# Qudratic Surfaces and Tangents

## Question 1:

Given a plane $P$ with equation $2x-5y+7z=9$, and a line with the parametric form $L:\begin{cases} x(t)= & 1+4t \\ y(t)= & 2+kt \\ z(t)= & 1+6t \end{cases}$,

find a value of $k$ such that $L$ is parallel to $P$.

A direction that is parallel to $P$ must be perpendicular to the normal vector of $P$: $\mathbf{n}=\begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$.

The direction vector of $L$ is: $\mathbf{v}=\begin{bmatrix} 4 \\ k \\ 6 \end{bmatrix}$.

$$\mathbf{n}\perp\mathbf{v} \iff \mathbf{n}\cdot\mathbf{v}=0 \iff \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ k \\ 6 \end{bmatrix}=0 \iff 8-5k+42=0 \iff k=10$$

Therefore: $k=10$

## Question 2:

Identify the quadratic surface, and find the point on which it is centered:

$$-x^2-4y^2-4z^2-2x+24z=33$$

$$\begin{aligned} &-x^2-4y^2-4z^2-2x+24z=33 \iff -(x^2+2x)-4y^2-4(z^2-6z)=33 \\ &\iff -((x+1)^2-1)-4y^2-4((z-3)^2-9)=33 \iff -(x+1)^2-4y^2-4(z-3)^2+1+36=33 \\ &\iff -(x+1)^2-4y^2-4(z-3)^2=-4 \iff \left(\frac{x-(-1)}{2}\right)^2+y^2+(z-3)^2=1 \end{aligned}$$

This surface is achieved by starting from the unit sphere $x^2+y^2+z^2=1$, and first applying the respective stretches of 2, 1, and 1 to the $x$, $y$, and $z$ directions, and then applying the respective translations of $-1$, 0, and 3 to the $x$, $y$, and $z$ directions.

This surface is an **ellipsoid** centered on the point $(-1,0,3)$.

## Question 3:

Identify the quadratic surface, and find the point on which it is centered:

$$-9x^2 + 9y^2 - z^2 + 72x + 2z = 136$$

$$\begin{aligned}
& -9x^2 + 9y^2 - z^2 + 72x + 2z = 136 \iff -9(x^2 - 8x) + 9y^2 - (z^2 - 2z) = 136 \\
& \iff -9((x-4)^2 - 16) + 9y^2 - ((z-1)^2 - 1) = 136 \iff -9(x-4)^2 + 9y^2 - (z-1)^2 + 144 + 1 = 136 \\
& \iff -9(x-4)^2 + 9y^2 - (z-1)^2 = -9 \iff (x-4)^2 - y^2 + \left(\frac{z-1}{3}\right)^2 = 1
\end{aligned}$$

This surface is achieved by starting from the one-sheet hyperboloid that is oriented along the $y$-axis $x^2 - y^2 + z^2 = 1$, and first applying the respective stretches of 1, 1, and 3 to the $x$, $y$, and $z$ directions, and then applying the respective translations of 4, 0, and 1 to the $x$, $y$, and $z$ directions.

This surface is a **one-sheet hyperboloid** that is oriented along the $y$-axis and centered on the point $(4, 0, 1)$.

## Question 4:

Identify the quadratic surface, and find the point on which it is centered:

$$-y^2 - 4z^2 + 4x + 2y - 24z = 41$$

$$\begin{aligned}
& -y^2 - 4z^2 + 4x + 2y - 24z = 41 \iff -(y^2 - 2y) - 4(z^2 + 6z) + 4x = 41 \\
& \iff -((y-1)^2 - 1) - 4((z+3)^2 - 9) + 4x = 41 \iff -(y-1)^2 - 4(z+3)^2 + 4x + 1 + 36 = 41 \\
& \iff -(y-1)^2 - 4(z+3)^2 + 4x - 4 = 0 \iff x - 1 = \left(\frac{y-1}{2}\right)^2 + (z - (-3))^2
\end{aligned}$$

This surface is achieved by starting from the paraboloid that is oriented along the +ve $x$-axis $x = y^2 + z^2$, and first applying the respective stretches of 1, 2, and 1 to the $x$, $y$, and $z$ directions, and then applying the respective translations of 1, 1, and $-3$ to the $x$, $y$, and $z$ directions.

This surface is a **paraboloid** that is oriented along the +ve $x$-axis and centered on the point $(1, 1, -3)$.

## Question 5:

The two curves $C_1$ and $C_2$ defined by:

$$C_1 : y = x^3 - 9x^2 + 24x - 15$$

and

$$C_2 : y = x^3 - 6x^2 + 6x + 9$$

intersect at the point $P(2, 5)$.

## part 5a:

Derive parametric equations for the tangent lines to $C_1$ and $C_2$ at the intersection point $P$.

For curve $C_1$, the derivative is $\frac{dy}{dx} = 3x^2 - 18x + 24$ so $\left.\frac{dy}{dx}\right|_{x=2} = 12 - 36 + 24 = 0$.

The tangent to curve $C_1$ is $T_1 : \begin{cases} x(t) = & 2 + t \\ y(t) = & 5 \end{cases}$

For curve $C_2$, the derivative is $\frac{dy}{dx} = 3x^2 - 12x + 6$ so $\left.\frac{dy}{dx}\right|_{x=2} = 12 - 24 + 6 = -6$.

The tangent to curve $C_2$ is $T_2 : \begin{cases} x(t) = & 2 + t \\ y(t) = & 5 - 6t \end{cases}$

## part 5b:

Derive the angle between the tangent lines from the previous section.

The direction vectors of $T_1$ and $T_2$ are respectively $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$.

The angle between $T_1$ are $T_2$ is:

$$\theta = \arccos\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|\,|\mathbf{v}_2|}\right) = \arccos\left(\frac{1}{1 \cdot \sqrt{37}}\right) = \arccos\left(\frac{1}{\sqrt{37}}\right)$$