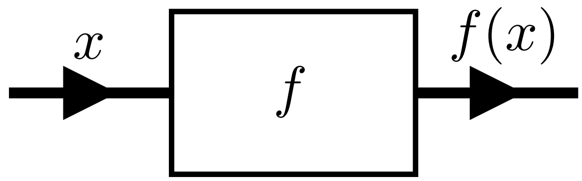
1 Functions

A function is a mathematical object that accepts one or more quantities as "input", and then returns a quantity as "output". In the image to the right, a function f takes a single quantity x and returns the quantity $f(x)$. The quantity x is often referred to as a "parameter" of function f. The set of input values x for which $f(x)$ is defined is referred to as the **domain** of f.



Functions are generally defined by the notation:

$$\textbf{function\_name}(\textbf{parameter}) = \textbf{expression}$$

The "**parameter**" is a symbol (often x) that is used as a placeholder for the input value in the expression that computes the output/return value. The "**expression**" is an expression that involves the parameter symbol. The value of this expression when the parameter symbol is replaced with the input value is the output/return value.

Examples: As an example function, let f be defined as $f(X) = 2X + 5$ for every $X \in \mathbb{R}$. The domain of f is $\mathbb{R}$. Some examples of using this function are:

- When the input is $X = -4$, the return value is $f(-4) = 2(-4) + 5 = -8 + 5 = -3$
- When the input is $X = 0$, the return value is $f(0) = 2(0) + 5 = 5$
- When the input is $X = 15$, the return value is $f(15) = 2(15) + 5 = 30 + 5 = 35$
- When the input is $X = y + 2$, the return value is $f(y + 2) = 2(y + 2) + 5 = (2y + 4) + 5 = 2y + 9$

The choice of parameter symbol does not change the function. The definition $f(y) = 2y + 5$ defines the same function as $f(X) = 2X + 5$. It is more convenient to use a symbol that is not used for other purposes. We will often use capital letters X, Y, etc. to distinguish the parameter from other variables. The lowercase x for example, often sees other uses, and having a symbol other than x for the parameter name helps reduce ambiguity.

Use another function g defined by $g(X) = \frac{8-3X}{7X+2}$. The domain of g is all real numbers except for the value of X where $7X + 2 = 0$.

- When the input is $X = 2$, the return value is $g(2) = \frac{8-3(2)}{7(2)+2} = \frac{8-6}{14+2} = \frac{2}{16} = \frac{1}{8}$
- When the input is $X = -1$, the return value is $g(-1) = \frac{8-3(-1)}{7(-1)+2} = \frac{8+3}{-7+2} = \frac{11}{-5} = -\frac{11}{5}$
- When the input is $X = 0$, the return value is $g(0) = \frac{8-3(0)}{7(0)+2} = \frac{8}{2} = 4$
- When the input is $X = -2/7$, the return value is $g(-2/7) = \frac{8-3(-2/7)}{7(-2/7)+2} = \frac{56+6}{-14+14} = \frac{62}{0}$ which is undefined. $-2/7$ is not part of the domain of $g(X)$.

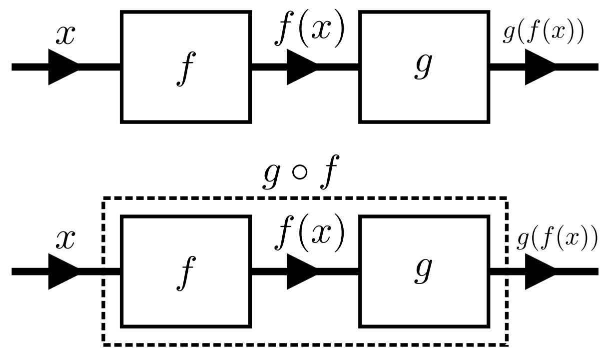
1.1 Function Domains

Given a function f, the set of input values X for which $f(X)$ is defined is referred to as the **domain** of f.

Not every function has a domain that is the entire set of real numbers. For example, the function $f(X) = \frac{1}{X}$ is not defined when $X = 0$, so the domain is the set of all real numbers except 0: $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$. Also, (not considering complex numbers) the function $f(X) = \sqrt{X}$ is not defined for negative values of X so the domain is the set of non-negative numbers: $[0, +\infty)$

1.2 Function Composition

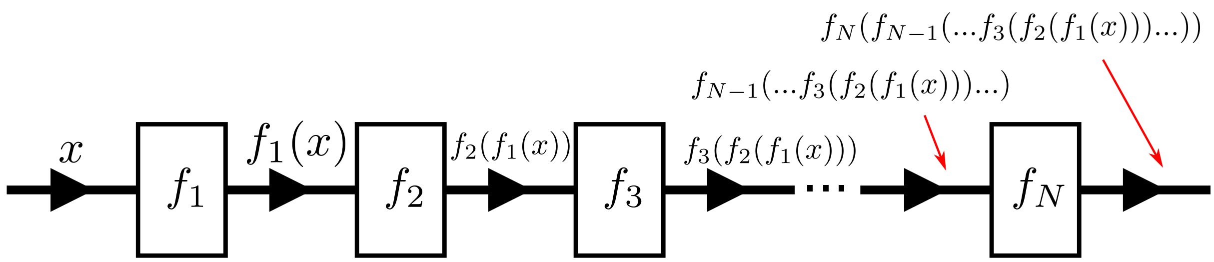
Linking two functions together is referred to as "function composition". The composition of two functions f followed by g, denoted by $g \circ f$ (this notation will not be used often), is a function that does the following: starting with the input parameter X, function f is used to get $f(X)$, and then function g is used to get $g(f(X))$ which is the return value: $(g \circ f)(X) = g(f(X))$



As in example of function composition, let function f be defined by $f(X) = 6X - 7$, and function g be defined by $g(X) = \frac{1}{2}X + \frac{1}{2}$. The composition of f followed by g is:

$$(g \circ f)(X) = g(f(X)) = g(6X - 7) = \frac{1}{2}(6X - 7) + \frac{1}{2} = 3X - 3$$

Below is shown the process of chaining together a large number of functions $f_1, f_2, ..., f_N$ to get $f_N \circ f_{N-1} \circ \cdots \circ f_3 \circ f_2 \circ f_1$. Composing functions is how large expressions are created.

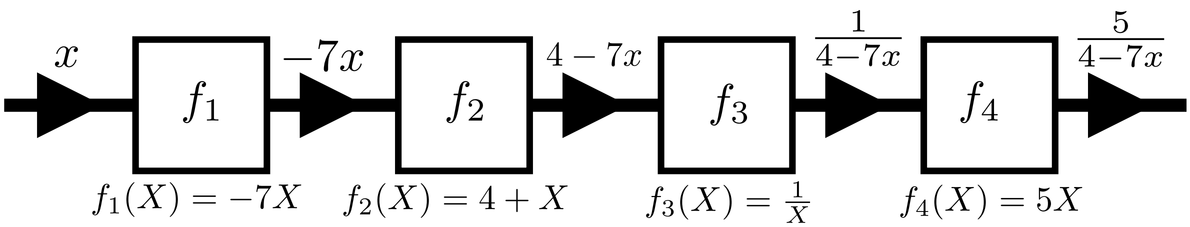


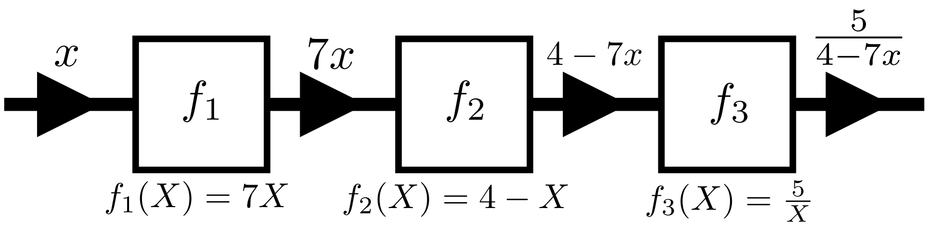
To assist with understanding what functions actually are, every arithmetic operator is a two-input function, and every expression is essentially a composition of simpler functions that are arithmetic operators.

In the image below, the expression $\frac{5}{4-7x}$ is expressed by two different compositions of simple arithmetic steps. In the upper flow chart, x is first multiplied by -7 via the function $f_1(X) = -7X$ to get $-7x$. $-7x$ is then added to 4 via the function $f_2(X) = 4 + X$ to get $4 - 7x$. The reciprocal of $4 - 7x$ is computed via the function $f_3(X) = \frac{1}{X}$ to get $\frac{1}{4-7x}$. Lastly, $\frac{1}{4-7x}$ is multiplied by 5 via the function $f_4(X) = 5X$ to get $\frac{5}{4-7x}$. As an example, if $x = 2$, then $f_1(2) = -14$, $f_2(-14) = -10$, $f_3(-10) = -\frac{1}{10}$, and lastly $f_4(-\frac{1}{10}) = -\frac{1}{2}$ so $\frac{5}{4-7x} = -\frac{1}{2}$.

In the lower flow chart, x is first multiplied by 7 via the function $f_1(X) = 7X$ to get $7x$. $7x$ is then subtracted from 4 via the function $f_2(X) = 4 - X$ to get $4 - 7x$. Lastly, 5 is then divided by $4 - 7x$ via the function $f_3(X) = \frac{5}{X}$ to get $\frac{5}{4-7x}$. As an example, if $x = 2$, then $f_1(2) = 14$, $f_2(14) = -10$, and lastly $f_3(-10) = -\frac{1}{2}$ so $\frac{5}{4-7x} = -\frac{1}{2}$.

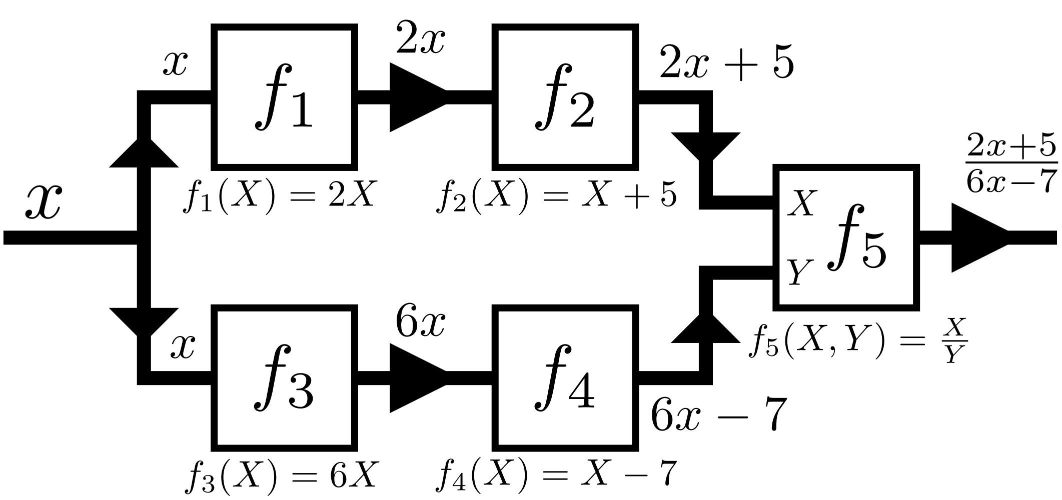
$$\frac{5}{4-7x}$$





In the image below, x appears twice in the expression $\frac{2x+5}{6x-7}$. The expressions $2x+5$ and $6x-7$ need to be separately computed before being divided via the two input function $f_5(X,Y) = \frac{X}{Y}$. To compute $2x+5$ from x, x is first multiplied by 2 via the function $f_1(X) = 2X$ to get $2x$. 5 is then added to $2x$ via the function $f_2(X) = X+5$ to get $2x+5$. To compute $6x-7$ from x, x is first multiplied by 6 via the function $f_3(X) = 6X$ to get $6x$. 7 is then subtracted from $6x$ via the function $f_4(X) = X-7$ to get $6x-7$. With $2x+5$ and $6x-7$ computed, the two input function $f_5(X,Y) = \frac{X}{Y}$ computes the quotient $\frac{2x+5}{6x-7}$. As an example, if $x = 1$, then $f_1(1) = 2$, $f_2(2) = 7$, $f_3(1) = 6$, $f_4(6) = -1$, and lastly $f_5(7,-1) = -7$ so $\frac{2x+5}{6x-7} = -7$

$$\frac{2x+5}{6x-7}$$



2 Inverse Functions

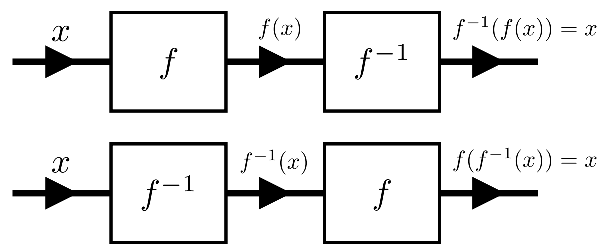
The **range** of a function f is the set of all possible output/return values:

$$\textbf{range} = \{f(x) | x \in \textbf{domain}\}$$

If a function is "1 to 1", then no two different input values result in the same output value:

$$\forall x, y \in \textbf{domain} : (x \neq y \implies f(x) \neq f(y))$$

For each value y from the range, there is exactly one input value x from the domain of f that generates y. When a function f is "1 to 1", the function f has an inverse. The inverse function of f, denoted by f^{-1}, "reverses" function f. The domain of f^{-1} is the range of f, while the range of f^{-1} is the domain of f. Given any value x from the range of f, the value of $f^{-1}(x)$ is the unique value from the domain of f such that $f(f^{-1}(x)) = x$. For every value x from the domain of f, applying f followed by f^{-1} brings x full circle:



$$\forall x \in \textbf{domain} : f^{-1}(f(x)) = x$$

For every value x from the range of f, applying f^{-1} followed by f also brings x full circle:

$$\forall x \in \textbf{range} : f(f^{-1}(x)) = x$$

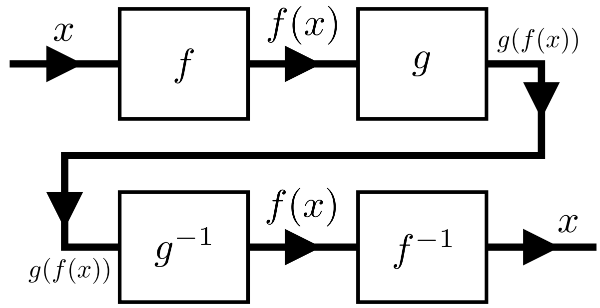
Examples

- Given for example the function $f(X) = 3X$, the inverse function is $f^{-1}(X) = \frac{X}{3}$.
- Given for example the function $f(X) = X - 9$, the inverse function is $f^{-1}(X) = X + 9$.
- The function $f(X) = X^2$ is not 1 to 1, since $f(x) = f(-x)$ for all values of x. However, if the domain of $f(X) = X^2$ is restricted to $[0, +\infty)$, then f has the inverse $f^{-1}(X) = \sqrt{X}$, since the values of X that yield duplicate values have been excluded. If the domain were instead restricted to $(-\infty, 0]$, then f has the inverse $f^{-1}(X) = -\sqrt{X}$.
- The function $f(X) = \sqrt{X}$ is 1 to 1, but the range is limited to $[0, +\infty)$. Therefore the domain of the inverse function $f^{-1}(X) = X^2$ is restricted to $[0, +\infty)$.

2.1 Inverting more complicated functions

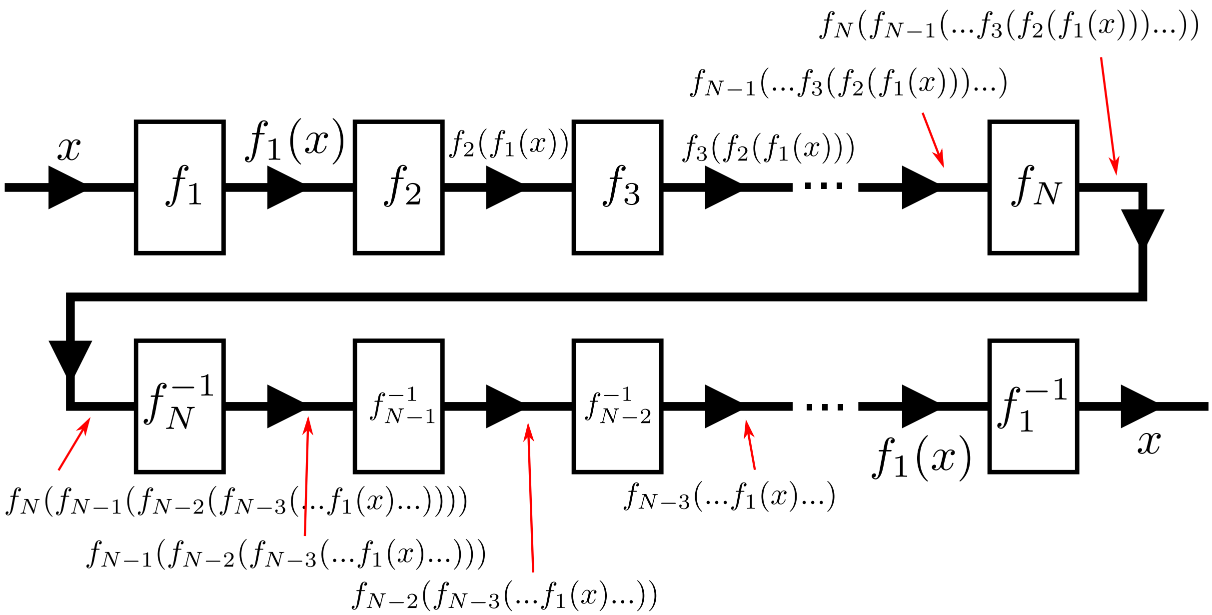
Given the composition of two functions f followed by g, denoted by $g \circ f$ (for all X, $(g \circ f)(X) = g(f(X))$), to invert the combined function, the inverses are applied in the reverse order. If f is applied first, and then g, to reverse the process, g is reversed first, followed by reversing f:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$



Consider a long chain of the functions f_1, f_2, f_3, ..., f_N applied in the given order: $f_N \circ ... \circ f_3 \circ f_2 \circ f_1$, the inverse is formed by applying the inverses of f_1, f_2, f_3, ..., f_N in the reversed order:

$$(f_N \circ ... \circ f_3 \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ ... \circ f_N^{-1}$$



Examples

- The function $g(X) = 3X - 7$ is the function $f_1(X) = 3X$ followed by the function $f_2(X) = X - 7$: $g(X) = f_2(f_1(X))$. The inverses of f_1 and f_2 are respectively $f_1^{-1}(X) = \frac{X}{3}$ and $f_2^{-1}(X) = X + 7$. The inverse of g is:

$$g^{-1}(X) = f_1^{-1}(f_2^{-1}(X)) = f_1^{-1}(X + 7) = \frac{X + 7}{3}$$

- The function $g(X) = \frac{3}{-2X-5}$ is the function $f_1(X) = -2X$ followed by the function $f_2(X) = X - 5$ followed by $f_3(X) = \frac{1}{X}$ followed by $f_4(X) = 3X$: $g(X) = f_4(f_3(f_2(f_1(X))))$. The inverses of these functions are respectively $f_1^{-1}(X) = \frac{X}{-2}$ and $f_2^{-1}(X) = X + 5$ and $f_3^{-1}(X) = \frac{1}{X}$ and $f_4^{-1}(X) = \frac{X}{3}$. The inverse of g is:

$$g^{-1}(X) = f_1^{-1}(f_2^{-1}(f_3^{-1}(f_4^{-1}(X)))) = f_1^{-1}(f_2^{-1}(f_3^{-1}(\frac{X}{3}))) = f_1^{-1}(f_2^{-1}(\frac{3}{X})) = f_1^{-1}(\frac{3}{X} + 5) = -\frac{3}{2X} - \frac{5}{2}$$

3 Solving Equations

Solving equations involves taking an equation with the form:

$$\textbf{expression}_1 = \textbf{expression}_2$$

and manipulating the equation to have the form

$$\textbf{variable} = \textbf{expression}$$

where the expression "**expression**" computes the value of the variable "**variable**". An equation is manipulated by deriving equations that are true if and only if the original equation is true. Given the equation $A = B$ and any single input function f whose domain includes A and B, then

$$A = B \implies f(A) = f(B)$$

The truth of $A = B$ implies the truth of $f(A) = f(B)$. However the reverse may not be true. It is not always the the case that the truth of $f(A) = f(B)$ implies the truth of $A = B$. A clear example uses the function $f(x) = x^2$. The truth of $x = 2$ implies that $x^2 = 4$, but the truth of $x^2 = 4$ does not imply that $x = 2$, since $x = -2$ is also a valid alternative. For the truth of $f(A) = f(B)$ to imply $A = B$, f must be 1 to 1 and have an inverse. When f is 1 to 1,

$$A = B \iff f(A) = f(B)$$

and the equation $f(A) = f(B)$ can replace $A = B$ with no loss of information.

Given a 1 to 1 function g and a value c from the range of g, the equation $g(x) = c$ is solved for x by simply applying the inverse of function g to both sides of the equation:

$$g(x) = c \iff g^{-1}(g(x)) = g^{-1}(c) \iff x = g^{-1}(x)$$

Now imagine that g is the composition of 1 to 1 single variable functions: $g(x) = f_N(...f_3(f_2(f_1(x)))...)$. x can only appear once in the expression for $g(x)$, or else functions with two or more inputs are needed to express $g(x)$. The equation $g(x) = c$ is equivalent to

$$f_N(...f_3(f_2(f_1(x)))...) = c$$

Variable x is "unwrapped" by applying the inverse functions of f_1, f_2, f_3, ..., f_N in the reverse order:

$$\begin{aligned}
& f_N(f_{N-1}(f_{N-2}(...f_1(x)...))) = c \\
\iff & f_{N-1}(f_{N-2}(...f_1(x)...)) = f_N^{-1}(c) \\
\iff & f_{N-2}(...f_1(x)...) = f_{N-1}^{-1}(f_N^{-1}(c)) \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\iff & x = f_1^{-1}(...f_{N-2}^{-1}(f_{N-1}^{-1}(f_N^{-1}(c)))...)
\end{aligned}$$

Examples:
Consider the equation:

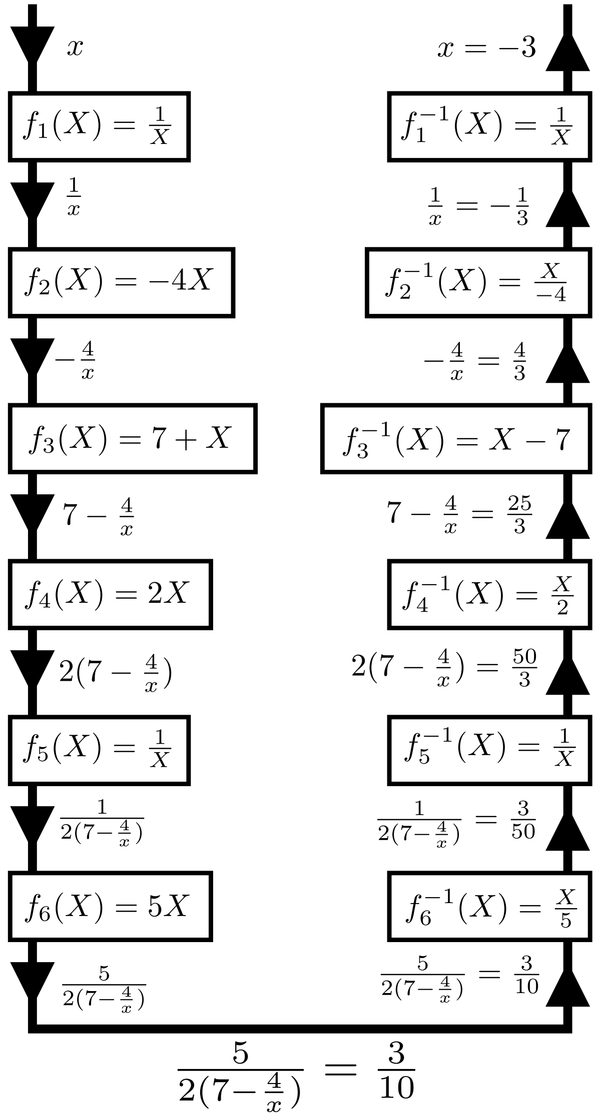
$$\frac{5}{2(7 - \frac{4}{x})} = \frac{3}{10}$$

In the image to the right, the expression $\frac{5}{2(7-\frac{4}{x})}$ is the composition of the functions $f_1(X) = \frac{1}{X}$, followed by $f_2(X) = -4X$, followed by $f_3(X) = 7 + X$, followed by $f_4(X) = 2X$, followed by $f_5(X) = \frac{1}{X}$, lastly followed by $f_6(X) = 5X$.

$$\frac{5}{2(7-\frac{4}{x})} = f_6(f_5(f_4(f_3(f_2(f_1(x))))))$$

The inverses of the component functions are respectively $f_1^{-1}(X) = \frac{1}{X}$; $f_2^{-1}(X) = \frac{X}{-4}$; $f_3^{-1}(X) = X - 7$; $f_4^{-1}(X) = 2X$; $f_5^{-1}(X) = \frac{1}{X}$; and $f_6^{-1}(X) = 5X$. Applying the inverse functions in the reverse order to the equation $\frac{5}{2(7-\frac{4}{x})} = \frac{3}{10}$ yields:

$$\begin{aligned}
&\frac{5}{2(7-\frac{4}{x})} = \frac{3}{10} \iff \frac{1}{2(7-\frac{4}{x})} = \frac{3}{50} \\
&\iff 2(7-\frac{4}{x}) = \frac{50}{3} \iff 7 - \frac{4}{x} = \frac{25}{3} \\
&\iff -\frac{4}{x} = \frac{4}{3} \iff \frac{1}{x} = -\frac{1}{3} \\
&\iff x = -3
\end{aligned}$$



The functions that comprise the expression that contains the variable to be solved for should be easily invertible. Easy functions to invert include:

- $f(X) = X + a$ (simply add a) with the inverse $f^{-1}(X) = X - a$ (simply subtract a)
- $f(X) = a - X$ (simply subtract from a) with the inverse $f^{-1}(X) = a - X$ (simply subtract from a)
- $f(X) = aX$ where $a \neq 0$ (simply multiply by a) with the inverse $f^{-1}(X) = \frac{X}{a}$ (simply divide by a)
- $f(X) = \frac{1}{X}$ (compute the reciprocal of X) with the inverse $f^{-1}(X) = \frac{1}{X}$ (compute the reciprocal)
- $f(X) = \frac{a}{X}$ where $a \neq 0$ (divide a by X) with the inverse $f^{-1}(X) = \frac{a}{X}$ (divide a by X)
- $f(X) = X^k$ where k is an odd natural number (raise to the k^{th} power) with the inverse $f^{-1}(X) = \sqrt[k]{X}$ (compute the k^{th} root).

When multiple instances of the desired variable are present, the equation must be manipulated until only one instance is present. Consider for example, the equation:

$$2a + 6 = 7a - 14$$

Subtracting $2a$ from both sides gives $6 = 5a - 14$, where a now only appears once. The expression that includes a can now be undone sequentially:

$$5a - 14 = 6 \iff 5a = 20 \iff a = 4$$

Consider the equation

$$\frac{4y - 3}{5 + 2y} = -2$$

The variable y appears twice, so to solve for y, both sides must be multiplied by $5 + 2y$ to eliminate the denominator:

$$\frac{4y - 3}{5 + 2y} = -2 \iff 4y - 3 = -2(5 + 2y) \iff 4y - 3 = -10 - 4y$$

Collecting y on the left side is done by adding $4y$ to both sides and then adding the coefficients of y:

$$4y - 3 = -10 - 4y \iff 8y - 3 = -10$$

Now y only appears once so the expression that includes y can now be undone sequentially:

$$8y - 3 = -10 \iff 8y = -7 \iff y = -\frac{7}{8}$$

3.1 Solving equations with placeholder values

Not every variable in an equation is to be solved for. Some values are left as symbols because:

- Their values have not yet been chosen.
- Or the values are left as symbols to avoid making approximations at this stage.

Consider the equation

$$\frac{1}{(x(5 - a) + 3b)^3 - x} = \frac{1}{2x}$$

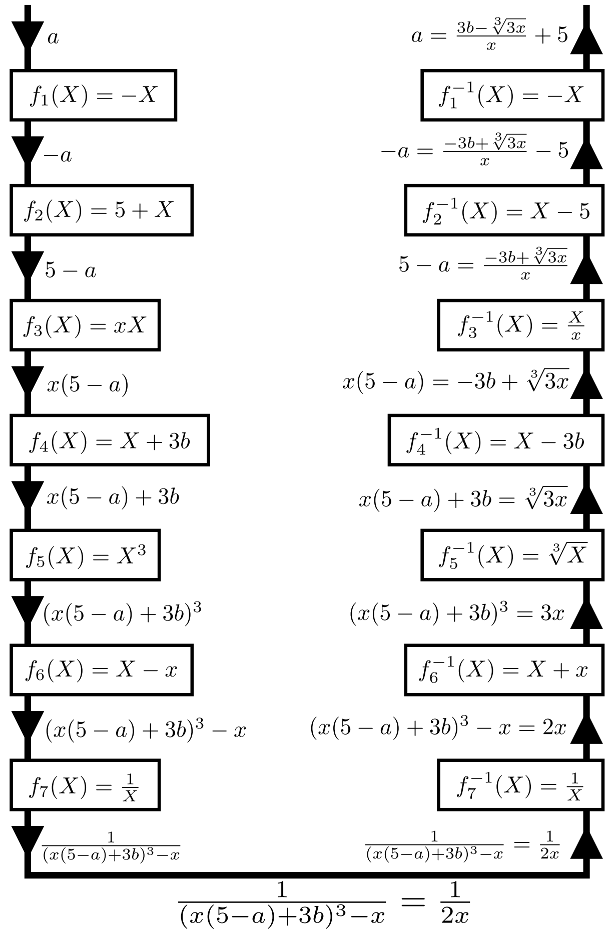
Treating b and x as fixed quantities, the equation will first be solved for a:

a appears only once, and the functions that are used to generate the final expression that contains a are: $f_1(X) = -X$, followed by $f_2(X) = 5+X$, followed by $f_3(X) = xX$, followed by $f_4(X) = X + 3b$, followed by $f_5(X) = X^3$, followed by $f_6(X) = X - x$, lastly followed by $f_7(X) = \frac{1}{X}$.

$$\frac{1}{(x(5-a)+3b)^3 - x}$$
$$= f_7(f_6(f_5(f_4(f_3(f_2(f_1(a)))))))$$

The inverses of the component functions are respectively $f_1^{-1}(X) = -X$; $f_2^{-1}(X) = X - 5$; $f_3^{-1}(X) = \frac{X}{x}$; $f_4^{-1}(X) = X - 3b$; $f_5^{-1}(X) = \sqrt[3]{X}$; $f_6^{-1}(X) = X + x$; and $f_7^{-1}(X) = \frac{1}{X}$. Applying the inverse functions in the reverse order to the equation yields:

$$\frac{1}{(x(5-a)+3b)^3 - x} = \frac{1}{2x}$$
$$\iff (x(5-a)+3b)^3 - x = 2x$$
$$\iff (x(5-a)+3b)^3 = 3x$$
$$\iff x(5-a)+3b = \sqrt[3]{3x}$$
$$\iff x(5-a) = -3b + \sqrt[3]{3x}$$
$$\iff 5 - a = \frac{-3b + \sqrt[3]{3x}}{x}$$
$$\iff -a = \frac{-3b + \sqrt[3]{3x}}{x} - 5$$
$$\iff a = \frac{3b - \sqrt[3]{3x}}{x} + 5$$



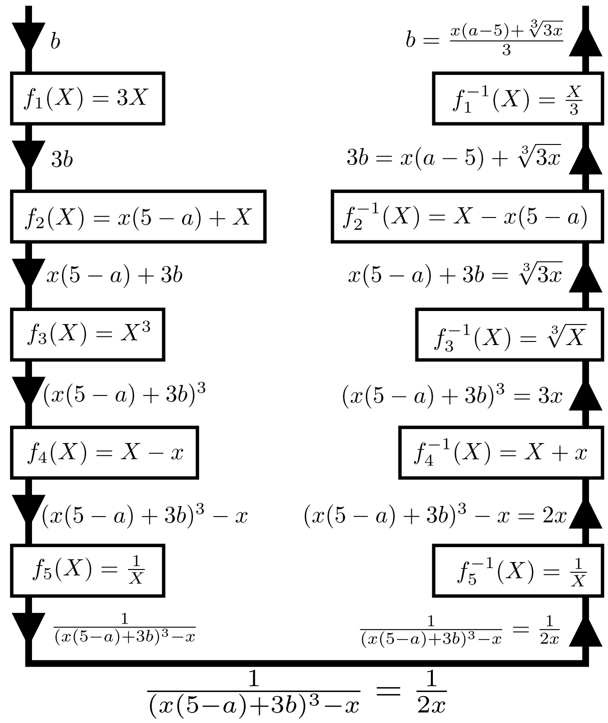
Now treating a and x as fixed quantities, the equation will next be solved for b:

b appears only once, and the functions that are used to generate the final expression that contains b are: $f_1(X) = 3X$, followed by $f_2(X) = x(5-a) + X$, followed by $f_3(X) = X^3$, followed by $f_4(X) = X - x$, lastly followed by $f_5(X) = \frac{1}{X}$.

$$\frac{1}{(x(5-a)+3b)^3 - x}$$
$$= f_5(f_4(f_3(f_2(f_1(b)))))$$

The inverses of the component functions are respectively $f_1^{-1}(X) = \frac{X}{3}$; $f_2^{-1}(X) = X - x(5-a)$; $f_3^{-1}(X) = \sqrt[3]{X}$; $f_4^{-1}(X) = X + x$; and $f_5^{-1}(X) = \frac{1}{X}$. Applying the inverse functions in the reverse order to the equation yields:

$$\frac{1}{(x(5-a)+3b)^3 - x} = \frac{1}{2x}$$
$$\iff (x(5-a)+3b)^3 - x = 2x$$
$$\iff (x(5-a)+3b)^3 = 3x$$
$$\iff x(5-a)+3b = \sqrt[3]{3x}$$
$$\iff 3b = x(a-5) + \sqrt[3]{3x}$$
$$\iff b = \frac{x(a-5)+\sqrt[3]{3x}}{3}$$



Another example:
Consider the equation

$$3ac + 5a - 4c = 2c$$

Solving for a, while treating c as a placeholder symbol gives:

$$3ac + 5a - 4c = 2c$$
$$\iff (3c+5)a - 4c = 2c \iff (3c+5)a = 6c \iff a = \frac{6c}{3c+5}$$

Solving for c, while treating a as a placeholder symbol gives:

$$3ac + 5a - 4c = 2c \iff 3ac + 5a - 6c = 0$$
$$\iff (3a-6)c + 5a = 0 \iff (3a-6)c = -5a \iff c = \frac{-5a}{3a-6}$$

Another example:
Consider the equation

$$4a^3 + \frac{1}{x} = 7 - 2a^3 - \frac{3}{x}$$

Solving for a, while treating x as a placeholder symbol gives:

$$
\begin{aligned}
&4a^3 + \frac{1}{x} = 7 - 2a^3 - \frac{3}{x} \\
\iff &6a^3 + \frac{1}{x} = 7 - \frac{3}{x} \iff 6a^3 = 7 - \frac{4}{x} \iff a^3 = \frac{7}{6} - \frac{2}{3x} \iff a = \sqrt[3]{\frac{7}{6} - \frac{2}{3x}}
\end{aligned}
$$

Solving for x, while treating a as a placeholder symbol gives:

$$
\begin{aligned}
&4a^3 + \frac{1}{x} = 7 - 2a^3 - \frac{3}{x} \\
\iff &4a^3 + \frac{4}{x} = 7 - 2a^3 \iff \frac{4}{x} = 7 - 6a^3 \iff \frac{1}{x} = \frac{7 - 6a^3}{4} \iff x = \frac{4}{7 - 6a^3}
\end{aligned}
$$

Another example:
Consider the equation

$$\frac{-9x + 5y - 7}{b + 2} = x(b + 2)^2$$

Solving for b, while treating x and y as placeholder symbols gives:

$$
\begin{aligned}
&\frac{-9x + 5y - 7}{b + 2} = x(b + 2)^2 \\
\iff &x(b + 2)^3 = -9x + 5y - 7 \iff (b + 2)^3 = \frac{-9x + 5y - 7}{x} \iff b + 2 = \sqrt[3]{\frac{-9x + 5y - 7}{x}} \\
\iff &b = -2 + \sqrt[3]{\frac{-9x + 5y - 7}{x}}
\end{aligned}
$$

Solving for x, while treating b and y as placeholder symbols gives:

$$
\begin{aligned}
&\frac{-9x + 5y - 7}{b + 2} = x(b + 2)^2 \iff \frac{5y - 7}{b + 2} = \frac{9x}{b + 2} + x(b + 2)^2 \\
\iff &\left(\frac{9}{b + 2} + (b + 2)^2\right) x = \frac{5y - 7}{b + 2} \iff x = \frac{5y - 7}{(b + 2)\left(\frac{9}{b+2} + (b + 2)^2\right)} \iff x = \frac{5y - 7}{9 + (b + 2)^3}
\end{aligned}
$$

Solving for y, while treating b and x as placeholder symbols gives:

$$
\begin{aligned}
&\frac{-9x + 5y - 7}{b + 2} = x(b + 2)^2 \iff -9x + 5y - 7 = x(b + 2)^3 \iff 5y = 9x + 7 + x(b + 2)^3 \\
\iff &y = \frac{9x + 7 + x(b + 2)^3}{5}
\end{aligned}
$$

3.2 Traveling Example

Consider the following scenario. With two available speeds, a slow speed v_{slow} and a fast speed v_{fast}, the aim is to traverse a total distance D in *exactly* total time T. As of now, the quantities v_{slow}, v_{fast}, D, and T have not yet been chosen, and are expressed using their respective symbols for now. The first leg of the journey will involve traveling at the slow speed v_{slow} for an unknown time t, followed by traveling at the fast speed for the remaining time $T - t$. The desired quantity to be solved for is the time t for which the slow speed is used. To find t, the value of t will be used to derive various quantities until a limitation is apparent which will constitute the equation that will be solved for t. The distance traveled in the first leg is $v_{\text{slow}}t$, while the distance traveled in the remaining time $T-t$ is $v_{\text{fast}}(T-t)$. The total traveled distance is required to be D, which yields the equation:

$$v_{\text{slow}}t + v_{\text{fast}}(T-t) = D$$

Solving this equation for t yields:

$$\begin{aligned} v_{\text{slow}}t + v_{\text{fast}}(T-t) = D &\iff v_{\text{slow}}t + v_{\text{fast}}T - v_{\text{fast}}t = D \\ \iff (v_{\text{slow}} - v_{\text{fast}})t + v_{\text{fast}}T = D &\iff (v_{\text{slow}} - v_{\text{fast}})t = D - v_{\text{fast}}T \iff t = \frac{D - v_{\text{fast}}T}{v_{\text{slow}} - v_{\text{fast}}} \end{aligned}$$

Therefore:

$$t = \frac{D - v_{\text{fast}}T}{v_{\text{slow}} - v_{\text{fast}}}$$

is the time that should be spent traveling at the slower speed.

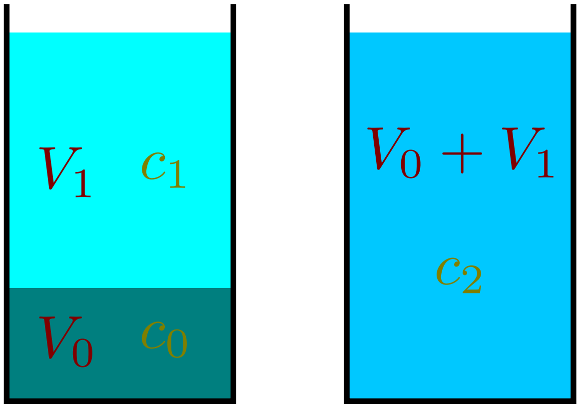
Now for some numbers. Let $v_{\text{slow}} = 1.561\text{m/s}$, $v_{\text{fast}} = 5.342\text{m/s}$, $D = 4000\text{m}$, and $T = 2000\text{s}$. Substituting these quantities into the computed expression for t gives:

$$t = \frac{D - v_{\text{fast}}T}{v_{\text{slow}} - v_{\text{fast}}} = \frac{4000\text{m} - (5.342\text{m/s})(2000\text{s})}{1.561\text{m/s} - 5.342\text{m/s}} \approx \frac{-6684\text{m}}{-3.781\text{m/s}} \approx 1768\text{s}$$

3.3 Mixing Example

Given a volume V_0 of salt water with a salt concentration of c_0, and another volume V_1 of salt water with concentration c_1, mixing the two volumes gives a mixture with volume $V_0 + V_1$ and concentration of c_2. The total mass of salt before mixing is $c_0 \cdot V_0 + c_1 \cdot V_1$, and the total mass of salt after mixing is $c_2(V_0 + V_1)$. Since the mass does not change during mixing, the following equation holds:

$$c_0 \cdot V_0 + c_1 \cdot V_1 = c_2(V_0 + V_1)$$



If we know the volumes V_0 and V_1, and the concentrations c_0 and c_1, and are interested in the final concentration c_2, the above equation can be solved for c_2 which yields:

$$c_0 \cdot V_0 + c_1 \cdot V_1 = c_2(V_0 + V_1) \iff c_2 = \frac{c_0 \cdot V_0 + c_1 \cdot V_1}{V_0 + V_1}$$

If we know volume V_0 and concentrations c_0 and c_1, but are instead interested in finding the volume V_1 needed to achieve a target final concentration of c_2, the equation can instead be solved for V_1 which yields:

$$\begin{aligned} & c_0 \cdot V_0 + c_1 \cdot V_1 = c_2(V_0 + V_1) \\ \iff & c_0 \cdot V_0 + c_1 \cdot V_1 = c_2 \cdot V_0 + c_2 \cdot V_1 \\ \iff & c_0 \cdot V_0 - c_2 \cdot V_0 = c_2 \cdot V_1 - c_1 \cdot V_1 \\ \iff & (c_2 - c_1)V_1 = (c_0 - c_2)V_0 \\ \iff & V_1 = \frac{c_0 - c_2}{c_2 - c_1}V_0 \end{aligned}$$

If $V_0 = 3.5$L; $V_1 = 2.5$L; $c_0 = 0.1$g/mL; $c_1 = 0.01$g/mL; then to compute c_2 we must first standardize the units: Replace g/mL with $\frac{\mathrm{g}}{10^{-3}L} = 10^3$g/L, or replace L with 10^3mL. It is simpler to replace L with 10^3mL, so volume is measured in mL's, and concentration is measured in g/mL. Hence

$$V_0 = 3.500 \times 10^3\mathrm{mL}\ ;\ V_1 = 2.500 \times 10^3\mathrm{mL}\ ;\ c_0 = 1.000 \times 10^{-1}\mathrm{g/mL}\ ;\ c_1 = 1.000 \times 10^{-2}\mathrm{g/mL}$$

$$\begin{aligned} c_2 =& \frac{c_0 \cdot V_0 + c_1 \cdot V_1}{V_0 + V_1} = \frac{(3.500 \times 10^2\mathrm{g}) + (2.500 \times 10^1\mathrm{g})}{6.000 \times 10^3\mathrm{mL}} = \frac{3.750 \times 10^2\mathrm{g}}{6.000 \times 10^3\mathrm{mL}} \approx 6.250 \times 10^{-2}\mathrm{g/mL} \\ =& 0.06250\mathrm{g/mL} \end{aligned}$$

If $V_0 = 3.5$L; $c_0 = 0.1$g/mL; $c_1 = 0.01$g/mL; and $c_2 = 0.05$g/mL; then to compute V_1 we must first standardize the units: replace L with 10^3mL, so volume is measured in mL's, and concentration is measured in g/mL. Hence

$$V_0 = 3.500 \times 10^3\mathrm{mL}\ ;\ c_0 = 1.000 \times 10^{-1}\mathrm{g/mL}\ ;\ c_1 = 1.000 \times 10^{-2}\mathrm{g/mL}\ ;\ c_2 = 5.000 \times 10^{-2}\mathrm{g/mL}$$

$$\begin{aligned} V_1 =& \frac{c_0 - c_2}{c_2 - c_1} V_0 = \frac{5.000 \times 10^{-2}\mathrm{g/mL}}{4.000 \times 10^{-2}\mathrm{g/mL}}(3.500 \times 10^3\mathrm{mL}) = (1.250)(3.500 \times 10^3\mathrm{mL}) \approx 4.375 \times 10^3\mathrm{mL} \\ =& 4.375\mathrm{L} \end{aligned}$$