Cartesian and Polar Double Integrals

Question 1:

part 1a:

For the following double integrals, convert the function that is being integrated over (the integrand) from a function of Cartesian coordinates to a function of Polar coordinates:

- $\iint_\sigma (x+y)dA$
- $\iint_\sigma xydA$
- $\iint_\sigma \frac{y}{x}dA$
- $\iint_\sigma \frac{dA}{(x^2+y^2)^{3/2}}$
- $\iint_\sigma \frac{x^2-y^2}{2xy}dA$

Solution:

- $\iint_\sigma (x+y)dA = \iint_\sigma (r\cos\theta + r\sin\theta)dA = \iint_\sigma r(\cos\theta+\sin\theta)dA$
- $\iint_\sigma xydA = \iint_\sigma (r\cos\theta)(r\sin\theta)dA = \iint_\sigma \frac{1}{2}r^2\sin(2\theta)dA$
- $\iint_\sigma \frac{y}{x}dA = \iint_\sigma \frac{r\sin\theta}{r\cos\theta}dA = \iint_\sigma \tan\theta dA$
- $\iint_\sigma \frac{dA}{(x^2+y^2)^{3/2}} = \iint_\sigma \frac{dA}{(r^2\cos^2\theta+r^2\sin^2\theta)^{3/2}} = \iint_\sigma \frac{dA}{r^3}$
- $\iint_\sigma \frac{x^2-y^2}{2xy}dA = \iint_\sigma \frac{r^2\cos^2\theta-r^2\sin^2\theta}{2(r\cos\theta)(r\sin\theta)}dA = \iint_\sigma \frac{\cos(2\theta)}{\sin(2\theta)}dA = \iint_\sigma \cot(2\theta)dA$

part 1b:

For the following double integrals, convert the function that is being integrated over (the integrand) from a function of Polar coordinates to a function of Cartesian coordinates:

- $\iint_\sigma r\cos\theta dA$
- $\iint_\sigma r^2\sin\theta dA$
- $\iint_\sigma r^{-3}dA$
- $\iint_\sigma \frac{\cos\theta-3\sin\theta}{2\cos\theta+\sin\theta}dA$
- $\iint_\sigma r\tan\theta dA$

- $\iint_\sigma \cos(2\theta)dA$
- $\iint_\sigma \sin(2\theta)dA$
- $\iint_\sigma \tan(2\theta)dA$

Solution:

- $\iint_\sigma r\cos\theta dA = \iint_\sigma r(x/r)dA = \iint_\sigma xdA$
- $\iint_\sigma r^2\sin\theta dA = \iint_\sigma r^2(y/r)dA = \iint_\sigma yrdA = \iint_\sigma y\sqrt{x^2+y^2}\cdot dA$
- $\iint_\sigma r^{-3}dA = \iint_\sigma \frac{1}{(x^2+y^2)^{3/2}}dA$
- $\iint_\sigma \frac{\cos\theta-3\sin\theta}{2\cos\theta+\sin\theta}dA = \iint_\sigma \frac{x/r-3(y/r)}{2(x/r)+y/r}dA = \iint_\sigma \frac{x-3y}{2x+y}dA$
- $\iint_\sigma r\tan\theta dA = \iint_\sigma r(y/x)dA = \iint_\sigma \frac{y\sqrt{x^2+y^2}}{x}dA$
- $\iint_\sigma \cos(2\theta)dA = \iint_\sigma(\cos^2\theta-\sin^2\theta)dA = \iint_\sigma((x/r)^2-(y/r)^2)dA = \iint_\sigma \frac{x^2-y^2}{r^2}dA = \iint_\sigma \frac{x^2-y^2}{x^2+y^2}dA$
- $\iint_\sigma \sin(2\theta)dA = \iint_\sigma 2\cos\theta\sin\theta dA = \iint_\sigma 2(x/r)(y/r)dA = \iint_\sigma \frac{2xy}{r^2}dA = \iint_\sigma \frac{2xy}{x^2+y^2}dA$
- $\iint_\sigma \tan(2\theta)dA = \iint_\sigma \frac{\sin(2\theta)}{\cos(2\theta)}dA = \iint_\sigma \frac{2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta}dA = \iint_\sigma \frac{2(x/r)(y/r)}{(x/r)^2-(y/r)^2}dA = \iint_\sigma \frac{2xy}{x^2-y^2}dA$

Question 2:

part 2a:

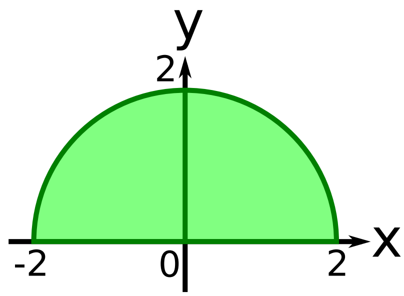
For the following regions characterized using Cartesian coordinates, express these regions using Polar coordinates:

- $\sigma = \{(x,y)| -2 \le x \le 2 \text{ and } 0 \le y \le \sqrt{4-x^2}\}$
- $\sigma = \{(x,y)|0 \le x \le 2 \text{ and } -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\}$
- $\sigma = \{(x,y)|0 \le x \le 1 \text{ and } -\frac{x}{\sqrt{3}} \le y \le x\}$
- $\sigma = \{(x,y)| -2 \le x \le 2 \text{ and } 0 \le y \le 4-x^2\}$
- $\sigma = \{(x,y)|0 \le y \le 3 \text{ and } -5+(5/3)y \le x \le 0\}$
- $\sigma = \{(x,y)| -1 \le x \le 1 \text{ and } 1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}\}$
- $\sigma = \{(x,y)|1-\sqrt{2} \le x \le 1+\sqrt{2} \text{ and } 1-\sqrt{-x^2+2x+1} \le y \le 1+\sqrt{-x^2+2x+1}\}$
- $\sigma = \{(x,y)| -6 \le x \le 0 \text{ and } 2x^2+12x \le y \le 0\}$

Solution:

For $\sigma = \{(x, y)| -2 \leq x \leq 2 \text{ and } 0 \leq y \leq \sqrt{4 - x^2}\}$,

The region can easily be plotted on the right.
The curve $y = \sqrt{4 - x^2}$ is equivalent to:



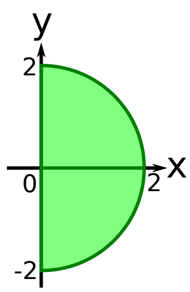
$$y = \sqrt{4 - x^2} \implies y^2 = 4 - x^2 \iff r^2 \sin^2\theta = 4 - r^2\cos^2\theta$$
$$\iff r^2 = 4 \iff r = 2$$

$r = 2$, along with $y \geq 0$, gives the region σ on the right:

Therefore $\sigma = \{(r, \theta)|0 \leq \theta \leq \pi \text{ and } 0 \leq r \leq 2\}$

For $\sigma = \{(x, y)|0 \leq x \leq 2 \text{ and } -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\}$,

The region can easily be plotted on the right.
The curve $y = \pm\sqrt{4 - x^2}$ is equivalent to:



$$y = \pm\sqrt{4 - x^2} \iff y^2 = 4 - x^2 \iff r^2 \sin^2\theta = 4 - r^2\cos^2\theta$$
$$\iff r^2 = 4 \iff r = 2$$

$r = 2$, along with $x \geq 0$, gives the region σ on the right:

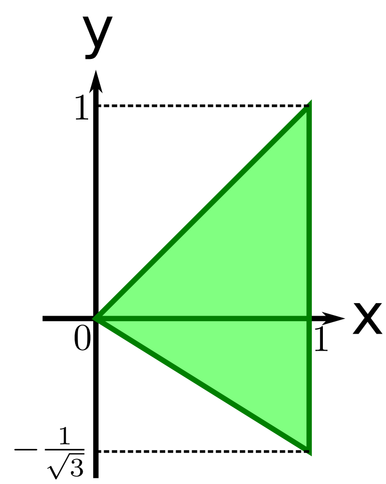
Therefore $\sigma = \left\{(r, \theta)\middle|-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 2\right\}$

For $\sigma = \{(x,y)|0 \leq x \leq 1 \text{ and } -\frac{x}{\sqrt{3}} \leq y \leq x\}$,

The region can easily be plotted on the right.
The counterclockwise angle of line $y = -\frac{x}{\sqrt{3}}$ relative to the positive x-axis is $\operatorname{atan}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$, so the lower bound on θ is $-\frac{\pi}{6}$.
The counterclockwise angle of line $y = x$ relative to the positive x-axis is $\operatorname{atan}(1) = \frac{\pi}{4}$, so the upper bound on θ is $\frac{\pi}{4}$.
The line $x = 1$ is equivalent to:

$$x = 1 \iff r\cos\theta = 1 \iff r = \sec\theta$$

This line gives the upper bound on r.



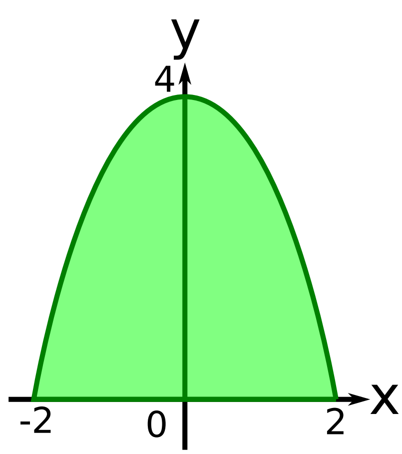
Therefore $\sigma = \left\{(r,\theta)\middle|-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq \sec\theta\right\}$

For $\sigma = \{(x,y)| -2 \leq x \leq 2 \text{ and } 0 \leq y \leq 4 - x^2\}$,

The region can easily be plotted on the right.
The curve $y = 4 - x^2$ is equivalent to:

$$\begin{aligned} y = 4 - x^2 &\iff r\sin\theta = 4 - r^2\cos^2\theta \\ &\iff (\cos^2\theta)r^2 + (\sin\theta)r - 4 = 0 \\ &\iff r = \frac{-\sin\theta + \sqrt{\sin^2\theta + 16\cos^2\theta}}{2\cos^2\theta} \\ &\iff r = \frac{-\sin\theta + \sqrt{1 + 15\cos^2\theta}}{2\cos^2\theta} \end{aligned}$$

The other root is omitted since r must be nonnegative.

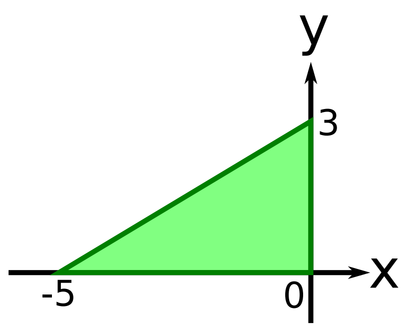


Therefore $\sigma = \left\{(r,\theta)\middle|0 \leq \theta \leq \pi \text{ and } 0 \leq r \leq \frac{-\sin\theta + \sqrt{1+15\cos^2\theta}}{2\cos^2\theta}\right\}$

For $\sigma = \{(x,y)|0 \le y \le 3 \text{ and } -5 + (5/3)y \le x \le 0\}$,

The region can easily be plotted on the right.
The line $x = -5 + (5/3)y$ is equivalent to:

$$\begin{aligned} x = -5 + (5/3)y &\iff r\cos\theta = -5 + \frac{5}{3} r\sin\theta \\ &\iff ((5/3)\sin\theta - \cos\theta)r = 5 \\ &\iff r = \frac{15}{5\sin\theta - 3\cos\theta} \end{aligned}$$

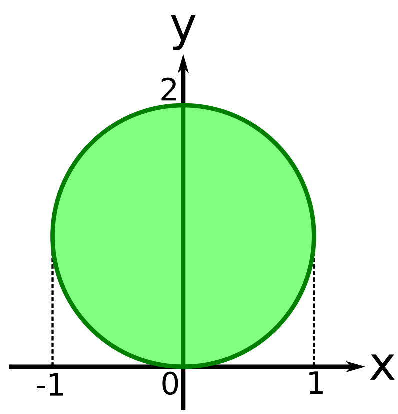


Therefore $\sigma = \left\{(r,\theta)\middle| \frac{\pi}{2} \le \theta \le \pi \text{ and } 0 \le r \le \frac{15}{5\sin\theta - 3\cos\theta}\right\}$

For $\sigma = \{(x,y)| -1 \le x \le 1 \text{ and } 1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}\}$,

The region can easily be plotted on the right.
The curve $y = 1 \pm \sqrt{1-x^2}$ is equivalent to:

$$\begin{aligned} &y = 1 \pm \sqrt{1-x^2} \iff (y-1)^2 = 1 - x^2 \\ &\iff y^2 - 2y + 1 = 1 - x^2 \iff x^2 + y^2 - 2y = 0 \\ &\iff r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0 \iff r^2 = 2r\sin\theta \\ &\iff r = 0, 2\sin\theta \end{aligned}$$



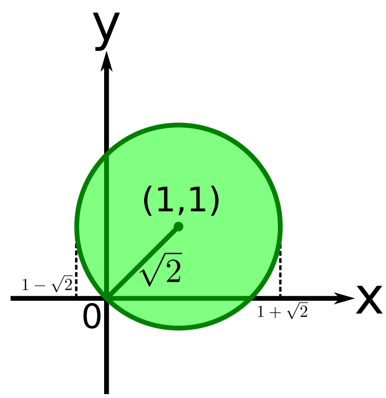
Therefore $\sigma = \{(r,\theta)|0 \le \theta \le \pi \text{ and } 0 \le r \le 2\sin\theta\}$

For $\sigma = \{(x,y)|1-\sqrt{2} \le x \le 1+\sqrt{2}$ and $1-\sqrt{-x^2+2x+1} \le y \le 1+\sqrt{-x^2+2x+1}\}$,
The curve $y = 1 \pm \sqrt{-x^2+2x+1}$ is equivalent to:

$$y = 1 \pm \sqrt{-x^2+2x+1} \iff (y-1)^2 = -x^2+2x+1$$
$$\iff (x^2-2x)+(y-1)^2 = 1 \iff (x-1)^2+(y-1)^2 = 2$$

This curve is a circle centered on the point $(1,1)$ and has radius of $\sqrt{2}$. The region σ is plotted on the right.
Converting the circle to polar coordinates gives:

$$(x-1)^2+(y-1)^2 = 2 \iff x^2+y^2-2x-2y = 0$$
$$\iff r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta - 2r\sin\theta = 0$$
$$\iff r^2 = 2r(\cos\theta+\sin\theta) \iff r = 0, 2(\cos\theta+\sin\theta)$$

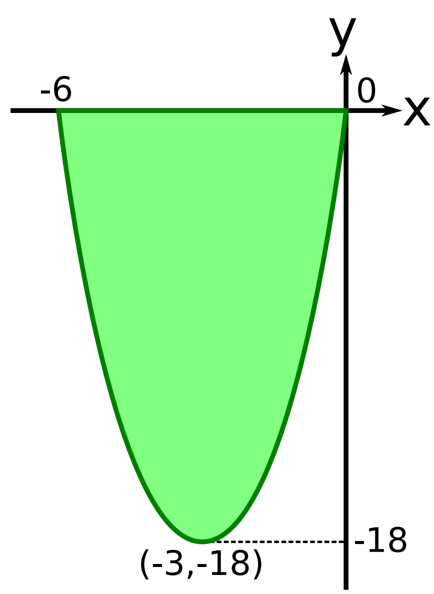


The bounds for θ can easily be determined from the drawing of σ.
Therefore $\sigma = \left\{(r,\theta)\middle|-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \text{ and } 0 \le r \le 2(\cos\theta+\sin\theta)\right\}$

For $\sigma = \{(x,y)|-6 \le x \le 0$ and $2x^2+12x \le y \le 0\}$,

The region σ can easily be plotted on the right, and is entirely contained in the $-x,-y$ quadrant.
The curve $y = 2x^2+12x$ is equivalent to:

$$y = 2x^2+12x$$
$$\iff r\sin\theta = 2r^2\cos^2\theta + 12r\cos\theta$$
$$\iff r^2 = r\frac{\sin\theta-12\cos\theta}{2\cos^2\theta} \iff r = 0, \frac{\sin\theta-12\cos\theta}{2\cos^2\theta}$$



The lower bound for θ is clearly $-\pi$. The upper bound for θ is determined by the angle that the parabola $y = 2x^2+12x$ makes with the x-axis at the origin. At the origin, the parabola has a slope of 12, so the angle made with the x-axis is atan(12). On the left side of the y-axis ($x \le 0$), this angle induces the upper bound of $-\pi+\text{atan}(12)$ for θ.

Therefore $\sigma = \left\{(r,\theta)\middle|-\pi \le \theta \le -\pi+\text{atan}(12) \text{ and } 0 \le r \le \frac{\sin\theta-12\cos\theta}{2\cos^2\theta}\right\}$

part 2b:

For the following regions characterized using Polar coordinates, express these regions using Cartesian coordinates:

- $\sigma = \left\{(r,\theta)\middle| -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq \frac{3}{2\cos\theta - \sin\theta}\right\}$
- $\sigma = \left\{(r,\theta)\middle| 0 \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq 2\sin\theta\right\}$
- $\sigma = \left\{(r,\theta)\middle| -\text{atan}(2) \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq \frac{2\cos\theta + \sin\theta}{\cos^2\theta}\right\}$
- $\sigma = \left\{(r,\theta)\middle| -\frac{\pi}{2} \leq \theta \leq \text{atan}(\frac{3}{2}) \text{ and } 0 \leq r \leq \frac{\sin\theta + \sqrt{1+3\cos^2\theta}}{2\cos^2\theta}\right\}$
- $\sigma = \left\{(r,\theta)\middle| -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \text{ and } 2\cos\theta - \sqrt{4\cos^2\theta - 3} \leq r \leq 2\cos\theta + \sqrt{4\cos^2\theta - 3}\right\}$

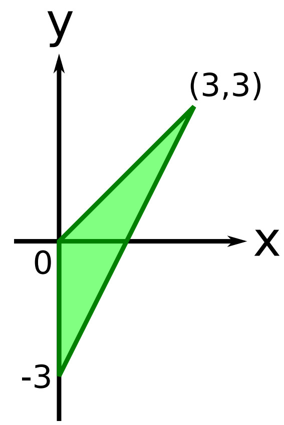
Solution:

For $\sigma = \left\{(r,\theta)\middle| -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq \frac{3}{2\cos\theta - \sin\theta}\right\}$,

The curve $r = \frac{3}{2\cos\theta - \sin\theta}$ is equivalent to:

$$r = \frac{3}{2\cos\theta - \sin\theta} \iff r = \frac{3}{2(x/r) - y/r} \iff 2x - y = 3$$
$$\iff y = 2x - 3$$

The lower bound of $\theta = -\frac{\pi}{2}$ generates the line $x = 0$, while the upper bound of $\theta = \frac{\pi}{4}$ generates the line $y = x$. Line $x = 0$ intersects line $y = x$ at $(0,0)$ and intersects line $y = 2x - 3$ at $(0,-3)$. Line $y = x$ intersects $y = 2x - 3$ at $(3,3)$. This generates the triangular region on the right.



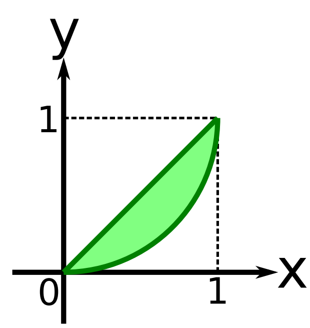
Therefore $\sigma = \{(x,y) | 0 \leq x \leq 3 \text{ and } 2x - 3 \leq y \leq x\}$

For $\sigma = \left\{(r,\theta)\middle| 0 \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq 2\sin\theta\right\}$,

The curve $r = 2\sin\theta$ is equivalent to:

$$r = 2\sin\theta \iff r = 2(y/r) \iff r^2 = 2y$$
$$\iff x^2 + y^2 = 2y \iff x^2 + (y-1)^2 = 1$$

which is a circle centered on the point $(0,1)$ and has a radius of 1. The lower bound of $\theta = 0$ generates the line $y = 0$, while the upper bound of $\theta = \frac{\pi}{4}$ generates the line $y = x$. The portion of the aforementioned circle that is sandwiched between these two lines is shown on the right.



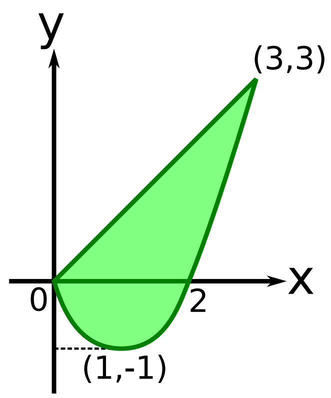
Therefore $\sigma = \{(x,y) | 0 \leq x \leq 1 \text{ and } 1 - \sqrt{1 - x^2} \leq y \leq x\}$

For $\sigma = \left\{(r,\theta)\middle| -\operatorname{atan}(2) \le \theta \le \frac{\pi}{4} \text{ and } 0 \le r \le \frac{2\cos\theta+\sin\theta}{\cos^2\theta}\right\}$,
The curve $r = \frac{2\cos\theta+\sin\theta}{\cos^2\theta}$ is equivalent to:

$$r = \frac{2\cos\theta + \sin\theta}{\cos^2\theta} \iff r = \frac{2(x/r) + y/r}{(x/r)^2}$$

$$\iff 1 = \frac{2x+y}{x^2} \iff y = x^2 - 2x$$

which is a parabola with a turning point at $(1,-1)$ and which passes through $(0,0)$. The lower bound of $\theta = -\operatorname{atan}(2)$ generates the line $y = -2x$, while the upper bound of $\theta = \frac{\pi}{4}$ generates the line $y = x$. The line $y = 2x$ is tangent to the parabola at $(0,0)$, while the line $y = x$ intersects the parabola at $(0,0)$ and $(3,3)$. The portion of the parabola that forms σ is shown on the right.



Therefore $\sigma = \{(x,y)|0 \le x \le 3 \text{ and } x^2 - 2x \le y \le x\}$

For $\sigma = \left\{(r,\theta)\middle| -\frac{\pi}{2} \le \theta \le \operatorname{atan}(\frac{3}{2}) \text{ and } 0 \le r \le \frac{\sin\theta+\sqrt{1+3\cos^2\theta}}{2\cos^2\theta}\right\}$,

The curve $r = \frac{\sin\theta+\sqrt{1+3\cos^2\theta}}{2\cos^2\theta}$ is equivalent to:

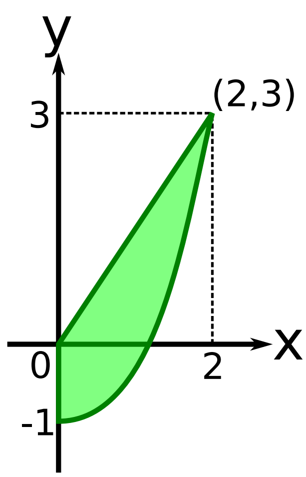
$$r = \frac{\sin\theta + \sqrt{1+3\cos^2\theta}}{2\cos^2\theta} \iff r = \frac{y/r + \sqrt{1+3(x/r)^2}}{2(x/r)^2}$$

$$\iff 1 = \frac{y + \sqrt{r^2+3x^2}}{2x^2} \iff 2x^2 = y + \sqrt{4x^2+y^2}$$

$$\iff 2x^2 - y = \sqrt{4x^2+y^2} \iff 4x^4 - 4x^2y + y^2 = 4x^2 + y^2$$

$$\iff 4x^2y = 4x^4 - 4x^2 \iff y = x^2 - 1$$

The lower bound of $\theta = -\frac{\pi}{2}$ generates the line $x = 0$. The upper bound of $\theta = \operatorname{atan}(\frac{3}{2})$ generates the line $y = \frac{3}{2}x$. The line $y = \frac{3}{2}x$ intersects the parabola $y = x^2 - 1$ at the points $(2,3)$ and $(-1/2,-3/4)$, though only the point $(2,3)$ is relevant to σ. σ is displayed on the right.



Therefore $\sigma = \{(x,y)|0 \le x \le 2 \text{ and } x^2 - 1 \le y \le \frac{3}{2}x\}$

For $\sigma = \left\{(r,\theta) \middle| -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \text{ and } 2\cos\theta - \sqrt{4\cos^2\theta - 3} \leq r \leq 2\cos\theta + \sqrt{4\cos^2\theta - 3}\right\}$,
The curve $r = 2\cos\theta \pm \sqrt{4\cos^2\theta - 3}$ is equivalent to:

$$\begin{aligned} & r = 2\cos\theta \pm \sqrt{4\cos^2\theta - 3} \iff r = 2(x/r) \pm \sqrt{4(x/r)^2 - 3} \\ \iff & r^2 = 2x \pm \sqrt{4x^2 - 3r^2} \iff x^2 + y^2 = 2x \pm \sqrt{x^2 - 3y^2} \\ \iff & x^2 + y^2 - 2x = \pm\sqrt{x^2 - 3y^2} \\ \iff & x^4 + y^4 + 4x^2 + 2x^2y^2 - 4x^3 - 4xy^2 = x^2 - 3y^2 \\ \iff & y^4 + (2x^2 - 4x + 3)y^2 + (x^4 - 4x^3 + 3x^2) = 0 \end{aligned}$$

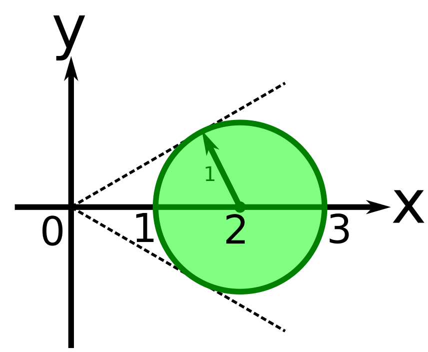
using the quadratic formula gives:

$$\begin{aligned} \iff & y^2 = \frac{(-2x^2 + 4x - 3) \pm \sqrt{16x^2 - 24x + 9}}{2} \\ \iff & y^2 = \frac{(-2x^2 + 4x - 3) \pm (4x - 3)^2}{2} \\ \iff & y^2 = -x^2 + 4x - 3, -x^2 \end{aligned}$$

since $y^2 \geq 0$, the option of $y^2 = -x^2$ is not allowed. Therefore:

$$y^2 = -x^2 + 4x - 3 \iff (x-2)^2 + y^2 = 1$$

which is a circle centered on the point $(2, 0)$ with a radius of 1. The lower bound of $\theta = -\frac{\pi}{6}$ gives the line $y = -\frac{x}{\sqrt{3}}$. The upper bound of $\theta = \frac{\pi}{6}$ gives the line $y = \frac{x}{\sqrt{3}}$. Both lines $y = -\frac{x}{\sqrt{3}}$ and $y = \frac{x}{\sqrt{3}}$ intersect the circle at exactly one point and are therefore tangent to the circle, and do not clip the circle. σ is shown on the right.



Therefore $\sigma = \{(x,y) | 1 \leq x \leq 3 \text{ and } -\sqrt{1-(x-2)^2} \leq y \leq \sqrt{1-(x-2)^2}\}$

Question 3:

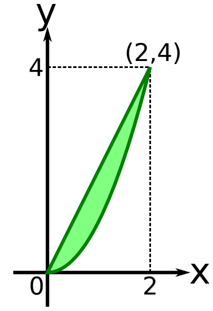
For the following iterated integrals, reverse the order of integration:

- $\int_{x=0}^{2} \int_{y=x^2}^{2x} f(x,y)dydx$
- $\int_{y=0}^{3} \int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)dxdy$
- $\int_{x=-5}^{1} \int_{y=-4}^{-x^2-4x+1} f(x,y)dydx$

Solution:

For $\int_{x=0}^{2}\int_{y=x^2}^{2x} f(x,y)dydx$,
the region of integration is: $\sigma = \{(x,y)|0 \le x \le 2 \text{ and } x^2 \le y \le 2x\}$

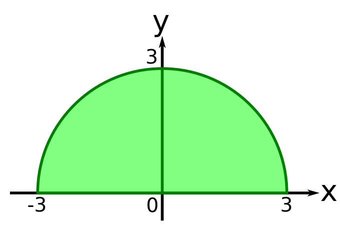
The line $y = 2x$ and the parabola $y = x^2$ intersect at the points $(0,0)$ and $(2,4)$. These points form the "endpoints" of σ. The line $y = 2x$ can be rearranged to give $x = y/2$. The parabola $y = x^2$ can be arranged to give $x = \pm\sqrt{y}$, and since $x \ge 0$, only $x = \sqrt{y}$ matters. The region σ is shown on the right.



Therefore $\sigma = \{(x,y)|0 \le y \le 4 \text{ and } y/2 \le x \le \sqrt{y}\}$
and $\int_{x=0}^{2}\int_{y=x^2}^{2x} f(x,y)dydx = \int_{y=0}^{4}\int_{x=y/2}^{\sqrt{y}} f(x,y)dxdy$

For $\int_{y=0}^{3}\int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)dxdy$,
the region of integration is: $\sigma = \{(x,y)|0 \le y \le 3 \text{ and } -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}\}$

σ is a semicircle with a radius of 3 that is centered at $(0,0)$ and is above the line $y = 0$. σ is shown on the right.



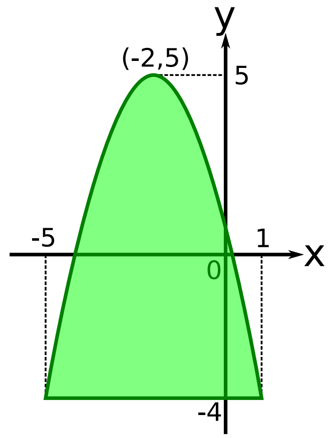
Therefore $\sigma = \{(x,y)|-3 \le x \le 3 \text{ and } 0 \le y \le \sqrt{9-x^2}\}$
and $\int_{y=0}^{3}\int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)dxdy = \int_{x=-3}^{3}\int_{y=0}^{\sqrt{9-x^2}} f(x,y)dydx$

For $\int_{x=-5}^{1}\int_{y=-4}^{-x^2-4x+1} f(x,y)dydx$,
the region of integration is: $\sigma = \{(x,y)| -5 \le x \le 1 \text{ and } -4 \le y \le -x^2-4x+1\}$

σ is bounded from below by the line $y=-4$, and from above by the parabola $y=-x^2-4x+1$. The line and parabola intersect at the points $(-5,-4)$ and $(1,-4)$, which matches exactly the given range for x. Region σ is shown on the right. The equation $y=-x^2-4x+1$ is equivalent to:

$$y=-x^2-4x+1 \iff y=-(x+2)^2+5$$
$$\iff x=-2\pm\sqrt{5-y}$$

The minimum value of y is -4, while the maximum value of y is 5.



Therefore $\sigma = \{(x,y)| -4 \le y \le 5 \text{ and } -2-\sqrt{5-y} \le x \le -2+\sqrt{5-y}\}$
and $\int_{x=-5}^{1}\int_{y=-4}^{-x^2-4x+1} f(x,y)dydx = \int_{y=-4}^{5}\int_{x=-2-\sqrt{5-y}}^{-2+\sqrt{5-y}} f(x,y)dxdy$