Double and Triple Integrals

Question 1:

Find the minimum and maximum values of the function:

$$f(x, y) = x + y$$

subject to the constraint:

$$x^2 + 4y^2 - 4x - 16y = -16$$

Solution:

Let $g(x, y) = x^2 + 4y^2 - 4x - 16y$ so $\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x - 4 \\ 8y - 16 \end{bmatrix}$. To find the candidate points for minimum and maximum, the following system must be solved:

$$\begin{cases} 1 = \lambda(2x - 4) \\ 1 = \lambda(8y - 16) \\ x^2 + 4y^2 - 4x - 16y = -16 \end{cases}$$

The first equation can be solved to give $\lambda = \frac{1}{2x-4}$, but before this can be done, the scenario where $2x - 4 = 0$ must be considered. $2x - 4 = 0 \iff x = 2$ so this condition causes the system to become: $\begin{cases} 1 = 0 \\ 1 = \lambda(8y - 16) \\ 4 + 4y^2 - 8 - 16y = -16 \end{cases}$ which is a contradiction. So we now know that $x \neq 2$ and that $\lambda = \frac{1}{2x-4}$. The second equation becomes:

$$1 = \frac{8y - 16}{2x - 4} \iff 8y - 16 = 2x - 4 \iff y = (1/4)x + 3/2$$

The last equation becomes:

$$\begin{aligned} & x^2 + 4y^2 - 4x - 16y = -16 \\ \iff & x^2 + 4((1/4)x + 3/2)^2 - 4x - 16((1/4)x + 3/2) = -16 \\ \iff & x^2 + ((1/4)x^2 + 3x + 9) - 4x + (-4x - 24) = -16 \\ \iff & (5/4)x^2 - 5x = -1 \iff x^2 - 4x = -4/5 \iff (x - 2)^2 = 16/5 \\ \iff & x = 2 \pm 4/\sqrt{5} \end{aligned}$$

which in turn gives: $y = (1/4)(2 \pm 4/\sqrt{5}) + 3/2 = 2 \pm 1/\sqrt{5}$ so the candidate points are $(2 + 4/\sqrt{5}, 2 + 1/\sqrt{5})$ and $(2 - 4/\sqrt{5}, 2 - 1/\sqrt{5})$.

- $f(2 + 4/\sqrt{5}, 2 + 1/\sqrt{5}) = 4 + \sqrt{5}$ (which is the maximum)
- $f(2 - 4/\sqrt{5}, 2 - 1/\sqrt{5}) = 4 - \sqrt{5}$ (which is the minimum)

Question 2:

Find the minimum and maximum values of the function:

$$f(x, y) = 2x + y$$

subject to the constraint:

$$x^2 + y^2 - 2x - 2y = 2$$

Solution:

Let $g(x, y) = x^2 + y^2 - 2x - 2y$ so $\nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x - 2 \\ 2y - 2 \end{bmatrix}$. To find the candidate points for minimum and maximum, the following system must be solved:

$$\begin{cases} 2 = \lambda(2x - 2) \\ 1 = \lambda(2y - 2) \\ x^2 + y^2 - 2x - 2y = 2 \end{cases}$$

The first equation can be solved to give $\lambda = \frac{2}{2x-2}$, but before this can be done, the scenario where $2x - 2 = 0$ must be considered. $2x - 2 = 0 \iff x = 1$ so this condition causes the system to become: $\begin{cases} 2 = 0 \\ 1 = \lambda(2y - 2) \\ 1 + y^2 - 2 - 2y = 2 \end{cases}$ which is a contradiction. So we now know that $x \neq 1$ and that $\lambda = \frac{2}{2x-2} = \frac{1}{x-1}$. The second equation becomes:

$$1 = \frac{2y - 2}{x - 1} \iff 2y - 2 = x - 1 \iff y = (1/2)x + 1/2$$

The last equation becomes:

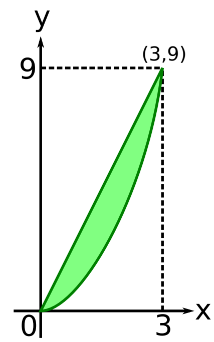
$$\begin{aligned} & x^2 + y^2 - 2x - 2y = 2 \\ \iff & x^2 + ((1/2)x + 1/2)^2 - 2x - 2((1/2)x + 1/2) = 2 \\ \iff & x^2 + ((1/4)x^2 + (1/2)x + 1/4) - 2x + (-x - 1) = 2 \\ \iff & (5/4)x^2 - (5/2)x = 11/4 \iff x^2 - 2x = 11/5 \iff (x - 1)^2 = 16/5 \\ \iff & x = 1 \pm 4/\sqrt{5} \end{aligned}$$

which in turn gives: $y = (1/2)(1 \pm 4/\sqrt{5}) + 1/2 = 1 \pm 2/\sqrt{5}$ so the candidate points are $(1 + 4/\sqrt{5}, 1 + 2/\sqrt{5})$ and $(1 - 4/\sqrt{5}, 1 - 2/\sqrt{5})$.

- $f(1 + 4/\sqrt{5}, 1 + 2/\sqrt{5}) = 3 + 2\sqrt{5}$ (which is the maximum)
- $f(1 - 4/\sqrt{5}, 1 - 2/\sqrt{5}) = 3 - 2\sqrt{5}$ (which is the minimum)

Question 3:

The leaf shaped region σ on the right is bounded from below by a parabola and from above by a straight line. Express σ as a Type I Cartesian region; a Type II Cartesian region; and a Polar region. In addition, given an arbitrary function $f(x,y)$, or $f(r,\theta)$ in polar coordinates, express the double integral $\iint_\sigma f(x,y)dA$ as a nested (iterated) integral using each of the 3 different forms of σ. Lastly, use all of the 3 forms to compute the double integral $\iint_\sigma x dA$.



Solution:

The line has the equation $y = 3x$ which is equivalent to $x = y/3$. The parabola has the equation $y = x^2$ which is equivalent to $x = \pm\sqrt{y}$.

The Type I formulation is:

$$\sigma = \{(x,y)|0 \le x \le 3 \text{ and } x^2 \le y \le 3x\}$$

The Type II formulation is:

$$\sigma = \{(x,y)|0 \le y \le 9 \text{ and } y/3 \le x \le \sqrt{y}\}$$

For the polar formulation, the equation $y = x^2$ must be converted to polar coordinates:

$$y = x^2 \iff r\sin\theta = r^2\cos^2\theta \iff r^2 = r\frac{\sin\theta}{\cos^2\theta}$$
$$\iff r = 0, \frac{\sin\theta}{\cos^2\theta}$$

Since the parabola is tangent to the x-axis, the lower bound on θ is 0. The upper bound on θ is the counterclockwise angle that the line $y = 3x$ makes with the positive x-axis which is atan(3). The polar formulation of σ is:

$$\sigma = \{(r,\theta)|0 \le \theta \le \text{atan}(3) \text{ and } 0 \le r \le \frac{\sin\theta}{\cos^2\theta}\}$$

The double integrals over σ are:

$$\iint_\sigma f(x,y)dA = \int_{x=0}^{3}\int_{y=x^2}^{3x} f(x,y)dydx$$

$$\iint_\sigma f(x,y)dA = \int_{y=0}^{9}\int_{x=y/3}^{\sqrt{y}} f(x,y)dxdy$$

$$\iint_\sigma f(r,\theta)dA = \int_{\theta=0}^{\text{atan}(3)}\int_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} f(r,\theta)\cdot r\cdot drd\theta$$

Evaluating $\iint_\sigma xdA$ with each of the 3 forms gives:

$$\iint_\sigma xdA = \int_{x=0}^{3}\int_{y=x^2}^{3x} xdydx = \int_{x=0}^{3} xy|_{y=x^2}^{3x}\,dx = \int_{x=0}^{3}(3x^2 - x^3)dx = (x^3 - \frac{1}{4}x^4)\Big|_{x=0}^{3}$$
$$=(27 - \frac{81}{4}) - 0 = \frac{108-81}{4} = \frac{27}{4}$$

and

$$\iint_\sigma x dA = \int_{y=0}^{9}\int_{x=y/3}^{\sqrt{y}} x dx dy = \int_{y=0}^{9} \frac{1}{2}x^2\Big|_{x=y/3}^{\sqrt{y}} dy = \int_{y=0}^{9}(\frac{1}{2}y - \frac{1}{18}y^2)dy = (\frac{1}{4}y^2 - \frac{1}{54}y^3)\Big|_{y=0}^{9}$$
$$=(\frac{81}{4} - \frac{9^3}{6\cdot 9}) - 0 = \frac{81}{4} - \frac{54}{4} = \frac{27}{4}$$

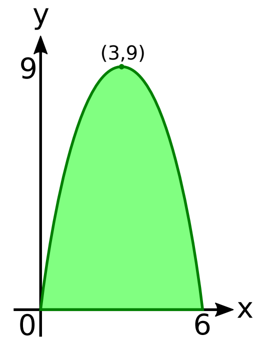
and

$$\iint_\sigma x dA = \iint_\sigma r\cos\theta dA = \int_{\theta=0}^{\text{atan}(3)}\int_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} r^2\cos\theta\cdot dr d\theta = \int_{\theta=0}^{\text{atan}(3)} \frac{r^3}{3}\cos\theta\Big|_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} d\theta$$
$$= \int_{\theta=0}^{\text{atan}(3)} \frac{\sin^3\theta}{3\cos^5\theta}d\theta = \int_{\theta=0}^{\text{atan}(3)} \frac{\cos^2\theta - 1}{3\cos^5\theta}(-\sin\theta)d\theta$$
$$= \int_{\theta=0}^{\text{atan}(3)} \frac{1}{3}((\cos\theta)^{-3} - (\cos\theta)^{-5})(-\sin\theta)d\theta = \frac{1}{3}(-\frac{1}{2}(\cos\theta)^{-2} + \frac{1}{4}(\cos\theta)^{-4})\Big|_{\theta=0}^{\text{atan}(3)}$$
$$=\frac{1}{3}(-\frac{1}{2(1/\sqrt{1+3^2})^2} + \frac{1}{4(1/\sqrt{1+3^2})^4}) - \frac{1}{3}(-\frac{1}{2} + \frac{1}{4})$$
$$=\frac{1}{3}(-5+25) + \frac{1}{12} = \frac{81}{12} = \frac{27}{4}$$

It can be readily seen that all approaches give the same result.

Question 4:

For the parabolic region σ on the right, express σ as: a Type I Cartesian region; a Type II Cartesian region; and a Polar region. In addition, given an arbitrary function $f(x,y)$, or $f(r,\theta)$ in polar coordinates, express the double integral $\iint_\sigma f(x,y)dA$ as a nested (iterated) integral using each of the 3 different forms of σ. Lastly, choose one of the 3 forms to compute the double integral $\iint_\sigma \frac{dA}{x}$.



Solution:

The parabola has the equation $y = 9-(x-3)^2 = 6x - x^2$ which is also equivalent to $x = 3\pm\sqrt{9-y}$. The Type I formulation is:

$$\sigma = \{(x,y)|0\le x\le 6 \text{ and } 0\le y\le 6x - x^2\}$$

The Type II formulation is:

$$\sigma = \{(x,y)|0\le y\le 9 \text{ and } 3-\sqrt{9-y}\le x\le 3+\sqrt{9-y}\}$$

For the polar formulation, the equation $y = 6x - x^2$ must be converted to polar coordinates:

$$y = 6x - x^2 \iff r\sin\theta = 6r\cos\theta - r^2\cos^2\theta \iff r^2 = r\frac{6\cos\theta - \sin\theta}{\cos^2\theta}$$
$$\iff r = 0, \frac{6-\tan\theta}{\cos\theta}$$

The lower bound on θ is clearly 0. The upper bound on θ occurs when $\frac{6-\tan\theta}{\cos\theta}$ becomes 0 after θ increases from 0. This occurs when $\theta = \text{atan}(6)$. The polar formulation of σ is:

$$\sigma = \{(r,\theta)|0 \le \theta \le \text{atan}(6) \text{ and } 0 \le r \le \frac{6-\tan\theta}{\cos\theta}\}$$

The double integrals over σ are:

$$\iint_\sigma f(x,y)dA = \int_{x=0}^{6}\int_{y=0}^{6x-x^2} f(x,y)dydx$$

$$\iint_\sigma f(x,y)dA = \int_{y=0}^{9}\int_{x=3-\sqrt{9-y}}^{3+\sqrt{9-y}} f(x,y)dxdy$$

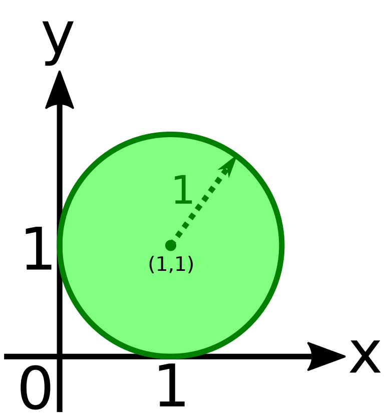
$$\iint_\sigma f(r,\theta)dA = \int_{\theta=0}^{\text{atan}(6)}\int_{r=0}^{\frac{6-\tan\theta}{\cos\theta}} f(r,\theta)\cdot r\cdot drd\theta$$

The double integral $\iint_\sigma \frac{dA}{x}$ is easiest to evaluate using the type I Cartesian formulation. This is determined through experimentation:

$$\iint_\sigma \frac{dA}{x} = \int_{x=0}^{6}\int_{y=0}^{6x-x^2}\frac{1}{x}dydx = \int_{x=0}^{6}\frac{y}{x}\Big|_{y=0}^{6x-x^2}dx = \int_{x=0}^{6}(6-x)dx$$
$$= (6x - \frac{1}{2}x^2)\Big|_{x=0}^{6} = (36-18) - 0 = 18$$

Question 5:

For the circular region σ on the right, express σ as a polar region, and then express the double integral $\iint_\sigma f(r,\theta)dA$ as a nested integral. Lastly, evaluate the double integral $\iint_\sigma \frac{\sqrt{\sin(2\theta)}\cdot dA}{r}$.



Solution:

The circle that bounds σ has the Cartesian equation $(x-1)^2+(y-1)^2=1$ which needs to converted to polar coordinates:

$$\begin{aligned}(x-1)^2+(y-1)^2=1 &\iff x^2+y^2-2x-2y=-1\\ &\iff r^2\cos^2\theta+r^2\sin^2\theta-2r\cos\theta-2r\sin\theta=-1\\ &\iff r^2-2r(\cos\theta+\sin\theta)+1=0\\ &\iff r=\frac{2(\cos\theta+\sin\theta)\pm\sqrt{4(\cos\theta+\sin\theta)^2-4}}{2}\\ &\iff r=(\cos\theta+\sin\theta)\pm\sqrt{2\cos\theta\sin\theta}\\ &\iff r=(\cos\theta+\sin\theta)\pm\sqrt{\sin(2\theta)}\end{aligned}$$

The bound on θ are clearly 0 and $\pi/2$. Therefore:

$$\sigma=\{(r,\theta)|0\le\theta\le\pi/2 \text{ and } (\cos\theta+\sin\theta)-\sqrt{\sin(2\theta)}\le r\le(\cos\theta+\sin\theta)+\sqrt{\sin(2\theta)}\}$$

$$\iint_\sigma f(r,\theta)dA=\int_{\theta=0}^{\pi/2}\int_{r=(\cos\theta+\sin\theta)-\sqrt{\sin(2\theta)}}^{(\cos\theta+\sin\theta)+\sqrt{\sin(2\theta)}} f(r,\theta)\cdot r\cdot drd\theta$$

Lastly,

$$\begin{aligned}\iint_\sigma\frac{\sqrt{\sin(2\theta)}\cdot dA}{r}&=\int_{\theta=0}^{\pi/2}\int_{r=(\cos\theta+\sin\theta)-\sqrt{\sin(2\theta)}}^{(\cos\theta+\sin\theta)+\sqrt{\sin(2\theta)}}\frac{\sqrt{\sin(2\theta)}}{r}\cdot r\cdot drd\theta\\ &=\int_{\theta=0}^{\pi/2}\sqrt{\sin(2\theta)}\cdot r\Big|_{r=(\cos\theta+\sin\theta)-\sqrt{\sin(2\theta)}}^{(\cos\theta+\sin\theta)+\sqrt{\sin(2\theta)}}d\theta\\ &=\int_{\theta=0}^{\pi/2}2\sin(2\theta)d\theta=-\cos(2\theta)|_{\theta=0}^{\pi/2}=1-(-1)=2\end{aligned}$$

Question 6:

Given the polar nested integral:

$$\int_{\theta=-\text{atan}(2)}^{\pi/4}\int_{r=0}^{\frac{2\cos\theta+\sin\theta}{\cos^2\theta}} r^2\cos\theta drd\theta$$

Sketch the region covered by this double integral, convert it to Cartesian coordinates, and lastly evaluate the integral.

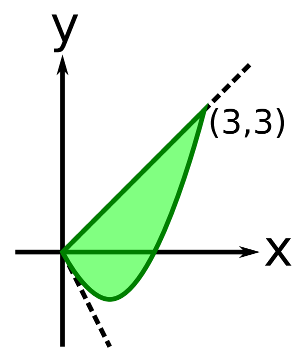
Solution:

The domain of integration is: $\sigma=\{(r,\theta)|-\text{atan}(2)\le\theta\le\pi/4 \text{ and } 0\le r\le\frac{2\cos\theta+\sin\theta}{\cos^2\theta}\}$.
To convert this domain to Cartesian coordinates, the equation $r=\frac{2\cos\theta+\sin\theta}{\cos^2\theta}$ must be converted to Cartesian coordinates:

$$r=\frac{2\cos\theta+\sin\theta}{\cos^2\theta}\iff r=\frac{2(x/r)+y/r}{(x/r)^2}\iff 1=\frac{2x+y}{x^2}\iff y=x^2-2x$$

Hence σ is bounded by the parabola $y = x^2 - 2x$.

The lower bound of $\theta = -\text{atan}(2)$ results in the line $y = x \cdot \tan(-\text{atan}(2)) = -2x \iff y = -2x$, while the upper bound of $\theta = \pi/4$ results in the line $y = x \cdot \tan(\pi/4) = x \iff y = x$. The line $y = -2x$ intersects the parabola when $x^2 - 2x = -2x \iff x = 0$. This single intersection point at the origin means that this line is tangent to the parabola at the origin, and hence lies outside of the parabola. The line $y = x$ intersects that parabola when $x^2 - 2x = x \iff x(x-3) = 0 \iff x = 0, 3$ so the intersection points are $(0,0)$ and $(3,3)$. The region σ is sketched to the right.



σ using the Type I Cartesian formulation is:

$$\sigma = \{(x,y) | 0 \le x \le 3 \text{ and } x^2 - 2x \le y \le x\}$$

so hence:

$$\int_{\theta=-\text{atan}(2)}^{\pi/4} \int_{r=0}^{\frac{2\cos\theta + \sin\theta}{\cos^2\theta}} r^2 \cos\theta dr d\theta = \iint_\sigma r\cos\theta dA = \iint_\sigma x dA = \int_{x=0}^{3} \int_{y=x^2-2x}^{x} x dy dx$$

Therefore:

$$\iint_\sigma x dA = \int_{x=0}^{3} \int_{y=x^2-2x}^{x} x dy dx = \int_{x=0}^{3} xy|_{y=x^2-2x}^{x} dx = \int_{x=0}^{3} (3x^2 - x^3) dx$$
$$= (x^3 - \frac{1}{4}x^4)\Big|_{x=0}^{3} = (27 - \frac{81}{4}) - 0 = \frac{27}{4}$$

Question 7:

Given the polar nested integral:

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{\frac{6}{2\cos\theta + 3\sin\theta}} r^2 \cos\theta dr d\theta$$

Sketch the region covered by this double integral, convert it to Cartesian coordinates, and lastly evaluate the integral.

Solution:

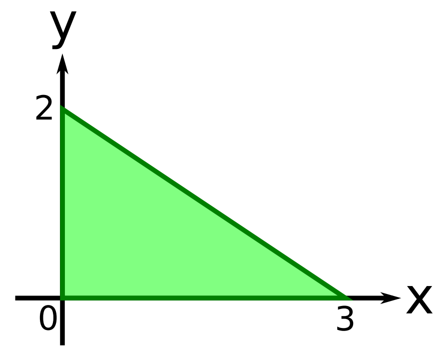
The domain of integration is: $\sigma = \{(r,\theta) | 0 \le \theta \le \pi/2 \text{ and } 0 \le r \le \frac{6}{2\cos\theta + 3\sin\theta}\}$.

To convert this domain to Cartesian coordinates, the equation $r = \frac{6}{2\cos\theta + 3\sin\theta}$ must be converted to Cartesian coordinates:

$$r = \frac{6}{2\cos\theta + 3\sin\theta} \iff r = \frac{6}{2(x/r) + 3(y/r)} \iff 1 = \frac{6}{2x + 3y} \iff y = 2 - (2/3)x$$

Hence σ is bounded by the line $y = 2 - (2/3)x$.

The lower bound of $\theta = 0$ results in the line $y = 0$, while the upper bound of $\theta = \pi/2$ results in the line $x = 0$. The line $y = 0$ intersects the line at $(3, 0)$. The line $x = 0$ intersects the line at $(0, 2)$. The region σ is sketched to the right.



σ using the Type I Cartesian formulation is:

$$\sigma = \{(x, y)|0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2 - (2/3)x\}$$

so hence:

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{\frac{6}{2\cos\theta+3\sin\theta}} r^2 \cos\theta dr d\theta = \iint_\sigma r\cos\theta dA = \iint_\sigma x dA = \int_{x=0}^{3} \int_{y=0}^{2-(2/3)x} x dy dx$$

Therefore:

$$\iint_\sigma x dA = \int_{x=0}^{3} \int_{y=0}^{2-(2/3)x} x dy dx = \int_{x=0}^{3} xy|_{y=0}^{2-(2/3)x} dx = \int_{x=0}^{3} (2x - (2/3)x^2) dx$$
$$= (x^2 - \frac{2}{9}x^3)\Big|_{x=0}^{3} = (9 - 6) - 0 = 3$$

Question 8:

Given the Cartesian nested integral:

$$\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} y\sqrt{x^2 + y^2} \cdot dy dx$$

Evaluate this integral directly, and then convert this integral to polar coordinates and evaluate that integral to demonstrate that you get the same result.

Solution:
Evaluating this integral directly gives:

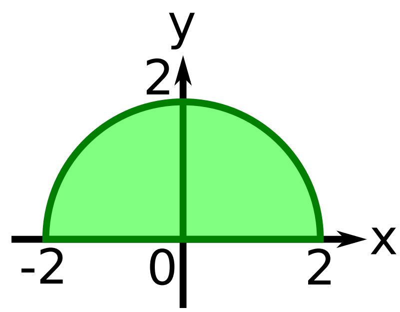
$$\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} y\sqrt{x^2 + y^2} \cdot dy dx = \int_{x=-2}^{2} \frac{1}{3}(x^2 + y^2)^{3/2}\Big|_{y=0}^{\sqrt{4-x^2}} dx$$
$$= \int_{x=-2}^{2} (\frac{8}{3} - \frac{1}{3}|x|^3) dx = \int_{x=-2}^{0} (\frac{8}{3} - \frac{1}{3}(-x)^3) dx + \int_{x=0}^{2} (\frac{8}{3} - \frac{1}{3}x^3) dx$$
$$= \int_{x=-2}^{0} (\frac{8}{3} + \frac{1}{3}x^3) dx + \int_{x=0}^{2} (\frac{8}{3} - \frac{1}{3}x^3) dx = (\frac{8}{3}x + \frac{1}{12}x^4)\Big|_{x=-2}^{0} + (\frac{8}{3}x - \frac{1}{12}x^4)\Big|_{x=0}^{2}$$
$$= (0 - (-\frac{16}{3} + \frac{4}{3})) + ((\frac{16}{3} - \frac{4}{3}) - 0) = \frac{12}{3} + \frac{12}{3} = 8$$

The domain of integration is: $\sigma = \{(x,y) | -2 \le x \le 2 \text{ and } 0 \le y \le \sqrt{4-x^2}\}$.
To convert this domain to polar coordinates, the equation $y = \sqrt{4-x^2}$ must be converted to polar coordinates:

$$\begin{aligned} y = \sqrt{4-x^2} &\iff r\sin\theta = \sqrt{4 - r^2\cos^2\theta} \\ &\implies r^2\sin^2\theta = 4 - r^2\cos^2\theta \iff r^2 = 4 \iff r = 2 \end{aligned}$$

Therefore σ is bounded by the circle $r = 2$.

The bounds of -2 and 2 on x correspond to the left and right extremes of the circle and have no impact on the circle's interior. In addition, the lower bound of $y = 0$ confines σ to the semicircle above the x-axis. The region σ is sketched on the right.



σ using the polar formulation is:

$$\sigma = \{(r,\theta) | 0 \le \theta \le \pi \text{ and } 0 \le r \le 2\}$$

so hence:

$$\int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} y\sqrt{x^2+y^2} \cdot dydx = \iint_\sigma y\sqrt{x^2+y^2} \cdot dA = \iint_\sigma r^2\sin\theta \cdot dA = \int_{\theta=0}^{\pi}\int_{r=0}^{2} r^3\sin\theta \cdot drd\theta$$

Therefore:

$$\begin{aligned} \iint_\sigma r^2\sin\theta \cdot dA = \int_{\theta=0}^{\pi}\int_{r=0}^{2} r^3\sin\theta \cdot drd\theta = \int_{\theta=0}^{\pi} \frac{1}{4}r^4\sin\theta\Big|_{r=0}^{2} d\theta = \int_{\theta=0}^{\pi} 4\sin\theta d\theta \\ = -4\cos\theta|_{\theta=0}^{\pi} = 4 - (-4) = 8 \end{aligned}$$

Both formulations of the double integral give the same value of 8 as expected.

Question 9:

Compute the volume between the two surfaces $z_1(r,\theta) = \sqrt{R^2 - r^2}$ and $z_2(r,\theta) = -\sqrt{R^2-r^2}$ over the region $\sigma = \{(r,\theta) | 0 \le \theta \le 2\pi \text{ and } 0 \le r \le R\}$ where $R > 0$ is a fixed constant. What is the significance of this volume?

Solution:
$z_1(r,\theta) \ge z_2(r,\theta)$ for all $(r,\theta) \in \sigma$. The volume between the two surfaces is:

$$\begin{aligned} V &= \iint_\sigma (z_1(r,\theta) - z_2(r,\theta))dA = \iint_\sigma 2\sqrt{R^2-r^2}dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} 2\sqrt{R^2-r^2} \cdot r \cdot drd\theta \\ &= \int_{\theta=0}^{2\pi} -\frac{2}{3}(R^2-r^2)^{3/2}\Big|_{r=0}^{R} d\theta = \int_{\theta=0}^{2\pi} \frac{2}{3}R^3 d\theta = \frac{2}{3}R^3\theta\Big|_{\theta=0}^{2\pi} = \frac{4}{3}\pi R^3 \end{aligned}$$

This is the volume of a sphere of radius R, which is exactly the shape of the volume sandwiched between surfaces $z_1(r,\theta)$ and $z_2(r,\theta)$.

Question 10:

Given the volume $\Omega = \{(x,y,z)|0 \le x \le 1 \text{ and } 0 \le y \le 2x \text{ and } 0 \le z \le 3y\}$, compute the triple integral:

$$\iiint_\Omega xyzdV$$

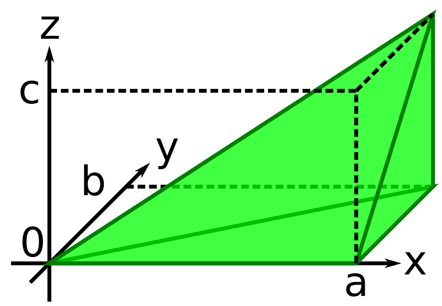
Solution:
The triple integral is the three-fold nested integral:

$$\iiint_\Omega xyzdV = \int_{x=0}^{1}\int_{y=0}^{2x}\int_{z=0}^{3y} xyz \cdot dzdydx = \int_{x=0}^{1}\int_{y=0}^{2x} \frac{1}{2}xyz^2\Big|_{z=0}^{3y} dydx = \int_{x=0}^{1}\int_{y=0}^{2x} \frac{9}{2}xy^3 dydx$$
$$= \int_{x=0}^{1} \frac{9}{8}xy^4\Big|_{y=0}^{2x} dx = \int_{x=0}^{1} 18x^5 dx = 3x^6\big|_{x=0}^{1} = 3$$

Question 11:

For the tetrahedron Ω on the right, compute the center of mass assuming a uniform mass density m. The center of mass is the weighted average position $\langle x,y,z\rangle$ of the points in Ω where the "weight" assigned to each point is the density:

$$\mathbf{r}_{\text{CM}} = \frac{\iiint_\Omega m\langle x,y,z\rangle\, dV}{\iiint_\Omega mdV} = \frac{\iiint_\Omega \langle x,y,z\rangle\, dV}{\iiint_\Omega dV}$$



Solution:
The region Ω is

$$\Omega = \{(x,y,z)|0 \le x \le a \text{ and } 0 \le y \le \frac{b}{a}x \text{ and } 0 \le z \le \frac{c}{b}y\}$$

The total volume of Ω is:

$$\iiint_\Omega dV = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x}\int_{z=0}^{\frac{c}{b}y} dzdydx = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x} z\big|_{z=0}^{\frac{c}{b}y} dydx = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x} \frac{c}{b}ydydx$$
$$= \int_{x=0}^{a} \frac{c}{2b}y^2\Big|_{y=0}^{\frac{b}{a}x} dx = \int_{x=0}^{a} \frac{bc}{2a^2}x^2 dx = \frac{bc}{6a^2}x^3\Big|_{x=0}^{a} = \frac{abc}{6}$$

The total position in Ω is:

$$\begin{aligned}
\iiint_\Omega \begin{bmatrix} x \\ y \\ z \end{bmatrix} dV &= \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}x} \int_{z=0}^{\frac{c}{b}y} \begin{bmatrix} x \\ y \\ z \end{bmatrix} dz dy dx = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}x} \left. \begin{bmatrix} xz \\ yz \\ (1/2)z^2 \end{bmatrix} \right|_{z=0}^{\frac{c}{b}y} dy dx \\
&= \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}x} \begin{bmatrix} (c/b)xy \\ (c/b)y^2 \\ (c^2/(2b^2))y^2 \end{bmatrix} dy dx = \int_{x=0}^{a} \left. \begin{bmatrix} (c/(2b))xy^2 \\ (c/(3b))y^3 \\ (c^2/(6b^2))y^3 \end{bmatrix} \right|_{y=0}^{\frac{b}{a}x} dx \\
&= \int_{x=0}^{a} \begin{bmatrix} ((bc)/(2a^2))x^3 \\ ((b^2c)/(3a^3))x^3 \\ ((bc^2)/(6a^3))x^3 \end{bmatrix} dx = \left. \begin{bmatrix} ((bc)/(8a^2))x^4 \\ ((b^2c)/(12a^3))x^4 \\ ((bc^2)/(24a^3))x^4 \end{bmatrix} \right|_{x=0}^{a} = \begin{bmatrix} (a^2bc)/8 \\ (ab^2c)/12 \\ (abc^2)/24 \end{bmatrix}
\end{aligned}$$

Therefore:

$$\mathbf{r}_{\mathrm{CM}} = \frac{\iiint_\Omega \langle x, y, z \rangle \, dV}{\iiint_\Omega dV} = \begin{bmatrix} (3/4)a \\ (1/2)b \\ (1/4)c \end{bmatrix}$$