Line Intersections

Given vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{proj_u}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$$

$$|\mathbf{proj_u}(\mathbf{v})| = \frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|}$$

$$\mathbf{perp_u}(\mathbf{v}) = \mathbf{v} - \mathbf{proj_u}(\mathbf{v})$$

$$|\mathbf{perp_u}(\mathbf{v})| = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|}$$

$\mathbf{u}||\mathbf{v}$ denotes that $\mathbf{u}$ and $\mathbf{v}$ are **parallel**.
$\mathbf{u} \perp \mathbf{v}$ denotes that $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular/orthogonal**.

Classifying the interaction between two lines

Given two straight lines L_1 and L_2, there are 4 possible relationships between these lines:

- L_1 and L_2 are **equivalent**.
- L_1 and L_2 are **parallel but not equal**.
- L_1 and L_2 **intersect** at a single point.
- L_1 and L_2 are **skew**.

The relationship between L_1 and L_2 can be determined via the following algorithm: L_1 will be described by the parametric line:

$$\mathbf{r}(t) = \mathbf{r}_{0,1} + t\mathbf{v}_1$$

and L_2 will be described by the parametric line:

$$\mathbf{r}(t) = \mathbf{r}_{0,2} + t\mathbf{v}_2$$

if $\mathbf{v}_1 || \mathbf{v}_2$ **then**
  Let $d = \left|\mathbf{perp}_{\mathbf{v}_1}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})\right| = \frac{|\mathbf{v}_1 \times (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{v}_1|}$
  if $d = 0$ **then**
    L_1 and L_2 are **equivalent**
  else
    L_1 and L_2 are **parallel but not equal**, and have a separation of d.
  end if
else
  Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$
  Let $d = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_{0,2} - \mathbf{r}_{0,1})| = \frac{|\mathbf{n} \cdot (\mathbf{r}_{0,2} - \mathbf{r}_{0,1})|}{|\mathbf{n}|}$
  if $d = 0$ **then**
    L_1 and L_2 **intersect** at a single point
  else
    L_1 and L_2 are **skew**, and have a closest distance of d.
  end if
end if

For the following pairs of lines, determine if these lines are **equivalent**; are **parallel but not equal**; **intersect** at a single point; or are **skew**. For parallel lines, give the separation, and for skew lines, give the closest distance. **Show all of your work.**

Line pair 1:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & 4 - t \\ y(t) = & -4 + 2t \\ z(t) = & -2 + 5t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & -1 - 2t \\ y(t) = & 6 + 4t \\ z(t) = & 23 + 10t \end{cases}$$

Line pair 2:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & -5 + t \\ y(t) = & -1 + 2t \\ z(t) = & 2 + t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & -8 + 2t \\ y(t) = & -1 + t \\ z(t) = & -1 + 2t \end{cases}$$

Line pair 3:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & -3 + 7t \\ y(t) = & 1 + 2t \\ z(t) = & 2 + 5t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & 4 - 21t \\ y(t) = & 2 - 6t \\ z(t) = & 7 - 15t \end{cases}$$

Line pair 4:

$$L_1 \text{ is parameterized by } \begin{cases} x(t) = & -1 + t \\ y(t) = & 2t \\ z(t) = & 1 + 3t \end{cases} \text{ and } L_2 \text{ is parameterized by } \begin{cases} x(t) = & -1 \\ y(t) = & 1 + 2t \\ z(t) = & 1 + 3t \end{cases}$$