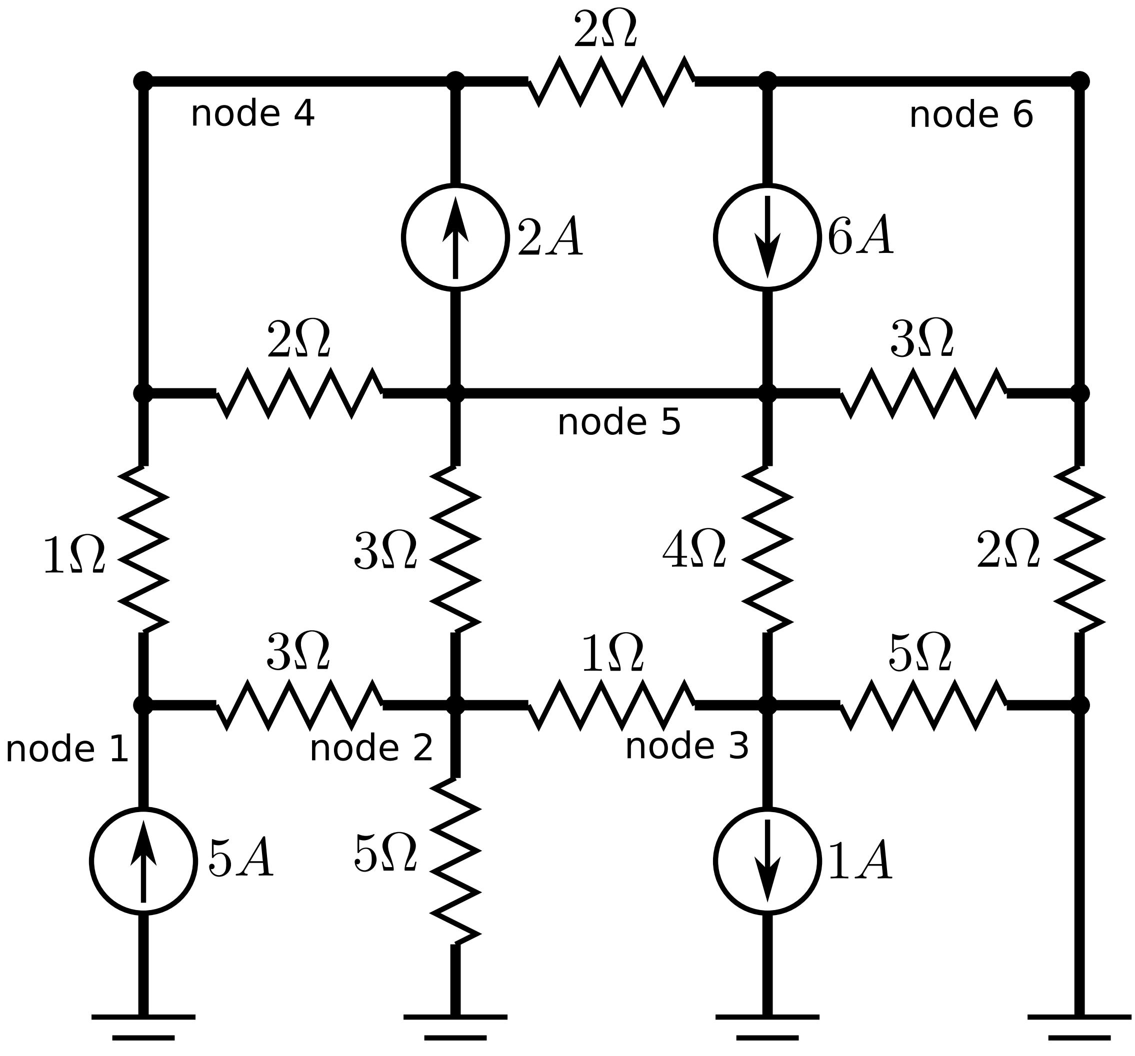
You are provided with an incomplete Matlab file. You must complete the Matlab file so that the file can be used for the following nodal analysis. The following is a description of the input variables used in the Matlab file:

- `num_of_nodes` is the number of nodes n in the circuit. The nodes are hence indexed from 1 to n.
- `num_of_current_sources` is the number k_I of current sources.
- `current_source_end_nodes` is an $k_I \times 2$ array where for each $i = 1, 2, ..., k_I$, entry $(i, 1)$ stores the node index of the start of current source i, and entry $(i, 2)$ stores the node index of the end of current source i. The ground node is indexed by 0.
- `current_source_values` is an $k_I \times 1$ array where for each $i = 1, 2, ..., k_I$, entry $(i, 1)$ stores the current being pumped by current source i.
- `num_of_resistors` is the number k_R of resistors.
- `resistor_end_nodes` is an $k_R \times 2$ array where for each $i = 1, 2, ..., k_R$, entries $(i, 1)$ and $(i, 2)$ store the node indices of the terminals of resistor i. The ground node is indexed by 0.
- `resistor_values` is an $k_R \times 1$ array where for each $i = 1, 2, ..., k_R$, entry $(i, 1)$ stores the resistance of resistor i.

Everywhere `???` appears you must complete the code.

In the circuit below, there are 6 nodes. Using Matlab compute the following. **Submit your Matlab script.** Your Matlab script must compute the asked for values.

- Φ_i at each node.
- The 6×6 coefficient matrix A
- Solve the system $A \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{bmatrix}$ for the voltage at each node.


2Ω
node 4
node 6
2A
6A
2Ω
3Ω
node 5
1Ω
3Ω
4Ω
2Ω
3Ω
1Ω
5Ω
node 1
node 2
node 3
5A
5Ω
1A