Vector Valued Functions

Question 1:

part 1a:

Given the vector valued function $\mathbf{r}(t) = \begin{bmatrix} t^2 - 1/t \\ \sqrt{t+5} \\ t^3 \end{bmatrix}$, compute its derivative $\frac{d\mathbf{r}}{dt}$.

Solution:

$\frac{d\mathbf{r}}{dt} = \begin{bmatrix} 2t + 1/t^2 \\ 1/(2\sqrt{t+5}) \\ 3t^2 \end{bmatrix}$

part 1b:

Given a vector valued function $\mathbf{r}_1(t)$ where $\mathbf{r}_1(2) = \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix}$ and $\frac{d\mathbf{r}_1}{dt}\big|_{t=2} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$; and a vector valued function $\mathbf{r}_2(t)$ where $\mathbf{r}_2(2) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\frac{d\mathbf{r}_2}{dt}\big|_{t=2} = \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$; use the **product rule** to compute $\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))\big|_{t=2}$ and $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t))\big|_{t=2}$.

Solution:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))\bigg|_{t=2} &= \frac{d\mathbf{r}_1}{dt}\bigg|_{t=2} \cdot \mathbf{r}_2(2) + \mathbf{r}_1(2) \cdot \frac{d\mathbf{r}_2}{dt}\bigg|_{t=2} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix} \\
&= (4 - 9 + 0) + (0 + 7 + 9) = -5 + 16 = 11
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t))\bigg|_{t=2} &= \frac{d\mathbf{r}_1}{dt}\bigg|_{t=2} \times \mathbf{r}_2(2) + \mathbf{r}_1(2) \times \frac{d\mathbf{r}_2}{dt}\bigg|_{t=2} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix} \\
&= \begin{bmatrix} (-3)(0) - (5)(3) \\ (5)(1) - (4)(0) \\ (4)(3) - (-3)(1) \end{bmatrix} + \begin{bmatrix} (-7)(9) - (1)(-1) \\ (1)(0) - (2)(9) \\ (2)(-1) - (-7)(0) \end{bmatrix} = \begin{bmatrix} 0 - 15 \\ 5 - 0 \\ 12 + 3 \end{bmatrix} + \begin{bmatrix} -63 + 1 \\ 0 - 18 \\ -2 - 0 \end{bmatrix} \\
&= \begin{bmatrix} -15 \\ 5 \\ 15 \end{bmatrix} + \begin{bmatrix} -62 \\ -18 \\ -2 \end{bmatrix} = \begin{bmatrix} -77 \\ -13 \\ 13 \end{bmatrix}
\end{aligned}$$

part 1c:

Given a vector valued function $\mathbf{r}(t)$ where $\frac{d\mathbf{r}}{dt}\big|_{t=3} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$ and $\frac{d\mathbf{r}}{dt}\big|_{t=-5} = \begin{bmatrix} -7 \\ 2 \\ 8 \end{bmatrix}$; and a scalar valued function $f(t)$ where $f(3) = -5$ and $\frac{df}{dt}\Big|_{t=3} = 4$; use the **chain rule** to compute $\frac{d}{dt}(\mathbf{r}(f(t)))\big|_{t=3}$.

Solution:

$$\frac{d}{dt}(\mathbf{r}(f(t)))\Big|_{t=3} = \frac{d\mathbf{r}}{dt}\Big|_{t=f(3)} \cdot \frac{df}{dt}\Big|_{t=3} = \frac{d\mathbf{r}}{dt}\Big|_{t=-5} \cdot \frac{df}{dt}\Big|_{t=3} = \begin{bmatrix} -7 \\ 2 \\ 8 \end{bmatrix} \cdot (4) = \begin{bmatrix} -28 \\ 8 \\ 32 \end{bmatrix}$$

part 1d:

Given the vector valued function $\mathbf{r}(t) = \begin{bmatrix} t\sqrt{t^2+1} \\ 1/t \\ \sin^2(t) \end{bmatrix}$, compute the definite integral $\int_{t=1}^{2} \mathbf{r}(t)dt$.

Solution:

$$\int_{t=1}^{2} \mathbf{r}(t)dt = \int_{t=1}^{2} \begin{bmatrix} t\sqrt{t^2+1} \\ 1/t \\ \sin^2(t) \end{bmatrix} dt = \begin{bmatrix} \int_{t=1}^{2} t\sqrt{t^2+1}dt \\ \int_{t=1}^{2} dt/t \\ \int_{t=1}^{2} \sin^2(t)dt \end{bmatrix} = \begin{bmatrix} (1/2)\int_{t=1}^{2}(2t)\sqrt{t^2+1}dt \\ \ln(t)|_{t=1}^{2} \\ \int_{t=1}^{2}(1/2)(1-\cos(2t))dt \end{bmatrix}$$
$$= \begin{bmatrix} (1/2)\ (2/3)(t^2+1)^{3/2}|_{t=1}^{2} \\ \ln(2) \\ (1/2)(1-(1/2)\sin(2t))|_{t=1}^{2} \end{bmatrix} = \begin{bmatrix} (1/3)(5^{3/2}-2^{3/2}) \\ \ln(2) \\ -(1/4)(\sin(4)-\sin(2)) \end{bmatrix}$$

Question 2:

Consider the curve $\mathbf{r}(t) = \begin{bmatrix} e^{-t}\cos(t) \\ e^{-t}\sin(t) \\ 0 \end{bmatrix}$.

Compute the velocity $\mathbf{v}(t)$; the acceleration $\mathbf{a}(t)$; the speed $u(t)$; the unit length tangent vector $\mathbf{T}(t)$; the curvature $\kappa(t)$; and the unit length normal vector $\mathbf{N}(t)$.

Solution:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \begin{bmatrix} -e^{-t}\cos(t) - e^{-t}\sin(t) \\ -e^{-t}\sin(t) + e^{-t}\cos(t) \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t}(-\cos(t)-\sin(t)) \\ e^{-t}(\cos(t)-\sin(t)) \\ 0 \end{bmatrix}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \begin{bmatrix} -e^{-t}(-\cos(t) - \sin(t)) + e^{-t}(\sin(t) - \cos(t)) \\ -e^{-t}(\cos(t) - \sin(t)) + e^{-t}(-\sin(t) - \cos(t)) \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t}\sin(t) \\ -2e^{-t}\cos(t) \\ 0 \end{bmatrix}$$

$$\begin{aligned} u(t) =& |\mathbf{v}(t)| = \sqrt{e^{-2t}(\cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)) + e^{-2t}(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)) + 0} \\ =& \sqrt{2e^{-2t}} = \sqrt{2} \cdot e^{-t} \end{aligned}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{u(t)} = \begin{bmatrix} (-\cos(t) - \sin(t))/\sqrt{2} \\ (\cos(t) - \sin(t))/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\frac{d\mathbf{T}}{dt} = \begin{bmatrix} (\sin(t) - \cos(t))/\sqrt{2} \\ (-\sin(t) - \cos(t))/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} (-\cos(t) + \sin(t))/\sqrt{2} \\ (-\cos(t) - \sin(t))/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{aligned} \left|\frac{d\mathbf{T}}{dt}\right| =& \sqrt{(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t))/2 + (\cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t))/2 + 0} \\ =& 1 \end{aligned}$$

$$\kappa(t) = \frac{1}{u(t)}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{e^t}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \begin{bmatrix} (-\cos(t) + \sin(t))/\sqrt{2} \\ (-\cos(t) - \sin(t))/\sqrt{2} \\ 0 \end{bmatrix}$$

Question 3:

For the curve $\mathbf{r}(t) = \begin{bmatrix} 3t^2 \\ 5t^2 \\ -4t^2 \end{bmatrix}$, compute the arc-length parameterization.

Solution:

$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \begin{bmatrix} 6t \\ 10t \\ -8t \end{bmatrix}$

$u(t) = |\mathbf{v}(t)| = \sqrt{36t^2 + 100t^2 + 64t^2} = \sqrt{200} \cdot t = 2\sqrt{50} \cdot t$ so the arc-length between 0 and t is $s(t) = \int_{t_1=0}^{t} u(t_1)dt_1 = \int_{t_1=0}^{t} 2\sqrt{50} \cdot t_1 dt_1 = \sqrt{50} \cdot t_1^2\big|_{t_1=0}^{t} = \sqrt{50} \cdot t^2$

$s = \sqrt{50} \cdot t^2$ so replacing t^2 with $s/\sqrt{50}$ gives the arc-length parameterization: $\mathbf{r}(s) = \begin{bmatrix} (3/\sqrt{50})s \\ (5/\sqrt{50})s \\ -(4/\sqrt{50})s \end{bmatrix}$

Question 4 (hard):

For the spiral $\mathbf{r}(t) = \begin{bmatrix} (1+t^2)\cos(\ln(1+t^2)) \\ (1+t^2)\sin(\ln(1+t^2)) \\ 0 \end{bmatrix}$, compute the arc-length parameterization.

Solution:

$$\begin{aligned} \mathbf{v}(t) = &\frac{d\mathbf{r}}{dt} = \begin{bmatrix} 2t\cos(\ln(1+t^2)) + (1+t^2)(-\sin(\ln(1+t^2)))(1/(1+t^2))(2t) \\ 2t\sin(\ln(1+t^2)) + (1+t^2)(+\cos(\ln(1+t^2)))(1/(1+t^2))(2t) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2t\cos(\ln(1+t^2)) - 2t\sin(\ln(1+t^2)) \\ 2t\sin(\ln(1+t^2)) + 2t\cos(\ln(1+t^2)) \\ 0 \end{bmatrix} = \begin{bmatrix} 2t(\cos(\ln(1+t^2)) - \sin(\ln(1+t^2))) \\ 2t(\sin(\ln(1+t^2)) + \cos(\ln(1+t^2))) \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} u(t) = &\Big(4t^2(\cos^2(\ln(1+t^2)) - 2\cos(\ln(1+t^2))\sin(\ln(1+t^2)) + \sin^2(\ln(1+t^2))) \\ &\quad 4t^2(\sin^2(\ln(1+t^2)) + 2\sin(\ln(1+t^2))\cos(\ln(1+t^2)) + \cos^2(\ln(1+t^2))) + 0 \Big)^{1/2} \\ = &\sqrt{4t^2 + 4t^2} = (2\sqrt{2})t \end{aligned}$$

so the arc-length between 0 and t is $s(t) = \int_{t_1=0}^{t} u(t_1)dt_1 = \int_{t_1=0}^{t} (2\sqrt{2})t_1 dt_1 = 2\sqrt{2}\left(\frac{1}{2}t^2 - \frac{1}{2}0^2\right) = \sqrt{2} \cdot t^2$

$s = \sqrt{2} \cdot t^2$ so replacing t^2 with $s/\sqrt{2}$ gives the arc-length parameterization:

$\mathbf{r}(s) = \begin{bmatrix} (1+s/\sqrt{2})\cos(\ln(1+s/\sqrt{2})) \\ (1+s/\sqrt{2})\sin(\ln(1+s/\sqrt{2})) \\ 0 \end{bmatrix}$