Orthogonality

Consider the set of n component vectors $\mathbb{R}^n$.

$$\|\mathbf{u}\| = \left\| \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right\| = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$$

$$\mathbf{u} \bullet \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$$

Note that $\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}}$.

$\mathbf{u} \bullet \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, also referred to as being orthogonal.

Given a set of k vectors $A = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, A is **orthogonal** if and only if $\mathbf{u}_i \bullet \mathbf{u}_j = 0$ for all $i \neq j$. If it is also the case that every vector from A has a magnitude of 1, then A is **orthonormal**.

Determining orthogonality

A set of k vectors $A = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, A is **orthogonal** if and only if $\mathbf{u}_i \bullet \mathbf{u}_j = 0$ for all $i \neq j$. This condition can be directly tested using an $n \times k$ matrix B where the vectors from A form the columns of B:

$$B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix}$$

It is then the case that:

$$B^T B = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \cdots & \mathbf{u}_1^T\mathbf{u}_k \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \cdots & \mathbf{u}_2^T\mathbf{u}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}_k^T\mathbf{u}_1 & \mathbf{u}_k^T\mathbf{u}_2 & \cdots & \mathbf{u}_k^T\mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \bullet \mathbf{u}_1 & \mathbf{u}_1 \bullet \mathbf{u}_2 & \cdots & \mathbf{u}_1 \bullet \mathbf{u}_k \\ \mathbf{u}_2 \bullet \mathbf{u}_1 & \mathbf{u}_2 \bullet \mathbf{u}_2 & \cdots & \mathbf{u}_2 \bullet \mathbf{u}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}_k \bullet \mathbf{u}_1 & \mathbf{u}_k \bullet \mathbf{u}_2 & \cdots & \mathbf{u}_k \bullet \mathbf{u}_k \end{bmatrix}$$

If A is an orthogonal set of vectors, then all non diagonal entries of $B^T B$ should be 0. If A additionally contains no zero vectors, then the diagonal entries of $B^T B$ are all nonzero. If A is an orthonormal set of vectors, then $B^T B$ is the $k \times k$ identity matrix $B^T B = I$.

If $A = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is an orthogonal set of n nonzero vectors, then the $n \times n$ matrix B:

$$B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix}$$

has the following inverse:

$$B^{-1} = \begin{bmatrix} \frac{1}{\mathbf{u}_1 \bullet \mathbf{u}_1} \mathbf{u}_1^T \\ \frac{1}{\mathbf{u}_2 \bullet \mathbf{u}_2} \mathbf{u}_2^T \\ \vdots \\ \frac{1}{\mathbf{u}_n \bullet \mathbf{u}_n} \mathbf{u}_n^T \end{bmatrix}$$

If $A = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is an orthonormal set of n vectors, then the $n \times n$ matrix B:

$$B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$$

has the following inverse:

$$B^{-1} = B^T = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

In general, if an orthogonal set does not contain any zero vectors, then the set is linearly independent and is hence a basis for its span.

Examples:*

- Given the set of vectors $A = \left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$, let $B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$.

$$B^T B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

  $B^T B$ is diagonal, so the vectors in set A are orthogonal. $B^T B \neq I$ however so the vectors in set A are not orthonormal however. Since no vectors in set A are zero, matrix B is invertible and has the inverse:

$$B^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1/4 & 0 & 1/4 \\ 0 & 1/5 & 0 \end{bmatrix}$$

- Given the set of vectors $A = \left\{ \mathbf{u}_1 = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} \right\}$, let $B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix}$.

$$B^T B = \begin{bmatrix} 1/5 & 1/5 & 1/5 \\ -1/2 & 1/2 & 0 \\ 1/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix} = \begin{bmatrix} 3/25 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

  $B^T B$ is diagonal, so the vectors in set A are orthogonal. $B^T B \neq I$ however so the vectors in set A are not orthonormal however. Since no vectors in set A are zero, matrix B is invertible and has the inverse:

$$B^{-1} = \begin{bmatrix} 5/3 & 5/3 & 5/3 \\ -1 & 1 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}$$

- Given the set of vectors $A = \left\{ \mathbf{u}_1 = \begin{bmatrix} -3/5 \\ 4/5 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4/5 \\ 3/5 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, let $B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$B^T B = \begin{bmatrix} -3/5 & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

B^TB is diagonal, so the vectors in set A are orthogonal. Moreover, $B^TB = I$ so the vectors in set A are also orthonormal. Matrix B is invertible and has the inverse:

$$B^{-1} = B^T = \begin{bmatrix} -3/5 & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Examples from the problem set of chapter 6.3 of the textbook:
Anton, Howard; Rorres, Chris, *Elementary Linear Algebra 11th edition, Applications Version*, Wiley, 2014.

Projections and perpendicular components

Given a set of nonzero orthogonal vectors $A = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, the advantage of orthogonality is that there is an easy approach to determining the coefficients needed to express any vector from span(A) as a linear combination of the vectors from A.

Given any vector $\mathbf{v}$ from the span of the orthogonal nonzero vectors span$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, the unique coefficients c_1, c_2, ..., c_k such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_k\mathbf{u}_k$ can be easily determined as follows:

Starting with

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_k\mathbf{u}_k = \mathbf{v}$$

To find coefficient c_i, multiply using the dot product both sides by $\mathbf{u}_i$:

$$\begin{aligned}
& c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_k\mathbf{u}_k = \mathbf{v} \\
\Longrightarrow\ & \mathbf{u}_i \bullet (c_1\mathbf{u}_1 + ... + c_i\mathbf{u}_i + ... + c_k\mathbf{u}_k) = \mathbf{u}_i \bullet \mathbf{v} \\
\Longleftrightarrow\ & c_1(\mathbf{u}_i \bullet \mathbf{u}_1) + ... + c_i(\mathbf{u}_i \bullet \mathbf{u}_i) + ... + c_k(\mathbf{u}_i \bullet \mathbf{u}_k) = \mathbf{u}_i \bullet \mathbf{v} \\
\Longleftrightarrow\ & c_1(0) + ... + c_i(\mathbf{u}_i \bullet \mathbf{u}_i) + ... + c_k(0) = \mathbf{u}_i \bullet \mathbf{v} \\
\Longleftrightarrow\ & c_i(\mathbf{u}_i \bullet \mathbf{u}_i) = \mathbf{u}_i \bullet \mathbf{v} \\
\Longleftrightarrow\ & c_i = \frac{\mathbf{u}_i \bullet \mathbf{v}}{\mathbf{u}_i \bullet \mathbf{u}_i}
\end{aligned}$$

Vector $\mathbf{v}$ can hence be expressed as the following linear combination:

$$\mathbf{v} = \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 + ... + \frac{\mathbf{u}_k \bullet \mathbf{v}}{\mathbf{u}_k \bullet \mathbf{u}_k}\mathbf{u}_k$$

For any vector $\mathbf{v}$ (in the span or not) the linear combination:

$$\text{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k) = \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 + ... + \frac{\mathbf{u}_k \bullet \mathbf{v}}{\mathbf{u}_k \bullet \mathbf{u}_k}\mathbf{u}_k$$

is referred to as the "projection" of $\mathbf{v}$ onto the span of A. The projection is equal to $\mathbf{v}$ if and only if $\mathbf{v}$ is in the span of A, otherwise part of $\mathbf{v}$, referred to as the "perpendicular component" is removed by the projection.

The part of $\mathbf{v}$ removed by the projection is:

$$\text{perp}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k) = \mathbf{v} - \text{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k) = \mathbf{v} - \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 - ... - \frac{\mathbf{u}_k \bullet \mathbf{v}}{\mathbf{u}_k \bullet \mathbf{u}_k}\mathbf{u}_k$$

and is referred to as the "perpendicular component" of $\mathbf{v}$ relative to the span of A. The perpendicular component is $\mathbf{0}$ if and only if $\mathbf{v}$ is in the span of A.

Examples:*

- Let $\mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$; $\mathbf{u}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$; and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

  It can easily be checked that $\mathbf{u}_1 \bullet \mathbf{u}_2 = 2/9 + 2/9 - 4/9 = 0$ so $A = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Computing the projection and perpendicular component of $\mathbf{v}$ relative to the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ is performed as follows. First compute:

  * $\mathbf{u}_1 \bullet \mathbf{u}_1 = 1/9 + 4/9 + 4/9 = 1$
  * $\mathbf{u}_2 \bullet \mathbf{u}_2 = 4/9 + 1/9 + 4/9 = 1$
  * $\mathbf{u}_1 \bullet \mathbf{v} = 4/3 + 4/3 - 2/3 = 2$
  * $\mathbf{u}_2 \bullet \mathbf{v} = 8/3 + 2/3 + 2/3 = 4$

  so the projection is:

$$\mathrm{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \frac{2}{1}\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} + \frac{4}{1}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ -4/3 \end{bmatrix} + \begin{bmatrix} 8/3 \\ 4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 8/3 \\ 4/3 \end{bmatrix}$$

  and the perpendicular component is:

$$\mathrm{perp}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \mathbf{v} - \mathrm{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 8/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$$

- Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$; and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

  It can easily be checked that $\mathbf{u}_1 \bullet \mathbf{u}_2 = 2 - 2 + 0 = 0$ so $A = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Computing the projection and perpendicular component of $\mathbf{v}$ relative to the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ is performed as follows. First compute:

  * $\mathbf{u}_1 \bullet \mathbf{u}_1 = 1 + 4 + 1 = 6$
  * $\mathbf{u}_2 \bullet \mathbf{u}_2 = 4 + 1 + 0 = 5$
  * $\mathbf{u}_1 \bullet \mathbf{v} = 1 + 0 + 3 = 4$
  * $\mathbf{u}_2 \bullet \mathbf{v} = 2 + 0 + 0 = 2$

  so the projection is:

$$\mathrm{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \frac{4}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 22/15 \\ -14/15 \\ 2/3 \end{bmatrix}$$

  and the perpendicular component is:

$$\mathrm{perp}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \mathbf{v} - \mathrm{proj}(\mathbf{v}|\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 22/15 \\ -14/15 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -7/15 \\ 14/15 \\ 7/3 \end{bmatrix}$$

- Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$; and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$.

  It can easily be checked that $\mathbf{u}_1 \bullet \mathbf{u}_2 = 1 + 1 - 1 - 1 = 0$ so $A = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Computing the projection and perpendicular component of $\mathbf{v}$ relative to the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ is performed as follows. First compute:

* $\mathbf{u}_1 \bullet \mathbf{u}_1 = 1+1+1+1 = 4$
* $\mathbf{u}_2 \bullet \mathbf{u}_2 = 1+1+1+1 = 4$
* $\mathbf{u}_1 \bullet \mathbf{v} = 1+2+0-2 = 1$
* $\mathbf{u}_2 \bullet \mathbf{v} = 1+2+0+2 = 5$

so the projection is:

$$\operatorname{proj}(\mathbf{v}|\mathbf{u}_1,\mathbf{u}_2) = \frac{\mathbf{u}_1 \bullet \mathbf{v}}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \bullet \mathbf{v}}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{5}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix} + \begin{bmatrix}5/4\\5/4\\-5/4\\-5/4\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\-1\\-1\end{bmatrix}$$

and the perpendicular component is:

$$\operatorname{perp}(\mathbf{v}|\mathbf{u}_1,\mathbf{u}_2) = \mathbf{v} - \operatorname{proj}(\mathbf{v}|\mathbf{u}_1,\mathbf{u}_2) = \begin{bmatrix}1\\2\\0\\-2\end{bmatrix} - \begin{bmatrix}3/2\\3/2\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\\-1\end{bmatrix}$$

*Examples from the problem set of chapter 6.3 of the textbook:
Anton, Howard; Rorres, Chris, *Elementary Linear Algebra 11th edition, Applications Version*, Wiley, 2014.

Gram-Schmidt orthogonalization

Given an arbitrary linearly independent set of vectors $A = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$, an orthogonal basis for span(A) can be derived by recursively eliminating from each vector the component that is parallel to the span of the previous vectors. Basis vector $\mathbf{u}_i$ is derived from $\mathbf{v}_i$ by computing the component of $\mathbf{v}_i$ that is perpendicular to the span of the previous vectors: $\mathbf{u}_i = \operatorname{perp}(\mathbf{v}_i|\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{i-1})$

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_2 &= \operatorname{perp}(\mathbf{v}_2|\mathbf{u}_1) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 \\
\mathbf{u}_3 &= \operatorname{perp}(\mathbf{v}_3|\mathbf{u}_1,\mathbf{u}_2) = \mathbf{v}_3 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 \\
&\vdots \\
\mathbf{u}_k &= \operatorname{perp}(\mathbf{v}_k|\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_{k-1}) = \mathbf{v}_k - \frac{\mathbf{u}_1 \bullet \mathbf{v}_k}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_k}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 - ... - \frac{\mathbf{u}_{k-1} \bullet \mathbf{v}_k}{\mathbf{u}_{k-1} \bullet \mathbf{u}_{k-1}}\mathbf{u}_{k-1}
\end{aligned}$$

The orthogonal basis set is $A' = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$

If the set A is linearly dependent, then for any vector that is a linear combination of the previous vectors, the perpendicular component will evaluate to $\mathbf{0}$. This zero vector should be discarded, and the resultant orthogonal basis set A' will contain less vectors than A.

Examples:

Example 1: Let:

$$A = \left\{ \mathbf{v}_1 = \begin{bmatrix}-6\\4\sqrt{5}\\3\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\7\sqrt{5}\\-1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}21\\2\sqrt{5}\\-13\end{bmatrix} \right\}$$

Start with:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix}-6\\4\sqrt{5}\\3\end{bmatrix}$$

Now compute:

* $\mathbf{u}_1 \bullet \mathbf{u}_1 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} = 36 + 80 + 9 = 125$
* $\mathbf{u}_1 \bullet \mathbf{v}_2 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 7\sqrt{5} \\ -1 \end{bmatrix} = -12 + 140 - 3 = 125$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1} = 1$
* $\mathbf{u}_1 \bullet \mathbf{v}_3 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 21 \\ 2\sqrt{5} \\ -13 \end{bmatrix} = -126 + 40 - 39 = -125$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1} = -1$

$$\mathbf{u}_2 = \text{perp}(\mathbf{v}_2|\mathbf{u}_1) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 2 \\ 7\sqrt{5} \\ -1 \end{bmatrix} - 1 \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7\sqrt{5} \\ -1 \end{bmatrix} - \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_2 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix} = -48 + 60 - 12 = 0$$

Now compute:

* $\mathbf{u}_2 \bullet \mathbf{u}_2 = \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix} = 64 + 45 + 16 = 125$
* $\mathbf{u}_2 \bullet \mathbf{v}_3 = \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 21 \\ 2\sqrt{5} \\ -13 \end{bmatrix} = 168 + 30 + 52 = 250$
* $\frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2} = 2$

$$\mathbf{u}_3 = \text{perp}(\mathbf{v}_3|\mathbf{u}_1, \mathbf{u}_2) = \mathbf{v}_3 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 21 \\ 2\sqrt{5} \\ -13 \end{bmatrix} - (-1) \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix}$$
$$= \begin{bmatrix} 21 \\ 2\sqrt{5} \\ -13 \end{bmatrix} - \begin{bmatrix} 6 \\ -4\sqrt{5} \\ -3 \end{bmatrix} - \begin{bmatrix} 16 \\ 6\sqrt{5} \\ -8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_3 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = 6 + 0 - 6 = 0$$

and

$$\mathbf{u}_2 \bullet \mathbf{u}_3 = \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = -8 + 0 + 8 = 0$$

An orthogonal basis is therefore:

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} -6 \\ 4\sqrt{5} \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 8 \\ 3\sqrt{5} \\ -4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

Example 2: Let:

$$A = \left\{ \mathbf{v}_1 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -\sqrt{2} \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2-\sqrt{2} \\ -6 \\ 2+\sqrt{2} \end{bmatrix} \right\}$$

Start with:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix}$$

Now compute:

* $\mathbf{u}_1 \bullet \mathbf{u}_1 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} = 2 + 4 + 2 = 8$
* $\mathbf{u}_1 \bullet \mathbf{v}_2 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -\sqrt{2} \\ 2 \end{bmatrix} = -2\sqrt{2} - 2\sqrt{2} - 2\sqrt{2} = -6\sqrt{2}$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1} = -\frac{3\sqrt{2}}{4}$
* $\mathbf{u}_1 \bullet \mathbf{v}_3 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} 2-\sqrt{2} \\ -6 \\ 2+\sqrt{2} \end{bmatrix} = (-2 + 2\sqrt{2}) - 12 + (-2 - 2\sqrt{2}) = -16$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1} = -2$

$$\mathbf{u}_2 = \text{perp}(\mathbf{v}_2|\mathbf{u}_1) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} -2 \\ -\sqrt{2} \\ 2 \end{bmatrix} - (-\frac{3\sqrt{2}}{4}) \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2 \\ -\sqrt{2} \\ 2 \end{bmatrix} - \begin{bmatrix} -3/2 \\ -3\sqrt{2}/2 \\ 3/2 \end{bmatrix}$$
$$= \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_2 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix} = -\sqrt{2}/2 + \sqrt{2} - \sqrt{2}/2 = 0$$

Now compute:

* $\mathbf{u}_2 \bullet \mathbf{u}_2 = \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix} \bullet \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix} = 1/4 + 1/2 + 1/4 = 1$

* $\mathbf{u}_2 \bullet \mathbf{v}_3 = \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix} \bullet \begin{bmatrix} 2-\sqrt{2} \\ -6 \\ 2+\sqrt{2} \end{bmatrix} = (-1+\sqrt{2}/2) - 3\sqrt{2} + (1+\sqrt{2}/2) = -2\sqrt{2}$
* $\frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2} = -2\sqrt{2}$

$$\mathbf{u}_3 = \text{perp}(\mathbf{v}_3|\mathbf{u}_1, \mathbf{u}_2) = \mathbf{v}_3 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 2-\sqrt{2} \\ -6 \\ 2+\sqrt{2} \end{bmatrix} - (-2)\begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} - (-2\sqrt{2})\begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\sqrt{2} \\ -6 \\ 2+\sqrt{2} \end{bmatrix} - \begin{bmatrix} -2\sqrt{2} \\ -4 \\ 2\sqrt{2} \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ -2 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_3 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\sqrt{2} + 0 - 2\sqrt{2} = 0$$

and

$$\mathbf{u}_2 \bullet \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = -1 + 0 + 1 = 0$$

An orthogonal basis is therefore:

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} \sqrt{2} \\ 2 \\ -\sqrt{2} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Example 3: Let:

$$A = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Start with:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Now compute:

* $\mathbf{u}_1 \bullet \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = 1 + 4 + 4 + 0 = 9$
* $\mathbf{u}_1 \bullet \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \\ -2 \\ -4 \end{bmatrix} = 3 - 8 - 4 + 0 = -9$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1} = -1$

* $\mathbf{u}_1 \bullet \mathbf{v}_3 = \begin{bmatrix}1\\2\\2\\0\end{bmatrix} \bullet \begin{bmatrix}1\\3\\1\\4\end{bmatrix} = 1+6+2+0 = 9$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1} = 1$

$$\mathbf{u}_2 = \text{perp}(\mathbf{v}_2|\mathbf{u}_1) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix}3\\-4\\-2\\-4\end{bmatrix} - (-1)\begin{bmatrix}1\\2\\2\\0\end{bmatrix} = \begin{bmatrix}3\\-4\\-2\\-4\end{bmatrix} - \begin{bmatrix}-1\\-2\\-2\\0\end{bmatrix}$$
$$= \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_2 = \begin{bmatrix}1\\2\\2\\0\end{bmatrix} \bullet \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix} = 4-4+0+0 = 0$$

Now compute:

* $\mathbf{u}_2 \bullet \mathbf{u}_2 = \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix} \bullet \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix} = 16+4+0+16 = 36$
* $\mathbf{u}_2 \bullet \mathbf{v}_3 = \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix} \bullet \begin{bmatrix}1\\3\\1\\4\end{bmatrix} = 4-6+0-16 = -18$
* $\frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2} = -1/2$

$$\mathbf{u}_3 = \text{perp}(\mathbf{v}_3|\mathbf{u}_1, \mathbf{u}_2) = \mathbf{v}_3 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix}1\\3\\1\\4\end{bmatrix} - (1)\begin{bmatrix}1\\2\\2\\0\end{bmatrix} - (-1/2)\begin{bmatrix}4\\-2\\0\\-4\end{bmatrix}$$
$$= \begin{bmatrix}1\\3\\1\\4\end{bmatrix} - \begin{bmatrix}1\\2\\2\\0\end{bmatrix} - \begin{bmatrix}-2\\1\\0\\2\end{bmatrix} = \begin{bmatrix}2\\0\\-1\\2\end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_3 = \begin{bmatrix}1\\2\\2\\0\end{bmatrix} \bullet \begin{bmatrix}2\\0\\-1\\2\end{bmatrix} = 2+0-2+0 = 0$$

and

$$\mathbf{u}_2 \bullet \mathbf{u}_3 = \begin{bmatrix}4\\-2\\0\\-4\end{bmatrix} \bullet \begin{bmatrix}2\\0\\-1\\2\end{bmatrix} = 8+0+0-8 = 0$$

An orthogonal basis is therefore:

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ -2 \\ 0 \\ -4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Example 4*: Let:

$$A = \left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Start with:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Now compute:

* $\mathbf{u}_1 \bullet \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = 0 + 4 + 1 + 0 = 5$
* $\mathbf{u}_1 \bullet \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 0 - 2 + 0 + 0 = -2$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1} = -2/5$
* $\mathbf{u}_1 \bullet \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = 0 + 4 + 0 + 0 = 4$
* $\frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1} = 4/5$

$$\mathbf{u}_2 = \text{perp}(\mathbf{v}_2 | \mathbf{u}_1) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_2}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - (-\frac{2}{5}) \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -4/5 \\ -2/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} = 0 - 2/5 + 2/5 + 0 = 0$$

Now compute:

* $\mathbf{u}_2 \bullet \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} = 1 + 1/25 + 4/25 + 0 = 6/5$
* $\mathbf{u}_2 \bullet \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = 1 - 2/5 + 0 + 0 = 3/5$
* $\frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2} = 1/2$

$$\mathbf{u}_3 = \text{perp}(\mathbf{v}_3 | \mathbf{u}_1, \mathbf{u}_2) = \mathbf{v}_3 - \frac{\mathbf{u}_1 \bullet \mathbf{v}_3}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \bullet \mathbf{v}_3}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 8/5 \\ 4/5 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/10 \\ 1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix}$$

It can be verified that:

$$\mathbf{u}_1 \bullet \mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix} = 0 + 1 - 1 + 0 = 0$$

and

$$\mathbf{u}_2 \bullet \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix} = 1/2 - 1/10 - 2/5 = 0$$

An orthogonal basis is therefore:

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix} \right\}$$

*Examples from the problem set of chapter 6.3 of the textbook:
Anton, Howard; Rorres, Chris, *Elementary Linear Algebra 11th edition, Applications Version*, Wiley, 2014.

Least squares solutions

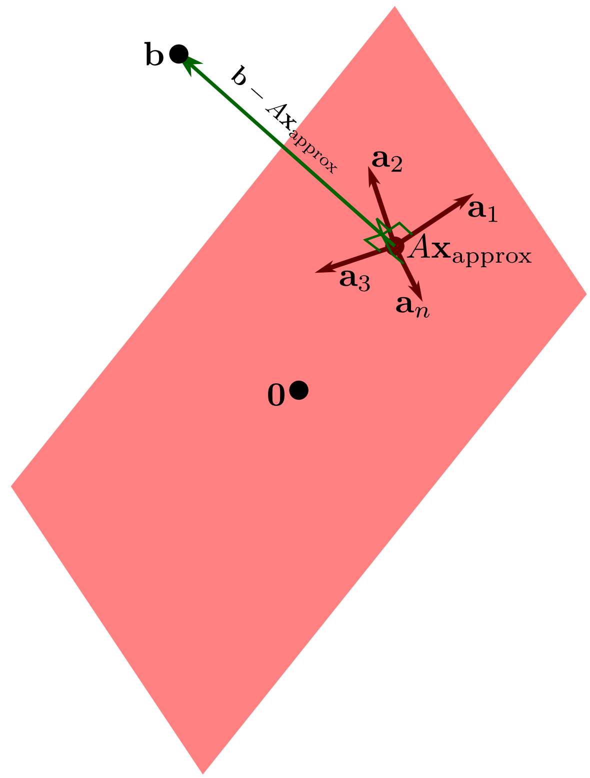
Consider the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ is the unknown quantity. It will be assumed that $\mathbf{b}$ is outside of the column space of A, which means that $\mathbf{b}$ is outside of the range of the linear mapping denoted by matrix A. The system $A\mathbf{x} = \mathbf{b}$ is **inconsistent**. While this system has no solutions, an approximate solution $\mathbf{x}_{\text{approx}}$ can still be solved for that makes $A\mathbf{x}_{\text{approx}}$ as close as possible to $\mathbf{b}$. This solution is referred to as the "least squares solution", and it is the solution that minimizes the magnitude of the "error" vector:

$$\|\mathbf{b} - A\mathbf{x}_{\text{approx}}\|$$

The value attained by $A\mathbf{x}_{\text{approx}}$ will be as close as possible to $\mathbf{b}$, and this will make the "error" $\mathbf{b} - A\mathbf{x}_{\text{approx}}$ orthogonal to the column space of A as depicted on the right. Let

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

The error $\mathbf{b} - A\mathbf{x}_{\text{approx}}$ is perpendicular to the column space of A if and only if the error $\mathbf{b} - A\mathbf{x}_{\text{approx}}$ is perpendicular to each column of A:



$$\left\{\begin{array}{c} \mathbf{a}_1 \bullet (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \\ \mathbf{a}_2 \bullet (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \\ \vdots \\ \mathbf{a}_n \bullet (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \end{array}\right. \iff \left\{\begin{array}{c} \mathbf{a}_1^T (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \\ \mathbf{a}_2^T (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \\ \vdots \\ \mathbf{a}_n^T (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = 0 \end{array}\right. \iff \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} (\mathbf{b} - A\mathbf{x}_{\text{approx}}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\iff A^T(\mathbf{b} - A\mathbf{x}_{\text{approx}}) = \mathbf{0} \iff A^T A\mathbf{x}_{\text{approx}} = A^T\mathbf{b}$$

The system

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

which is the **normal system**, will always be consistent (will have solutions). While the solution $\mathbf{x}$ may not be unique, the closest point $A\mathbf{x}$ will always be unique. The solution is unique if and only if the linear mapping denoted by A is 1 to 1, which occurs if and only if the null space of A is trivial: $\text{nullity}(A) = 0$.

Examples:*

- Consider the linear system:

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

Attempting to directly solve this linear system yields:

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 4 & 5 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 9 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 - (9/5)R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 0 & 6 \end{array}\right]$$

A pivot is in the last column so the system is therefore inconsistent. A least squares solution is now what is sought. Multiplying both sides of the system on the left by $A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ gives:

$$\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Solving this linear system gives:

$$\left[\begin{array}{cc|c} 21 & 25 & 20 \\ 25 & 35 & 20 \end{array}\right] \xrightarrow{R_2 \to R_2 - (25/21)R_1} \left[\begin{array}{cc|c} 21 & 25 & 20 \\ 0 & 110/21 & -80/21 \end{array}\right]$$

$$\xrightarrow{R_2 \to (21/110)R_2} \left[\begin{array}{cc|c} 21 & 25 & 20 \\ 0 & 1 & -8/11 \end{array}\right] \xrightarrow{R_1 \to R_1 - 25R_2} \left[\begin{array}{cc|c} 21 & 0 & 420/11 \\ 0 & 1 & -8/11 \end{array}\right]$$

$$\xrightarrow{R_1 \to (1/21)R_1} \left[\begin{array}{cc|c} 1 & 0 & 20/11 \\ 0 & 1 & -8/11 \end{array}\right]$$

The least squares solution is therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20/11 \\ -8/11 \end{bmatrix}$$

- Consider the linear system:

$$\begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Attempting to directly solve this linear system yields:

$$\left[\begin{array}{cc|c} 2 & -2 & 2 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - (3/2)R_1]{R_2 \to R_2 - (1/2)R_1} \left[\begin{array}{cc|c} 2 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 4 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cc|c} 2 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{array}\right]$$

A pivot is in the last column so the system is therefore inconsistent. A least squares solution is now what is sought. Multiplying both sides of the system on the left by $A^T = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$ gives:

$$\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

Solving this linear system gives:

$$\left[\begin{array}{cc|c} 14 & 0 & 6 \\ 0 & 6 & -4 \end{array}\right] \xrightarrow{R_2 \to (1/6)R_2} \left[\begin{array}{cc|c} 14 & 0 & 6 \\ 0 & 1 & -2/3 \end{array}\right]$$

$$\xrightarrow{R_1 \to (1/14)R_1} \left[\begin{array}{cc|c} 1 & 0 & 3/7 \\ 0 & 1 & -2/3 \end{array}\right]$$

The least squares solution is therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -2/3 \end{bmatrix}$$

- Consider the linear system:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

Attempting to directly solve this linear system yields:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 2 & 1 & -2 & 0 \\ 1 & 1 & 0 & 9 \\ 1 & 1 & -1 & 3 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 0 & 1 & 0 & -12 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -3 \end{array}\right] \xrightarrow{\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

A pivot is in the last column so the system is therefore inconsistent. A least squares solution is now what is sought. Multiplying both sides of the system on the left by $A^T = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}$ gives:

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

Solving this linear system gives:

$$\left[\begin{array}{ccc|c} 7 & 4 & -6 & 18 \\ 4 & 3 & -3 & 12 \\ -6 & -3 & 6 & -9 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - (4/7)R_1 \\ R_3 \to R_3 + (6/7)R_1}} \left[\begin{array}{ccc|c} 7 & 4 & -6 & 18 \\ 0 & 5/7 & 3/7 & 12/7 \\ 0 & 3/7 & 6/7 & 45/7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (3/5)R_2} \left[\begin{array}{ccc|c} 7 & 4 & -6 & 18 \\ 0 & 5/7 & 3/7 & 12/7 \\ 0 & 0 & 3/5 & 27/5 \end{array}\right] \xrightarrow{R_3 \to (5/3)R_3} \left[\begin{array}{ccc|c} 7 & 4 & -6 & 18 \\ 0 & 5/7 & 3/7 & 12/7 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 - (3/7)R_3}} \left[\begin{array}{ccc|c} 7 & 4 & 0 & 72 \\ 0 & 5/7 & 0 & -15/7 \\ 0 & 0 & 1 & 9 \end{array}\right] \xrightarrow{R_2 \to (7/5)R_2} \left[\begin{array}{ccc|c} 7 & 4 & 0 & 72 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc|c} 7 & 0 & 0 & 84 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \to (1/7)R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

The least squares solution is therefore:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix}$$

*Examples from the problem set of chapter 6.4 of the textbook:
Anton, Howard; Rorres, Chris, *Elementary Linear Algebra 11th edition, Applications Version*, Wiley, 2014.