Addition vs Multiplication

Multiplication bears many similarities to addition:

- Addition and multiplication are both commutative.

$$\forall a, b \in \mathbb{R} : (a + b = b + a) \ \wedge \ (a \cdot b = b \cdot a)$$

  "for all choices of real numbers a and b, it is the case that $a + b = b + a$ and $a \cdot b = b \cdot a$"

- Addition and multiplication are both associative.

$$\forall a, b, c \in \mathbb{R} : ((a + b) + c = a + (b + c)) \ \wedge \ ((a \cdot b) \cdot c = a \cdot (b \cdot c))$$

  "for all choices of real numbers a, b, and c, it is the case that $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$"

- Adding 0 to a number does not change the number, and multiplying 1 onto a number does not change the number. 1 is the multiplication equivalent of 0.

$$\forall a \in \mathbb{R} : (a + 0 = a) \ \wedge \ (a \cdot 1 = a)$$

  "for all choices of real number a, it is the case that $a + 0 = a$ and $a \cdot 1 = a$"

- Every real number as a negative, and every nonzero number has a reciprocal. The negative of a number when added to the original number results in 0:

$$\forall a \in \mathbb{R} : (a + (-a) = 0)$$

  "for all choices of real number a, it is the case that $a + (-a) = 0$"

  The reciprocal of a nonzero number when multiplied by the original number results in 1:

$$\forall a \in \mathbb{R} : a \neq 0 \implies (a \cdot \frac{1}{a} = 1)$$

  "for all choices of real number a, if $a \neq 0$ then $a \cdot \frac{1}{a} = 1$"

- Moreover, subtraction is the addition of a negative:

$$\forall a, b \in \mathbb{R} : a - b = a + (-b)$$

  "for all choices of real numbers a and b, it is the case that $a - b = a + (-b)$"

  and division is the multiplication of a reciprocal:

$$\forall a, b \in \mathbb{R} : b \neq 0 \implies (a/b = a \cdot \frac{1}{b})$$

  "for all choices of real numbers a and b, if $b \neq 0$ then $a/b = a \cdot \frac{1}{b}$"

The difficulties of multiplication

Despite multiplication and addition being similar in many respects, multiplication is a much more compli-
cated operation to perform by hand than is addition. Before the widespread use of calculators, multiplying
together to numbers directly is prohibitively difficult.

Consider for example the two 4-digit numbers $a = 7430$ and $b = 1956$.

The addition of the two numbers simply involves adding the digits from least significant to most signif-
icant, propagating carries if necessary. The **number of steps** is proportional to the **sum** of the lengths
(number of digits) of a and b.

$$
\begin{array}{r}
7\ \ 4\ \ 3\ \ 0 \\
+\ \ 1\ \ 9\ \ 5\ \ 6 \\
\hline
0\ \ 9\ \ 3\ \ 8\ \ 6
\end{array}
$$

In contrast, the multiplication of the two numbers involves the computing of partial products for each
pair of digits, and the summation of the results. The **number of steps** is proportional to the **product** of
the lengths (number of digits) of a and b. This becomes prohibitively large as a and b increase in length.

$$
\begin{array}{r}
7\ \ 4\ \ 3\ \ 0 \\
\times\ \ 1\ \ 9\ \ 5\ \ 6 \\
\hline
4\ \ 4\ \ 5\ \ 8\ \ 0 \\
3\ \ 7\ \ 1\ \ 5\ \ 0\ \ 0 \\
6\ \ 6\ \ 8\ \ 7\ \ 0\ \ 0\ \ 0 \\
0\ \ 7\ \ 4\ \ 3\ \ 0\ \ 0\ \ 0\ \ 0 \\
\hline
1\ \ 4\ \ 5\ \ 3\ \ 3\ \ 0\ \ 8\ \ 0
\end{array}
$$

The logarithm

To address the challenges of multiplication, the similarities of multiplication to addition can be used to
assign to each number a "multiplicative essence". **The term "multiplicative essence" is used only in
this document, and is not recognized officially.** The multiplicative essence can be extracted through
use of the **logarithm**. Given an arbitrary **positive** number x, $\log(x)$ is the "multiplicative essence" of x.
When multiplying two positive real numbers x and y, the multiplicative essence of the product is the **sum**
of the multiplicative essences of the factors, the sum of two numbers being much easier to compute. Using
logarithms turns multiplication into addition:

- Multiplying two numbers can be performed by adding their logarithms. For any choice of positive
  numbers x and y,
$$\log(x \cdot y) = \log(x) + \log(y)$$

- 1 is the multiplication analog to 0, so
$$\log(1) = 0$$

- The reciprocal is the multiplication analog to the negative, so for any positive number x,
$$\log(1/x) = -\log(x)$$

- Division is the multiplication analog to subtraction. For any choice of positive numbers x and y,
$$\log(x/y) = \log(x) - \log(y)$$

- Multiplying a number x by itself n times is defined as x^n. The logarithm of x^n is the logarithm of x **added** to itself n times. For any positive number x, and natural number n,

$$\log(x^n) = \log(\underbrace{x \cdot x \cdot \ldots \cdot x}_{n}) = \underbrace{\log(x) + \log(x) + \ldots + \log(x)}_{n} = n \cdot \log(x)$$

If r is an arbitrary real number, then

$$\log(x^r) = r \cdot \log(x)$$

also.

- The nth root of a positive number x, denoted by $\sqrt[n]{x}$, is a number that when multiplied by itself n times yields x. The logarithm of $\sqrt[n]{x}$ when added to itself n times yields the logarithm of x, so $n \cdot \log(\sqrt[n]{x}) = \log(x)$ which then yields:

$$\log(\sqrt[n]{x}) = \frac{1}{n}\log(x)$$

Even with the above properties, the logarithm is not yet fully defined. The choice that remains is to choose a "base" for the logarithm. The base b is the number whose multiplicative essence is 1 (the multiplicative essence of 1 is 0): $\log(b) = 1$. The base b should always be a positive real number greater than 1. The notation $\log_b(x)$ is the logarithm of x using base b. $\log_b(x)$ is the multiplicative essence of x where the multiplicative essence of b is calibrated to 1. It goes by definition that $\log_b(b) = 1$. Moreover for any real number r the multiplicative essence of b^r is r:

$$\log_b(b^r) = r \cdot \log_b(b) = r \cdot 1 = r$$

Given an arbitrary multiplicative essence r, the number that has a multiplicative essence of r is b^r. b^r is sometimes referred to as the **antilogarithm** of r. If function $f(X)$ is defined as $f(X) = \log_b(X)$, then the inverse of f is $f^{-1}(Y) = b^Y$.

The graph on the right illustrates how the base b logarithm of x changes with x. The logarithm only exists for positive values of x. The logarithm increases with x, and every real number is in the range of possible logarithms.

- When $x = 1$, $\log_b(1) = 0$, since 1 is the multiplication analog to 0.

- When x decreases from 1 to 0, $\log_b(x)$ decreases from 0 to $-\infty$. When $0 < x < 1$, multiplying a number by x decreases the magnitude of said number, so a negative logarithm makes sense.

- When x increases from 1 to b, $\log_b(x)$ increases from 0 to 1. When $1 < x < b$, multiplying a number by x increases its magnitude, but not as much as a multiplication by b or higher. Hence a positive logarithm that is less than 1 makes sense.
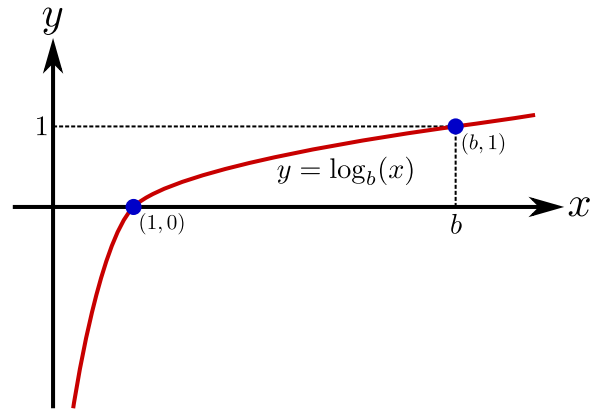
- When $x = b$, $\log_b(b) = 1$ by definition of the base.

- When x increases past b, $\log_b(x)$ increases past 1 and very slowly increases to $+\infty$. When $x > b$, multiplying a number by x increases its magnitude more than a multiplication by b, so a logarithm greater than 1 makes sense.

Common bases include 10, 2, and "e". The number "e" is Euler's constant and will be covered later.

Base 2 is often used when measuring information as a number of "bits" (binary digits).

Base 10 is what was used to quicken the process of multiplication before the use of calculators, as will be covered below.

Base e has a very important property that is invaluable to algebra and Calculus, and will be covered later.

To help further illustrate logarithm values, the base 10 logarithms for several numbers are shown below:

x	0.1	0.2	0.3	0.4	0.5
$\log_{10}(x)$	-1	-0.698970	-0.522879	-0.397940	-0.301030

x	0.6	0.7	0.8	0.9	1.0
$\log_{10}(x)$	-0.221849	-0.154902	-0.0969100	-0.0457575	0

x	2.0	3.0	4.0	5.0	6.0
$\log_{10}(x)$	0.301030	0.477121	0.602060	0.698970	0.778151

x	7.0	8.0	9.0	10.0	20.0
$\log_{10}(x)$	0.845098	0.903090	0.954243	1	1.30103

Before there were calculators...

As noted in the previous sections, multiplying numbers by hand before the use of calculators was prohibitively difficult. Fortunately, if one is willing to allow some approximations, logarithms can be used to convert the difficult task of multiplication into the much simpler task of addition. Base 10 logarithms are used. Base 10 logarithms are referred to as "common logarithms" and are the default for the "`log`" key found on most calculators. Why are base 10 logarithms used to simplify multiplication and not some other base? The answer is that finding the base 10 logarithm of a number in **scientific notation** is easy when paired with a "log table".

A "log table" is a lengthy book that lists the logarithms for a vast range of numbers. A fragment of a such a log table is displayed on the right. For the purpose of this discussion, we will assume that the log table will allow the quick determination of the logarithm $\log_{10}(x)$ of numbers from the range $[1, 10)$, and the quick determination of the antilogarithm 10^x of numbers from the range $[0, 1)$.

No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857332	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

Image credit: Wikimedia Commons, URL: <https://commons.wikimedia.org/wiki/File:Tables_of_six-figure_logarithms_(IA_cu31924031296795).pdf >

- To use the log table to compute the logarithm of an arbitrary positive number x, the number will be expressed in scientific notation:

$$x = +a \times 10^n$$

where n is an integer and $a \in [1, 10)$ is the "mantissa". The base 10 logarithm of x is

$$\log_{10}(x) = \log_{10}(+a \times 10^n) = \log_{10}(a) + \log_{10}(10^n) = n + \log_{10}(a)$$

where $\log_{10}(a)$ is computed from the log table.

- To use the log table to compute the antilogarithm of an arbitrary number y, the number will be expressed in the form:

$$y = n + r$$

where n is an integer and $r \in [0, 1)$. The base 10 antilogarithm is

$$10^y = 10^{n+r} = +10^r \times 10^n$$

where 10^r is computed from the log table.

This is how log tables allow the quick calculations of base 10 logarithms and antilogarithms.

Example 1:
Consider the process of multiplying the numbers $x = 5.67128 \times 10^8$ and $y = 1.04690 \times 10^3$.
The logarithm (multiplication essence) of x is:

$$\log_{10}(x) = \log_{10}(5.67128 \times 10^8) = 8 + \log_{10}(5.67128) \approx 8 + 0.753681 = 8.753681$$

The logarithm (multiplication essence) of y is:

$$\log_{10}(y) = \log_{10}(1.04690 \times 10^3) = 3 + \log_{10}(1.04690) \approx 3 + 0.019905 = 3.019905$$

With the logarithms of x and y now known (to an approximation), the logarithm of the product xy is the sum of the logarithms:

$$\log_{10}(xy) = \log_{10}(x) + \log_{10}(y) \approx 8.753681 + 3.019905 = 11.773586$$

Now is time to compute the "antilogarithm" and extract the final product xy. The logarithm of the product is,

$$11.773586 = 11 + 0.773586$$

so

$$xy \approx 10^{11+0.773586} = 10^{0.773586} \times 10^{11} \approx 5.93726 \times 10^{11}$$

Therefore

$$(5.67128 \times 10^8)(1.04690 \times 10^3) \approx 5.93726 \times 10^{11}$$

Example 2:
As another example, let $x = 2.45207 \times 10^2$ and $y = 6.89641 \times 10^4$, and then compute the quotient x/y.
The logarithm (multiplication essence) of x is:

$$\log_{10}(x) = \log_{10}(2.45207 \times 10^2) = 2 + \log_{10}(2.45207) \approx 2 + 0.389533 = 2.389533$$

The logarithm (multiplication essence) of y is:

$$\log_{10}(y) = \log_{10}(6.89641 \times 10^4) = 4 + \log_{10}(6.89641) \approx 4 + 0.838623 = 4.838623$$

With the logarithms of x and y now known (to an approximation), the logarithm of the quotient x/y is the difference of the logarithms:

$$\log_{10}(x/y) = \log_{10}(x) - \log_{10}(y) \approx 2.389533 - 4.838623 = -2.449090$$

Now is time to compute the "antilogarithm" and extract the final quotient x/y. The logarithm of the product is,

$$-2.449090 = -3 + 0.550910$$

so

$$\frac{x}{y} \approx 10^{-3+0.550910} = 10^{0.550910} \times 10^{-3} \approx 3.55558 \times 10^{-3}$$

Therefore

$$\frac{2.45207 \times 10^2}{6.89641 \times 10^4} \approx 3.55558 \times 10^{-3}$$

Example 3:
As a last example, let $x = 2.68194 \times 10^7$ and $y = -4.57891$, and then compute x^y.
The logarithm (multiplication essence) of x is:

$$\log_{10}(x) = \log_{10}(2.68194 \times 10^7) = 7 + \log_{10}(2.68194) \approx 7 + 0.428449 = 7.428449$$

$\log_{10}(x)$ is now to be multiplied by y. Once again there is a tough multiplication problem. The logarithm of $\log_{10}(x)$ is:

$$\log_{10}(\log_{10}(x)) \approx \log_{10}(7.428449) \approx 0.870898$$

y is negative, so the logarithm of $|y|$ is computed instead. The sign will be reincorporated later.

$$\log_{10}(|y|) = \log_{10}(4.57891) \approx 0.660762$$

The logarithm of the product $|y| \log_{10}(x)$ is:

$$\log_{10}(|y| \log_{10}(x)) = \log_{10}(|y|) + \log_{10}(\log_{10}(x)) \approx 0.660762 + 0.870898 = 1.531660$$
$$= 1 + 0.531660$$

Computing the antilogarithm gives the product:

$$|y| \log_{10}(x) \approx 10^{1+0.531660} = 10^{0.531660} \times 10^1 \approx 3.401418 \times 10^1$$

so the logarithm of x^y is:

$$y \log_{10}(x) \approx -3.401418 \times 10^1 = -34.01418 = -35 + 0.98582$$

therefore:

$$x^y \approx 10^{-35+0.98582} = 10^{0.98582} \times 10^{-35} \approx 9.67877 \times 10^{-35}$$
$$(2.68194 \times 10^7)^{-4.57891} \approx 9.67877 \times 10^{-35}$$

Natural logarithms

The natural logarithm, which is denoted by the symbol ln instead of log, is a logarithm where the following approximation holds:
For a number ϵ that is very close to 0, the **relative error** in the approximation

$$\ln(1 + \epsilon) \approx \epsilon$$

is very close to 0.
As a reminder, the relative error is an error measure where the magnitude of the difference is divided by either the approximation or true value to get a number which measures the fraction (commonly expressed

as a percentage) of the approximation that is actually error, or by what fraction of the true value the approximation misses its mark.

$$\textbf{relative\_error} = \frac{|\textbf{approximation} - \textbf{true\_value}|}{|\textbf{approximation}|} \quad \text{or} \quad \textbf{relative\_error} = \frac{|\textbf{approximation} - \textbf{true\_value}|}{|\textbf{true\_value}|}$$

Why is the approximation $\ln(1 + \epsilon) \approx \epsilon$ important? This approximation enables the approximation of the logarithm for values that are close to 1, 1 being the multiplication equivalent of 0.

The base that the natural logarithm uses is denoted by "e", referred to as "Euler's constant". To home-in on the actual value of e, let N denote a very large positive number. $\frac{1}{N}$ is close to 0, so

$$\ln\left(1 + \frac{1}{N}\right) \approx \frac{1}{N} \implies N \cdot \ln\left(1 + \frac{1}{N}\right) \approx N \cdot \frac{1}{N} \implies \ln\left(\left(1 + \frac{1}{N}\right)^N\right) \approx 1$$

The base e of the natural logarithm is approximately $\left(1 + \frac{1}{N}\right)^N$, the approximation becoming more accurate as N increases. The table below shows the value of $\left(1 + \frac{1}{N}\right)^N$ converging on e as N increases:

N	1	2	3	4	5	6	7	20	$+\infty$
$\left(1 + \frac{1}{N}\right)^N$	2	2.25	2.37037	2.44141	2.48832	2.52163	2.54650	2.65330	$e = 2.71828...$

Given a series of small numbers $\epsilon_1, \epsilon_2, ..., \epsilon_N$, the product

$$P = (1 + \epsilon_1)(1 + \epsilon_2)...(1 + \epsilon_N)$$

can be simplified through the use of the natural logarithm. For each $i = 1, 2, ..., N$, the natural logarithm of the ith factor is $\ln(1 + \epsilon_i) \approx \epsilon_i$. The logarithm of the entire product is:

$$\ln(P) = \ln((1 + \epsilon_1)(1 + \epsilon_2)...(1 + \epsilon_N)) = \ln(1 + \epsilon_1) + \ln(1 + \epsilon_2) + \cdots + \ln(1 + \epsilon_N)$$
$$\approx \epsilon_1 + \epsilon_2 + \cdots + \epsilon_N$$

Therefore the product is:
$$(1 + \epsilon_1)(1 + \epsilon_2)...(1 + \epsilon_N) = P \approx e^{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_N}$$

Example application

As an example application, consider a machine that is known, on average, to fail λ times over a single unit of time. λ, preferably less than 1, is referred to as the failure rate. Given an arbitrary time t, what is the probability of avoiding a failure over a time interval of t?

Let N be a very large natural number and divide the time interval of t into N tiny fragments of $\Delta t = \frac{t}{N}$. The average number of failures in the tiny time interval Δt is $\lambda \Delta t$. Since Δt is tiny, the probability of scenarios of 2 or more failures in the time interval of Δt is negligible compared to the probability of a single failure. The average number of failures $\lambda \Delta t$ becomes the probability of any failure occurring in the time interval Δt. The probability of no failure is $1 - \lambda \Delta t$. The probability of no failure over N consecutive intervals of Δt is $(1 - \lambda \Delta t)^N$. N consecutive intervals of Δt is the desired time interval t. The probability of no failure over a time interval of t is

$$P = (1 - \lambda \Delta t)^N$$

This is the product of N quantities that are close to 1. To multiply these quantities that are close to 1, the natural logarithm of each factor of $1 - \lambda \Delta t$ will be computed and then summed. Since $-\lambda \Delta t$ is close to 0,

$$\ln(1 - \lambda \Delta t) \approx -\lambda \Delta t$$

The natural logarithm of the product is:

$$\ln(P) = \ln((1 - \lambda \Delta t)^N) = N \ln(1 - \lambda \Delta t) \approx N(-\lambda \Delta t) = -\lambda \cdot N \Delta t = -\lambda \cdot t$$

With the natural logarithm of the probability of no failure being $-\lambda \cdot t$, the probability of no failure over a time interval of t is:

$$P = e^{-\lambda \cdot t}$$

As an example, if $\lambda = 0.000001$ failures per second, the probability of there being no failure for an entire year, which is $\sim 3.15576 \times 10^7$s, is

$$P = e^{-0.000001 \times (3.15576 \times 10^7)} \approx 1.97110 \times 10^{-14}$$

With such low odds, a failure is basically certain.