# About division

To "divide" a number a by another number b, is to either split a into b equal parts, or to count the number of copies of b required to sum to a. There are two views of division, that give the exact same result:

Partition Division After splitting a into b equal parts, the quotient a/b is the size of a single part.

Quotition Division The quotient a/b is the number of copies of b that need to be summed to get a total of a.
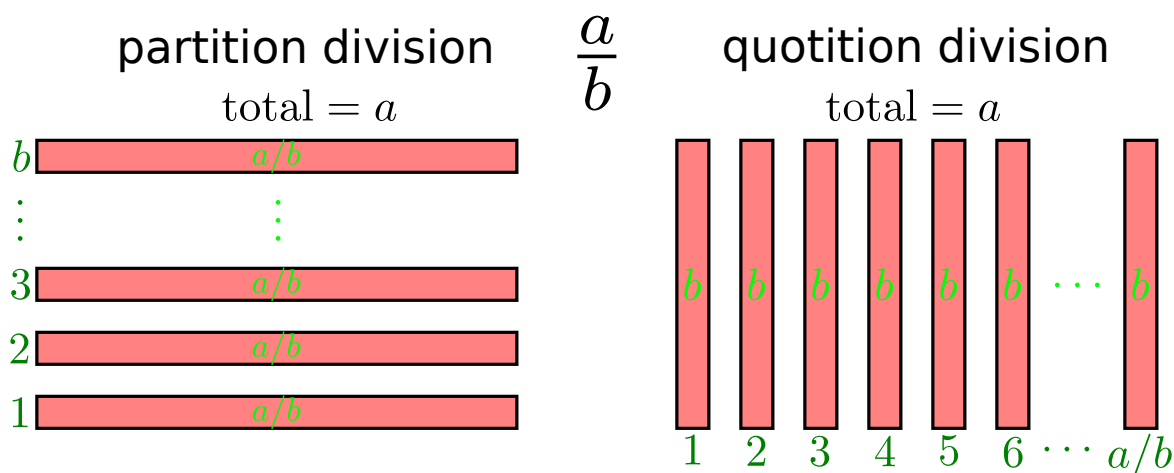
It is important to note that no matter which view of division is being used, the result is always the same. Below is given a series of example quotients where the partition and quotition interpretations of division are presented side by side. It should be noted that the interpretation, partition or quotition, has no effect on the value of the quotient. Why is the distinction important? The distinction helps in deciding when to use division, and it also helps in the understanding of the algebraic rules surrounding division.
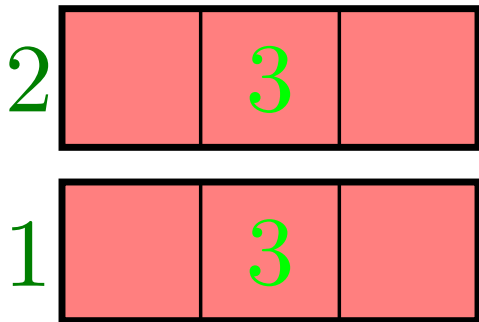
When it comes to **computing** the quotients, the quotition interpretation is the interpretation that is almost always used. The quotient a/b is computed by counting the number of copies of b that can be taken from a. Despite this, the partition interpretation still sees use in identifying scenarios where division is to be used, establishes the notion of fractions, is the literal interpretation of the word "division", and is the first interpretation of division that is taught to students.

In the image below, the quotient $\frac{6}{2} = 3$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 6 into 2 equal parts. The size of a single part is 3.

- On the right, quotitioning is used. 3 is the number of copies of 2 that need to be summed to total 6.

partition division

quotition division

$$\text{total} = 6 \qquad \frac{6}{2} = 3 \qquad \text{total} = 6$$

2 | 3

1 | 3

2 2 2

1 2 3

In the image below, the quotient $\frac{12}{3} = 4$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 12 into 3 equal parts. The size of a single part is 4.

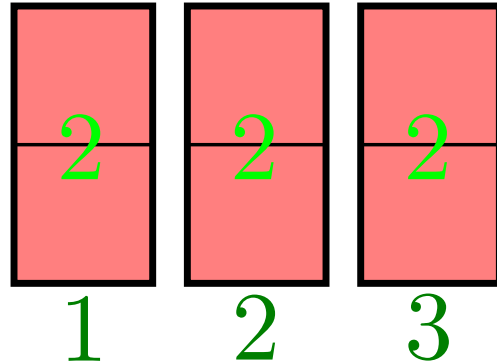- On the right, quotitioning is used. 4 is the number of copies of 3 that need to be summed to total 12.

partition division

quotition division

$$\text{total} = 12 \qquad \frac{12}{3} = 4 \qquad \text{total} = 12$$

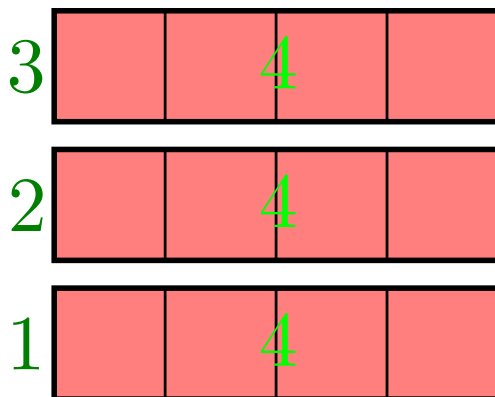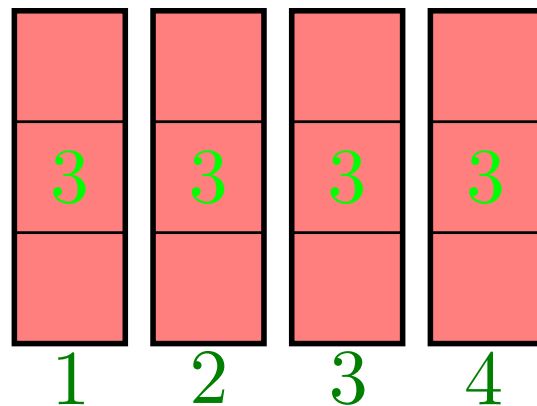3 | 4

2 | 4

1 | 4

3 3 3 3

1 2 3 4

In the image below, the quotient $\frac{7}{2} = 3.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 7 into 2 equal parts. The size of a single part is 3.5.

- On the right, quotitioning is used. 3.5 is the number of copies of 2 that need to be summed to total 7. With the quotitioning, note that a 0.5 copy of 2 is need to finish the total.

partition division
total = 7

$$\frac{7}{2} = 3.5$$

quotition division
total = 7

In the image to the right, the quotient $\frac{7}{3.5} = 2$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 7 units into 3.5 equal parts. Note that 0.5 of a part is 50% the size of a whole part. The quotient is the size of a full part which is 2 units.

- On the right, quotitioning is used. 2 is the number of copies of 3.5 that need to be summed to total 7.

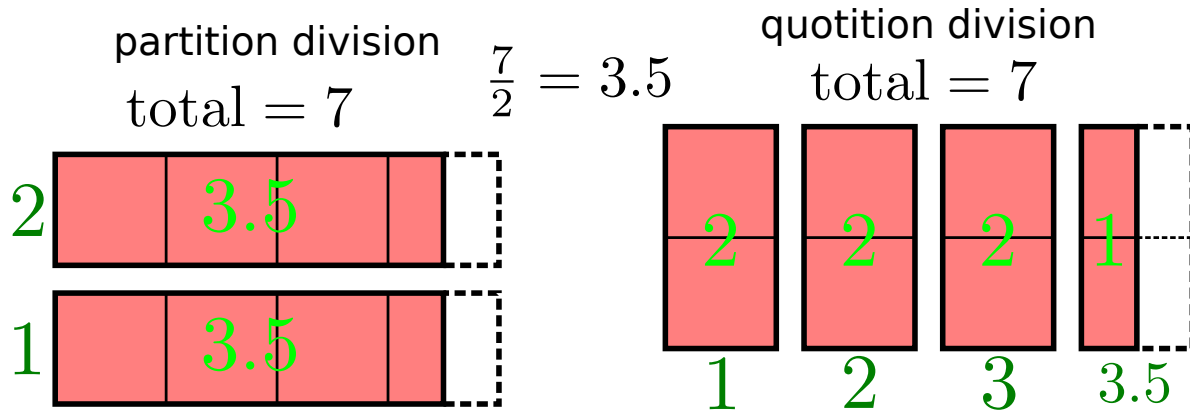### partition division
total = 7

$$\frac{7}{3.5} = 2$$

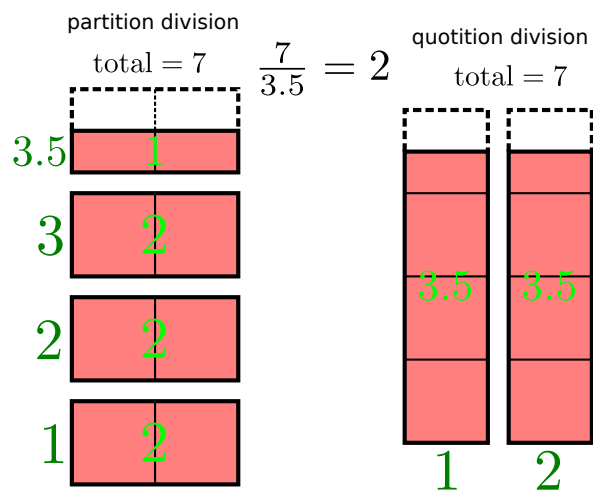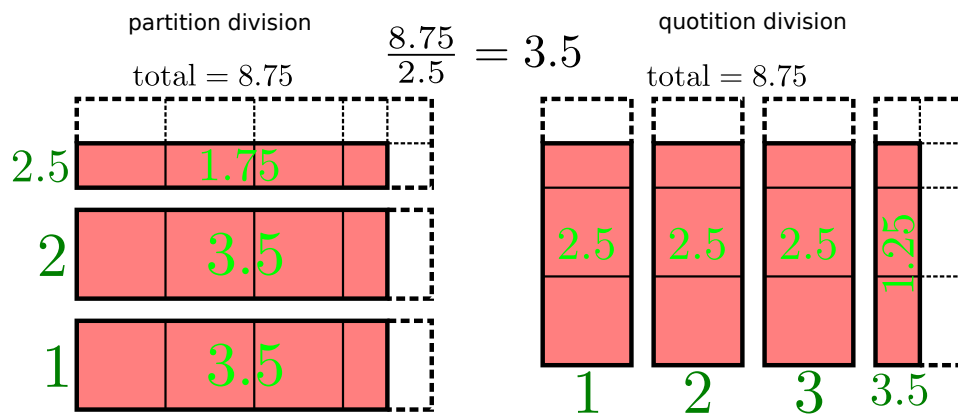### quotition division
total = 7

In the image below, the quotient $\frac{8.75}{2.5} = 3.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 8.75 units into 2.5 equal parts. Note that 0.5 of a part is 50% the size of a whole part. The quotient is the size of a full part which is 3.5 units.

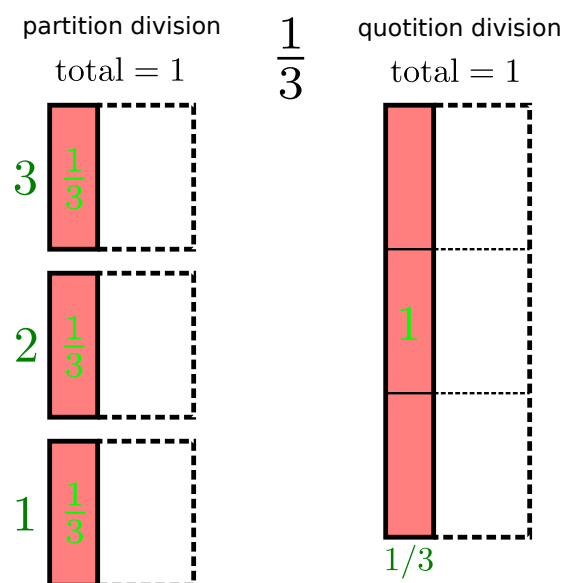- On the right, quotitioning is used. 3.5 is the number of copies of 2.5 that need to be summed to total 7. With the quotitioning, note that a 0.5 copy of 2.5 is need to finish the total.

partition division

total = 8.75

$$\frac{8.75}{2.5} = 3.5$$

quotition division

total = 8.75

2.5 1.75

2 3.5

1 3.5

2.5 2.5 2.5 1.25

1 2 3 3.5

partition division

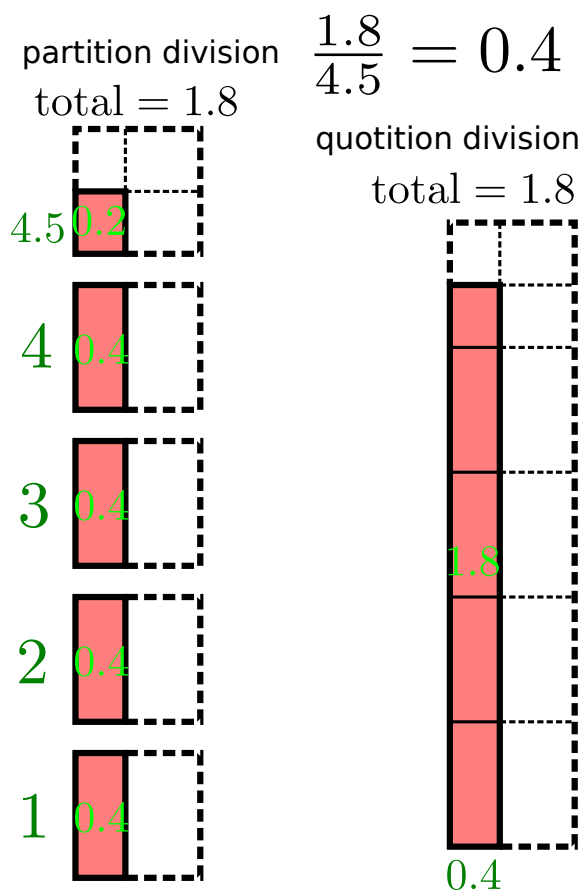total = 1

$$\frac{1}{3}$$

quotition division

total = 1

In the image to the right, the quotient $\frac{1}{3}$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 1 into 3 equal parts.

- On the right, quotitioning is used. 1/3 of a copy of 3 is needed to total to 1.

3 $\frac{1}{3}$

2 $\frac{1}{3}$

1 $\frac{1}{3}$

1

1/3

partition division $\dfrac{1.8}{4.5} = 0.4$

total $= 1.8$

quotition division

total $= 1.8$

4.5 0.2

4 0.4

3 0.4

2 0.4

1 0.4

1.8

0.4

In the image to the right, the quotient $\frac{1.8}{4.5} = 0.4$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 1.8 units into 4.5 equal parts. Note that 0.5 of a part is 50% the size of a whole part. The quotient is the size of a full part which is 0.4 units.

- On the right, quotitioning is used. 0.4 of a copy of 4.5 is needed to total to 1.8.

When b becomes less then 1, splitting a into b equal parts becomes less intuitive. Below, the quotient $\frac{1.5}{0.5} = 3$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 1.5 units into 0.5 equal parts. Normally the numerator of $a = 1.5$ units is split between multiple parts. Here instead, the 0.5 part receives all of the $a = 1.5$ units. The quotient is the size of a whole part however, and a whole part is 2 times the size of a 0.5 part. The size of a "whole part" is therefore $2 \times 1.5 = 3$ units.

- On the right, quotitioning is used. 3 is the number of copies of 0.5 is needed to total to 1.5.

$$\frac{1.5}{0.5} = 3$$

partition division

quotition division

total = 1.5

total = 1.5
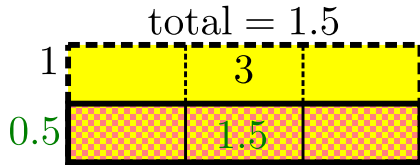
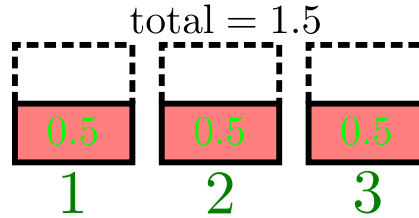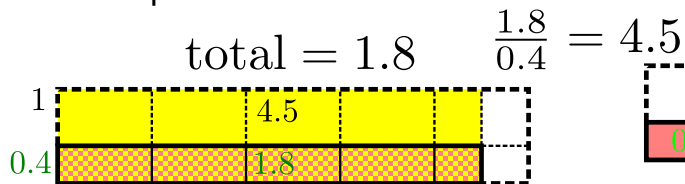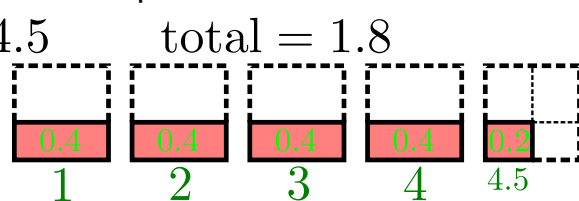Below, the quotient $\frac{1.8}{0.4} = 4.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 1.8 units into 0.4 equal parts. Normally the numerator $a = 1.8$ units is split between multiple parts. Here instead, the 0.4 part receives all of $a = 1.8$ units. The quotient is the size of a whole part however, and a whole part is multiple copies of a 0.4 part (2.5 copies to be exact). The size of a "whole part" is 4.5 units.

- On the right, quotitioning is used. 4.5 is the number of copies of 0.4 is needed to total to 1.8. Note that a 0.5 copy of 0.4 is needed to finish the total.

partition division

quotition division

total = 1.8 $\frac{1.8}{0.4} = 4.5$ total = 1.8

Below, the quotient $\frac{-5}{2} = -2.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split -5 into 2 equal parts. The size of a single part is -2.5.

- On the right, quotitioning is used. 2.5 negative copies of 2 are required to form the total of $a = -5$ units. Each negative copy subtracts 1 from the total number of copies, so the number of copies is -2.5. Note that a -0.5 copy of 2 is needed to finish the total.

$$\frac{-5}{2} = -2.5$$

partition division

total $= -5$

2   -2.5

1   -2.5
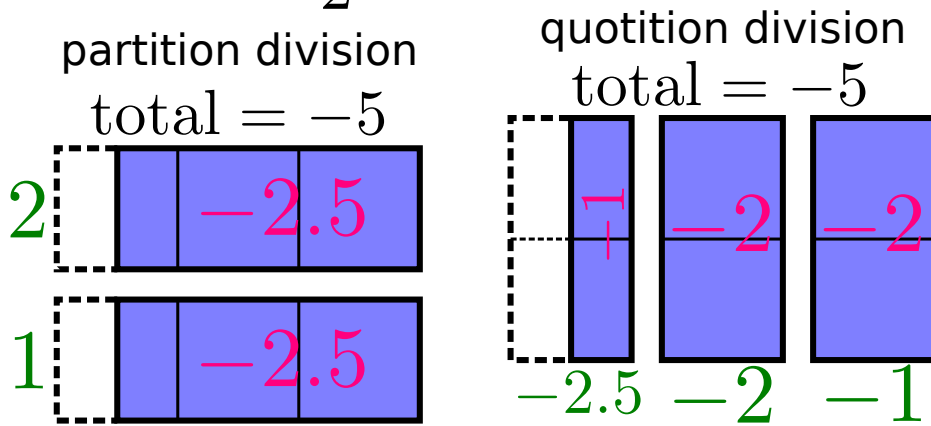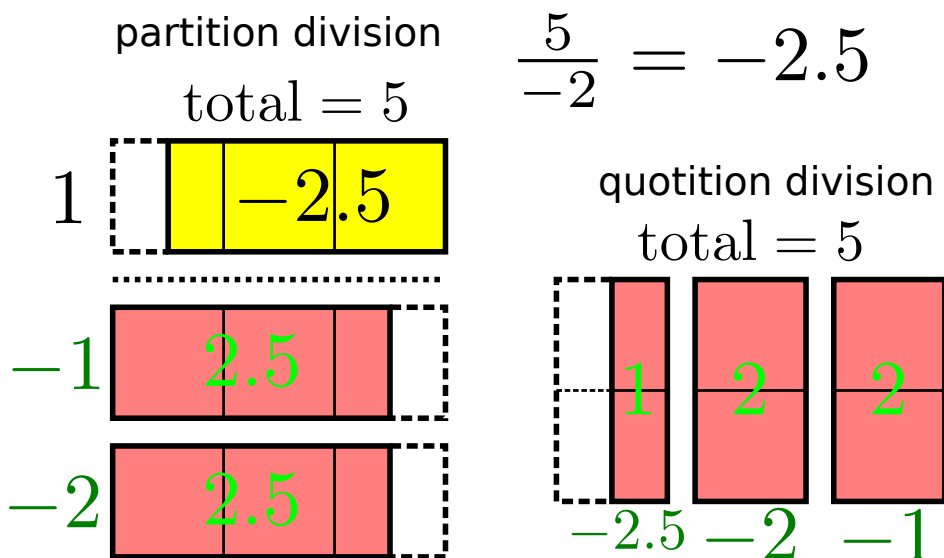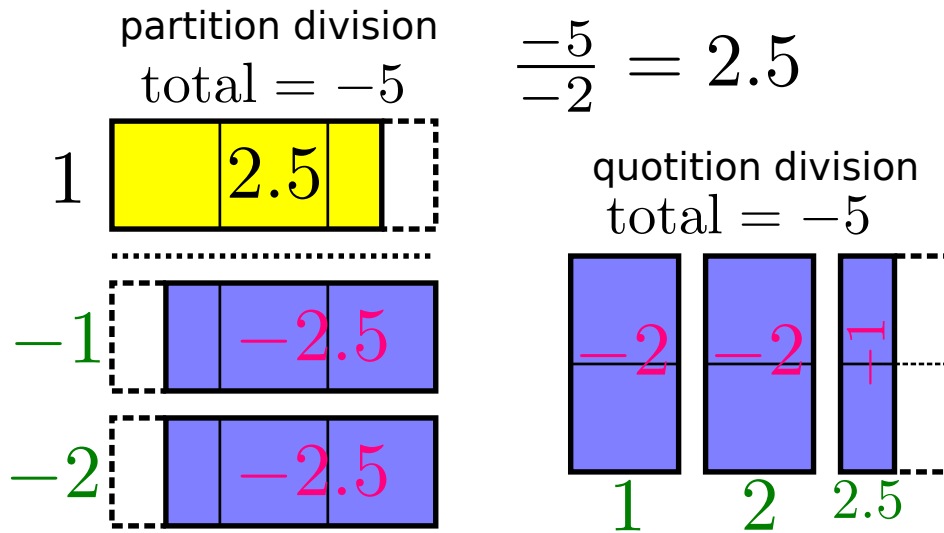
quotition division

total $= -5$

-1 -2 -2

-2.5 -2 -1

Below, the quotient $\frac{5}{-2} = -2.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split 5 into -2 equal parts. This is done by splitting the $a = 5$ units evenly between 2 parts that each have a negative weight of -1. Each negative part gets 2.5 units. The quotient however is the number of units that would be in a positive whole part. A positive whole part has a weight of $+1$. A part with a weight of -1 is getting 2.5 units, so a part with a weight of $+1$ would get -2.5 units, hence the quotient is -2.5.

- On the right, quotitioning is used. 2.5 negative copies of -2 are required to form the total of $a = 5$ units. Each negative copy subtracts 1 from the total number of copies, so the number of copies is -2.5. Note that a -0.5 copy of -2 is needed to finish the total.

partition division

total $= 5$

$$\frac{5}{-2} = -2.5$$

1   -2.5

-1 2.5

-2 2.5

quotition division

total $= 5$

1 2 2

-2.5 -2 -1

Below, the quotient $\frac{-5}{-2} = 2.5$ is depicted via both partition and quotition.

- On the left, partitioning is used to split -5 into -2 equal parts. This is done by splitting the $a = -5$ units evenly between 2 parts that each have a negative weight of -1. Each negative part gets -2.5 units. The quotient however is the number of units that would be in a positive whole part. A positive whole part has a weight of $+1$. A part with a weight of -1 is getting -2.5 units, so a part with a weight of $+1$ would get 2.5 units, hence the quotient is 2.5.

- On the right, quotitioning is used. 2.5 copies of -2 are required to form the total of $a = -5$ units. Note that a 0.5 copy of -2 is needed to finish the total.

In summary,

- With partition division, the quotient a/b is the value of a **whole part** after a has been split into b equal parts. When b is not a positive integer, the following must clarified:

    * When b has a fractional part, the value of the fractional part is the corresponding proportion of a full part. For instance, a 0.5 part has $1/2$ the value of a full part, and a 0.1 part has $1/10$ the value of a full part. a/b is the value of a full part, and not of any fractional part.

    * When b is negative, the value of a negative part is the negative of the value of a full part. If a negative part has a value of 3, then a full part has a value of -3. a/b is the value of a full part, and not of any negative part.

- With quotition division, the quotient a/b is the **number of copies** of b that need to be summed to get a. When a/b is not a positive integer, the following must clarified:

    * When a/b has a fractional part, the contents of the fractional copy of b has the corresponding proportion of a full copy of b. For instance, a 0.5 copy of b has $1/2$ the value of b, and a 0.1 copy has $1/10$ the value of b.

    * When a/b is negative, the value of a negative copy of b is $-b$.

Division identities

The various algebraic rules (identities) concerning division can be understood using a mixture of the partition interpretation and the quotition interpretation of division.

Given any real numbers a, b, and c, the division of the product ab by c is equivalent to the product $a \cdot \frac{b}{c}$:

$$\frac{ab}{c} = a \cdot \frac{b}{c}$$

This equation can be understood in the following ways:

- **Partition division:** The product ab is a copies of b. $\frac{ab}{c}$ is the product ab split into c equal pieces. Splitting ab into c equal pieces can be done by splitting each of the a copies of b into c equal pieces to get a copies of $\frac{b}{c}$. This understanding is depicted in the image below on the left where an area of ab is split into c equal columns, and each column is a copies of $\frac{b}{c}$.

- **Quotition division:** $\frac{ab}{c}$ is the number of copies of c that need to be summed to get ab. $\frac{b}{c}$ is the number of copies of c that need to be summed to get b. a copies of $\frac{b}{c}$ is a number of copies of c that when summed gives a copies of b which is ab. This understanding is depicted in the image below on the right where $a \cdot \frac{b}{c}$ copies of c sum to ab.

For example, the expression $\frac{a}{b}$ is taught in elementary school to mean a copies of $\frac{1}{b}$. The above identity makes it clear that $\frac{a}{b}$ is also a divided into b equal parts. This is why the fraction bar "/" replaces the division symbol $\div$.

Given any real numbers a, b, and c, dividing a by b, and then dividing by c is equivalent to a straightforwards division of a by the product bc:

$$\frac{a/b}{c} = \frac{a}{bc}$$

This equation is easiest to understand using **partition** division:

$\frac{a}{b}$ is a divided into b equal pieces. $\frac{a/b}{c}$ is a further division of $\frac{a}{b}$ into c equal pieces. c copies of $\frac{a/b}{c}$ is $\frac{b}{a}$. b copies of c copies of $\frac{a/b}{c}$ is b copies of $\frac{a}{b}$ which in turn is a. Therefore bc copies of $\frac{a/b}{c}$ is a. $\frac{a/b}{c}$ is a divided into bc equal pieces.

partitioning
$$\frac{a/b}{c} = \frac{a}{bc}$$
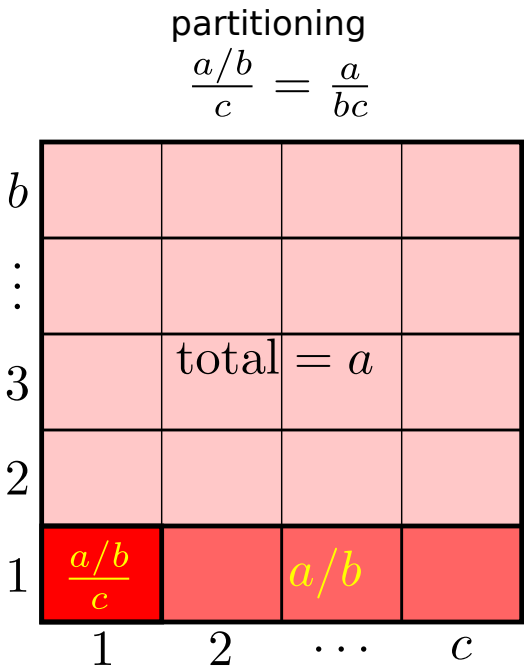
Given any real numbers a and b, the division of a by $1/b$ is the product ab:

$$\frac{a}{1/b} = ab$$

The division of a into "$\frac{1}{b}$" equal parts can be difficult to grasp, so the **quotition** interpretation of division will used instead. b copies of $\frac{1}{b}$ sum to 1 since $\frac{1}{b}$ is 1 split into b equal pieces. ab copies of $\frac{1}{b}$ sum to a copies of 1 which is a. The product ab is the number of copies of $\frac{1}{b}$ that need to be summed to get a. This is illustrated in the image below.

quotitioning

$$\frac{a}{1/b} = ab$$
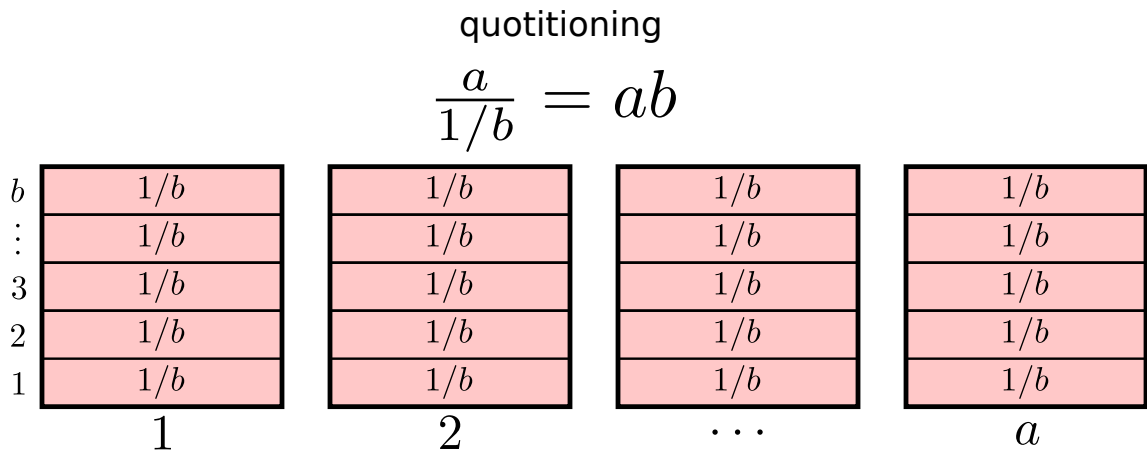
About ratios

When two quantities Q_1 and Q_2 (concrete examples will be given later) are "proportional" to each other, there are numbers a and b such that any collection of a units of quantity Q_1 "corresponds" to b units of quantity Q_2. The **ratio** of quantity Q_1 to quantity Q_2 is represented by:

$$a : b$$

Given any real number k, k copies of a units of Q_1, ka, "corresponds" to k copies of b units of Q_2, kb, so the ratio $a : b$ is unchanged if both a and b are multiplied by the common multiplier of k:

$$ka : kb = a : b$$

As an example, consider a mystery metal where a volume of 2.5L has a mass of 10kg. The first quantity Q_1 is "volume", and the second quantity Q_2 is "mass". The volume-mass ratio is denoted by

$$2.5\text{L} : 10\text{kg}$$

3 copies of 2.5L of the mystery metal has a mass equivalent to 3 copies of 10kg. The ratio can also be expressed as:

$$3 \times 2.5\text{L} : 3 \times 10\text{kg} = 7.5\text{L} : 30\text{kg}$$

A very common problem involving ratios is the following:

If a units of quantity Q_1 correspond to b units of quantity Q_2, then c units of Q_1 correspond to how many units of Q_2?

There are two approaches to solving this problem: the "partition approach" and the "quotition approach".

The partition approach

The partition approach involves first computing the number of units of Q_2 that correspond to a single unit of Q_1. The number of units of Q_2 that correspond to 1 unit of Q_1 is determined by distributing the b units of Q_2 evenly amongst the a units of Q_1. This is to invoke the partition interpretation of division and split b into a equal parts, a single part corresponding to a single unit of Q_1. With 1 unit of Q_1 corresponding to b/a units of Q_2, c units of Q_1 corresponds to $c(b/a)$ units of Q_2.

The quotition approach

The quotition approach involves first invoking the quotition interpretation of division and compute the number of copies of a units of Q_1 needed to build up c units of Q_1. Once c/a has been computed, each block of a units of Q_1 is converted to b units of Q_2. c/a blocks of a units of Q_1 becomes c/a blocks of b units of Q_2. c/a blocks of b units of Q_2 is $(c/a)b$ units of Q_2.

Both approaches yield the final expression of bc/a.

Example 1:

Consider the previous example. A mystery metal where a volume of 2.5L has a mass of 10kg. In this context, Q_1 is volume, and Q_2 is mass. Two questions will be asked:

Question 1: What is the mass of 75L of this mystery metal?

Partition approach: The mass per 1L of metal is: $\frac{10\text{kg}}{2.5\text{L}} = 4\text{kg/L}$. This is also referred to as the "density". The mass of 75L of metal is $75\text{L} \times 4\text{kg/L} = 300\text{kg}$.

Quotition approach: The number of copies of 2.5L needed to build up 75L is $\frac{75\text{L}}{2.5\text{L}} = 30$. 30 copies of 10kg is a total of 300kg.

Question 2: What is the volume of 80kg of this mystery metal?

Partition approach: The volume per 1kg of metal is: $\frac{2.5\text{L}}{10\text{kg}} = 0.25\text{L/kg}$. The volume of 80kg of metal is $80\text{kg} \times 0.25\text{L/kg} = 20\text{L}$.

Quotition approach: The number of copies of 10kg needed to build up 80kg is $\frac{80\text{kg}}{10\text{kg}} = 8$. 8 copies of 2.5L is a total of 20L.

Example 2:

A vehicle is traveling at a constant speed. In a time 1.76hr, distance of 43.8km is traversed. In this context, Q_1 is time, and Q_2 is distance. Two questions will be asked:

Question 1: What is the distance traversed in a time of 0.45hrs?

> **Partition approach:** The distance traversed over a time of 1hr is: $\frac{43.8\text{km}}{1.76\text{hr}} \approx 24.8864\text{km/hr}$. This is also referred to as the "speed". The distance traversed in 0.45hr is $0.45\text{hr} \times 24.8864\text{km/hr} \approx 11.1989\text{km}$.

> **Quotition approach:** The number of copies of 1.76hr needed to build up 0.45hr is $\frac{0.45\text{hr}}{1.76\text{hr}} \approx 0.255682$. 0.255682 copies of 43.8km is a total of approximately 11.1989km.

Question 2: What is the time elapsed in traversing a distance of 105km?

> **Partition approach:** The time elapsed per 1km is: $\frac{1.76\text{hr}}{43.8\text{km}} \approx 0.0401826\text{hr/km}$. The time elapsed over a distance of 105km is $105\text{km} \times 0.0401826\text{hr/km} \approx 4.21917\text{hr}$.

> **Quotition approach:** The number of copies of 43.8km needed to build up 105km is $\frac{105\text{kg}}{43.8\text{km}} \approx 2.39726$. 2.39726 copies of 1.76hr is a total of approximately 4.21918hr.