Multi-variable Functions

Question 1:

Draw the domains of the following multi-variable functions. For curves, use solid lines to include the curve as part of the domain, and use dashed lines to exclude the curve from the domain.

- $f(x, y) = \frac{\sqrt{-x^2+4x}}{\sqrt{9-x^2-y^2}}$
- $f(x, y) = \ln(x + y - x^2)$
- $f(x, y) = \frac{\ln(x)}{xy+2x-3y-6}$

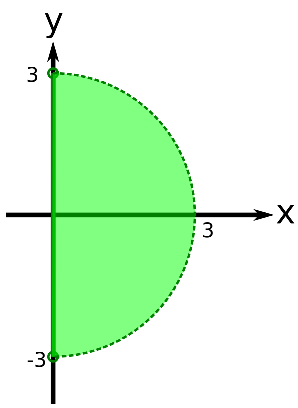
Solution:

For $f(x, y) = \frac{\sqrt{-x^2+4x}}{\sqrt{9-x^2-y^2}}$, the required conditions on the input pair are $-x^2 + 4x \geq 0$ so the numerator is real valued, and $9 - x^2 - y^2 > 0$ so the denominator is both real valued and nonzero.

The condition $-x^2 + 4x \geq 0$ is equivalent to $x(x - 4) \leq 0 \iff (0 \leq x)$ and $(x \leq 4)$

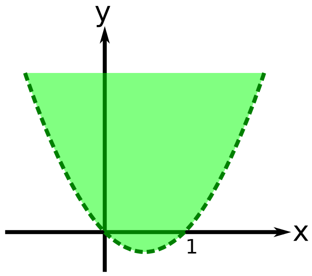
The condition $9 - x^2 - y^2 > 0$ is equivalent to $\sqrt{x^2 + y^2} < 3$

The domain can then be drawn from the conditions $(0 \leq x)$ and $(x \leq 4)$ and $\sqrt{x^2 + y^2} < 3$:



For $f(x, y) = \ln(x + y - x^2)$, the required condition on the input pair is $x + y - x^2 > 0$ so that ln returns a real value. $x + y - x^2 > 0$ is equivalent to $y > x^2 - x$.

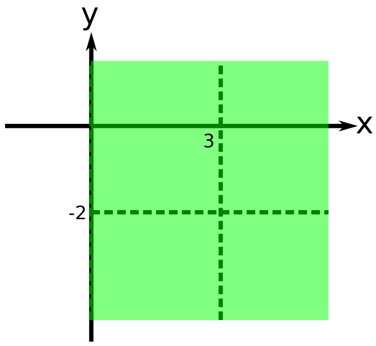
The domain can be drawn from the condition $y > x^2 - x$:



For $f(x,y) = \frac{\ln(x)}{xy+2x-3y-6}$, the required conditions on the input pair are $x > 0$ so the numerator is real valued, and $xy + 2x - 3y - 6 \neq 0$ so the denominator is nonzero.

The condition $xy + 2x - 3y - 6 \neq 0$ is equivalent to $(x-3)(y+2) \neq 0 \iff (x \neq 3)$ and $(y \neq -2)$

The domain can then be drawn from the conditions $(x > 0)$ and $(x \neq 3)$ and $(y \neq -2)$:



Question 2:

Compute the following limits:

- $\lim_{t\to-1} \begin{bmatrix} \sqrt{t+3} \\ \frac{t^2}{t+2} \\ \ln(t+5) \end{bmatrix}$
- $\lim_{(x,y)\to(-1,-2)} \frac{\sqrt{x+y+5}}{x+y+4}$

Solution:

For $\begin{bmatrix} \sqrt{t+3} \\ \frac{t^2}{t+2} \\ \ln(t+5) \end{bmatrix}$, there are no discontinuous singularities at $t = -1$. Therefore $\lim_{t\to-1} \begin{bmatrix} \sqrt{t+3} \\ \frac{t^2}{t+2} \\ \ln(t+5) \end{bmatrix} = \begin{bmatrix} \sqrt{-1+3} \\ \frac{(-1)^2}{-1+2} \\ \ln(-1+5) \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \\ \ln(4) \end{bmatrix}$

For $\frac{\sqrt{x+y+5}}{x+y+4}$, there are no discontinuous singularities at $(x,y)=(-1,-2)$. Therefore $\lim_{(x,y)\to(-1,-2)} \frac{\sqrt{x+y+5}}{x+y+4} = \frac{\sqrt{(-1)+(-2)+5}}{(-1)+(-2)+4} = \sqrt{2}$

Question 3:

Define the two-variable function $f(x,y) = \frac{xy}{x^2+y^2}$.

part 3a:

Compute the following partial derivatives:

$$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial^2 f}{\partial x^2} \quad \frac{\partial^2 f}{\partial y^2} \quad \frac{\partial^2 f}{\partial y \partial x}$$

Solution:

Using the quotient rule, the first order partial derivatives can be computed:

$$\frac{\partial f}{\partial x} = \frac{(y)(x^2+y^2)-(xy)(2x)}{(x^2+y^2)^2} = \frac{-x^2y+y^3}{(x^2+y^2)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{(x)(x^2+y^2)-(xy)(2y)}{(x^2+y^2)^2} = \frac{x^3-xy^2}{(x^2+y^2)^2}$$

Further using the quotient rule, the second order partial derivatives can be computed:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{(-2xy)(x^2+y^2)^2-(-x^2y+y^3)(2(x^2+y^2)(2x))}{(x^2+y^2)^4}\\ &= \frac{-2xy(x^2+y^2)-4x(-x^2y+y^3)}{(x^2+y^2)^3} = \frac{2x^3y-6xy^3}{(x^2+y^2)^3}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{(-2xy)(x^2+y^2)^2-(x^3-xy^2)(2(x^2+y^2)(2y))}{(x^2+y^2)^4}\\ &= \frac{-2xy(x^2+y^2)-4y(x^3-xy^2)}{(x^2+y^2)^3} = \frac{-6x^3y+2xy^3}{(x^2+y^2)^3}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{(-x^2+3y^2)(x^2+y^2)^2-(-x^2y+y^3)(2(x^2+y^2)(2y))}{(x^2+y^2)^4}\\ &= \frac{(-x^2+3y^2)(x^2+y^2)-4y(-x^2y+y^3)}{(x^2+y^2)^3}\\ &= \frac{(-x^4+2x^2y^2+3y^4)+(4x^2y^2-4y^4)}{(x^2+y^2)^3} = \frac{-x^4+6x^2y^2-y^4}{(x^2+y^2)^3}\end{aligned}$$

part 3b:

Compute the gradient ∇f at the point $(x_0, y_0) = (3, 1)$. Compute the equation of the tangent plane to the surface $z = f(x, y)$ that passes through the point $(x_0, y_0, f(x_0, y_0))$. Given a direction of $\mathbf{v} = \langle 3, -4 \rangle$, what is the "directional derivative" of $f(x, y)$ at (x_0, y_0) in the direction of $\mathbf{v}$?

Solution:

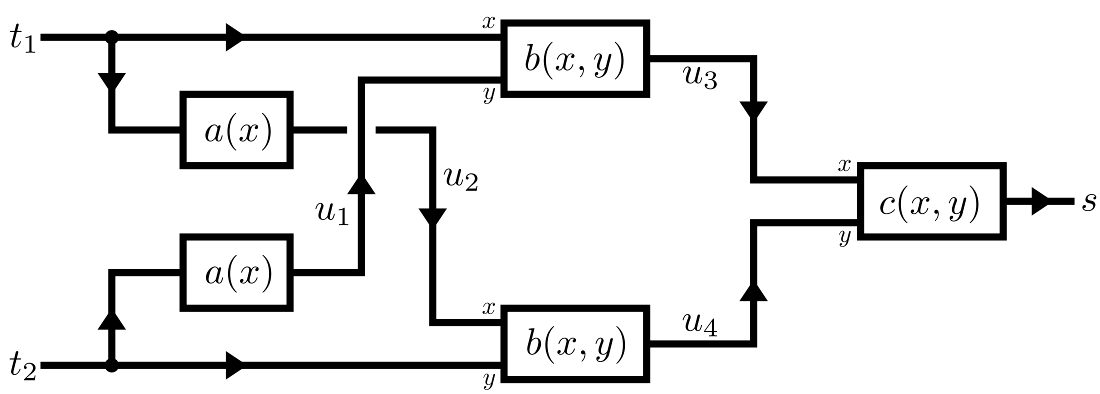
$$\nabla f|_{(3,1)} = \left\langle \frac{-9+1}{(9+1)^2}, \frac{27-3}{(9+1)^2} \right\rangle = \left\langle \frac{-8}{100}, \frac{24}{100} \right\rangle = \left\langle \frac{-2}{25}, \frac{6}{25} \right\rangle$$

From $f(3, 1) = \frac{3}{9+1} = \frac{3}{10}$ and $\nabla f|_{(3,1)} = \langle \frac{-2}{25}, \frac{6}{25} \rangle$, the tangent plane that passes through the point $(3, 1, 3/10)$ is

$$z = \frac{3}{10} + \frac{-2}{25}(x-3) + \frac{6}{25}(y-1) \iff z = -\frac{2}{25}x + \frac{6}{25}y + \frac{3}{10}$$

A unit vector that shares the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{9+16}}\langle 3, -4 \rangle = \langle \frac{3}{5}, -\frac{4}{5} \rangle$. The directional derivative at $(3, 1)$ in the direction of $\mathbf{v}$ is $\mathbf{u} \cdot (\nabla f|_{(3,1)}) = \frac{3}{5} \cdot \frac{-2}{25} + (-\frac{4}{5}) \cdot \frac{6}{25} = \frac{-6}{125} + \frac{-24}{125} = \frac{-30}{125} = \frac{-6}{25}$.

Question 4:



In the flow-chart (arithmetic circuit) above, the output quantity s is being computed from input quantities t_1 and t_2. There are the internal variables u_1, u_2, u_3, and u_4.

part 4a:

Build expressions for u_1, u_2, u_3, u_4, and s from the input parameters t_1 and t_2, and the functions $a(x)$, $b(x, y)$, and $c(x, y)$.

Solution:

- $u_1 = a(t_2)$

- $u_2 = a(t_1)$
- $u_3 = b(t_1, u_1) = b(t_1, a(t_2))$
- $u_4 = b(u_2, t_2) = b(a(t_1), t_2)$
- $s = c(u_3, u_4) = c(b(t_1, a(t_2)), b(a(t_1), t_2))$

part 4b:

Without any knowledge of $a(x)$, $b(x,y)$, or $c(x,y)$, derive expressions for the following partial derivatives: $\frac{\partial u_1}{\partial t_1}$, $\frac{\partial u_1}{\partial t_2}$, $\frac{\partial u_2}{\partial t_1}$, and $\frac{\partial u_2}{\partial t_2}$.

Derive expressions for the following partial derivatives: $\frac{\partial u_3}{\partial t_1}$, $\frac{\partial u_3}{\partial t_2}$, $\frac{\partial u_4}{\partial t_1}$, and $\frac{\partial u_4}{\partial t_2}$ in terms of the partial derivatives computed previously.

Derive expressions for the following partial derivatives: $\frac{\partial s}{\partial t_1}$, and $\frac{\partial s}{\partial t_2}$ in terms of the partial derivatives computed previously.

Solution:

u_1 is the output of function a with input t_2 so $\frac{\partial u_1}{\partial t_1} = \frac{da}{dx}\Big|_{t_2}(0) = 0$ and $\frac{\partial u_1}{\partial t_2} = \frac{da}{dx}\Big|_{t_2}(1) = \frac{da}{dx}\Big|_{t_2}$
u_2 is the output of function a with input t_1 so $\frac{\partial u_2}{\partial t_1} = \frac{da}{dx}\Big|_{t_1}(1) = \frac{da}{dx}\Big|_{t_1}$ and $\frac{\partial u_2}{\partial t_2} = \frac{da}{dx}\Big|_{t_1}(0) = 0$
u_3 is the output of function b with respective inputs t_1 and u_1 so

$$\frac{\partial u_3}{\partial t_1} = \frac{\partial b}{\partial x}\bigg|_{(t_1,u_1)}(1) + \frac{\partial b}{\partial y}\bigg|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_1} = \frac{\partial b}{\partial x}\bigg|_{(t_1,u_1)} + \frac{\partial b}{\partial y}\bigg|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_1}$$

and

$$\frac{\partial u_3}{\partial t_2} = \frac{\partial b}{\partial x}\bigg|_{(t_1,u_1)}(0) + \frac{\partial b}{\partial y}\bigg|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_2} = \frac{\partial b}{\partial y}\bigg|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_2}$$

u_4 is the output of function b with respective inputs u_2 and t_2 so

$$\frac{\partial u_4}{\partial t_1} = \frac{\partial b}{\partial x}\bigg|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_1} + \frac{\partial b}{\partial y}\bigg|_{(u_2,t_2)}(0) = \frac{\partial b}{\partial x}\bigg|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_1}$$

and

$$\frac{\partial u_4}{\partial t_2} = \frac{\partial b}{\partial x}\bigg|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_2} + \frac{\partial b}{\partial y}\bigg|_{(u_2,t_2)}(1) = \frac{\partial b}{\partial x}\bigg|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_2} + \frac{\partial b}{\partial y}\bigg|_{(u_2,t_2)}$$

s is the output of function c with respective inputs u_3 and u_4 so

$$\frac{\partial s}{\partial t_1} = \frac{\partial c}{\partial x}\bigg|_{(u_3,u_4)} \cdot \frac{\partial u_3}{\partial t_1} + \frac{\partial c}{\partial y}\bigg|_{(u_3,u_4)} \cdot \frac{\partial u_4}{\partial t_1}$$

and

$$\frac{\partial s}{\partial t_2} = \frac{\partial c}{\partial x}\bigg|_{(u_3,u_4)} \cdot \frac{\partial u_3}{\partial t_2} + \frac{\partial c}{\partial y}\bigg|_{(u_3,u_4)} \cdot \frac{\partial u_4}{\partial t_2}$$

In summary:

- $\frac{\partial u_1}{\partial t_1} = 0$ and $\frac{\partial u_1}{\partial t_2} = \frac{da}{dx}\Big|_{t_2}$
- $\frac{\partial u_2}{\partial t_1} = \frac{da}{dx}\Big|_{t_1}$ and $\frac{\partial u_2}{\partial t_2} = 0$

- $\frac{\partial u_3}{\partial t_1} = \frac{\partial b}{\partial x}\Big|_{(t_1,u_1)} + \frac{\partial b}{\partial y}\Big|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_1}$ and $\frac{\partial u_3}{\partial t_2} = \frac{\partial b}{\partial y}\Big|_{(t_1,u_1)} \cdot \frac{\partial u_1}{\partial t_2}$
- $\frac{\partial u_4}{\partial t_1} = \frac{\partial b}{\partial x}\Big|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_1}$ and $\frac{\partial u_4}{\partial t_2} = \frac{\partial b}{\partial x}\Big|_{(u_2,t_2)} \cdot \frac{\partial u_2}{\partial t_2} + \frac{\partial b}{\partial y}\Big|_{(u_2,t_2)}$
- $\frac{\partial s}{\partial t_1} = \frac{\partial c}{\partial x}\Big|_{(u_3,u_4)} \cdot \frac{\partial u_3}{\partial t_1} + \frac{\partial c}{\partial y}\Big|_{(u_3,u_4)} \cdot \frac{\partial u_4}{\partial t_1}$ and $\frac{\partial s}{\partial t_2} = \frac{\partial c}{\partial x}\Big|_{(u_3,u_4)} \cdot \frac{\partial u_3}{\partial t_2} + \frac{\partial c}{\partial y}\Big|_{(u_3,u_4)} \cdot \frac{\partial u_4}{\partial t_2}$

part 4c:

Now let $a(x) = 1 - x$, $b(x, y) = xy$, and $c(x, y) = x + y - xy$. Compute all first-order derivatives: $\frac{da}{dx}$, $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$, $\frac{\partial c}{\partial x}$, and $\frac{\partial c}{\partial y}$.

Solution:

- $\frac{da}{dx} = -1$
- $\frac{\partial b}{\partial x} = y$ and $\frac{\partial b}{\partial y} = x$
- $\frac{\partial c}{\partial x} = 1 - y$ and $\frac{\partial c}{\partial y} = 1 - x$

part 4d:

From the results of the previous sections, compute at $(t_1, t_2) = (3/4, 1/4)$ the output s, as well as the partial derivatives $\frac{\partial s}{\partial t_1}$ and $\frac{\partial s}{\partial t_2}$.

Solution:

From $t_1 = 3/4$ and $t_2 = 1/4$, $u_1 = a(t_2) = 3/4$, $u_2 = a(t_1) = 1/4$, $u_3 = b(t_1, u_1) = (3/4)(3/4) = 9/16$, $u_4 = b(u_2, t_2) = (1/4)(1/4) = 1/16$, and $s = c(u_3, u_4) = 9/16 + 1/16 - (9/16)(1/16) = 10/16 - 9/256 = (160 - 9)/256 = 151/256$.

$\frac{\partial u_1}{\partial t_1} = 0$ and $\frac{\partial u_1}{\partial t_2} = \frac{da}{dx}\Big|_{1/4} = -1$

$\frac{\partial u_2}{\partial t_1} = \frac{da}{dx}\Big|_{3/4} = -1$ and $\frac{\partial u_2}{\partial t_2} = 0$

$\frac{\partial u_3}{\partial t_1} = \frac{\partial b}{\partial x}\Big|_{(3/4,3/4)} + \frac{\partial b}{\partial y}\Big|_{(3/4,3/4)} \cdot \frac{\partial u_1}{\partial t_1} = 3/4 + (3/4)(0) = 3/4$
and $\frac{\partial u_3}{\partial t_2} = \frac{\partial b}{\partial y}\Big|_{(3/4,3/4)} \cdot \frac{\partial u_1}{\partial t_2} = (3/4)(-1) = -3/4$

$\frac{\partial u_4}{\partial t_1} = \frac{\partial b}{\partial x}\Big|_{(1/4,1/4)} \cdot \frac{\partial u_2}{\partial t_1} = (1/4)(-1) = -1/4$
and $\frac{\partial u_4}{\partial t_2} = \frac{\partial b}{\partial x}\Big|_{(1/4,1/4)} \cdot \frac{\partial u_2}{\partial t_2} + \frac{\partial b}{\partial y}\Big|_{(1/4,1/4)} = (1/4)(0) + 1/4 = 1/4$

$\frac{\partial s}{\partial t_1} = \frac{\partial c}{\partial x}\Big|_{(9/16,1/16)} \cdot \frac{\partial u_3}{\partial t_1} + \frac{\partial c}{\partial y}\Big|_{(9/16,1/16)} \cdot \frac{\partial u_4}{\partial t_1} = (1 - 1/16)(3/4) + (1 - 9/16)(-1/4) = (15/16)(3/4) + (7/16)(-1/4) = (45 - 7)/64 = 38/64 = 19/32$

and $\frac{\partial s}{\partial t_2} = \frac{\partial c}{\partial x}\Big|_{(9/16,1/16)} \cdot \frac{\partial u_3}{\partial t_2} + \frac{\partial c}{\partial y}\Big|_{(9/16,1/16)} \cdot \frac{\partial u_4}{\partial t_2} = (1-1/16)(-3/4)+(1-9/16)(1/4) = (15/16)(-3/4)+(7/16)(1/4) = (-45+7)/64 = -38/64 = -19/32$

Therefore $s = 151/256$, $\frac{\partial s}{\partial t_1} = 19/32$ and $\frac{\partial s}{\partial t_2} = -19/32$.

Question 5:

For each of the following two variable functions $f(x,y)$, find and classify all of the critical points:

- $f(x,y) = -11x^2 + 6xy - 19y^2 + 78x - 94y - 211$
- $f(x,y) = \frac{1}{3}x^3 - \frac{7}{18}x^2 - \frac{2}{3}xy + y^2 - \frac{10}{9}x - \frac{8}{3}y + \frac{16}{9}$
- $f(x,y) = \frac{1}{3}x^3 + \frac{7}{16}x^2 - \frac{3}{2}xy - y^2 - \frac{3}{8}x + \frac{7}{2}y - \frac{49}{16}$
- $f(x,y) = \frac{1}{3}y^3 - x^2 - \frac{3}{2}xy + \frac{7}{16}y^2 - \frac{5}{2}x - \frac{39}{8}y - \frac{25}{16}$
- $f(x,y) = \frac{5}{4}x^4 - \frac{1}{3}x^3 - 2x^2y - 2x^2 + y^2 + 4x$

Solution:

For $f(x,y) = -11x^2 + 6xy - 19y^2 + 78x - 94y - 211$, the first and second order partial derivatives are: $\frac{\partial f}{\partial x} = -22x + 6y + 78$; $\frac{\partial f}{\partial y} = 6x - 38y - 94$; $\frac{\partial^2 f}{\partial x^2} = -22$; $\frac{\partial^2 f}{\partial y^2} = -38$; and $\frac{\partial^2 f}{\partial y \partial x} = 6$

Finding the critical points requires solving the system: $\begin{cases} \partial f/\partial x = 0 \\ \partial f/\partial y = 0 \end{cases} \iff \begin{cases} -22x + 6y + 78 = 0 \\ 6x - 38y - 94 = 0 \end{cases}$

The first equation gives $y = (11/3)x - 13$. Substituting into the second equation gives $6x + ((-418/3)x + 494) - 94 = 0 \iff (-400/3)x + 400 = 0 \iff x = 3$. This gives $y = -2$.

There is hence only one critical point $(x_c, y_c) = (3, -2)$.

The discriminant at $(3,-2)$ is $\Delta = (-22)(-38) - (6)^2 = 836 - 36 = 800 > 0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are negative, $(3,-2)$ is a local maximum.

For $f(x,y) = \frac{1}{3}x^3 - \frac{7}{18}x^2 - \frac{2}{3}xy + y^2 - \frac{10}{9}x - \frac{8}{3}y + \frac{16}{9}$, the first and second order partial derivatives are: $\frac{\partial f}{\partial x} = x^2 - \frac{7}{9}x - \frac{2}{3}y - \frac{10}{9}$; $\frac{\partial f}{\partial y} = -\frac{2}{3}x + 2y - \frac{8}{3}$; $\frac{\partial^2 f}{\partial x^2} = 2x - \frac{7}{9}$; $\frac{\partial^2 f}{\partial y^2} = 2$; and $\frac{\partial^2 f}{\partial y \partial x} = -\frac{2}{3}$

Finding the critical points requires solving the system: $\begin{cases} \partial f/\partial x = 0 \\ \partial f/\partial y = 0 \end{cases} \iff \begin{cases} x^2 - \frac{7}{9}x - \frac{2}{3}y - \frac{10}{9} = 0 \\ -\frac{2}{3}x + 2y - \frac{8}{3} = 0 \end{cases}$

The first equation gives $y = \frac{3}{2}x^2 - \frac{7}{6}x - \frac{5}{3}$. Substituting into the second equation gives $-\frac{2}{3}x + (3x^2 - \frac{7}{3}x - \frac{10}{3}) - \frac{8}{3} = 0 \iff 3x^2 - 3x - 6 = 0 \iff x^2 - x - 2 = 0 \iff (x+1)(x-2) = 0 \iff x = -1, 2$. $x = -1$ gives $y = \frac{3}{2} + \frac{7}{6} - \frac{5}{3} = \frac{9+7-10}{6} = 1$. $x = 2$ gives $y = 6 - \frac{7}{3} - \frac{5}{3} = 6 - \frac{12}{3} = 2$.

There are two critical points $(x_c, y_c) = (-1, 1); (2, 2)$.

The discriminant at $(-1,1)$ is $\Delta = (-\frac{25}{9})(2) - (-\frac{2}{3})^2 = -\frac{50}{9} - \frac{4}{9} = -6 < 0$. Hence $(-1,1)$ is a saddle point.

The discriminant at $(2,2)$ is $\Delta = (\frac{29}{9})(2) - (-\frac{2}{3})^2 = \frac{58}{9} - \frac{4}{9} = 6 > 0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are positive, $(2,2)$ is a local minimum.

For $f(x,y) = \frac{1}{3}x^3 + \frac{7}{16}x^2 - \frac{3}{2}xy - y^2 - \frac{3}{8}x + \frac{7}{2}y - \frac{49}{16}$, the first and second order partial derivatives are: $\frac{\partial f}{\partial x} = x^2 + \frac{7}{8}x - \frac{3}{2}y - \frac{3}{8}$; $\frac{\partial f}{\partial y} = -\frac{3}{2}x - 2y + \frac{7}{2}$; $\frac{\partial^2 f}{\partial x^2} = 2x + \frac{7}{8}$; $\frac{\partial^2 f}{\partial y^2} = -2$; and $\frac{\partial^2 f}{\partial y \partial x} = -\frac{3}{2}$

Finding the critical points requires solving the system: $\left\{\begin{array}{l}\partial f/\partial x=0\\ \partial f/\partial y=0\end{array}\right. \iff \left\{\begin{array}{c}x^2+\frac{7}{8}x-\frac{3}{2}y-\frac{3}{8}=0\\ -\frac{3}{2}x-2y+\frac{7}{2}=0\end{array}\right.$

The first equation gives $y=\frac{2}{3}x^2+\frac{7}{12}x-\frac{1}{4}$. Substituting into the second equation gives $-\frac{3}{2}x+(-\frac{4}{3}x^2-\frac{7}{6}x+\frac{1}{2})+\frac{7}{2}=0 \iff -\frac{4}{3}x^2-\frac{16}{6}x+4=0 \iff x^2+2x-3=0 \iff (x+3)(x-1)=0 \iff x=-3,1$. $x=-3$ gives $y=6-\frac{7}{4}-\frac{1}{4}=6-2=4$. $x=1$ gives $y=\frac{2}{3}+\frac{7}{12}-\frac{1}{4}=\frac{8+7-3}{12}=1$.

There are two critical points $(x_c,y_c)=(-3,4);(1,1)$.

The discriminant at $(-3,4)$ is $\Delta=(-\frac{41}{8})(-2)-(-\frac{3}{2})^2=\frac{41}{4}-\frac{9}{4}=8>0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are negative, $(-3,4)$ is a local maximum.

The discriminant at $(1,1)$ is $\Delta=(\frac{23}{8})(-2)-(-\frac{3}{2})^2=-\frac{23}{4}-\frac{9}{4}=\frac{-32}{4}=-8<0$. Hence $(1,1)$ is a saddle point.

For $f(x,y)=\frac{1}{3}y^3-x^2-\frac{3}{2}xy+\frac{7}{16}y^2-\frac{5}{2}x-\frac{39}{8}y-\frac{25}{16}$, the first and second order partial derivatives are: $\frac{\partial f}{\partial x}=-2x-\frac{3}{2}y-\frac{5}{2}$; $\frac{\partial f}{\partial y}=y^2-\frac{3}{2}x+\frac{7}{8}y-\frac{39}{8}$; $\frac{\partial^2 f}{\partial x^2}=-2$; $\frac{\partial^2 f}{\partial y^2}=2y+\frac{7}{8}$; and $\frac{\partial^2 f}{\partial y\partial x}=-\frac{3}{2}$

Finding the critical points requires solving the system: $\left\{\begin{array}{l}\partial f/\partial x=0\\ \partial f/\partial y=0\end{array}\right. \iff \left\{\begin{array}{c}-2x-\frac{3}{2}y-\frac{5}{2}=0\\ y^2-\frac{3}{2}x+\frac{7}{8}y-\frac{39}{8}=0\end{array}\right.$

The first equation gives $x=-\frac{5}{4}-\frac{3}{4}y$. Substituting into the second equation gives $y^2+(\frac{15}{8}+\frac{9}{8}y)+\frac{7}{8}y-\frac{39}{8}=0 \iff y^2+2y-3 \iff (y+3)(y-1)=0 \iff y=-3,1$. $y=-3$ gives $x=1$. $y=1$ gives $x=-2$.

There are two critical points $(x_c,y_c)=(1,-3);(-2,1)$.

The discriminant at $(1,-3)$ is $\Delta=(-2)(-\frac{41}{8})-(-\frac{3}{2})^2=\frac{41}{4}-\frac{9}{4}=8>0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are negative, $(1,-3)$ is a local maximum.

The discriminant at $(-2,1)$ is $\Delta=(-2)(\frac{23}{8})-(-\frac{3}{2})^2=-\frac{23}{4}-\frac{9}{4}=-8<0$. Hence $(-2,1)$ is a saddle point.

For $f(x,y)=\frac{5}{4}x^4-\frac{1}{3}x^3-2x^2y-2x^2+y^2+4x$, the first and second order partial derivatives are: $\frac{\partial f}{\partial x}=5x^3-x^2-4xy-4x+4$; $\frac{\partial f}{\partial y}=-2x^2+2y$; $\frac{\partial^2 f}{\partial x^2}=15x^2-2x-4y-4$; $\frac{\partial^2 f}{\partial y^2}=2$; and $\frac{\partial^2 f}{\partial y\partial x}=-4x$

Finding the critical points requires solving the system: $\left\{\begin{array}{l}\partial f/\partial x=0\\ \partial f/\partial y=0\end{array}\right. \iff \left\{\begin{array}{c}5x^3-x^2-4xy-4x+4=0\\ -2x^2+2y=0\end{array}\right.$

The second equation gives $y=x^2$. Substituting into the first equation gives $x^3-x^2-4x+4=0 \iff x^2(x-1)-4(x-1)=0 \iff (x-1)(x^2-4)=0 \iff (x+2)(x-1)(x-2)=0 \iff x=-2,1,2$. $x=-2$ gives $y=4$. $x=1$ gives $y=1$. $x=2$ gives $y=4$.

There are three critical points $(x_c,y_c)=(-2,4);(1,1);(2,4)$.

The discriminant at $(-2,4)$ is $\Delta=(44)(2)-(8)^2=88-64=24>0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are positive, $(-2,4)$ is a local minimum.

The discriminant at $(1,1)$ is $\Delta=(5)(2)-(-4)^2=10-16=-6<0$. Hence $(1,1)$ is a saddle point.

The discriminant at $(2,4)$ is $\Delta=(36)(2)-(-8)^2=72-64=8>0$. Since both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are positive, $(2,4)$ is a local minimum.