Theorems related to Vector Calculus

Question 1:

part 1a:

Evaluate the scalar line integral:

$$\int_C (6x^2 + 4y)ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} 3t+2 \\ -t+3 \end{bmatrix} \text{ and } t \in [-1, 1]$$

part 1b:

Evaluate the scalar line integral:

$$\int_C (x^2 + 2y)ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} 1-2t \\ 2-t \end{bmatrix} \text{ and } t \in [0, 3]$$

part 1c:

Evaluate the scalar line integral:

$$\int_C 2x \cdot \sin(y) \cdot ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} \cos(t) \\ t \end{bmatrix} \text{ and } t \in [0, \pi/2]$$

Question 2:

part 2a:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix} -6x \\ 4y \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} t^2 \\ 1/t \end{bmatrix} \text{ and } t \in [1, 2]$$

part 2b:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix} y \\ x \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} t^3 \\ t^4 \end{bmatrix} \text{ and } t \in [0, 2]$$

part 2c:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix} 2x \\ y^2 \\ -z \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} -t^2 \\ t+4 \\ 3t+2 \end{bmatrix} \text{ and } t \in [-1, 1]$$

Question 3:

part 3a:

Is the vector field $\mathbf{F}(x, y, z) = \begin{bmatrix} y^2 \sin(z) \\ 2xy \sin(z) \\ xy^2 \cos(z) \end{bmatrix}$ conservative? If yes, use the **gradient theorem** to evaluate the vector line integral:

$$\int_C \begin{bmatrix} y^2 \sin(z) \\ 2xy \sin(z) \\ xy^2 \cos(z) \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_{\text{initial}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{r}_{\text{final}} = \begin{bmatrix} 1 \\ 1 \\ \pi/2 \end{bmatrix}$$

part 3b:

Is the vector field $\mathbf{F}(x, y, z) = \begin{bmatrix} xy \\ xy \\ z \end{bmatrix}$ conservative? If yes, use the **gradient theorem** to evaluate the vector line integral:

$$\int_C \begin{bmatrix} xy \\ xy \\ z \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_{\text{initial}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{r}_{\text{final}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Question 4:

part 4a:

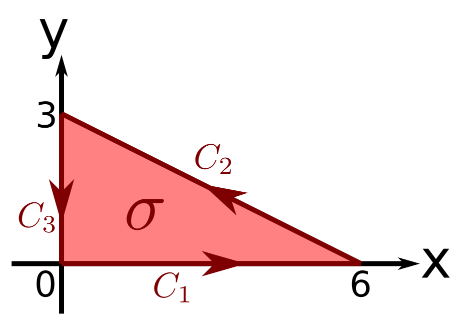
The region σ on the right is

$$\sigma = \{(x, y) | 0 \leq x \leq 6 \text{ and } 0 \leq y \leq 3 - (1/2)x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r}$; $\int_{C_2} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_3} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} 3x - 5y \\ x - 2y \end{bmatrix} \cdot d\mathbf{r}$$

part 4b:

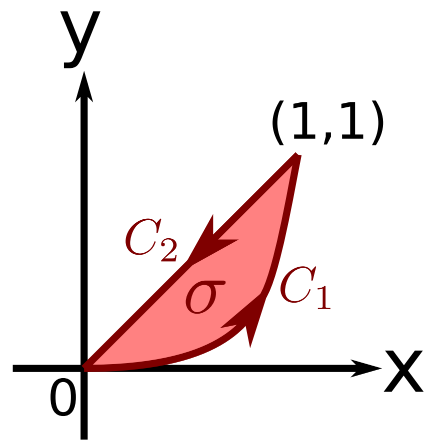
The region σ on the right is

$$\sigma = \{(x,y)|0 \leq x \leq 1 \text{ and } x^3 \leq y \leq x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_2} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$$

part 4c:

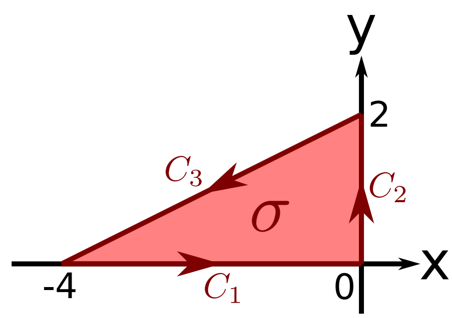
The region σ on the right is

$$\sigma = \{(x,y)| -4 \leq x \leq 0 \text{ and } 0 \leq y \leq 2 + (1/2)x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$$

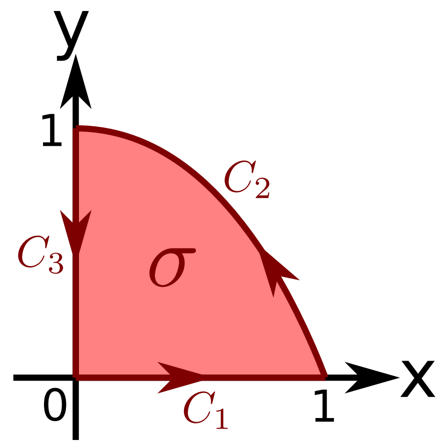
where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$; $\int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$$

Question 5:



The region σ on the right is

$$\sigma = \{(x, y) | 0 \le x \le 1 \text{ and } 0 \le y \le 1 - x^2\}$$

Use Gauss's divergence theorem to compute the flux integral

$$\int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.

Next, compute each of $\int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$; $\int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$; and $\int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} = \int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$$