Theorems related to Vector Calculus

Question 1:

part 1a:

Evaluate the scalar line integral:

$$\int_C (6x^2 + 4y) ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} 3t+2 \\ -t+3 \end{bmatrix} \text{ and } t \in [-1, 1]$$

Solution:

$$\begin{aligned}
\int_C (6x^2 + 4y) ds &= \int_{t=-1}^{1} (6x_C(t)^2 + 4y_C(t)) \left| \frac{d\mathbf{r}_C}{dt} \right| dt \\
&= \int_{t=-1}^{1} (6(3t+2)^2 + 4(-t+3)) \left| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right| dt \\
&= \int_{t=-1}^{1} ((54t^2 + 72t + 24) + (-4t + 12))\sqrt{10} \cdot dt \\
&= \int_{t=-1}^{1} (54t^2 + 68t + 36)\sqrt{10} \cdot dt \\
&= (18t^3 + 34t^2 + 36t)\sqrt{10} \Big|_{t=-1}^{1} \\
&= (18 + 34 + 36)\sqrt{10} - (-18 + 34 - 36)\sqrt{10} \\
&= 88\sqrt{10} - (-20)\sqrt{10} = 108\sqrt{10}
\end{aligned}$$

part 1b:

Evaluate the scalar line integral:

$$\int_C (x^2 + 2y) ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} 1-2t \\ 2-t \end{bmatrix} \text{ and } t \in [0, 3]$$

Solution:

$$
\begin{aligned}
\int_C (x^2+2y)ds &= \int_{t=0}^{3} (x_C(t)^2 + 2y_C(t)) \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{3} ((1-2t)^2 + 2(2-t)) \left|\begin{bmatrix}-2\\-1\end{bmatrix}\right| dt \\
&= \int_{t=0}^{3} ((4t^2-4t+1)+(-2t+4))\sqrt{5}\cdot dt \\
&= \int_{t=0}^{3} (4t^2-6t+5)\sqrt{5}\cdot dt \\
&= (\frac{4}{3}t^3 - 3t^2 + 5t)\sqrt{5}\Big|_{t=0}^{3} \\
&=(36-27+15)\sqrt{5} - 0 = 24\sqrt{5}
\end{aligned}
$$

part 1c:

Evaluate the scalar line integral:

$$\int_C 2x\cdot\sin(y)\cdot ds \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix}\cos(t)\\ t\end{bmatrix} \text{ and } t\in[0,\pi/2]$$

Solution:

$$
\begin{aligned}
\int_C 2x\cdot\sin(y)\cdot ds &= \int_{t=0}^{\pi/2} 2x_C(t)\cdot\sin(y_C(t)) \left|\frac{d\mathbf{r}_C}{dt}\right| dt \\
&= \int_{t=0}^{\pi/2} 2\cos(t)\sin(t) \left|\begin{bmatrix}-\sin(t)\\ 1\end{bmatrix}\right| dt \\
&= \int_{t=0}^{\pi/2} 2\cos(t)\sin(t)\sqrt{\sin^2(t)+1}dt \\
&= \frac{2}{3}(\sin^2(t)+1)^{3/2}\Big|_{t=0}^{\pi/2} = \frac{2}{3}(2^{3/2}-1)
\end{aligned}
$$

Question 2:

part 2a:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix}-6x\\ 4y\end{bmatrix}\cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix}t^2\\ 1/t\end{bmatrix} \text{ and } t\in[1,2]$$

Solution:

$$\int_C \begin{bmatrix} -6x \\ 4y \end{bmatrix} \cdot d\mathbf{r} = \int_{t=1}^{2} \begin{bmatrix} -6x_C(t) \\ 4y_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=1}^{2} \begin{bmatrix} -6t^2 \\ 4/t \end{bmatrix} \cdot \begin{bmatrix} 2t \\ -1/t^2 \end{bmatrix} dt$$

$$= \int_{t=1}^{2} (-12t^3 - \frac{4}{t^3})dt = (-3t^4 + \frac{2}{t^2})\Big|_{t=1}^{2}$$

$$=(-48 + \frac{1}{2}) - (-3+2) = -47 + \frac{1}{2} = -\frac{93}{2}$$

part 2b:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix} y \\ x \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} t^3 \\ t^4 \end{bmatrix} \text{ and } t \in [0,2]$$

Solution:

$$\int_C \begin{bmatrix} y \\ x \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^{2} \begin{bmatrix} y_C(t) \\ x_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=0}^{2} \begin{bmatrix} t^4 \\ t^3 \end{bmatrix} \cdot \begin{bmatrix} 3t^2 \\ 4t^3 \end{bmatrix} dt = \int_{t=0}^{2} 7t^6 dt = t^7\big|_{t=0}^{2} = 128$$

part 2c:

Evaluate the vector line integral:

$$\int_C \begin{bmatrix} 2x \\ y^2 \\ -z \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_C(t) = \begin{bmatrix} -t^2 \\ t+4 \\ 3t+2 \end{bmatrix} \text{ and } t \in [-1,1]$$

Solution:

$$\int_C \begin{bmatrix} 2x \\ y^2 \\ -z \end{bmatrix} \cdot d\mathbf{r} = \int_{t=-1}^{1} \begin{bmatrix} 2x_C(t) \\ y_C(t)^2 \\ -z_C(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_C}{dt} dt = \int_{t=-1}^{1} \begin{bmatrix} -2t^2 \\ t^2+8t+16 \\ -3t-2 \end{bmatrix} \cdot \begin{bmatrix} -2t \\ 1 \\ 3 \end{bmatrix} dt$$

$$= \int_{t=-1}^{1} (4t^3 + (t^2+8t+16) + (-9t-6))dt = \int_{t=-1}^{1} (4t^3 + t^2 - t + 10)dt$$

$$= (t^4 + \frac{1}{3}t^3 - \frac{1}{2}t^2 + 10t)\Big|_{t=-1}^{1} = (1 + \frac{1}{3} - \frac{1}{2} + 10) - (1 - \frac{1}{3} - \frac{1}{2} - 10)$$

$$=\frac{66+2-3}{6} - \frac{-54-2-3}{6} = \frac{65}{6} + \frac{59}{6} = \frac{124}{6} = \frac{62}{3}$$

Question 3:

part 3a:

Is the vector field $\mathbf{F}(x,y,z) = \begin{bmatrix} y^2\sin(z) \\ 2xy\sin(z) \\ xy^2\cos(z) \end{bmatrix}$ conservative? If yes, use the **gradient theorem** to evaluate the vector line integral:

$$\int_C \begin{bmatrix} y^2\sin(z) \\ 2xy\sin(z) \\ xy^2\cos(z) \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_{\text{initial}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{r}_{\text{final}} = \begin{bmatrix} 1 \\ 1 \\ \pi/2 \end{bmatrix}$$

Solution:

The curl (circulation density) of $\mathbf{F}(x,y,z) = \begin{bmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{bmatrix} = \begin{bmatrix} y^2\sin(z) \\ 2xy\sin(z) \\ xy^2\cos(z) \end{bmatrix}$ is:

$$\nabla\times\mathbf{F} = \begin{bmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{bmatrix} = \begin{bmatrix} 2xy\cos(z) - 2xy\cos(z) \\ y^2\cos(z) - y^2\cos(z) \\ 2y\sin(z) - 2y\sin(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

Since the curl is $\mathbf{0}$ everywhere $\mathbf{F}(x,y,z)$ is conservative, and the vector line integral $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ depends only on the endpoints of C, and not on any of the interior points of C. Now to find a function $f(x,y,z)$ where $\nabla f = \begin{bmatrix} y^2\sin(z) \\ 2xy\sin(z) \\ xy^2\cos(z) \end{bmatrix}$:

$$\begin{aligned} f(x,y,z) =& f_0 + \int_{x_1=0}^{x} F_x(x_1,0,0)dx_1 + \int_{y_1=0}^{y} F_y(x,y_1,0)dy_1 + \int_{z_1=0}^{z} F_z(x,y,z_1)dz_1 \\ =& f_0 + \int_{x_1=0}^{x} 0dx_1 + \int_{y_1=0}^{y} 0dy_1 + \int_{z_1=0}^{z} xy^2\cos(z_1)dz_1 \\ =& f_0 + xy^2\sin(z_1)\Big|_{z_1=0}^{z} = f_0 + xy^2\sin(z) \end{aligned}$$

Using the gradient theorem,

$$\int_C \begin{bmatrix} y^2\sin(z) \\ 2xy\sin(z) \\ xy^2\cos(z) \end{bmatrix} \cdot d\mathbf{r} = \int_C (\nabla f)\cdot d\mathbf{r} = f(1,1,\pi/2) - f(0,0,0) = (f_0+1) - f_0 = 1$$

part 3b:

Is the vector field $\mathbf{F}(x,y,z) = \begin{bmatrix} xy \\ xy \\ z \end{bmatrix}$ conservative? If yes, use the **gradient theorem** to evaluate the vector line integral:

$$\int_C \begin{bmatrix} xy \\ xy \\ z \end{bmatrix} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{r}_{\text{initial}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{r}_{\text{final}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

The curl (circulation density) of $\mathbf{F}(x,y,z) = \begin{bmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{bmatrix} = \begin{bmatrix} xy \\ xy \\ z \end{bmatrix}$ is:

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ y-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y-x \end{bmatrix}$$

Since the curl is not $\mathbf{0}$ everywhere $\mathbf{F}(x,y,z)$ is **not** conservative, and the vector line integral $\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r}$ depends on the interior points of C.

Question 4:

part 4a:

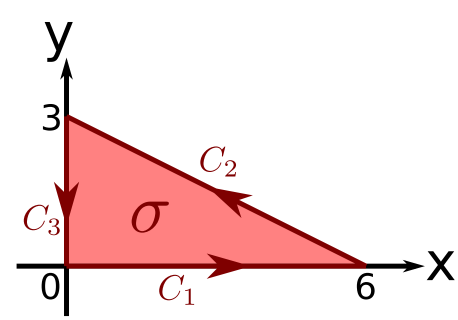
The region σ on the right is

$$\sigma = \{(x,y) | 0 \le x \le 6 \text{ and } 0 \le y \le 3 - (1/2)x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$; $\int_{C_2} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_3} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$$

Solution:
Using Green's theorem gives:

$$\begin{aligned}
\int_{\partial\sigma} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} &= \iint_\sigma \left(\frac{\partial}{\partial x}(x-2y) - \frac{\partial}{\partial y}(3x-5y)\right) dA = \iint_\sigma (1-(-5)) dA = \iint_\sigma 6 dA \\
&= \int_{x=0}^{6} \int_{y=0}^{3-(1/2)x} 6 dy dx = \int_{x=0}^{6} 6y\Big|_{y=0}^{3-(1/2)x} dx = \int_{x=0}^{6} (18-3x) dx \\
&= (18x - \frac{3}{2}x^2)\Big|_{x=0}^{6} = (108-54) - 0 = 54
\end{aligned}$$

One possible parameterization of C_1 is $\mathbf{r}_{C_1}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$ and $t \in [0,6]$ so

$$\begin{aligned}
\int_{C_1} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{6} \begin{bmatrix} 3x_{C_1}(t) - 5y_{C_1}(t) \\ x_{C_1}(t) - 2y_{C_1}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_1}}{dt} dt \\
&= \int_{t=0}^{6} \begin{bmatrix} 3t \\ t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = \int_{t=0}^{6} 3t dt = \frac{3}{2}t^2\Big|_{t=0}^{6} = 54
\end{aligned}$$

One possible parameterization of C_2 is $\mathbf{r}_{C_2}(t) = \begin{bmatrix} 6-t \\ 3-(1/2)(6-t) \end{bmatrix} = \begin{bmatrix} 6-t \\ (1/2)t \end{bmatrix}$ and $t \in [0,6]$ so

$$\begin{aligned}
\int_{C_2} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{6} \begin{bmatrix} 3x_{C_2}(t) - 5y_{C_2}(t) \\ x_{C_2}(t) - 2y_{C_2}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_2}}{dt} dt \\
&= \int_{t=0}^{6} \begin{bmatrix} (18-3t)-(5/2)t \\ (6-t)-t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} dt = \int_{t=0}^{6} \begin{bmatrix} 18-(11/2)t \\ 6-2t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} dt \\
&= \int_{t=0}^{6} (-15+(9/2)t)dt = (-15t+(9/4)t^2)\Big|_{t=0}^{6} \\
&= (-90+81) - 0 = -9
\end{aligned}$$

One possible parameterization of C_3 is $\mathbf{r}_{C_3}(t) = \begin{bmatrix} 0 \\ 3-t \end{bmatrix}$ and $t \in [0,3]$ so

$$\begin{aligned}
\int_{C_3} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{3} \begin{bmatrix} 3x_{C_3}(t) - 5y_{C_3}(t) \\ x_{C_2}(t) - 2y_{C_3}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_3}}{dt} dt \\
&= \int_{t=0}^{3} \begin{bmatrix} 5t-15 \\ 2t-6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} dt = \int_{t=0}^{3} (6-2t)dt = (6t-t^2)\Big|_{t=0}^{3} \\
&= (18-9) - 0 = 9
\end{aligned}$$

Therefore:

$$\int_{C_1} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r} = 54 + (-9) + 9 = 54 = \int_{\partial\sigma} \begin{bmatrix} 3x-5y \\ x-2y \end{bmatrix} \cdot d\mathbf{r}$$

part 4b:

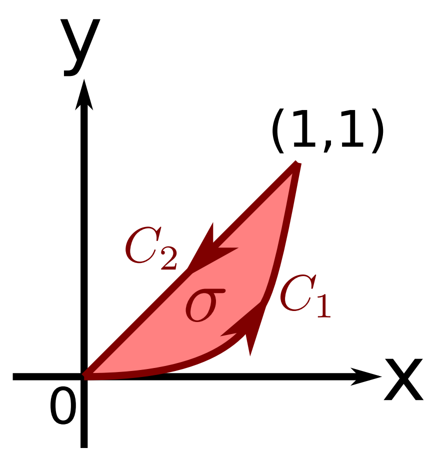
The region σ on the right is

$$\sigma = \{(x,y) | 0 \le x \le 1 \text{ and } x^3 \le y \le x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_2} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} 2xy \\ x+y \end{bmatrix} \cdot d\mathbf{r}$$

Solution:
Using Green's theorem gives:

$$\begin{aligned}\int_{\partial\sigma}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r} &= \iint_\sigma\left(\frac{\partial}{\partial x}(x+y)-\frac{\partial}{\partial y}(2xy)\right)dA\\ &= \iint_\sigma(1-2x)dA = \int_{x=0}^1\int_{y=x^3}^x(1-2x)dydx\\ &= \int_{x=0}^1 (1-2x)y\Big|_{y=x^3}^x dx = \int_{x=0}^1((-2x^2+x)-(-2x^4+x^3))dx\\ &= \int_{x=0}^1(2x^4-x^3-2x^2+x)dx = \left(\frac{2}{5}x^5-\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2\right)\Big|_{x=0}^1\\ &= \left(\frac{2}{5}-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)-0=\frac{24-15-40+30}{60}=\frac{9-10}{60}=-\frac{1}{60}\end{aligned}$$

One possible parameterization of C_1 is $\mathbf{r}_{C_1}(t)=\begin{bmatrix}t\\t^3\end{bmatrix}$ and $t\in[0,1]$ so

$$\begin{aligned}\int_{C_1}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r} &= \int_{t=0}^1\begin{bmatrix}2x_{C_1}(t)y_{C_1}(t)\\x_{C_1}(t)+y_{C_1}(t)\end{bmatrix}\cdot\frac{d\mathbf{r}_{C_1}}{dt}dt\\ &= \int_{t=0}^1\begin{bmatrix}2t^4\\t+t^3\end{bmatrix}\cdot\begin{bmatrix}1\\3t^2\end{bmatrix}dt = \int_{t=0}^1(3t^5+2t^4+3t^3)dt\\ &= \left(\frac{1}{2}t^6+\frac{2}{5}t^5+\frac{3}{4}t^4\right)\Big|_{t=0}^1 = \left(\frac{1}{2}+\frac{2}{5}+\frac{3}{4}\right)-0\\ &= \frac{10+8+15}{20}=\frac{33}{20}\end{aligned}$$

One possible parameterization of C_2 is $\mathbf{r}_{C_2}(t)=\begin{bmatrix}1-t\\1-t\end{bmatrix}$ and $t\in[0,1]$ so

$$\begin{aligned}\int_{C_2}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r} &= \int_{t=0}^1\begin{bmatrix}2x_{C_2}(t)y_{C_2}(t)\\x_{C_2}(t)+y_{C_2}(t)\end{bmatrix}\cdot\frac{d\mathbf{r}_{C_2}}{dt}dt\\ &= \int_{t=0}^1\begin{bmatrix}2(1-t)^2\\(1-t)+(1-t)\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\end{bmatrix}dt = \int_{t=0}^1\begin{bmatrix}2t^2-4t+2\\-2t+2\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\end{bmatrix}dt\\ &= \int_{t=0}^1(-2t^2+6t-4)dt = \left(-\frac{2}{3}t^3+3t^2-4t\right)\Big|_{t=0}^1\\ &= \left(-\frac{2}{3}+3-4\right)-0=-\frac{5}{3}\end{aligned}$$

Therefore:

$$\int_{C_1}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r}+\int_{C_2}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r}=\frac{33}{20}+-\frac{5}{3}=-\frac{1}{60}=\int_{\partial\sigma}\begin{bmatrix}2xy\\x+y\end{bmatrix}\cdot d\mathbf{r}$$

part 4c:

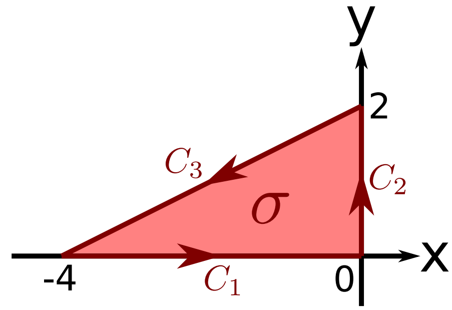
The region σ on the right is

$$\sigma = \{(x,y)| -4 \leq x \leq 0 \text{ and } 0 \leq y \leq 2 + (1/2)x\}$$

Use Green's theorem to compute the loop integral

$$\int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$; $\int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$; and $\int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} = \int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$$

Solution:
Using Green's theorem gives:

$$\begin{aligned}
\int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} &= \iint_\sigma (\frac{\partial}{\partial x}(5x+y) - \frac{\partial}{\partial y}(x+2y))dA = \iint_\sigma (5-2)dA = \iint_\sigma 3dA \\
&= \int_{x=-4}^{0} \int_{y=0}^{2+(1/2)x} 3dydx = \int_{x=-4}^{0} 3y|_{y=0}^{2+(1/2)x} dx = \int_{x=-4}^{0} (6+(3/2)x)dx \\
&= (6x+(3/4)x^2)\Big|_{x=-4}^{0} = 0 - (-24+12) = 12
\end{aligned}$$

One possible parameterization of C_1 is $\mathbf{r}_{C_1}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$ and $t \in [-4, 0]$ so

$$\begin{aligned}
\int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=-4}^{0} \begin{bmatrix} x_{C_1}(t) + 2y_{C_1}(t) \\ 5x_{C_1}(t) + y_{C_1}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_1}}{dt} dt \\
&= \int_{t=-4}^{0} \begin{bmatrix} t \\ 5t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = \int_{t=-4}^{0} tdt = \frac{1}{2}t^2\Big|_{t=-4}^{0} = -8
\end{aligned}$$

One possible parameterization of C_2 is $\mathbf{r}_{C_2}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}$ and $t \in [0, 2]$ so

$$\begin{aligned}
\int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} &= \int_{t=0}^{2} \begin{bmatrix} x_{C_2}(t) + 2y_{C_2}(t) \\ 5x_{C_2}(t) + y_{C_2}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_2}}{dt} dt \\
&= \int_{t=0}^{2} \begin{bmatrix} 2t \\ t \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt = \int_{t=0}^{2} tdt = \frac{1}{2}t^2\Big|_{t=0}^{2} = 2
\end{aligned}$$

One possible parameterization of C_3 is $\mathbf{r}_{C_3}(t) = \begin{bmatrix} -t \\ 2-(1/2)t \end{bmatrix}$ and $t \in [0,4]$ so

$$\begin{aligned}
\int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} =& \int_{t=0}^{4} \begin{bmatrix} x_{C_3}(t) + 2y_{C_3}(t) \\ 5x_{C_3}(t) + y_{C_3}(t) \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_3}}{dt} dt \\
=& \int_{t=0}^{4} \begin{bmatrix} -t + (4-t) \\ -5t + (2-(1/2)t) \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} dt = \int_{t=0}^{4} \begin{bmatrix} -2t+4 \\ -(11/2)t+2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} dt \\
=& \int_{t=0}^{4} ((2t-4) + ((11/4)t - 1)) dt = \int_{t=0}^{4} ((19/4)t - 5) dt = ((19/8)t^2 - 5t)\Big|_{t=0}^{4} \\
=& (38-20) - 0 = 18
\end{aligned}$$

Therefore:

$$\int_{C_1} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_2} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} + \int_{C_3} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r} = -8 + 2 + 18 = 12 = \int_{\partial\sigma} \begin{bmatrix} x+2y \\ 5x+y \end{bmatrix} \cdot d\mathbf{r}$$

Question 5:

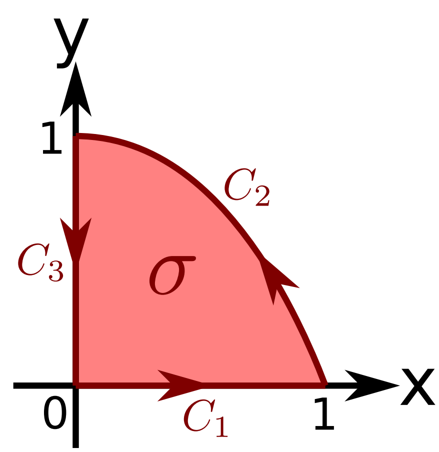
The region σ on the right is

$$\sigma = \{(x,y) | 0 \le x \le 1 \text{ and } 0 \le y \le 1 - x^2\}$$

Use Gauss's divergence theorem to compute the flux integral

$$\int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$$

where $\partial\sigma$ is the counterclockwise oriented boundary of σ.



Next, compute each of $\int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$; $\int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$; and $\int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$ and show that

$$\int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} = \int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$$

Solution:

Using Gauss's divergence theorem gives:

$$\begin{aligned}
\int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} &= \iint_\sigma (\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y))dA = \iint_\sigma (2x+1)dA \\
&= \int_{x=0}^1 \int_{y=0}^{1-x^2} (2x+1)dydx = \int_{x=0}^1 (2x+1)y|_{y=0}^{1-x^2} dx \\
&= \int_{x=0}^1 ((2x+1)(1-x^2) - 0)dx = \int_{x=0}^1 (-2x^3 - x^2 + 2x + 1)dx \\
&= (-\frac{1}{2}x^4 - \frac{1}{3}x^3 + x^2 + x)\Big|_{x=0}^1 = (-\frac{1}{2} - \frac{1}{3} + 2) - 0 \\
&= \frac{-3-2+12}{6} = \frac{7}{6}
\end{aligned}$$

One possible parameterization of C_1 is $\mathbf{r}_{C_1}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$ and $t \in [0,1]$ so

$$\begin{aligned}
\int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} &= \int_{C_1} \begin{bmatrix} -y \\ x^2 \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^1 \begin{bmatrix} -y_{C_1}(t) \\ x_{C_1}(t)^2 \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_1}}{dt} dt \\
&= \int_{t=0}^1 \begin{bmatrix} 0 \\ t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = \int_{t=0}^1 0dt = 0
\end{aligned}$$

One possible parameterization of C_2 is $\mathbf{r}_{C_2}(t) = \begin{bmatrix} 1-t \\ 1-(1-t)^2 \end{bmatrix} = \begin{bmatrix} -t+1 \\ -t^2+2t \end{bmatrix}$ and $t \in [0,1]$ so

$$\begin{aligned}
\int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} &= \int_{C_2} \begin{bmatrix} -y \\ x^2 \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^1 \begin{bmatrix} -y_{C_2}(t) \\ x_{C_2}(t)^2 \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_2}}{dt} dt \\
&= \int_{t=0}^1 \begin{bmatrix} t^2-2t \\ (-t+1)^2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2t+2 \end{bmatrix} dt = \int_{t=0}^1 \begin{bmatrix} t^2-2t \\ t^2-2t+1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2t+2 \end{bmatrix} dt \\
&= \int_{t=0}^1 ((-t^2+2t) + (-2t^3+6t^2-6t+2))dt = \int_{t=0}^1 (-2t^3+5t^2-4t+2)dt \\
&= (-\frac{1}{2}t^4 + \frac{5}{3}t^3 - 2t^2 + 2t)\Big|_{t=0}^1 = (-\frac{1}{2} + \frac{5}{3}) - 0 = \frac{-3+10}{6} = \frac{7}{6}
\end{aligned}$$

One possible parameterization of C_3 is $\mathbf{r}_{C_3}(t) = \begin{bmatrix} 0 \\ 1-t \end{bmatrix}$ and $t \in [0,1]$ so

$$\begin{aligned}
\int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} &= \int_{C_3} \begin{bmatrix} -y \\ x^2 \end{bmatrix} \cdot d\mathbf{r} = \int_{t=0}^1 \begin{bmatrix} -y_{C_3}(t) \\ x_{C_3}(t)^2 \end{bmatrix} \cdot \frac{d\mathbf{r}_{C_3}}{dt} dt \\
&= \int_{t=0}^1 \begin{bmatrix} t-1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} dt = \int_{t=0}^1 0dt = 0
\end{aligned}$$

Therefore:

$$\int_{C_1} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_2} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} + \int_{C_3} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix} = 0 + \frac{7}{6} + 0 = \frac{7}{6} = \int_{\partial\sigma} \begin{bmatrix} x^2 \\ y \end{bmatrix} \cdot \begin{bmatrix} dy \\ -dx \end{bmatrix}$$