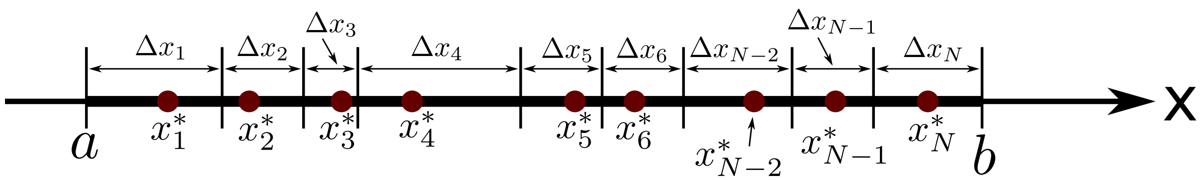
Single variable integrals

Given a single variable function $f(x)$, the definite integral of $f(x)$ over the interval $[a, b]$ is defined by the following Riemann sum:

Let N denote a large integer. Partition the interval $[a, b]$ into a series of N intervals as depicted below:



For each $i = 1, 2, ..., N$, the i^{th} interval has a length of Δx_i and contains the "representative" point x_i^*. The definite integral of $f(x)$ over the interval $[a, b]$ is the limit:

$$\int_{x=a}^{b} f(x)dx = \lim_{N\to+\infty} \sum_{i=1}^{N} f(x_i^*)\Delta x_i$$

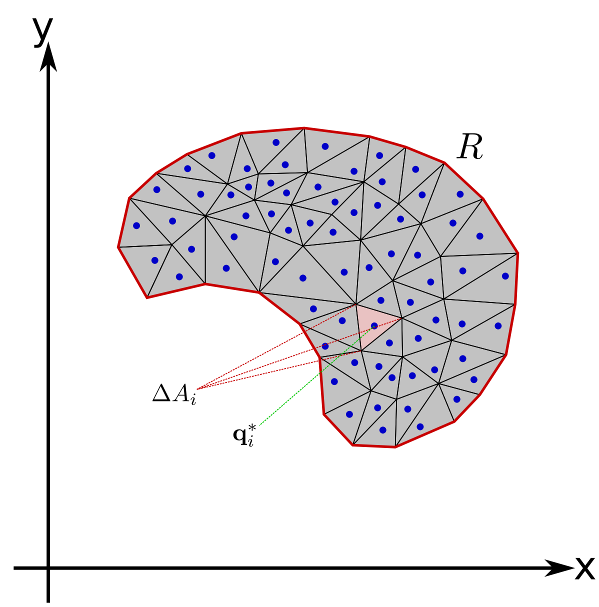
The concept will now be generalized in order to integrate multivariable functions.

Double integrals

Given a function $f(\mathbf{q})$ whose domain is a set of points in 2D space, the **double integral** of $f(\mathbf{q})$ over the 2D region $R \subseteq \mathbb{R}^2$ is defined by the following Riemann sum:

Let N denote a large integer. Partition the region R into a set of N tiny regions as depicted to the right. For each $i = 1, 2, ..., N$, the i^{th} section has an area of ΔA_i and contains the "representative" point $\mathbf{q}_i^*$. The double integral of $f(\mathbf{q})$ over the region R is the limit:

$$\iint_R f(\mathbf{q})dA = \lim_{N\to+\infty} \sum_{i=1}^{N} f(\mathbf{q}_i^*)\Delta A_i$$



If the integrand $f(\mathbf{q})$ is 1, then the double integral evaluates to:

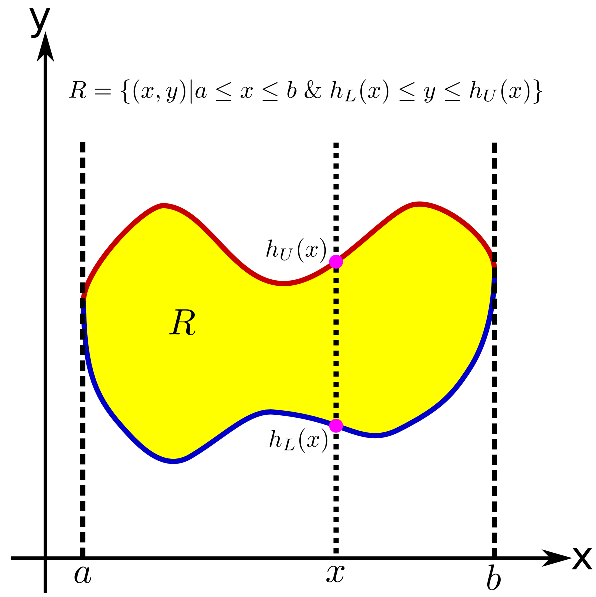
$$\iint_R dA = \lim_{N\to+\infty} \sum_{i=1}^{N} \Delta A_i = \text{area of } R$$

The area of a 2D region is the double integral of 1 over that region.

Double integrals using Cartesian coordinates

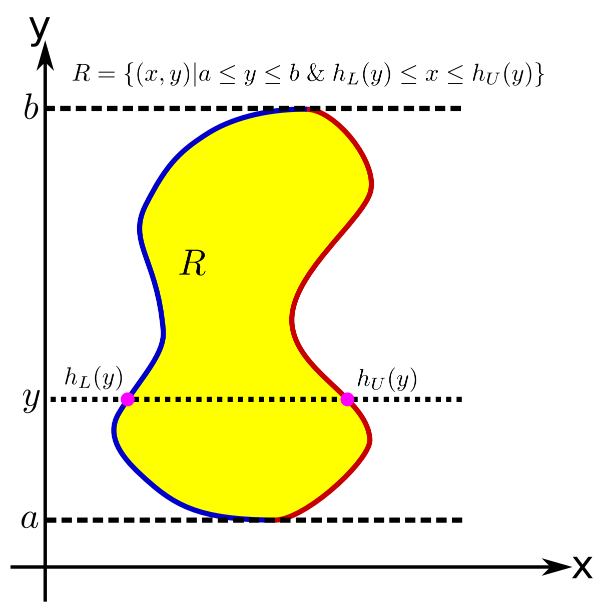
2D regions can be quantified by first establishing bounds on the x-coordinate, and then establishing bounds on the y-coordinate that are functions of x. This is referred to as a "Type I" region. In the image on the right, the bounds on x are a and b: $a \leq x \leq b$. The bounds on y are functions of x: $h_L(x) \leq y \leq h_U(x)$. The region itself is:

$$R = \{(x,y) | a \leq x \leq b \ \& \ h_L(x) \leq y \leq h_U(x)\}$$



2D regions can also be quantified by first establishing bounds on the y-coordinate, and then establishing bounds on the x-coordinate that are functions of y. This is referred to as a "Type II" region, which essentially a Type I region with the roles of x and y reversed. In the image on the right, the bounds on y are a and b: $a \leq y \leq b$. The bounds on x are functions of y: $h_L(y) \leq x \leq h_U(y)$. The region itself is:

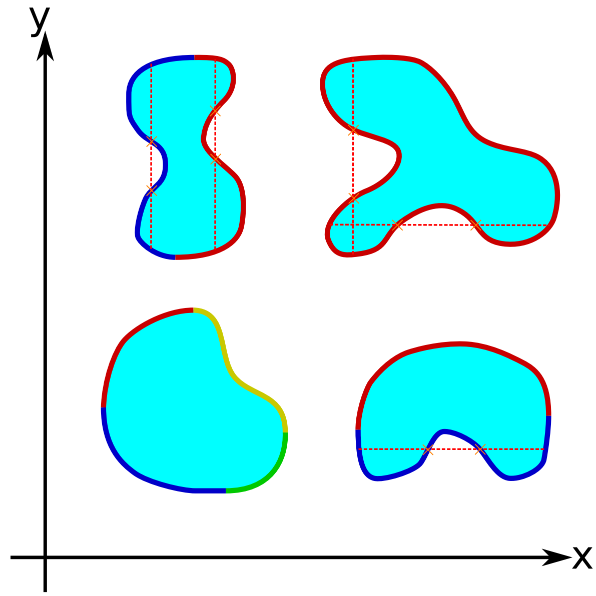
$$R = \{(x,y) | a \leq y \leq b \ \& \ h_L(y) \leq x \leq h_U(y)\}$$



Not every region is a type I or a type II region.

- To be a type I region, the range of x values must form a continuous interval $[a, b]$ with no gaps, and then fixing the value of x, the range of y values must form a continuous interval $[h_L(x), h_U(x)]$ with no gaps.
- To be a type II region, the range of y values must form a continuous interval $[a, b]$ with no gaps, and then fixing the value of y, the range of x values must form a continuous interval $[h_L(y), h_U(y)]$ with no gaps.

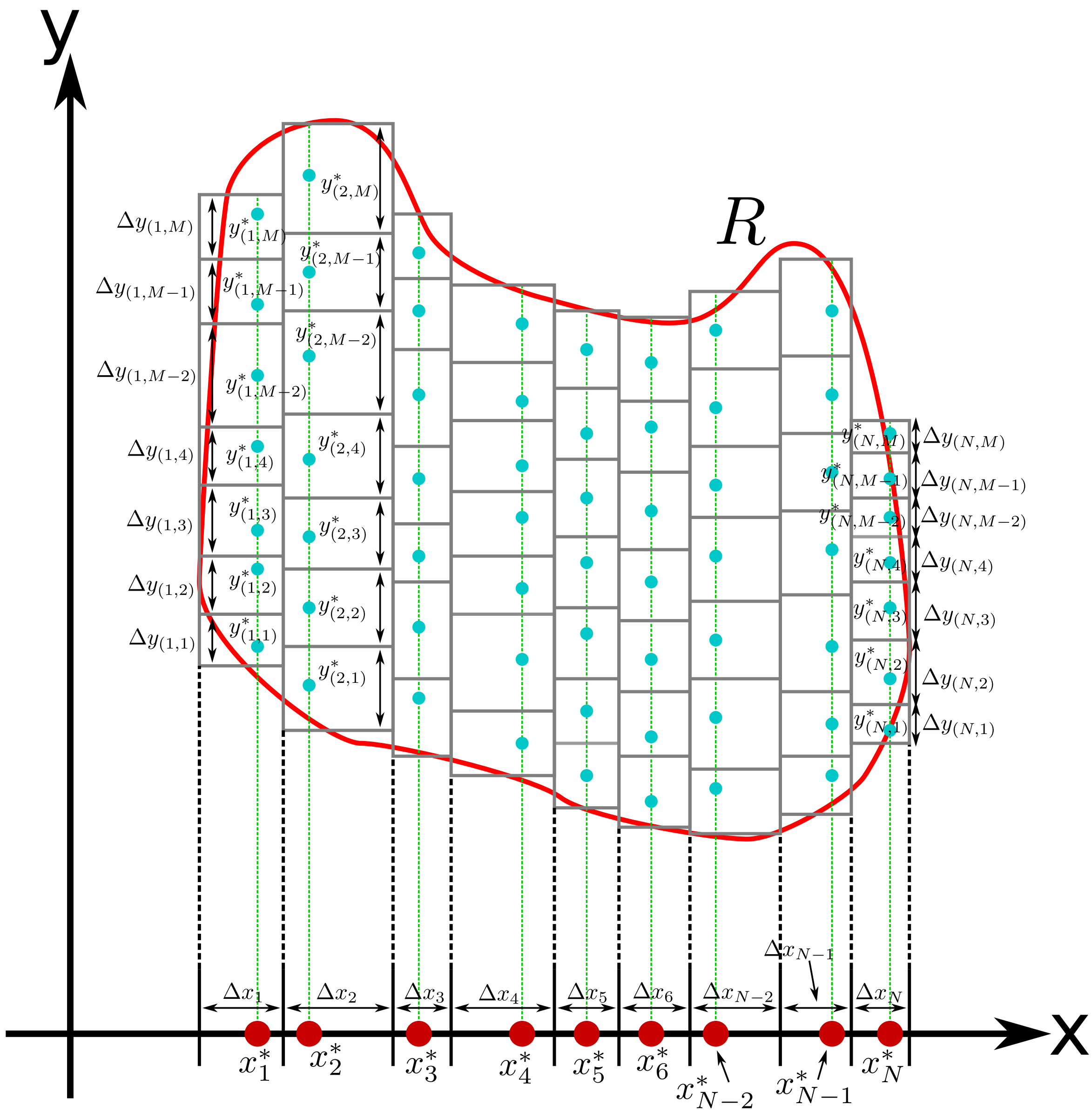
In the image to the right, the lower left region is both a type I and a type II region. The lower right region is a type I but not a type II region. The upper left region is a type II but not a type I region. The upper right region is neither a type I or type II region.



The double integral $\iint_R f(x,y)dA$ over the type I region:

$$R = \{(x,y)|a \leq x \leq b \;\&\; h_L(x) \leq y \leq h_U(x)\}$$

will now be evaluated using a series of two single variable integrals. To establish the single integral expression for $\iint_R f(x,y)dA$, region R will be decomposed into a series of N vertical slivers where N is a large number, as depicted in the image below. For each $i = 1, 2, ..., N$, the width of the i^{th} sliver is Δx_i, and x_i^* is a "representative" x value from the i^{th} sliver. Now for each vertical sliver, further partition the sliver into a series of M rectangles where M is a large number, as depicted in the image below. For each $i = 1, 2, ..., N$ and for each $j = 1, 2, ..., M$, the height of the j^{th} rectangle in the i^{th} sliver is $\Delta y_{(i,j)}$, and $y^*_{(i,j)}$ is a "representative" y value from the j^{th} rectangle in the i^{th} sliver. The area of the j^{th} rectangle in the i^{th} sliver is $\Delta A_{(i,j)} = \Delta x_i \Delta y_{(i,j)}$.



Evaluating the Riemann sum gives:

$$\iint_R f(x,y)dA = \lim_{N\to+\infty}\lim_{M\to+\infty}\sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i^*, y^*_{(i,j)})\Delta A_{(i,j)} = \lim_{N\to+\infty}\lim_{M\to+\infty}\sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i^*, y^*_{(i,j)})(\Delta x_i \Delta y_{(i,j)})$$

$$= \lim_{N\to+\infty}\sum_{i=1}^{N}\left(\lim_{M\to+\infty}\sum_{j=1}^{M} f(x_i^*, y^*_{(i,j)})\Delta y_{(i,j)}\right)\Delta x_i$$

$$= \lim_{N\to+\infty}\sum_{i=1}^{N}\left(\int_{y=h_L(x_i^*)}^{h_U(x_i^*)} f(x_i^*, y)dy\right)\Delta x_i = \int_{x=a}^{b}\left(\int_{y=h_L(x)}^{h_U(x)} f(x,y)dy\right)dx$$

Therefore:

$$\iint_R f(x,y)dA = \int_{x=a}^{b} \left(\int_{y=h_L(x)}^{h_U(x)} f(x,y)dy \right) dx$$

This expression involving the repeated single variable integrals is often referred to as an "**iterated integral**" or a "**nested integral**".

For type II regions where the roles of x and y are reversed, if:

$$R = \{(x,y) | a \leq y \leq b \;\&\; h_L(y) \leq x \leq h_U(y)\}$$

then

$$\iint_R f(x,y)dA = \int_{y=a}^{b} \left(\int_{x=h_L(y)}^{h_U(y)} f(x,y)dx \right) dy$$

Reversing the order of integration

Double integrals can be used to swap the order of integration for a nested integral. Let R have the type I and type II characterizations:

$$R = \{(x,y) | a \leq x \leq b \;\&\; h_L(x) \leq y \leq h_U(x)\} = \{(x,y) | c \leq y \leq d \;\&\; g_L(y) \leq x \leq g_U(y)\}$$

and let $f(x,y)$ be an arbitrary integrand. The following nested integrals are now equivalent:

$$\int_{x=a}^{b} \left(\int_{y=h_L(x)}^{h_U(x)} f(x,y)dy \right) dx = \iint_R f(x,y)dA = \int_{y=c}^{d} \left(\int_{x=g_L(y)}^{g_U(y)} f(x,y)dx \right) dy$$

Example 1:

Consider the triangular region R to the right. This region can be characterized as both a type I and a type II region.
For a type I characterization, the bounds on x are 0 and 3. The equation of the line that forms the hypotenuse of the triangle is:

$$y - 4 = \frac{0-4}{3-0}(x-0) \iff y - 4 = -\frac{4}{3}x$$
$$\iff y = 4 - \frac{4}{3}x$$

The bounds on y as functions of x are 0 and $4 - \frac{4}{3}x$. Therefore:

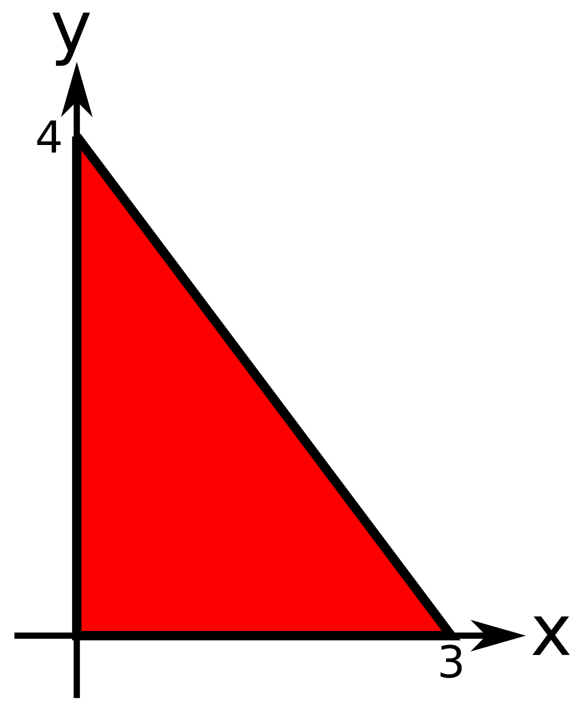
$$R = \left\{(x,y)\middle|0 \leq x \leq 3 \;\&\; 0 \leq y \leq 4 - \frac{4}{3}x\right\}$$

For a type II characterization, the bounds on y are 0 and 4. The equation of the line that forms the hypotenuse of the triangle can be rearranged to get:

$$y = 4 - \frac{4}{3}x \iff -\frac{4}{3}x = y - 4 \iff x = 3 - \frac{3}{4}y$$

The bounds on x as functions of y are 0 and $3 - \frac{3}{4}y$. Therefore:

$$R = \left\{(x,y)\middle|0 \leq y \leq 4 \;\&\; 0 \leq x \leq 3 - \frac{3}{4}y\right\}$$



Now consider the arbitrary integrand $f(x,y)$. The double integral $\iint_R f(x,y)dA$ can be computed by using either the type I characterization:

$$\iint_R f(x,y)dA = \int_{x=0}^{3}\left(\int_{y=0}^{4-\frac{4}{3}x} f(x,y)dy\right)dx$$

or via the type II characterization:

$$\iint_R f(x,y)dA = \int_{y=0}^{4}\left(\int_{x=0}^{3-\frac{3}{4}y} f(x,y)dx\right)dy$$

If the integrals in the nested integral need to be swapped, then:

$$\int_{x=0}^{3}\left(\int_{y=0}^{4-\frac{4}{3}x} f(x,y)dy\right)dx = \iint_R f(x,y)dA = \int_{y=0}^{4}\left(\int_{x=0}^{3-\frac{3}{4}y} f(x,y)dx\right)dy$$

Consider the specific integrand $f(x,y) = x$. The double integral $\iint_R x dA$ can now be evaluated via:

$$\iint_R x dA = \int_{x=0}^{3} \left(\int_{y=0}^{4-\frac{4}{3}x} x dy \right) dx = \int_{x=0}^{3} \left(xy \Big|_{y=0}^{4-\frac{4}{3}x} \right) dx = \int_{x=0}^{3} \left(x \left(4 - \frac{4}{3}x \right) - 0 \right) dx$$
$$= \int_{x=0}^{3} \left(4x - \frac{4}{3}x^2 \right) dx = \left(2x^2 - \frac{4}{9}x^3 \right) \Big|_{x=0}^{3} = (18 - 12) - 0 = 6$$

or via:

$$\iint_R x dA = \int_{y=0}^{4} \left(\int_{x=0}^{3-\frac{3}{4}y} x \cdot dx \right) dy = \int_{y=0}^{4} \left(\frac{1}{2}x^2 \Big|_{x=0}^{3-\frac{3}{4}y} \right) dy = \int_{y=0}^{4} \left(\frac{1}{2} \left(3 - \frac{3}{4}y \right)^2 - 0 \right) dy$$
$$= \int_{y=0}^{4} \left(\frac{9}{2} - \frac{9}{4}y + \frac{9}{32}y^2 \right) dy = \left(\frac{9}{2}y - \frac{9}{8}y^2 + \frac{3}{32}y^3 \right) \Big|_{y=0}^{4} = (18 - 18 + 6) - 0 = 6$$

Note that the final answer is the same either way.
Now consider the nested integral:

$$\int_{y=0}^{4} \left(\int_{x=0}^{3-\frac{3}{4}y} e^{4x-\frac{2}{3}x^2} dx \right) dy$$

This nested integral is difficult to directly evaluate. By reversing the order of the variables, the solution becomes clear:

$$\int_{y=0}^{4} \left(\int_{x=0}^{3-\frac{3}{4}y} e^{4x-\frac{2}{3}x^2} dx \right) dy = \iint_R e^{4x-\frac{2}{3}x^2} dA = \int_{x=0}^{3} \left(\int_{y=0}^{4-\frac{4}{3}x} e^{4x-\frac{2}{3}x^2} dy \right) dx$$
$$= \int_{x=0}^{3} \left(y \cdot e^{4x-\frac{2}{3}x^2} \Big|_{y=0}^{4-\frac{4}{3}x} \right) dx = \int_{x=0}^{3} \left((4 - \frac{4}{3}x) e^{4x-\frac{2}{3}x^2} \right) dx = e^{4x-\frac{2}{3}x^2} \Big|_{x=0}^{3}$$
$$= e^{12-6} - e^0 = e^6 - 1$$

Example 2:

Consider the parabolic region on the right. This region can be characterized as both a type I and a type II region.
For a type I characterization, the bounds on x are -4 and 0. The equation of the parabola is:

$$y = 5 - \frac{5}{16}x^2$$

The bounds on y as functions of x are 0 and $5 - \frac{5}{16}x^2$. Therefore:

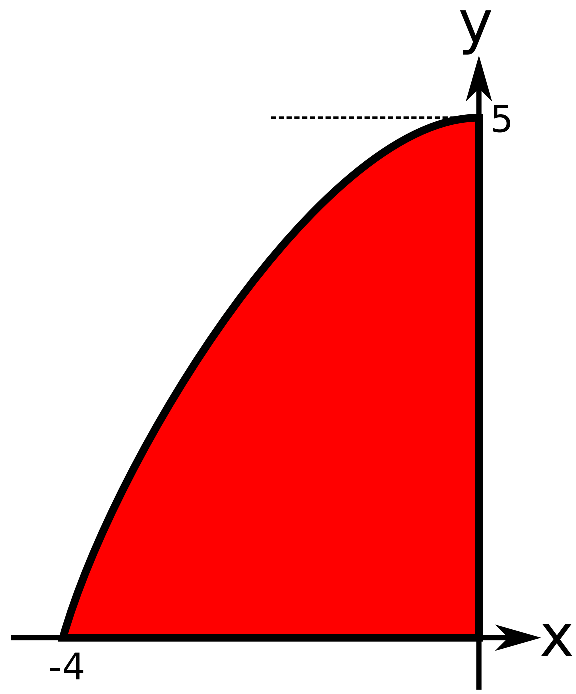
$$R = \left\{ (x, y) \middle| -4 \leq x \leq 0 \ \& \ 0 \leq y \leq 5 - \frac{5}{16}x^2 \right\}$$

For a type II characterization, the bounds on y are 0 and 5. The equation of the parabola can be rearranged to get:

$$y = 5 - \frac{5}{16}x^2 \iff -\frac{5}{16}x^2 = y - 5$$
$$\iff x^2 = 16 - \frac{16}{5}y \iff x = \pm 4\sqrt{1 - \frac{y}{5}}$$

The upper bound on x is 0, while the lower bound on x as a function of y must be ≤ 0, so this lower bound is $-4\sqrt{1 - \frac{y}{5}}$. Therefore:

$$R = \left\{ (x, y) \middle| 0 \leq y \leq 5 \ \& \ -4\sqrt{1 - \frac{y}{5}} \leq x \leq 0 \right\}$$



Now consider the arbitrary integrand $f(x, y)$. The double integral $\iint_R f(x, y)dA$ can be computed by using either the type I characterization:

$$\iint_R f(x, y)dA = \int_{x=-4}^{0} \left(\int_{y=0}^{5-\frac{5}{16}x^2} f(x, y)dy \right) dx$$

or via the type II characterization:

$$\iint_R f(x, y)dA = \int_{y=0}^{5} \left(\int_{x=-4\sqrt{1-\frac{y}{5}}}^{0} f(x, y)dx \right) dy$$

If the integrals in the nested integral need to be swapped, then:

$$\int_{x=-4}^{0} \left(\int_{y=0}^{5-\frac{5}{16}x^2} f(x, y)dy \right) dx = \int_{y=0}^{5} \left(\int_{x=-4\sqrt{1-\frac{y}{5}}}^{0} f(x, y)dx \right) dy$$

Consider the specific integrand $f(x, y) = \frac{1}{\sqrt{5-y}}$. The double integral $\iint_R \frac{1}{\sqrt{5-y}}dA$ can now be evaluated via:

$$\begin{aligned}\iint_R \frac{1}{\sqrt{5-y}}dA =& \int_{x=-4}^{0}\left(\int_{y=0}^{5-\frac{5}{16}x^2}\frac{1}{\sqrt{5-y}}dy\right)dx = \int_{x=-4}^{0}\left(-2\sqrt{5-y}\Big|_{y=0}^{5-\frac{5}{16}x^2}\right)dx\\ =& \int_{x=-4}^{0}\left(-2\sqrt{\frac{5}{16}x^2}-(-2\sqrt{5})\right)dx = \int_{x=-4}^{0}\left(-\frac{\sqrt{5}}{2}|x|+2\sqrt{5}\right)dx\\ =& \int_{x=-4}^{0}\left(\frac{\sqrt{5}}{2}x+2\sqrt{5}\right)dx = \left(\frac{\sqrt{5}}{4}x^2+(2\sqrt{5})x\right)\Big|_{x=-4}^{0}\\ =& 0-\left(4\sqrt{5}-8\sqrt{5}\right) = 4\sqrt{5}\end{aligned}$$

or via:

$$\begin{aligned}\iint_R \frac{1}{\sqrt{5-y}}dA =& \int_{y=0}^{5}\left(\int_{x=-4\sqrt{1-\frac{y}{5}}}^{0}\frac{1}{\sqrt{5-y}}dx\right)dy = \int_{y=0}^{5}\left(\frac{x}{\sqrt{5-y}}\Big|_{x=-4\sqrt{1-\frac{y}{5}}}^{0}\right)dy\\ =& \int_{y=0}^{5}\left(0-\frac{-4\sqrt{1-\frac{y}{5}}}{\sqrt{5-y}}\right)dy = \int_{y=0}^{5}\frac{(4/\sqrt{5})\sqrt{5-y}}{\sqrt{5-y}}dy = \int_{y=0}^{5}\frac{4}{\sqrt{5}}dy\\ =& \frac{4y}{\sqrt{5}}\Big|_{y=0}^{5} = 4\sqrt{5}\end{aligned}$$

Note that the final answer is the same either way.
Now consider the nested integral:

$$\int_{x=-4}^{0}\left(\int_{y=0}^{5-\frac{5}{16}x^2}\cos((5-y)^{3/2})dy\right)dx$$

This nested integral is difficult to directly evaluate. By reversing the order of the variables, the solution becomes clear:

$$\begin{aligned}&\int_{x=-4}^{0}\left(\int_{y=0}^{5-\frac{5}{16}x^2}\cos((5-y)^{3/2})dy\right)dx = \iint_R \cos((5-y)^{3/2})dA\\ &= \int_{y=0}^{5}\left(\int_{x=-4\sqrt{1-\frac{y}{5}}}^{0}\cos((5-y)^{3/2})dx\right)dy = \int_{y=0}^{5}\left(x\cos((5-y)^{3/2})\Big|_{x=-4\sqrt{1-\frac{y}{5}}}^{0}\right)dy\\ &= \int_{y=0}^{5}\frac{4}{\sqrt{5}}\cdot\sqrt{5-y}\cdot\cos((5-y)^{3/2})dy = \int_{y=0}^{5}\frac{-8}{3\sqrt{5}}\cdot\cos((5-y)^{3/2})\cdot\frac{3}{2}(5-y)^{1/2}\cdot(-1)dy\\ &= \frac{-8}{3\sqrt{5}}\cdot\sin((5-y)^{3/2})\Big|_{y=0}^{5} = 0-\frac{-8}{3\sqrt{5}}\cdot\sin(5^{3/2}) = \frac{8}{3\sqrt{5}}\cdot\sin(5^{3/2})\end{aligned}$$

Example 3:

Consider the parabolic region on the right. This region can be characterized as both a type I and a type II region.
For a type I characterization, the bounds on x are -2 and 0. The equation of the parabola is:

$$y = -(x+2)^2$$

The bounds on y as functions of x are $-(x+2)^2$ and 0. Therefore:

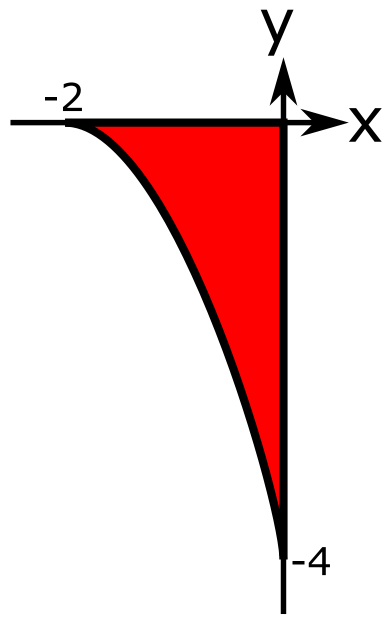
$$R = \left\{(x,y)\middle|-2 \leq x \leq 0 \ \& \ -(x+2)^2 \leq y \leq 0\right\}$$

For a type II characterization, the bounds on y are -4 and 0. The equation of the parabola can be rearranged to get:

$$y = -(x+2)^2 \iff (x+2)^2 = -y$$
$$\iff x+2 = \pm\sqrt{-y} \iff x = -2 \pm \sqrt{-y}$$

The upper bound on x is 0, while the lower bound on x as a function of y must be ≥ -2 , so this lower bound is $-2+\sqrt{-y}$. Therefore:

$$R = \left\{(x,y)\middle|-4 \leq y \leq 0 \ \& \ -2+\sqrt{-y} \leq x \leq 0\right\}$$



Now consider the arbitrary integrand $f(x,y)$. The double integral $\iint_R f(x,y)dA$ can be computed by using either the type I characterization:

$$\iint_R f(x,y)dA = \int_{x=-2}^{0}\left(\int_{y=-(x+2)^2}^{0} f(x,y)dy\right)dx$$

or via the type II characterization:

$$\iint_R f(x,y)dA = \int_{y=-4}^{0}\left(\int_{x=-2+\sqrt{-y}}^{0} f(x,y)dx\right)dy$$

If the integrals in the nested integral need to be swapped, then:

$$\int_{x=-2}^{0}\left(\int_{y=-(x+2)^2}^{0} f(x,y)dy\right)dx = \int_{y=-4}^{0}\left(\int_{x=-2+\sqrt{-y}}^{0} f(x,y)dx\right)dy$$

Consider the specific integrand $f(x,y) = x+2$. The double integral $\iint_R (x+2)dA$ can now be evaluated via:

$$\iint_R (x+2)dA = \int_{x=-2}^{0}\left(\int_{y=-(x+2)^2}^{0}(x+2)dy\right)dx = \int_{x=-2}^{0}\left((x+2)y\Big|_{y=-(x+2)^2}^{0}\right)dx$$
$$= \int_{x=-2}^{0}\left(0-(-(x+2)^3)\right)dx = \int_{x=-2}^{0}(x+2)^3dx$$
$$= \frac{1}{4}(x+2)^4\Big|_{x=-2}^{0} = \frac{1}{4}\cdot 2^4 - 0 = 4$$

or via:

$$\begin{aligned}
\iint_R (x+2)dA &= \int_{y=-4}^{0} \left(\int_{x=-2+\sqrt{-y}}^{0} (x+2)dx \right) dy = \int_{y=-4}^{0} \left((\frac{1}{2}x^2 + 2x)\bigg|_{x=-2+\sqrt{-y}}^{0} \right) dy \\
&= \int_{y=-4}^{0} \left(0 - (\frac{1}{2}(4 - 4\sqrt{-y} - y) + 2(-2 + \sqrt{-y})) \right) dy \\
&= \int_{y=-4}^{0} \left(-((2 - 2\sqrt{-y} - \frac{1}{2}y) + (-4 + 2\sqrt{-y})) \right) dy = \int_{y=-4}^{0} \left(2 + \frac{1}{2}y \right) dy \\
&= (2y + \frac{1}{4}y^2)\bigg|_{y=-4}^{0} = 0 - (-8 + 4) = 4
\end{aligned}$$

Note that the final answer is the same either way.
Now consider the specific integrand $f(x, y) = y$. The double integral $\iint_R ydA$ can now be evaluated via:

$$\begin{aligned}
\iint_R ydA &= \int_{x=-2}^{0} \left(\int_{y=-(x+2)^2}^{0} ydy \right) dx = \int_{x=-2}^{0} \left(\frac{1}{2}y^2 \bigg|_{y=-(x+2)^2}^{0} \right) dx \\
&= \int_{x=-2}^{0} \left(0 - \frac{1}{2}(x+2)^4 \right) dx = \int_{x=-2}^{0} -\frac{1}{2}(x+2)^4 dx \\
&= -\frac{1}{10}(x+2)^5 \bigg|_{x=-2}^{0} = -\frac{1}{10} \cdot 2^5 - 0 = -\frac{16}{5}
\end{aligned}$$

or via:

$$\begin{aligned}
\iint_R ydA &= \int_{y=-4}^{0} \left(\int_{x=-2+\sqrt{-y}}^{0} ydx \right) dy = \int_{y=-4}^{0} \left(xy\bigg|_{x=-2+\sqrt{-y}}^{0} \right) dy \\
&= \int_{y=-4}^{0} \left(0 - (-2y + y\sqrt{-y})\right) dy = \int_{y=-4}^{0} \left(2y + (-y)^{3/2} \right) dy = \left(y^2 - \frac{2}{5}(-y)^{5/2} \right) \bigg|_{y=-4}^{0} \\
&= 0 - (16 - \frac{2}{5} \cdot 4^{5/2}) = -(16 - \frac{64}{5}) = -\frac{16}{5}
\end{aligned}$$

Note that the final answer is the same either way.

Example 4:

Consider the quarter circular region on the right. This region can be characterized as both a type I and a type II region.
For a type I characterization, the bounds on x are -3 and 0. The lower bound on y is 0. The upper bound on y as a function x is the circle with a radius of 3. The equation of the circular arc is:

$$(x+3)^2 + y^2 = 9 \iff y^2 = 9 - (x+3)^2$$
$$\iff y = \pm\sqrt{9-(x+3)^2}$$

The upper bound on y is positive, and is $\sqrt{9-(x+3)^2}$. Therefore:

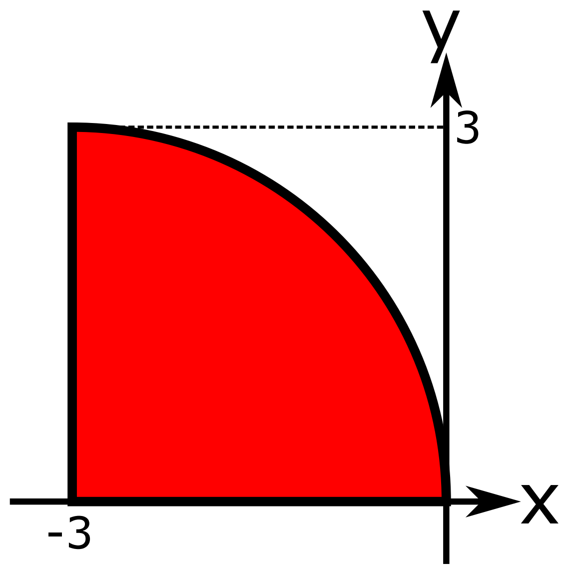
$$R = \left\{(x,y)\middle|-3 \le x \le 0 \;\&\; 0 \le y \le \sqrt{9-(x+3)^2}\right\}$$

For a type II characterization, the bounds on y are 0 and 3. The equation of the circular arc can be rearranged to get:

$$(x+3)^2 + y^2 = 9 \iff (x+3)^2 = 9 - y^2$$
$$\iff x+3 = \pm\sqrt{9-y^2} \iff x = -3 \pm \sqrt{9-y^2}$$

The lower bound on x is -3, while the upper bound on x as a function of y must be ≥ -3, so this upper bound is $-3+\sqrt{9-y^2}$. Therefore:

$$R = \left\{(x,y)\middle|0 \le y \le 3 \;\&\; -3 \le x \le -3+\sqrt{9-y^2}\right\}$$



Now consider the arbitrary integrand $f(x,y)$. The double integral $\iint_R f(x,y)dA$ can be computed by using either the type I characterization:

$$\iint_R f(x,y)dA = \int_{x=-3}^{0}\left(\int_{y=0}^{\sqrt{9-(x+3)^2}} f(x,y)dy\right)dx$$

or via the type II characterization:

$$\iint_R f(x,y)dA = \int_{y=0}^{3}\left(\int_{x=-3}^{-3+\sqrt{9-y^2}} f(x,y)dx\right)dy$$

If the integrals in the nested integral need to be swapped, then:

$$\int_{x=-3}^{0}\left(\int_{y=0}^{\sqrt{9-(x+3)^2}} f(x,y)dy\right)dx = \int_{y=0}^{3}\left(\int_{x=-3}^{-3+\sqrt{9-y^2}} f(x,y)dx\right)dy$$

Example 5:

Consider the region on the right. This region is bounded from below by the parabola $y = x^2$ and from above by the line $y = 3x$. This region can be characterized as both a type I and a type II region.
For a type I characterization, the bounds on x are 0 and 3. The bounds on y as functions of x are x^2 and $3x$. Therefore:

$$R = \left\{(x, y)\middle|0 \leq x \leq 3 \text{ \& } x^2 \leq y \leq 3x\right\}$$

For a type II characterization, the bounds on y are 0 and 9. The equation of the line can be rearranged to get the lower bound on x:

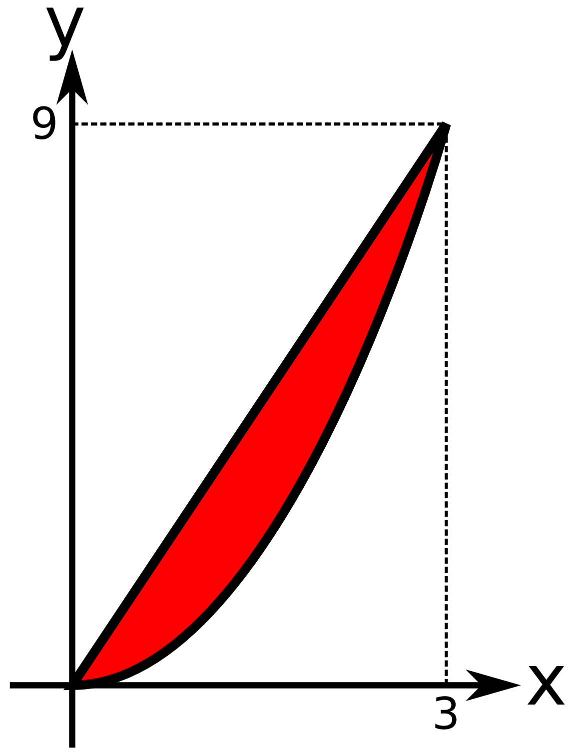
$$y = 3x \iff x = \frac{y}{3}$$

The equation of the parabola can be rearranged to get the upper bound on x:

$$y = x^2 \iff x = \pm\sqrt{y}$$

The upper bound of x must be ≥ 0, so the upper bound is $\sqrt{y}$. Therefore:

$$R = \left\{(x, y)\middle|0 \leq y \leq 9 \text{ \& } \frac{y}{3} \leq x \leq \sqrt{y}\right\}$$



Now consider the arbitrary integrand $f(x, y)$. The double integral $\iint_R f(x, y)dA$ can be computed by using either the type I characterization:

$$\iint_R f(x, y)dA = \int_{x=0}^{3} \left(\int_{y=x^2}^{3x} f(x, y)dy\right) dx$$

or via the type II characterization:

$$\iint_R f(x, y)dA = \int_{y=0}^{9} \left(\int_{x=y/3}^{\sqrt{y}} f(x, y)dx\right) dy$$

If the integrals in the nested integral need to be swapped, then:

$$\int_{x=0}^{3} \left(\int_{y=x^2}^{3x} f(x, y)dy\right) dx = \int_{y=0}^{9} \left(\int_{x=y/3}^{\sqrt{y}} f(x, y)dx\right) dy$$

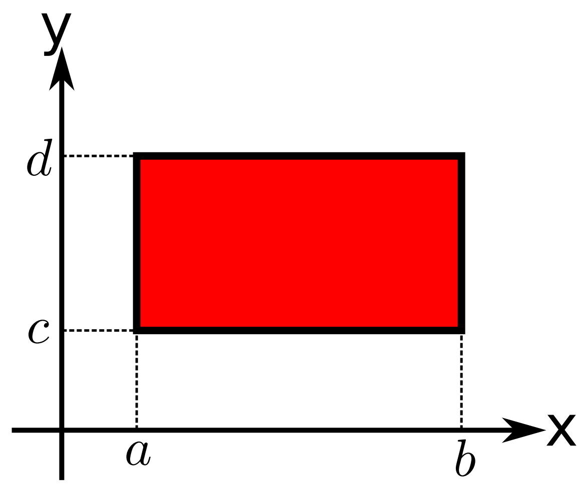
Rectangular regions

A rectangular region R is a region where the bounds on x and y are entirely independent of each other. The bounds are constant, and fixing x to different values does not change the bounds on y, and vice versa.

$$R = [a, b] \times [c, d] = \{(x, y) | a \leq x \leq b \ \& \ c \leq y \leq d\}$$

A rectangular region is both a type I and a type II region. Moreover, the order of integration in a nested integral over a rectangular region can be reversed by simply swapping the integral signs without any further considerations:

$$\int_{x=a}^{b} \left(\int_{y=c}^{d} f(x, y) dy \right) dx = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x, y) dx \right) dy$$



Given two single variable functions $g(x)$ and $h(x)$, and the definite integrals $\int_{x=a}^{b} g(x)dx$ and $\int_{x=c}^{d} h(x)dx$, then the product of these definite integrals is a double integral of the product $g(x)h(y)$ over the rectangular region $R = [a, b] \times [c, d] = \{(x, y) | a \leq x \leq b \ \& \ c \leq y \leq d\}$:

$$\left(\int_{x=a}^{b} g(x)dx \right) \left(\int_{x=c}^{d} h(x)dx \right)$$

$$= \left(\int_{x=a}^{b} g(x)dx \right) \left(\int_{y=c}^{d} h(y)dy \right)$$

Replace the local placeholder variable of x in the second integral with the distinct symbol y.

$$= \int_{x=a}^{b} g(x) \left(\int_{y=c}^{d} h(y)dy \right) dx$$

$\int_{y=c}^{d} h(y)dy$ is constant with respect to x.

$$= \int_{x=a}^{b} \left(\int_{y=c}^{d} g(x)h(y)dy \right) dx$$

$g(x)$ is constant with respect to y.

$$= \iint_{R} g(x)h(y)dA$$

Therefore:

$$\left(\int_{x=a}^{b} g(x)dx \right) \left(\int_{x=c}^{d} h(x)dx \right) = \iint_{R} g(x)h(y)dA$$

Examples:

- Consider the rectangle $R = \{(x, y) | -2 \leq x \leq 1 \ \& \ -1 \leq y \leq 1\}$ and the function $f(x, y) = xy + 3x + 2y + 6$.

  $f(x, y)$ can be factored to give $f(x, y) = (x+2)(y+3)$ so the double integral of $f(x, y)$ over the rectangle R is:

$$\iint_{R} (x + 2)(y + 3)dA = \left(\int_{x=-2}^{1} (x + 2)dx \right) \left(\int_{y=-1}^{1} (y + 3)dy \right) = (\frac{1}{2}x^2 + 2x)\Big|_{x=-2}^{1} \cdot (\frac{1}{2}y^2 + 3y)\Big|_{y=-1}^{1}$$

$$= ((\frac{1}{2} + 2) - (2 - 4)) \cdot ((\frac{1}{2} + 3) - (\frac{1}{2} - 3)) = (\frac{5}{2} + 2) \cdot (\frac{7}{2} + \frac{5}{2}) = \frac{9}{2} \cdot 6 = 27$$