Line Intersections

Given vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{proj_u}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$$

$$|\mathbf{proj_u}(\mathbf{v})| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|}$$

$$\mathbf{perp_u}(\mathbf{v}) = \mathbf{v} - \mathbf{proj_u}(\mathbf{v})$$

$$|\mathbf{perp_u}(\mathbf{v})| = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|}$$

$\mathbf{u}||\mathbf{v}$ denotes that $\mathbf{u}$ and $\mathbf{v}$ are **parallel**.
$\mathbf{u} \perp \mathbf{v}$ denotes that $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular/orthogonal**.

Classifying the interaction between two lines

Given two straight lines L_1 and L_2, there are 4 possible relationships between these lines:

- L_1 and L_2 are **equivalent**.
- L_1 and L_2 are **parallel but not equal**.
- L_1 and L_2 **intersect** at a single point.
- L_1 and L_2 are **skew**.

The relationship between L_1 and L_2 can be determined via the following algorithm: L_1 will be described by the parametric line:

$$\mathbf{r}(t) = \mathbf{r}_{0,1} + t\mathbf{v}_1$$

and L_2 will be described by the parametric line:

$$\mathbf{r}(t) = \mathbf{r}_{0,2} + t\mathbf{v}_2$$

if $\mathbf{v}_1 || \mathbf{v}_2$ **then**
Let $d = \left|\mathbf{perp}_{\mathbf{v}_1}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})\right| = \frac{|\mathbf{v}_1 \times (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{v}_1|}$
if $d = 0$ **then**
L_1 and L_2 are **equivalent**
else
L_1 and L_2 are **parallel but not equal**, and have a separation of d.
end if
else
Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$
Let $d = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})| = \frac{|\mathbf{n} \cdot (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{n}|}$
if $d = 0$ **then**
L_1 and L_2 **intersect** at a single point
else
L_1 and L_2 are **skew**, and have a closest distance of d.
end if
end if

For the following pairs of lines, determine if these lines are **equivalent**; are **parallel but not equal**; **intersect** at a single point; or are **skew**. For parallel lines, give the separation, and for skew lines, give the closest distance. **Show all of your work.**

Line pair 1:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & 4 - t \\ y(t) = & -4 + 2t \\ z(t) = & -2 + 5t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & -1 - 2t \\ y(t) = & 6 + 4t \\ z(t) = & 23 + 10t \end{cases}$$

$$\mathbf{r}_{0,1} = \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix};\ \text{and } \mathbf{r}_{0,2} = \begin{bmatrix} -1 \\ 6 \\ 23 \end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}$$

$\mathbf{v}_2 = 2\mathbf{v}_1$ so $\mathbf{v}_1 || \mathbf{v}_2$, and hence L_1 and L_2 are parallel.

$$d = \left|\mathbf{perp}_{\mathbf{v}_1}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})\right| = \frac{|\mathbf{v}_1 \times (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{v}_1|} = \frac{1}{\sqrt{1+4+25}} \left| \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \times \begin{bmatrix} -5 \\ 10 \\ 25 \end{bmatrix} \right|$$

$$= \frac{1}{\sqrt{30}} \left| \begin{bmatrix} (2)(25) - (5)(10) \\ (5)(-5) - (-1)(25) \\ (-1)(10) - (2)(-5) \end{bmatrix} \right| = \frac{1}{\sqrt{30}} \left| \begin{bmatrix} 50 - 50 \\ -25 + 25 \\ -10 + 10 \end{bmatrix} \right| = \frac{1}{\sqrt{30}} \left| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right| = 0$$

The perpendicular distance between L_1 and L_2 is $d = 0$, so L_1 and L_2 are **equivalent**.

Line pair 2:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & -5 + t \\ y(t) = & -1 + 2t \\ z(t) = & 2 + t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & -8 + 2t \\ y(t) = & -1 + t \\ z(t) = & -1 + 2t \end{cases}$$

$$\mathbf{r}_{0,1} = \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};\ \text{and } \mathbf{r}_{0,2} = \begin{bmatrix} -8 \\ -1 \\ -1 \end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$\mathbf{v}_1 \not\parallel \mathbf{v}_2$, and hence L_1 and L_2 are not parallel.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix}1\\2\\1\end{bmatrix} \times \begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}(2)(2)-(1)(1)\\(1)(2)-(1)(2)\\(1)(1)-(2)(2)\end{bmatrix} = \begin{bmatrix}4-1\\2-2\\1-4\end{bmatrix} = \begin{bmatrix}3\\0\\-3\end{bmatrix}$$

$$d = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})| = \frac{|\mathbf{n} \cdot (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{n}|} = \frac{1}{\sqrt{9+0+9}} \left| \begin{bmatrix}3\\0\\-3\end{bmatrix} \cdot \begin{bmatrix}-3\\0\\-3\end{bmatrix} \right|$$
$$= \frac{|-9+0+9|}{\sqrt{18}} = 0$$

The minimum distance between L_1 and L_2 is $d = 0$, so L_1 and L_2 **intersect** at a single point.

Line pair 3:

L_1 is parameterized by $\begin{cases} x(t) = & -3+7t \\ y(t) = & 1+2t \\ z(t) = & 2+5t \end{cases}$ and L_2 is parameterized by $\begin{cases} x(t) = & 4-21t \\ y(t) = & 2-6t \\ z(t) = & 7-15t \end{cases}$

$\mathbf{r}_{0,1} = \begin{bmatrix}-3\\1\\2\end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix}7\\2\\5\end{bmatrix}$; and $\mathbf{r}_{0,2} = \begin{bmatrix}4\\2\\7\end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix}-21\\-6\\-15\end{bmatrix}$
$\mathbf{v}_2 = -3\mathbf{v}_1$ so $\mathbf{v}_1 || \mathbf{v}_2$, and hence L_1 and L_2 are parallel.

$$d = \left|\mathbf{perp}_{\mathbf{v}_1}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})\right| = \frac{|\mathbf{v}_1 \times (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{v}_1|} = \frac{1}{\sqrt{49+4+25}} \left| \begin{bmatrix}7\\2\\5\end{bmatrix} \times \begin{bmatrix}7\\1\\5\end{bmatrix} \right|$$
$$= \frac{1}{\sqrt{78}} \left| \begin{bmatrix}(2)(5)-(5)(1)\\(5)(7)-(7)(5)\\(7)(1)-(2)(7)\end{bmatrix} \right| = \frac{1}{\sqrt{78}} \left| \begin{bmatrix}10-5\\35-35\\7-14\end{bmatrix} \right| = \frac{1}{\sqrt{78}} \left| \begin{bmatrix}5\\0\\-7\end{bmatrix} \right| = \frac{\sqrt{25+0+49}}{\sqrt{78}} = \frac{\sqrt{74}}{\sqrt{78}}$$

The perpendicular distance between L_1 and L_2 is $d = \frac{\sqrt{74}}{\sqrt{78}} > 0$, so L_1 and L_2 are **parallel but not equal**.

Line pair 4:

L_1 is parameterized by $\begin{cases} x(t) = & -1+t \\ y(t) = & 2t \\ z(t) = & 1+3t \end{cases}$ and L_2 is parameterized by $\begin{cases} x(t) = & -1 \\ y(t) = & 1+2t \\ z(t) = & 1+3t \end{cases}$

$\mathbf{r}_{0,1} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$; and $\mathbf{r}_{0,2} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix}0\\2\\3\end{bmatrix}$
$\mathbf{v}_1 \not\parallel \mathbf{v}_2$, and hence L_1 and L_2 are not parallel.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}0\\2\\3\end{bmatrix} = \begin{bmatrix}(2)(3)-(3)(2)\\(3)(0)-(1)(3)\\(1)(2)-(2)(0)\end{bmatrix} = \begin{bmatrix}6-6\\0-3\\2-0\end{bmatrix} = \begin{bmatrix}0\\-3\\2\end{bmatrix}$$

$$
\begin{aligned}
d = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})| = \frac{|\mathbf{n} \cdot (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{n}|} &= \frac{1}{\sqrt{0+9+4}} \left| \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right| \\
&= \frac{|0-3+0|}{\sqrt{13}} = \frac{3}{\sqrt{13}}
\end{aligned}
$$

The minimum distance between L_1 and L_2 is $d = \frac{3}{\sqrt{13}} > 0$, so L_1 and L_2 are **skew**.