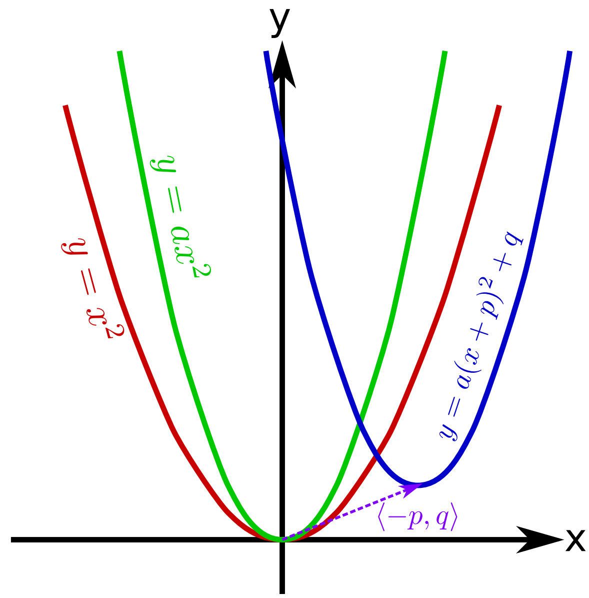
Parabolas



The general parabola has an equation with the form:

$$y = a(x+p)^2 + q$$

where a, p, and q are fixed constants.
The shape of this parabola is a "cup" as seen to the right. The **turning point** of a parabola is the apex of the parabola, the point at which the value of y transitions, as x increases, from a state of decreasing to increasing ($a > 0$), or from increasing to decreasing ($a < 0$). The turning point will always be located at the point $(-p, q)$.
The basic parabola has the equation

$$y = x^2$$

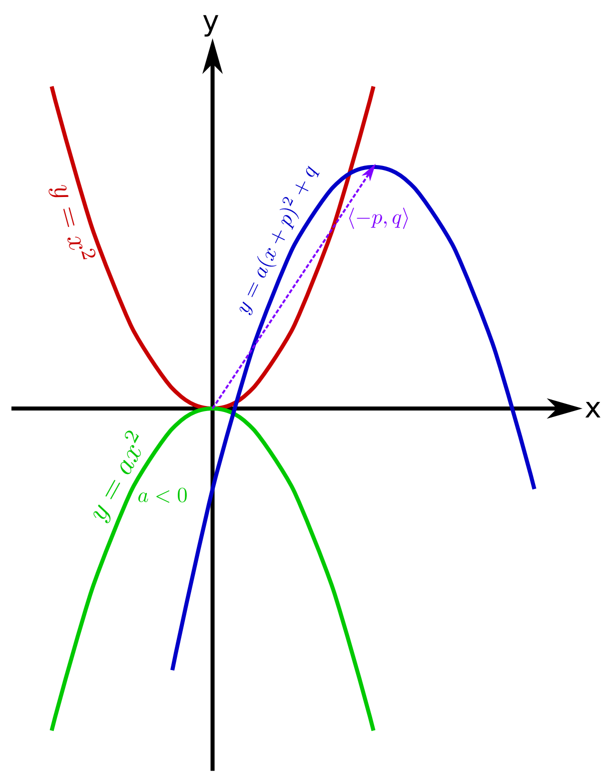
This parabola is the red curves on the right. In the case of $y = x^2$, the turning point is $(0,0)$.
Stretching this parabola vertically by a scale of a gives the equation:

$$y/a = x^2 \iff y = ax^2$$

These parabolas are the green curves on the right. The parabola is vertically flipped if $a < 0$ as seen in the lower image on the right.
Shifting the stretched parabola through a displacement of $\langle -p, q \rangle$ gives the equation:

$$y - q = a(x - (-p))^2 \iff y = a(x+p)^2 + q$$

These parabolas are the blue curves on the right. The turning point is $(-p, q)$.



Every equation of the form

$$y = ax^2 + bx + c$$

graphs a parabola. Completing the square is used to convert the equation $y = ax^2 + bx + c$ to the form:

$$y = a(x + p)^2 + q$$

Staring from

$$y = ax^2 + bx + c$$

Firstly, a is factored from the first two terms to give:

$$y = a(x^2 + \frac{b}{a}x) + c$$

Next, the square of a binomial is completed by adding and subtracting the final term:

$$y = a((x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2) - (\frac{b}{2a})^2) + c$$

Lastly collapsing the square expression $x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2$ to $(x + \frac{b}{2a})^2$ gives:

$$y = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$$

Several examples are given below:

Examples:

- Consider the quadratic equation:

  $$y = x^2 + 14x + 40$$

  $$y = x^2 + 14x + 40 = ((x^2 + 2 \cdot 7 \cdot x + 7^2) - 49) + 40 = (x + 7)^2 - 9$$

  Therefore:

  $$y = (x + 7)^2 - 9$$

  The parabola's turning point is $(-7, -9)$

- Consider the quadratic equation:

  $$y = 2x^2 - 20x + 48$$

  $$\begin{aligned} y =& 2x^2 - 20x + 48 = 2(x^2 - 10x) + 48 = 2((x^2 + 2 \cdot (-5) \cdot x + (-5)^2) - 25) + 48 \\ =& 2(x - 5)^2 - 2 \end{aligned}$$

  Therefore:

  $$y = 2(x - 5)^2 - 2$$

  The parabola's turning point is $(5, -2)$

- Consider the quadratic equation:

  $$y = -x^2 - 6x - 5$$

  $$\begin{aligned} y =& -x^2 - 6x - 5 = -(x^2 + 6x) - 5 = -((x^2 + 2 \cdot 3 \cdot x + 3^2) - 9) - 5 \\ =& -(x + 3)^2 + 4 \end{aligned}$$

  Therefore:

  $$y = -(x + 3)^2 + 4$$

  The parabola's turning point is $(-3, 4)$

- Consider the quadratic equation:
$$y = \frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3}$$

$$\begin{aligned} y =& \frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3} = \frac{1}{3}(x^2 + 2x) - \frac{8}{3} = \frac{1}{3}((x^2 + 2 \cdot 1 \cdot x + 1^2) - 1) - \frac{8}{3} \\ =& \frac{1}{3}(x+1)^2 - 3 \end{aligned}$$

  Therefore:
$$y = \frac{1}{3}(x+1)^2 - 3$$

  The parabola's turning point is $(-1, -3)$

- Consider the quadratic equation:
$$y = 5x^2 - 20x + 20$$

$$\begin{aligned} y =& 5x^2 - 20x + 20 = 5(x^2 - 4x) + 20 = 5((x^2 + 2 \cdot (-2) \cdot x + (-2)^2) - 4) + 20 \\ =& 5(x-2)^2 \end{aligned}$$

  Therefore:
$$y = 5(x-2)^2$$

  The parabola's turning point is $(2, 0)$

- Consider the quadratic equation:
$$y = -4x^2 + 16x - 32$$

$$\begin{aligned} y =& -4x^2 + 16x - 32 = -4(x^2 - 4x) - 32 = -4((x^2 + 2 \cdot (-2) \cdot x + (-2)^2) - 4) - 32 \\ =& -4(x-2)^2 - 16 \end{aligned}$$

  Therefore:
$$y = -4(x-2)^2 - 16$$

  The parabola's turning point is $(2, -16)$

- Consider the quadratic equation:
$$y = \frac{1}{5}x^2 + \frac{6}{5}x + 2$$

$$\begin{aligned} y =& \frac{1}{5}x^2 + \frac{6}{5}x + 2 = \frac{1}{5}(x^2 + 6x) + 2 = \frac{1}{5}((x^2 + 2 \cdot 3 \cdot x + 3^2) - 9) + 2 \\ =& \frac{1}{5}(x+3)^2 + \frac{1}{5} \end{aligned}$$

  Therefore:
$$y = \frac{1}{5}(x+3)^2 + \frac{1}{5}$$

  The parabola's turning point is $(-3, 1/5)$

- Consider the quadratic equation:

$$y = -2x^2 + 24x - 64$$

$$\begin{aligned} y =& -2x^2 + 24x - 64 = -2(x^2 - 12x) - 64 = -2((x^2 + 2 \cdot (-6) \cdot x + (-6)^2) - 36) - 64 \\ =& -2(x-6)^2 + 8 \end{aligned}$$

Therefore:

$$y = -2(x-6)^2 + 8$$

The parabola's turning point is $(6, 8)$

- Consider the quadratic equation:

$$y = 9x^2 - 6x - 35$$

$$\begin{aligned} y =& 9x^2 - 6x - 35 = 9(x^2 - \frac{2}{3}x) - 35 = 9((x^2 + 2 \cdot (-\frac{1}{3}) \cdot x + (-\frac{1}{3})^2) - \frac{1}{9}) - 35 \\ =& 9(x - \frac{1}{3})^2 - 36 \end{aligned}$$

Therefore:

$$y = 9(x - \frac{1}{3})^2 - 36$$

The parabola's turning point is $(1/3, -36)$

- Consider the quadratic equation:

$$y = 8x^2 + 40x + 32$$

$$\begin{aligned} y =& 8x^2 + 40x + 32 = 8(x^2 + 5x) + 32 = 8((x^2 + 2 \cdot \frac{5}{2} \cdot x + (\frac{5}{2})^2) - \frac{25}{4}) + 32 \\ =& 8(x + \frac{5}{2})^2 - 18 \end{aligned}$$

Therefore:

$$y = 8(x + \frac{5}{2})^2 - 18$$

The parabola's turning point is $(-5/2, -18)$

- Consider the quadratic equation:

$$y = -7x^2 + 21x + 70$$

$$\begin{aligned} y =& -7x^2 + 21x + 70 = -7(x^2 - 3x) + 70 = -7((x^2 + 2 \cdot (-\frac{3}{2}) \cdot x + (-\frac{3}{2})^2) - \frac{9}{4}) + 70 \\ =& -7(x - \frac{3}{2})^2 + \frac{343}{4} \end{aligned}$$

Therefore:

$$y = -7(x - \frac{3}{2})^2 + \frac{343}{4}$$

The parabola's turning point is $(3/2, 343/4)$

- Consider the quadratic equation:
$$y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$$
$$\begin{aligned} y =& -\frac{1}{2}x^2 + \frac{1}{2}x + 3 = -\frac{1}{2}(x^2 - x) + 3 = -\frac{1}{2}((x^2 + 2 \cdot (-\frac{1}{2}) \cdot x + (-\frac{1}{2})^2) - \frac{1}{4}) + 3 \\ =& -\frac{1}{2}(x - \frac{1}{2})^2 + \frac{25}{8} \end{aligned}$$
Therefore:
$$y = -\frac{1}{2}(x - \frac{1}{2})^2 + \frac{25}{8}$$
The parabola's turning point is $(1/2, 25/8)$

- Consider the quadratic equation:
$$y = -10x^2 + 10x$$
$$\begin{aligned} y =& -10x^2 + 10x = -10(x^2 - x) = -10((x^2 + 2 \cdot (-\frac{1}{2}) \cdot x + (-\frac{1}{2})^2) - \frac{1}{4}) \\ =& -10(x - \frac{1}{2})^2 + \frac{5}{2} \end{aligned}$$
Therefore:
$$y = -10(x - \frac{1}{2})^2 + \frac{5}{2}$$
The parabola's turning point is $(1/2, 5/2)$

- Consider the quadratic equation:
$$y = 4x^2 + 4x - 3$$
$$\begin{aligned} y =& 4x^2 + 4x - 3 = 4(x^2 + x) - 3 = 4((x^2 + 2 \cdot \frac{1}{2} \cdot x + (\frac{1}{2})^2) - \frac{1}{4}) - 3 \\ =& 4(x + \frac{1}{2})^2 - 4 \end{aligned}$$
Therefore:
$$y = 4(x + \frac{1}{2})^2 - 4$$
The parabola's turning point is $(-1/2, -4)$

- Consider the quadratic equation:
$$y = -6x^2 + 13x - 6$$
$$\begin{aligned} y =& -6x^2 + 13x - 6 = -6(x^2 - \frac{13}{6}x) - 6 = -6((x^2 + 2 \cdot (-\frac{13}{12}) \cdot x + (-\frac{13}{12})^2) - \frac{169}{144}) - 6 \\ =& -6(x - \frac{13}{12})^2 + \frac{169}{24} - 6 = -6(x - \frac{13}{12})^2 + \frac{25}{24} \end{aligned}$$
Therefore:
$$y = -6(x - \frac{13}{12})^2 + \frac{25}{24}$$
The parabola's turning point is $(13/12, 25/24)$

Systems using quadratic equations

Example 1:

$$\begin{cases} y - x = 5 \\ x^2 + y^2 - 14y = -39 \end{cases}$$

Solving the first equation for y gives:

$$y - x = 5 \iff y = x + 5$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 - 14y = -39 \iff x^2 + (x^2 + 10x + 25) + (-14x - 70) = -39$$
$$\iff 2x^2 - 4x - 6 = 0 \iff x^2 - 2x - 3 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = -2$; and $c = -3$. The discriminant is:

$$\Delta = b^2 - 4ac = 4 + 12 = 16$$

$\Delta > 0$ so there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 4}{2} = \frac{6}{2}, \frac{-2}{2} = 3, -1$$

With the values of x now known, the expression $y = x + 5$ can now be used to compute corresponding values of y.

- When $x = 3$, $y = 8$
- When $x = -1$, $y = 4$

Therefore there are two possible solutions:

$$(x, y) = (3, 8), (-1, 4)$$

Example 2:

$$\begin{cases} 7y - x = 10 \\ x^2 + y^2 - 2x + 4y = 20 \end{cases}$$

Solving the first equation for y gives:

$$7y - x = 10 \iff 7y = x + 10 \iff y = (1/7)x + 10/7$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 - 2x + 4y = 20 \iff x^2 + ((1/49)x^2 + (20/49)x + 100/49) - 2x + ((4/7)x + 40/7) = 20$$
$$\iff (50/49)x^2 - (50/49)x - 600/49 = 0 \iff x^2 - x - 12 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = -1$; and $c = -12$. The discriminant is:

$$\Delta = b^2 - 4ac = 1 + 48 = 49$$

$\Delta > 0$ so there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 7}{2} = \frac{8}{2}, \frac{-6}{2} = 4, -3$$

With the values of x now known, the expression $y = (1/7)x + 10/7$ can now be used to compute corresponding values of y.

- When $x = 4$, $y = 4/7 + 10/7 = 2$
- When $x = -3$, $y = -3/7 + 10/7 = 1$

Therefore there are two possible solutions:

$$(x, y) = (4, 2), (-3, 1)$$

Example 3:

$$\begin{cases} -3x + 2y = 1 \\ x^2 + y^2 + 4x - 8y = 6 \end{cases}$$

Solving the first equation for y gives:

$$-3x + 2y = 1 \iff 2y = 3x + 1 \iff y = (3/2)x + 1/2$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 + 4x - 8y = 6 \iff x^2 + ((9/4)x^2 + (3/2)x + 1/4) + 4x + (-12x - 4) = 6$$
$$\iff (13/4)x^2 - (13/2)x - 39/4 = 0 \iff x^2 - 2x - 3 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = -2$; and $c = -3$. The discriminant is:

$$\Delta = b^2 - 4ac = 4 + 12 = 16$$

$\Delta > 0$ so there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 4}{2} = \frac{6}{2}, \frac{-2}{2} = 3, -1$$

With the values of x now known, the expression $y = (3/2)x + 1/2$ can now be used to compute corresponding values of y.

- When $x = 3$, $y = 9/2 + 1/2 = 5$
- When $x = -1$, $y = -3/2 + 1/2 = -1$

Therefore there are two possible solutions:

$$(x, y) = (3, 5), (-1, -1)$$

Example 4:

$$\begin{cases} -3x + 2y = 1 \\ x^2 + y^2 + 4x - 8y = -7 \end{cases}$$

Solving the first equation for y gives:

$$-3x + 2y = 1 \iff 2y = 3x + 1 \iff y = (3/2)x + 1/2$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 + 4x - 8y = -7 \iff x^2 + ((9/4)x^2 + (3/2)x + 1/4) + 4x + (-12x - 4) = -7$$
$$\iff (13/4)x^2 - (13/2)x + 13/4 = 0 \iff x^2 - 2x + 1 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = -2$; and $c = 1$. The discriminant is:

$$\Delta = b^2 - 4ac = 4 - 4 = 0$$

$\Delta = 0$ so there is one solution:

$$x = \frac{-b}{2a} = \frac{2}{2} = 1$$

With the value of x now known, the expression $y = (3/2)x + 1/2$ can now be used to compute the corresponding value of y.

- When $x = 1$, $y = 3/2 + 1/2 = 2$

Therefore there is one solution:

$$(x, y) = (1, 2)$$

Example 5:

$$\begin{cases} -3x + 2y = 1 \\ x^2 + y^2 + 4x - 8y = -16 \end{cases}$$

Solving the first equation for y gives:

$$-3x + 2y = 1 \iff 2y = 3x + 1 \iff y = (3/2)x + 1/2$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 + 4x - 8y = -16 \iff x^2 + ((9/4)x^2 + (3/2)x + 1/4) + 4x + (-12x - 4) = -16$$
$$\iff (13/4)x^2 - (13/2)x + 49/4 = 0 \iff 13x^2 - 26x + 49 = 0$$

The coefficients in the quadratic equation are $a = 13$; $b = -26$; and $c = 49$. The discriminant is:

$$\Delta = b^2 - 4ac = 676 - 2548 = -1872$$

$\Delta < 0$ so there is **no solutions**

Example 6:

$$\begin{cases} x + 3y = 1 \\ x^2 + y^2 - 4x - 6y = 7 \end{cases}$$

Solving the first equation for y gives:

$$x + 3y = 1 \iff 3y = -x + 1 \iff y = -(1/3)x + 1/3$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 - 4x - 6y = 7 \iff x^2 + ((1/9)x^2 - (2/9)x + 1/9) - 4x + (2x - 2) = 7$$
$$\iff (10/9)x^2 - (20/9)x - 80/9 = 0 \iff x^2 - 2x - 8 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = -2$; and $c = -8$. The discriminant is:

$$\Delta = b^2 - 4ac = 4 + 32 = 36$$

$\Delta > 0$ so there are two solutions:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 6}{2} = \frac{8}{2}, \frac{-4}{2} = 4, -2$$

With the values of x now known, the expression $y = -(1/3)x+1/3$ can now be used to compute corresponding values of y.

- When $x = 4$, $y = -4/3 + 1/3 = -1$
- When $x = -2$, $y = 2/3 + 1/3 = 1$

Therefore there are two possible solutions:

$$(x, y) = (4, -1), (-2, 1)$$

Example 7:

$$\begin{cases} x + y + 6 = 0 \\ x^2 + y^2 - 2x - 6y = 40 \end{cases}$$

Solving the first equation for y gives:

$$x + y + 6 = 0 \iff y = -x - 6$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 - 2x - 6y = 40 \iff x^2 + (x^2 + 12x + 36) - 2x + (6x + 36) = 40$$
$$\iff 2x^2 + 16x + 32 = 0 \iff x^2 + 8x + 16 = 0$$

The coefficients in the quadratic equation are $a = 1$; $b = 8$; and $c = 16$. The discriminant is:

$$\Delta = b^2 - 4ac = 64 - 64 = 0$$

$\Delta = 0$ so there is one solution:

$$x = \frac{-b}{2a} = -8/2 = -4$$

With the value of x now known, the expression $y = -x - 6$ can now be used to compute the corresponding value of y.

- When $x = -4$, $y = 4 - 6 = -2$

Therefore there is one solution:

$$(x, y) = (-4, -2)$$

Example 8:

$$\begin{cases} 2x - y = 3 \\ x^2 + y^2 + 4x - 6y = -9 \end{cases}$$

Solving the first equation for y gives:

$$2x - y = 3 \iff -y = -2x + 3 \iff y = 2x - 3$$

With an expression for y in terms of x known, replacing y in the second equation with the expression of x gives:

$$x^2 + y^2 + 4x - 6y = -9 \iff x^2 + (4x^2 - 12x + 9) + 4x + (-12x + 18) = -9$$
$$\iff 5x^2 - 20x + 36 = 0$$

The coefficients in the quadratic equation are $a = 5$; $b = -20$; and $c = 36$. The discriminant is:

$$\Delta = b^2 - 4ac = 400 - 720 = -320$$

$\Delta < 0$ so there is **no solutions**