Determining Nonsingularity

Given and arbitrary quadratic polynomial

$$ax^2 + bx + c$$

the "discriminant", defined by

$$\Delta = b^2 - 4ac$$

identifies the number of roots that the polynomial has:

- If $\Delta > 0$, then there are exactly 2 real valued roots.
- If $\Delta = 0$, then there is exactly 1 root.
- If $\Delta < 0$, then there are exactly 2 complex valued roots.

Similar to the discriminant for quadratic polynomials, the **determinant** of an $n \times n$ matrix A is a quantity that is nonzero if and only if A is nonsingular (i.e. invertible).

Determinants are only defined for square matrices, and are nonzero if and only if the matrix is invertible.

Given an arbitrary $n \times n$ matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$, the determinant of A is denoted by the following notation:

$$\det(A) \qquad \text{or} \qquad |A| \qquad \text{or} \qquad \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}$$

1×1 matrices

Given an simple 1×1 matrix $A = \begin{bmatrix} a_{1,1} \end{bmatrix}$, it is clear that A is invertible if and only if $a_{1,1} \neq 0$. Therefore the determinant is defined as:

$$\det\left(\begin{bmatrix} a_{1,1} \end{bmatrix}\right) = a_{1,1}$$

2×2 matrices

For a 2×2 matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, the determinant is:

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

$n \times n$ matrices

The cofactor expansion

Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$ denote an arbitrary $n \times n$ matrix.

The **minor** $M_{i,j}$ at entry (i,j) is the determinant of A with the i^{th} row and the j^{th} column removed. The **cofactor** $C_{i,j}$ is $C_{i,j} = (-1)^{i+j} M_{i,j}$. The sign introduced by the cofactor alternates between $+$ and $-$ during transitions between adjacent cells, starting with $+$ at entry $(1,1)$:

$$\begin{bmatrix} + \end{bmatrix} \qquad \begin{bmatrix} + & - \\ - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

The cofactor expansion along row i is:

$$\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + ... + a_{i,n}C_{i,n} = (-1)^{i+1}a_{i,1}M_{i,1} + (-1)^{i+2}a_{i,2}M_{i,2} + ... + (-1)^{i+n}a_{i,n}M_{i,n}$$

The cofactor expansion along column j is:

$$\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + ... + a_{n,j}C_{n,j} = (-1)^{j+1}a_{1,j}M_{1,j} + (-1)^{j+2}a_{2,j}M_{2,j} + ... + (-1)^{j+n}a_{n,j}M_{n,j}$$

No matter which cofactor expansion is used, the final value of the determinant will always be the same. For example, given the 3×3 matrix:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

There are 6 possible cofactor expansions that will all return the same result:

The cofactor expansion along row 1 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

The cofactor expansion along row 2 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = -a_{2,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} + a_{2,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} - a_{2,3}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

The cofactor expansion along row 3 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{3,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

The cofactor expansion along column 1 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{2,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} + a_{3,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}$$

The cofactor expansion along column 2 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = -a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{2,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} - a_{3,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}$$

The cofactor expansion along column 3 gives:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} - a_{2,3} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} + a_{3,3} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

When choosing the row or column to apply the cofactor expansion along, choosing a row or column with the largest number of 0's is optimal.

Examples:*

- Evaluate the determinant:

$$\begin{vmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{vmatrix}$$

  Out of all of the rows and columns, column 2 has the most 0's. The cofactor expansion along column 2 gives:

$$\begin{aligned} \begin{vmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{vmatrix} &= -0 \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} + 5 \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 \begin{vmatrix} -3 & 7 \\ 2 & 1 \end{vmatrix} \\ &= 5((-3)(5) - (-1)(7)) = 5(-15 + 7) = 5(-8) = -40 \end{aligned}$$

  Now consider that instead of expanding along column 2, the determinant is expanded along row 2. This is a suboptimal choice since row 2 has no 0's.

$$\begin{aligned} \begin{vmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{vmatrix} &= -2 \begin{vmatrix} 0 & 7 \\ 0 & 5 \end{vmatrix} + 5 \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 1 \begin{vmatrix} -3 & 0 \\ -1 & 0 \end{vmatrix} \\ &= -2((0)(5) - (0)(7)) + 5((-3)(5) - (-1)(7)) - 1((-3)(0) - (-1)(0)) \\ &= -2(0 - 0) + 5(-15 + 7) - 1(0 - 0) = 5(-8) = -40 \end{aligned}$$

  The calculations are more complex, but the result is the same.

- Evaluate the determinant:

$$\begin{vmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

  Row 2 (or column 2) has the most 0's. Expanding along row 2 gives:

$$\begin{aligned} \begin{vmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{vmatrix} &= -1 \begin{vmatrix} 3 & 1 \\ -3 & 5 \end{vmatrix} + 0 \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} - (-4) \begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} \\ &= -1((3)(5) - (-3)(1)) - (-4)((3)(-3) - (1)(3)) \\ &= -(15 + 3) + 4(-9 - 3) = -18 + 4(-12) = -18 - 48 = -66 \end{aligned}$$

  Now consider that instead of expanding along row 2, the determinant is expanded along column 1. This is a suboptimal choice since column 1 has no 0's.

$$\begin{aligned} \begin{vmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{vmatrix} &= 3 \begin{vmatrix} 0 & -4 \\ -3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 0 & -4 \end{vmatrix} \\ &= 3((0)(5) - (-3)(-4)) - 1((3)(5) - (-3)(1)) + 1((3)(-4) - (0)(1)) \\ &= 3(0 - 12) - (15 + 3) + (-12 - 0) = -36 - 18 - 12 = -66 \end{aligned}$$

- Evaluate the determinant:

$$\begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix}$$

Column 3 has the most 0's. Expanding along column 3 gives:

$$\begin{aligned}
\begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} &= 0\begin{vmatrix} 2 & 2 & -2 \\ 4 & 1 & 0 \\ 2 & 10 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 3 & 5 \\ 4 & 1 & 0 \\ 2 & 10 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} \\
&= -3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix}
\end{aligned}$$

Each of the 3×3 determinants will be expanded along column 1:

$$\begin{aligned}
&-3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} \\
&= -3\left(3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 10 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 \\ 2 & -2 \end{vmatrix}\right) - 3\left(3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} + 4\begin{vmatrix} 3 & 5 \\ 2 & -2 \end{vmatrix}\right) \\
&= -3(3(24) - 2(-44) + 2(-16)) - 3(3(2) - 2(-5) + 4(-16)) \\
&= -3(72 + 88 - 32) - 3(6 + 10 - 64) \\
&= -3(160 - 32) - 3(16 - 64) = -3(128) - 3(-48) \\
&= -384 + 144 = -240
\end{aligned}$$

*Examples from the problem set of chapter 2.1 of the textbook:
Anton, Howard; Rorres, Chris, *Elementary Linear Algebra 11th edition, Applications Version*, Wiley, 2014.

Triangular matrices

The determinant of triangular matrices is simply the product of the diagonal entries.

Consider the upper triangular $n \times n$ matrix:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n} \end{bmatrix}$$

By repeatedly computing the cofactor expansion along column 1, or along row n, it is easy to see that:

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{vmatrix} = a_{1,1}a_{2,2}\begin{vmatrix} a_{3,3} & \cdots & a_{3,n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3}\cdots a_{n,n}$$

Now consider the lower triangular $n \times n$ matrix:

$$A = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}$$

By repeatedly computing the cofactor expansion along row 1, or along column n, it is easy to see that:

$$\begin{aligned} \det(A) = &\begin{vmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & 0 & \cdots & 0 \\ a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} = a_{1,1}a_{2,2} \begin{vmatrix} a_{3,3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\ =& a_{1,1}a_{2,2}a_{3,3}\cdots a_{n,n} \end{aligned}$$

Interpreting the determinant

Elementary row (and column) operations and determinants

The determinant of an $n \times n$ matrix A also