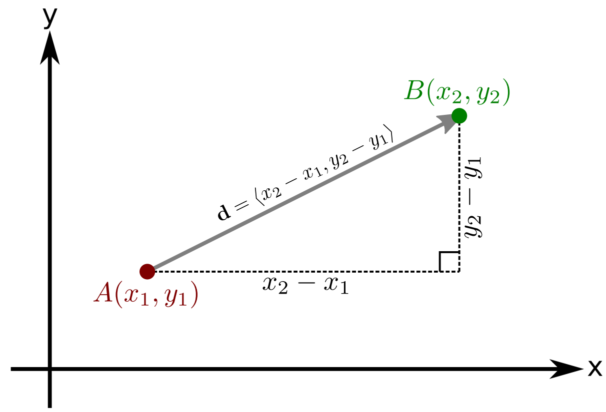
Vectors Concluded

Displacement between two points

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the displacement $\mathbf{d}$ from point A to point B is $\mathbf{d} = \langle x_2 - x_1, y_2 - y_1 \rangle$. The horizontal component of $\mathbf{d}$ is the change in the x-coordinate from A to B, while the vertical component of $\mathbf{d}$ is the change in the y-coordinate from A to B.
Examples:

- The displacement from $A(-3, 2)$ to $B(4, -1)$ is $\mathbf{d} = \langle 7, -3 \rangle$.
- The displacement from $A(11, -2)$ to $B(3, 5)$ is $\mathbf{d} = \langle -8, 7 \rangle$.
- The displacement from $A(5, 8)$ to $B(7, 0)$ is $\mathbf{d} = \langle 2, -8 \rangle$.



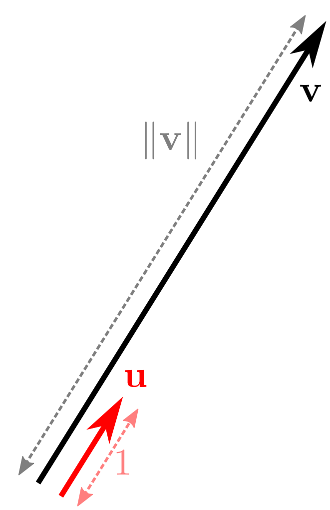
Also, given a point $A(x, y)$ and a displacement vector $\mathbf{d} = \langle d_x, d_y \rangle$, the resultant point B from moving through a displacement of $\mathbf{d}$ starting from point A is $B(x + d_x, y + d_y)$.
Examples:

- Starting from point $A(6, -2)$, moving a displacement of $\mathbf{d} = \langle -3, 5 \rangle$ arrives at the point $B(3, 3)$.
- Starting from point $A(-1, -7)$, moving a displacement of $\mathbf{d} = \langle 2, 4 \rangle$ arrives at the point $B(1, -3)$.
- Starting from point $A(-4, 7)$, moving a displacement of $\mathbf{d} = \langle 4, -1 \rangle$ arrives at the point $B(0, 6)$.

Unit vectors

A **unit vector** is a vector with a length of 1. Given an arbitrary vector $\mathbf{v}$, a commonly sought quantity is a unit vector $\mathbf{u}$ that shares the same direction as $\mathbf{v}$. Since $\mathbf{u}$ shares the same direction as $\mathbf{v}$, unit vector $\mathbf{u}$ is a multiple of $\mathbf{v}$. To change the length from $\|\mathbf{v}\|$ to 1, multiplication by $\frac{1}{\|\mathbf{v}\|}$ is needed. Therefore $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector that shares the same direction as $\mathbf{v}$.
Examples:

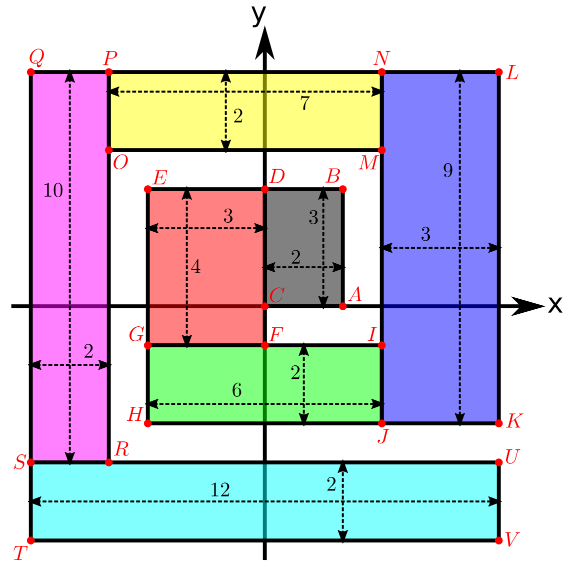
- If $\mathbf{v} = \langle -4, 3 \rangle$, the unit vector $\mathbf{u}$ that shares the same direction as $\mathbf{v}$ is sought. $\|\mathbf{v}\| = \sqrt{(-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ so $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle -0.8, 0.6 \rangle$.
- If $\mathbf{v} = \langle -23, -40 \rangle$, the unit vector $\mathbf{u}$ that shares the same direction as $\mathbf{v}$ is sought. $\|\mathbf{v}\| = \sqrt{(-23)^2 + (-40)^2} \approx 46.1411$ so $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \approx \langle -0.498471, -0.866906 \rangle$.



Review of Cartesian coordinates

In the image to the right, the coordinates of the labeled points A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, and V are:

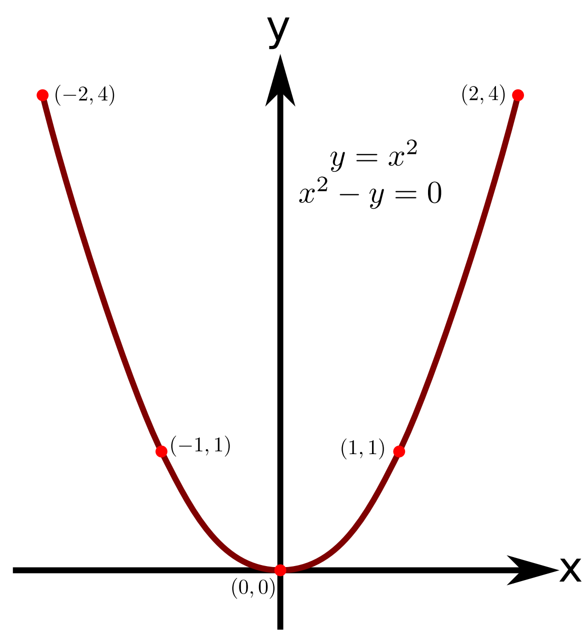
$A(2,0)$	$B(2,3)$	$C(0,0)$	$D(0,3)$
$E(-3,3)$	$F(0,-1)$	$G(-3,-1)$	$H(-3,-3)$
$I(3,-1)$	$J(3,-3)$	$K(6,-3)$	$L(6,6)$
$M(3,4)$	$N(3,6)$	$O(-4,4)$	$P(-4,6)$
$Q(-6,6)$	$R(-4,-4)$	$S(-6,-4)$	$T(-6,-6)$
$U(6,-4)$	$V(6,-6)$		



Consider an arbitrary equation $f(x,y) = g(x,y)$, where $f(x,y)$ and $g(x,y)$ are arbitrary expressions involving x and y. The set of points that satisfy this equation forms a curve, where a point (x,y) is included on the curve if and only if (x,y) satisfies $f(x,y) = g(x,y)$. This curve is referred to the **graph** of $f(x,y) = g(x,y)$ or the **locus** of points that satisfies $f(x,y) = g(x,y)$.

The purpose of the equation $f(x,y) = g(x,y)$ is to distinguish between points that are part of the curve, and points that are not part of the curve. Given an arbitrary point (x,y), the equation $f(x,y) = g(x,y)$ is true if and only if point (x,y) is part of the curve graphed by $f(x,y) = g(x,y)$.

In the image on the right, the graph of the equation $y = x^2$ is shown. In this case, $f(x,y) = y$ and $g(x,y) = x^2$. Given the point $(x,y) = (-1,1)$ for example, $y = 1$ and $x^2 = 1$ so the equation is satisfied, and $(-1,1)$ is part of the curve. Given the point $(x,y) = (2,1)$ for example, $y = 1$ and $x^2 = 4$ so the equation is falsified, and $(2,1)$ is not part of the curve. This equation $y = x^2$ is also equivalent to the equation $x^2 - y = 0$, where $f(x,y) = x^2 - y$ and $g(x,y) = 0$.



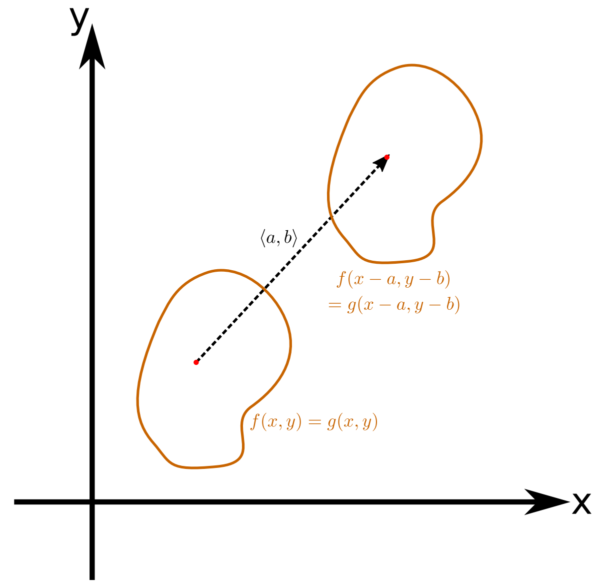
Consider a curve with the equation

$$f(x, y) = g(x, y)$$

Shifting this curve by a displacement of $\langle a, b\rangle$, the equation of the new shifted curve can be derived as follows: If point $(x_{\text{old}}, y_{\text{old}})$ lies on the original curve, then point $(x_{\text{new}}, y_{\text{new}}) = (x_{\text{old}} + a, y_{\text{old}} + b)$ lies on the shifted curve. Point $(x_{\text{old}}, y_{\text{old}})$ satisfies the equation $f(x_{\text{old}}, y_{\text{old}}) = g(x_{\text{old}}, y_{\text{old}})$. Since $(x_{\text{old}}, y_{\text{old}}) = (x_{\text{new}} - a, y_{\text{new}} - b)$, point $(x_{\text{new}}, y_{\text{new}})$ satisfies the equation $f(x_{\text{new}} - a, y_{\text{new}} - b) = g(x_{\text{new}} - a, y_{\text{new}} - b)$. Therefore:

$$f(x - a, y - b) = g(x - a, y - b)$$

is the equation of the graph of $f(x, y) = g(x, y)$ shifted by the displacement $\langle a, b\rangle$.



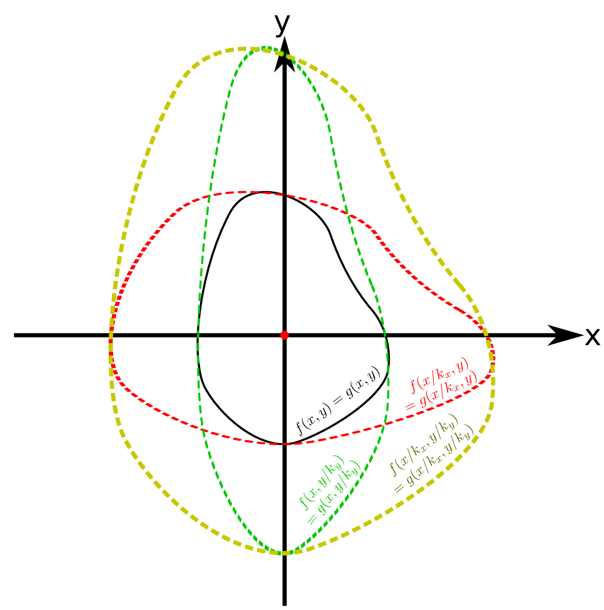
Consider a curve with the equation

$$f(x, y) = g(x, y)$$

Scaling the curve by a factor of k_x parallel to the x-axis, and by a factor of k_y parallel to the y-axis, the equation of the new scaled curve can be derived as follows: If point $(x_{\text{old}}, y_{\text{old}})$ lies on the original curve, then point $(x_{\text{new}}, y_{\text{new}}) = (k_x x_{\text{old}}, k_y y_{\text{old}})$ lies on the scaled curve. Point $(x_{\text{old}}, y_{\text{old}})$ satisfies the equation $f(x_{\text{old}}, y_{\text{old}}) = g(x_{\text{old}}, y_{\text{old}})$. Since $(x_{\text{old}}, y_{\text{old}}) = (x_{\text{new}}/k_x, y_{\text{new}}/k_y)$, point $(x_{\text{new}}, y_{\text{new}})$ satisfies the equation $f(x_{\text{new}}/k_x, y_{\text{new}}/k_y) = g(x_{\text{new}}/k_x, y_{\text{new}}/k_y)$. Therefore:

$$f(x/k_x, y/k_y) = g(x/k_x, y/k_y)$$

is the equation of the graph of $f(x, y) = g(x, y)$ scaled by a factor of k_x parallel to the x-axis, and scaled by a factor of k_y parallel to the y-axis.

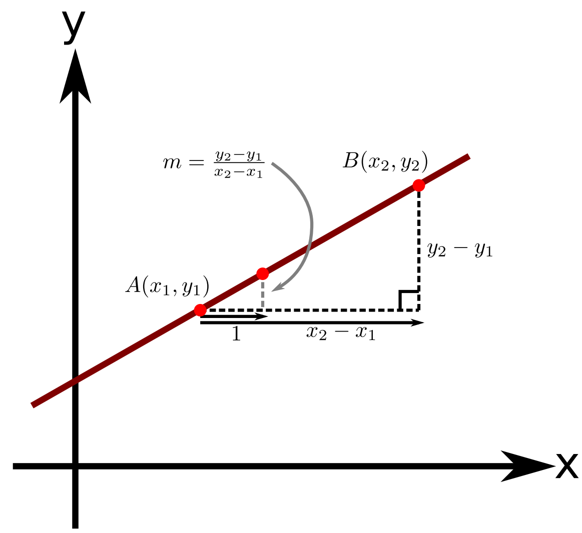


Lines

Now will be described the equations whose graphs are **lines**.

Important to lines is the "gradient" or "slope" of the line. The slope quantifies the "steepness" of the line, and is defined as the change in the y-coordinate for a change of 1 in the x coordinate. Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ and the line that passes through A and B, the change of $x_2 - x_1$ in the x-coordinate corresponds to a change of $y_2 - y_1$ in the y-coordinate. The change in the y-coordinate per change of 1 in the x-coordinate is the gradient:

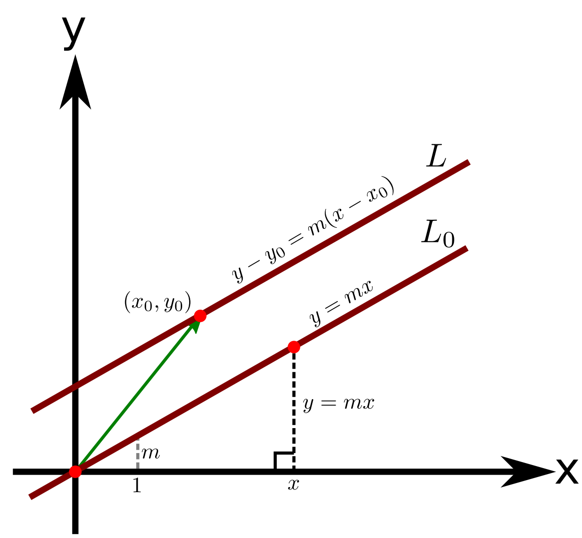
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



The equation of a line L_0 with a gradient of m and that passes through the origin point is $y = mx$. Starting at $(0, 0)$, increasing the x-coordinate to x changes the y-coordinate to $y = mx$. Now if it is known that line L passes through the point (x_0, y_0), then displacing line L_0 by a displacement of $\langle x_0, y_0\rangle$ so that the point at the origin has been moved to (x_0, y_0), the new equation for line L is $y - y_0 = m(x - x_0)$. This equation can also easily be observed by noting that if (x_0, y_0) lies on L, then increasing x_0 by the amount $x - x_0$ to x, will also increase y_0 by the amount $y - y_0 = m(x - x_0)$ by definition of the gradient. Therefore:

$$y - y_0 = m(x - x_0)$$

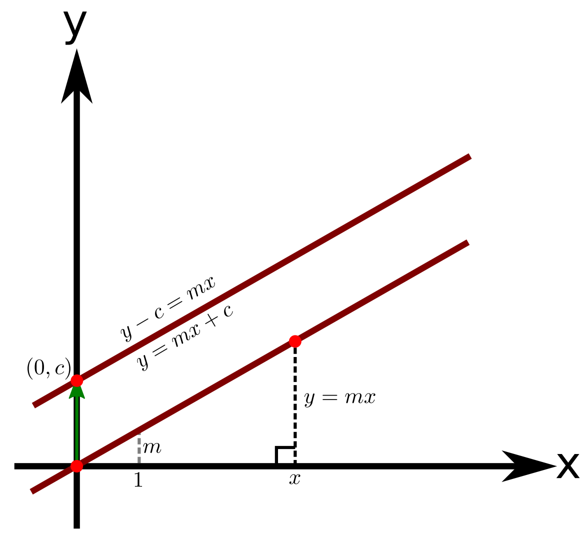
is the equation of line L.



If the point that line L passes through is the y-intercept $(0, c)$, then the equation of line L is:

$$y - c = m(x - 0) \iff y = mx + c$$

The equation $y = mx + c$ is the most common form of equation for a line. The equation $y - y_0 = m(x - x_0)$ will often be manipulated to $y = mx + (y_0 - mx_0)$.



The standard equation of the line that is being used is:

$$y = mx + c$$

where m is the gradient of the line, and c is the y-axis intercept. The slope m is the change in y *per unit change* in x: $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are points from the line. The y-axis intercept c is the value attained by y when the line crosses the y-axis, which occurs when $x = 0$.

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the process of finding the equation of the line L that passes through points A and B can be summarized as follows:

- Compute the slope m of line L:
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
- Start with $y = mx$, which is the equation of a line with a slope of m that contains the origin $(0, 0)$. Move this line through a displacement of $\langle x_1, y_1 \rangle$ so that the origin is moved to point A. This results in an equation for line L:
$$y - y_1 = m(x - x_1)$$
Alternately, the line $y = mx$ can be moved through the displacement $\langle x_2, y_2 \rangle$ so that the origin is moved to point B. This results in an **equivalent** equation for line L:
$$y - y_2 = m(x - x_2)$$
- Solve the equation for y as an expression of x. This equation will have the desired form
$$y = mx + c$$

Examples:

- Start with the two points $A(-2, -5)$ and $B(3, 8)$. The slope of the line that connects points A and B is $m = \frac{8-(-5)}{3-(-2)} = \frac{13}{5}$. Using point A, the equation of the line that contains both A and B is:
$$y - (-5) = \frac{13}{5}(x - (-2)) \iff y = -5 + (\frac{13}{5}x + \frac{26}{5}) \iff y = \frac{13}{5}x + \frac{1}{5}$$

The equation of the line that contains both A and B is:

$$y = \frac{13}{5}x + \frac{1}{5}$$

To check that A and B are on this line, for point $A(-2,-5)$ the equation becomes $-5 = \frac{13}{5}(-2)+\frac{1}{5} \iff -5 = \frac{-25}{5} \iff -5 = -5$. For point $B(3,8)$ the equation becomes $8 = \frac{13}{5}(3) + \frac{1}{5} \iff 8 = \frac{40}{5} \iff 8 = 8$.

- Start with the two points $A(-3,4)$ and $B(4,-2)$. The slope of the line that connects points A and B is $m = \frac{(-2)-4}{4-(-3)} = \frac{-6}{7}$. Using point A, the equation of the line that contains both A and B is:

$$y - 4 = -\frac{6}{7}(x - (-3)) \iff y = 4 + (-\frac{6}{7}x - \frac{18}{7}) \iff y = -\frac{6}{7}x + \frac{10}{7}$$

The equation of the line that contains both A and B is:

$$y = -\frac{6}{7}x + \frac{10}{7}$$

To check that A and B are on this line, for point $A(-3,4)$ the equation becomes $4 = -\frac{6}{7}(-3)+\frac{10}{7} \iff 4 = \frac{28}{7} \iff 4 = 4$. For point $B(4,-2)$ the equation becomes $-2 = -\frac{6}{7}(4) + \frac{10}{7} \iff -2 = -\frac{14}{7} \iff -2 = -2$.

- Start with the two points $A(1,-4)$ and $B(-5,2)$. The slope of the line that connects points A and B is $m = \frac{2-(-4)}{(-5)-1} = \frac{6}{-6} = -1$. Using point A, the equation of the line that contains both A and B is:

$$y - (-4) = -1 \cdot (x - 1) \iff y = -4 + (-x + 1) \iff y = -x - 3$$

The equation of the line that contains both A and B is:

$$y = -x - 3$$

To check that A and B are on this line, for point $A(1,-4)$ the equation becomes $-4 = -1 - 3 \iff -4 = -4$. For point $B(-5,2)$ the equation becomes $2 = -(-5) - 3 \iff 2 = 2$.

- Start with the two points $A(5,2)$ and $B(-4,-1)$. The slope of the line that connects points A and B is $m = \frac{(-1)-2}{(-4)-5} = \frac{-3}{-9} = \frac{1}{3}$. Using point A, the equation of the line that contains both A and B is:

$$y - 2 = \frac{1}{3}(x - 5) \iff y = 2 + (\frac{1}{3}x - \frac{5}{3}) \iff y = \frac{1}{3}x + \frac{1}{3}$$

The equation of the line that contains both A and B is:

$$y = \frac{1}{3}x + \frac{1}{3}$$

To check that A and B are on this line, for point $A(5,2)$ the equation becomes $2 = \frac{1}{3}(5) + \frac{1}{3} \iff 2 = \frac{6}{3} \iff 2 = 2$. For point $B(-4,-1)$ the equation becomes $-1 = \frac{1}{3}(-4) + \frac{1}{3} \iff -1 = \frac{-3}{3} \iff -1 = -1$.

Converting equations to $y = mx + c$

Sometimes, the equation of a line will not be given in the form $y = mx + c$. When this is the case, the equation must be manipulated until the form $y = mx + c$ is attained.

Examples:

- Given the equation $3x - 7y = 14$,

$$3x - 7y = 14 \iff 7y = 3x - 14 \iff y = \frac{3}{7}x - 2$$

so $3x - 7y = 14$ is equivalent to $y = \frac{3}{7}x - 2$

- Given the equation $-10x + 5y = 15$,

$$-10x + 5y = 15 \iff 5y = 10x + 15 \iff y = 2x + 3$$

so $-10x + 5y = 15$ is equivalent to $y = 2x + 3$

- Given the equation $6x - 7y + 13 = x + 3y - 7$,

$$6x - 7y + 13 = x + 3y - 7 \iff 5x + 20 = 10y \iff y = \frac{1}{2}x + 2$$

so $6x - 7y + 13 = x + 3y - 7$ is equivalent to $y = \frac{1}{2}x + 2$

Vertical lines

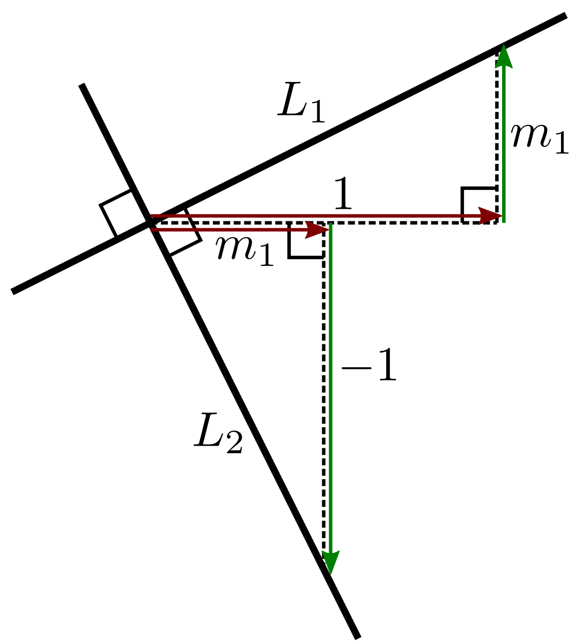
When a line is vertical, its slope is $m = \infty$. The equation that quantifies a vertical line cannot take the form $y = mx + c$, and instead has the form $x = a$ where a is the value that x is fixed to. Examples include:

- $x = -1$ is the equation of a vertical line where the x-coordinate is -1 for all points on the line.
- $x = 3$ is the equation of a vertical line where the x-coordinate is 3 for all points on the line.
- $x = 1$ is the equation of a vertical line where the x-coordinate is 1 for all points on the line.

Perpendicular lines

Consider two **perpendicular** lines L_1 and L_2. Let the slopes of lines L_1 and L_2 be respectively denoted by m_1 and m_2. In the image on the right, there are two triangles. The upper triangle illustrates the slope of line L_1, with a change in the x-coordinate of 1, and a change in the y-coordinate of $m_1 \cdot 1 = m_1$. The lower triangle is congruent to the upper triangle, albeit rotated by 90°. The lower triangle illustrates the slope of L_2, with a change in the x-coordinate of m_1, and a change in the y-coordinate of -1. The slope of L_2 is $m_2 = \frac{-1}{m_1}$. Therefore:

$$m_2 = -\frac{1}{m_1}$$



Examples:

- Consider the points $A(-3,-1)$; $B(5,3)$; and $C(4,-4)$. What is sought is the equation of a line L_2 that passes through point C and is **perpendicular** to the line L_1 that passes through points A and B. The slope m_1 of line L_1 is $m_1 = \frac{3-(-1)}{5-(-3)} = \frac{4}{8} = \frac{1}{2}$. The slope m_2 of line L_2 is $m_2 = -\frac{1}{m_1} = -2$. The line $y = -2x$ passes through the origin, and moving the origin to point C yields the following equation for L_2:
$$y - (-4) = -2(x - 4) \iff y = -4 + (-2x + 8) = -2x + 4$$
The equation for L_2 is:
$$y = -2x + 4$$

- Consider the points $A(3,-2)$; $B(-2,4)$; and $C(8,-9)$. What is sought is the equation of a line L_2 that passes through point C and is **perpendicular** to the line L_1 that passes through points A and B. The slope m_1 of line L_1 is $m_1 = \frac{4-(-2)}{-2-3} = \frac{6}{-5} = -\frac{6}{5}$. The slope m_2 of line L_2 is $m_2 = -\frac{1}{m_1} = \frac{5}{6}$. The line $y = \frac{5}{6}x$ passes through the origin, and moving the origin to point C yields the following equation for L_2:
$$y - (-9) = \frac{5}{6}(x - 8) \iff y = -9 + (\frac{5}{6}x - \frac{20}{3}) = \frac{5}{6}x - \frac{47}{3}$$
The equation for L_2 is:
$$y = \frac{5}{6}x - \frac{47}{3}$$

Plotting lines: finding the intercept points

It is often useful to draw a line from its equation to get a better understanding of its shape and behavior. One of the most efficient approaches to plotting a line to is to first compute the x and y intercepts, and then plot the intercept points and draw a line that passes through the two intercept points. Given a line whose equation is $y = mx + c$, the y intercept point is clearly $(0, c)$. To find the x intercept point, the value of x that causes $y = 0$ must be found:
$$0 = mx + c \iff x = -\frac{c}{m}$$
this yields the intercept point $(-c/m, 0)$.

Examples:

- The line $y = \frac{2}{3}x + 5$ intersects the y axis at the point $(0,5)$, and intersects the x axis when $y = 0$. $\frac{2}{3}x + 5 = 0 \iff x = -\frac{15}{2}$ so the x intercept is $(-15/2, 0)$. Draw a line through the points $(0,5)$ and $(-15/2, 0)$ to plot the line $y = \frac{2}{3}x + 5$.

- The line $y = -\frac{1}{2}x + 7$ intersects the y axis at the point $(0,7)$, and intersects the x axis when $y = 0$. $-\frac{1}{2}x + 7 = 0 \iff x = 14$ so the x intercept is $(14, 0)$. Draw a line through the points $(0,7)$ and $(14,0)$ to plot the line $y = -\frac{1}{2}x + 7$.

- The line $y = -\frac{3}{8}x + 6$ intersects the y axis at the point $(0,6)$, and intersects the x axis when $y = 0$. $-\frac{3}{8}x + 6 = 0 \iff x = 16$ so the x intercept is $(16, 0)$. Draw a line through the points $(0,6)$ and $(16,0)$ to plot the line $y = -\frac{3}{8}x + 6$.

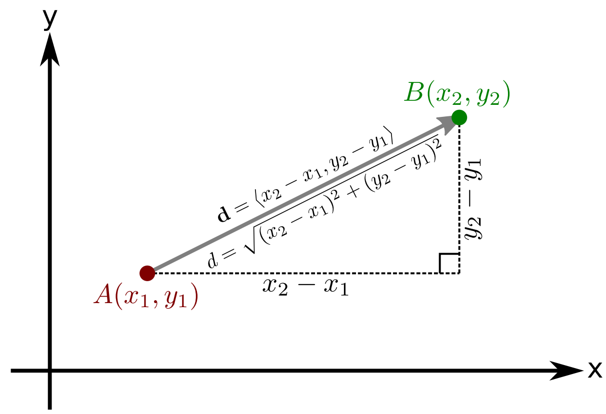
Circles

Distance between two points

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the displacement from A to B is $\mathbf{d} = \langle x_2 - x_1, y_2 - y_1 \rangle$. The distance from A to B is the length of this displacement which is $d = \|\mathbf{d}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Examples:

- The distance from $A(-3, 2)$ to $B(4, -1)$ is $d = \sqrt{7^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58} \approx 7.61577$
- The distance from $A(11, -2)$ to $B(3, 5)$ is $d = \sqrt{(-8)^2 + 7^2} = \sqrt{64 + 49} = \sqrt{113} \approx 10.6301$
- The distance from $A(5, 8)$ to $B(7, 0)$ is $d = \sqrt{2^2 + (-8)^2} = \sqrt{4 + 64} = \sqrt{68} = 2\sqrt{17} \approx 8.24621$



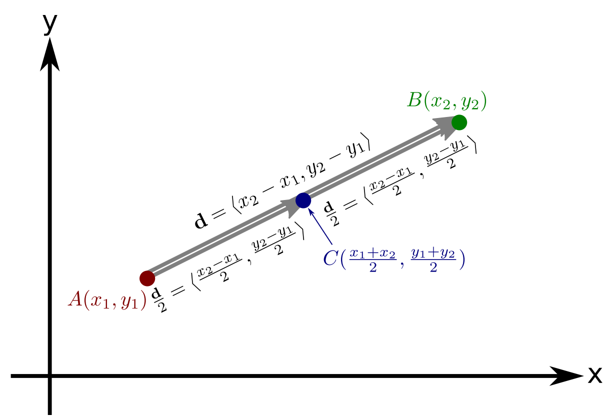
Midpoint between two points

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the "midpoint" C that is exactly halfway between points A and B is computed as follows: The displacement from point A to point B is $\mathbf{d} = \langle x_2 - x_1, y_2 - y_1 \rangle$. The "midpoint" C is reached from A by a displacement of $\frac{\mathbf{d}}{2} = \langle \frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2} \rangle$. Shifting A by the displacement of $\frac{\mathbf{d}}{2}$ gives $C(x_1 + \frac{x_2 - x_1}{2}, y_1 + \frac{y_2 - y_1}{2}) = C(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$. Therefore the midpoint is:

$$C(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Examples:

- The midpoint between $A(-3, 2)$ and $B(4, -1)$ is $C(\frac{1}{2}, \frac{1}{2})$
- The midpoint between $A(11, -2)$ and $B(3, 5)$ is $C(7, \frac{3}{2})$
- The midpoint between $A(5, 8)$ and $B(7, 0)$ is $C(6, 4)$



The equations of circles

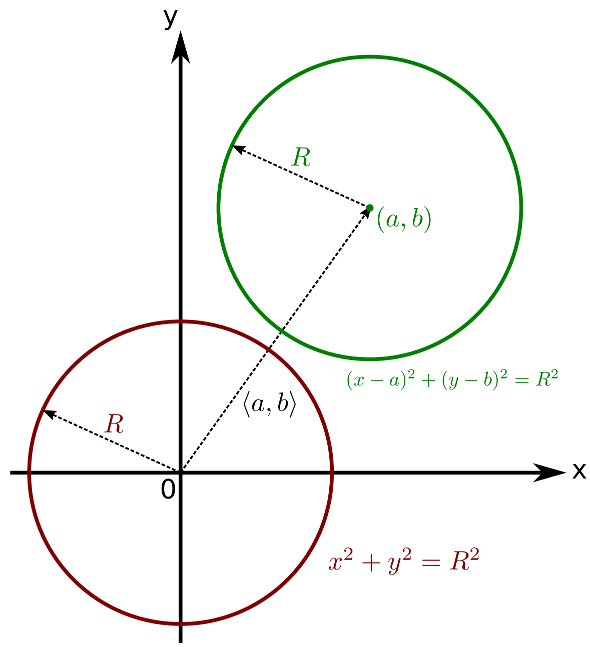
Given a radius $R > 0$, a circle of radius R that is centered on the origin is the set (locus) of all points that are a distance of R from the origin. A point (x, y) is a distance of R from the origin if and only if $\sqrt{x^2 + y^2} = R$ which is equivalent to $x^2 + y^2 = R^2$. The equation

$$x^2 + y^2 = R^2$$

defines a circle of radius R that is centered on the origin.
Centering a circle of radius R on the point (a, b) requires that the origin centered circle be shifted a displacement of $\langle a, b\rangle$. This yields the equation

$$(x - a)^2 + (y - b)^2 = R^2$$

which defines a circle of radius R that is centered on the point (a, b).



Examples:

- Given a point $A(1, 2)$ and radius of $R = 3$, a circle of radius R centered on point A has the equation:

$$(x - 1)^2 + (y - 2)^2 = 9$$

- Given a point $A(-3, 3)$ and radius of $R = 4$, a circle of radius R centered on point A has the equation:

$$(x + 3)^2 + (y - 3)^2 = 16$$

- Given points $A(2, 6)$ and $B(-1, -1)$, a circle that is centered of point A and that *passes through* point B is sought. The radius R of this circle is the distance between points A and B. $R = \sqrt{(-3)^2 + (-7)^2} = \sqrt{9 + 49} = \sqrt{58}$. The circle's equation is:

$$(x - 2)^2 + (y - 6)^2 = 58$$

- Given points $A(-4, 1)$ and $B(6, 9)$, a circle whose diameter is the line segment from A to B is sought. The radius R of the circle is half the distance between points A and B. $R = \frac{1}{2}\sqrt{10^2 + 8^2} = \frac{1}{2}\sqrt{100 + 64} = \frac{1}{2}\sqrt{164} = \sqrt{41}$. The circle is centered on the midpoint C between points A and B: $C(1, 5)$. The circle's equation is:

$$(x - 1)^2 + (y - 5)^2 = 41$$

- Given points $A(3, 8)$ and $B(-7, 10)$, a circle whose diameter is the line segment from A to B is sought. The radius R of the circle is half the distance between points A and B. $R = \frac{1}{2}\sqrt{(-10)^2 + 2^2} = \frac{1}{2}\sqrt{100 + 4} = \frac{1}{2}\sqrt{104} = \sqrt{26}$. The circle is centered on the midpoint C between points A and B: $C(-2, 9)$. The circle's equation is:

$$(x + 2)^2 + (y - 9)^2 = 26$$