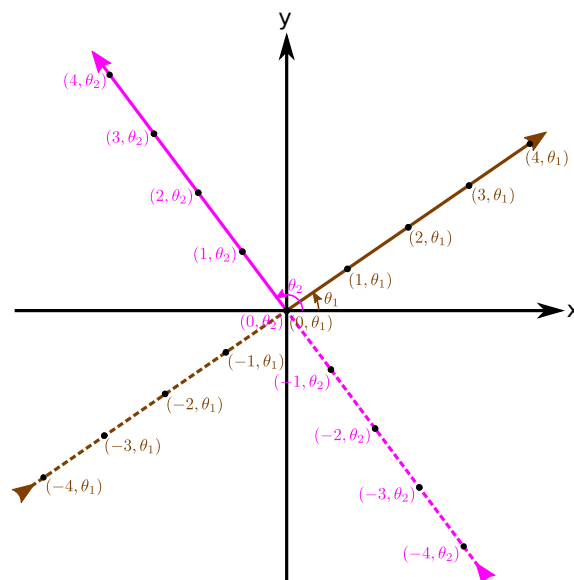# Polar Coordinates

As an alternative to Cartesian coordinates, polar coordinates provide another means of quantifying the position of a point using two numbers. Polar coordinates describe a point $P$'s location using a pair of numbers $(r, \theta)$ instead of $(x, y)$. $r$ denotes the "distance" of $P$ from the origin, while $\theta$ denotes the counterclockwise rotation from the positive $x$-axis required to aim at point $P$. Polar coordinates are illustrated by the image on the right.

- $\theta$ will always be measured in radians.

- The rotation $\theta$ is always measured relative to the positive $x$-axis.

- A clockwise rotation corresponds to a negative value of $\theta$.
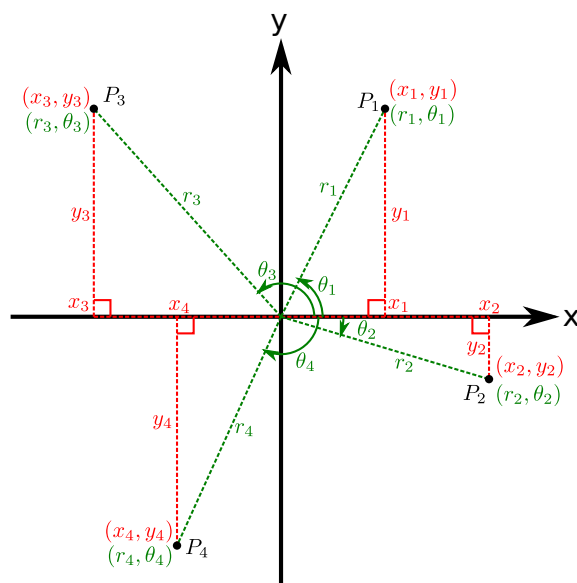
Despite $r$ denoting the "distance" required to reach point $P$, $r$ can still be negative. Initially facing in the direction of the positive $x$-axis, coordinate $\theta$ denotes the counterclockwise rotation required to lineup point $P$. $r$ is positive if forwards movement is necessary to reach point $P$, while $r$ is **negative** if **backwards** movement is required to reach point $P$.

Given polar coordinate $(r, \theta)$, for every integer $k \in \mathbb{Z}$, the polar coordinates $(r, \theta + k \cdot 2\pi) = (r, \theta + 2k\pi)$ and $(-r, \theta + \pi + k \cdot 2\pi) = (-r, \theta + (2k+1)\pi)$ describe the same point $(r, \theta)$. Adding or subtracting full revolutions from $\theta$ does not change the referenced point. In addition, adding or subtracting a half revolution followed by sign change of $r$ does not change the referenced point.

In polar coordinates,

$$\cdots = (-r, \theta - 3\pi) = (r, \theta - 2\pi) = (-r, \theta - \pi) = (r, \theta) = (-r, \theta + \pi) = (r, \theta + 2\pi) = (-r, \theta + 3\pi) = \cdots$$
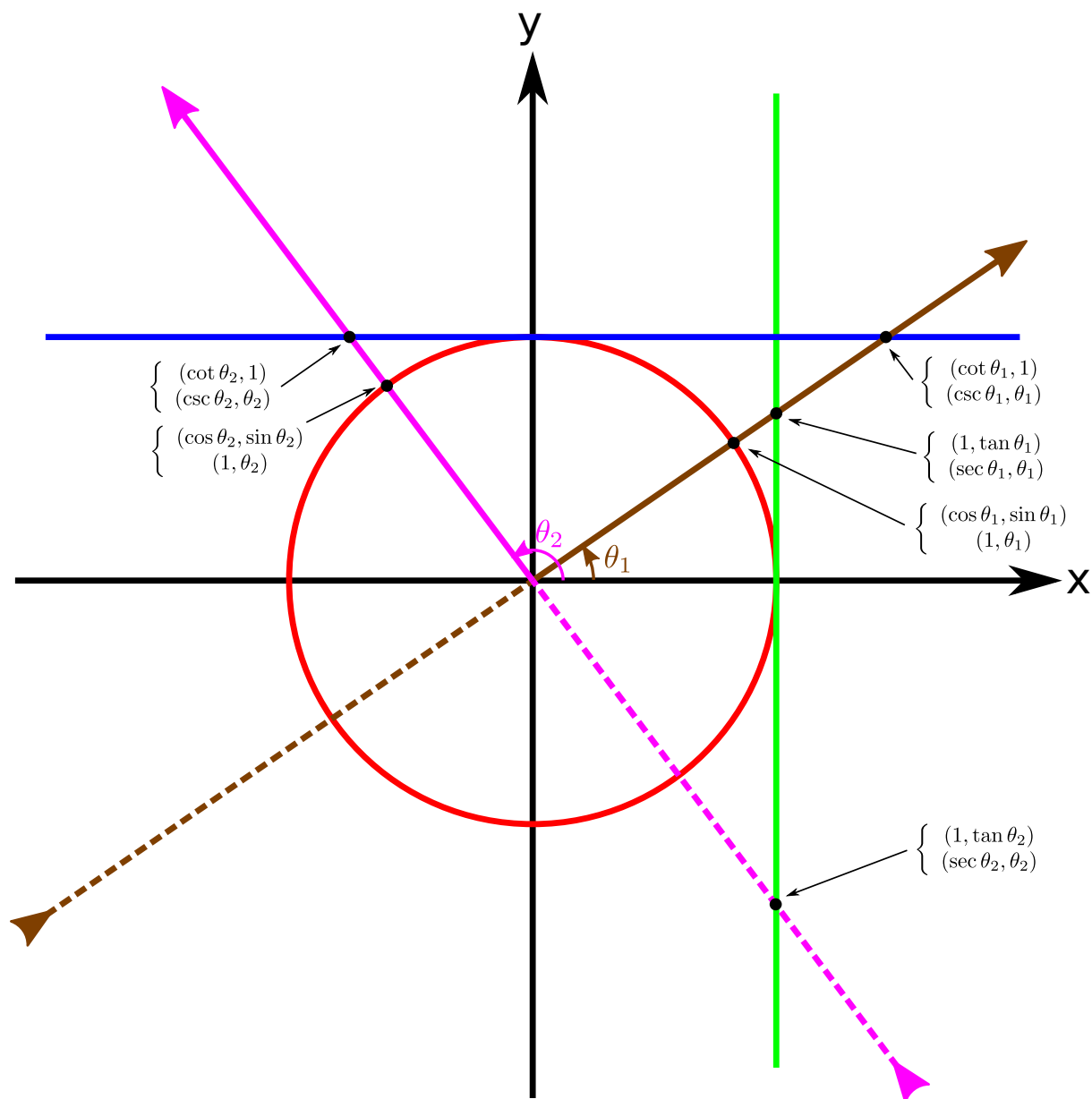
all denote the same point.

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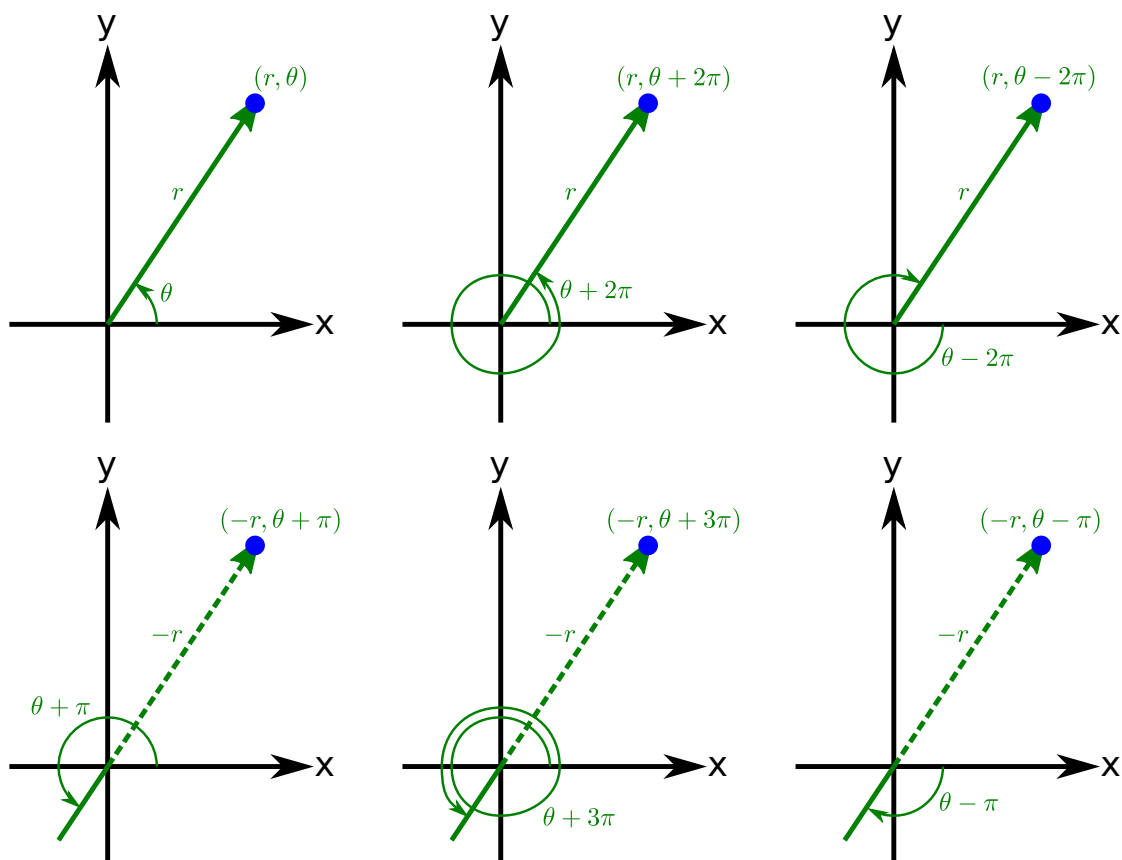
**Examples:**

- The polar coordinates $(3, 3\pi/4)$, $(3, 11\pi/4)$, $(3, -5\pi/4)$, $(-3, 7\pi/4)$, $(-3, 15\pi/4)$, and $(-3, -\pi/4)$ all describe the same point.

## Polar coordinates and the trigonometric ratios

In the image below, the red unit circle, the green tangent line defined by $x = 1$, and the blue tangent line defined by $y = 1$ are shown. Starting from the positive $x$ axis, rotate by a counterclockwise angle of $\theta$. A ray projected forwards corresponds to positive values of $r$. A ray projected backwards corresponds to negative values of $r$. The positive ray intersects the unit circle at the point with Cartesian coordinates $(\cos\theta, \sin\theta)$, which also has the polar coordinates $(1, \theta)$. The rays intersect the green tangent line at the point with Cartesian coordinates $(1, \tan\theta)$, which also has the polar coordinates $(\sec\theta, \theta)$. The rays intersect the blue tangent line at the point with Cartesian coordinates $(\cot\theta, 1)$, which also has the polar coordinates $(\csc\theta, \theta)$. These points are depicted in the image below for two different values of $\theta$, namely $\theta_1$ and $\theta_2$. $\theta_1$ is chosen from the interval $(0, \pi/2)$ and generates the brown rays, while $\theta_2$ is chosen from the interval $(\pi/2, \pi)$ and generates the pink rays.

$$\begin{cases} (\cot\theta_2, 1) \\ (\csc\theta_2, \theta_2) \end{cases}$$

$$\begin{cases} (\cos\theta_2, \sin\theta_2) \\ (1, \theta_2) \end{cases}$$

$$\begin{cases} (\cot\theta_1, 1) \\ (\csc\theta_1, \theta_1) \end{cases}$$

$$\begin{cases} (1, \tan\theta_1) \\ (\sec\theta_1, \theta_1) \end{cases}$$

$$\begin{cases} (\cos\theta_1, \sin\theta_1) \\ (1, \theta_1) \end{cases}$$

$$\begin{cases} (1, \tan\theta_2) \\ (\sec\theta_2, \theta_2) \end{cases}$$

$\theta_2$

$\theta_1$

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## Polar to Cartesian conversion

Given an arbitrary polar coordinate $(r, \theta)$, there is only one Cartesian coordinate $(x, y)$ for the point generated by $(r, \theta)$. Project the ray rotated a counterclockwise angle of $\theta$ from the positive $x$-axis. This ray intersects the unit circle at a point with Cartesian coordinates $(\cos \theta, \sin \theta)$ and polar coordinates $(1, \theta)$. Scaling every length uniformly by $r$ changes the Cartesian coordinate to $(r \cos \theta, r \sin \theta)$ and the polar coordinate to $(r, \theta)$. Therefore:

$$\left\{ \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array} \right.$$

is how the polar coordinate $(r, \theta)$ is converted to Cartesian coordinate $(x, y)$.
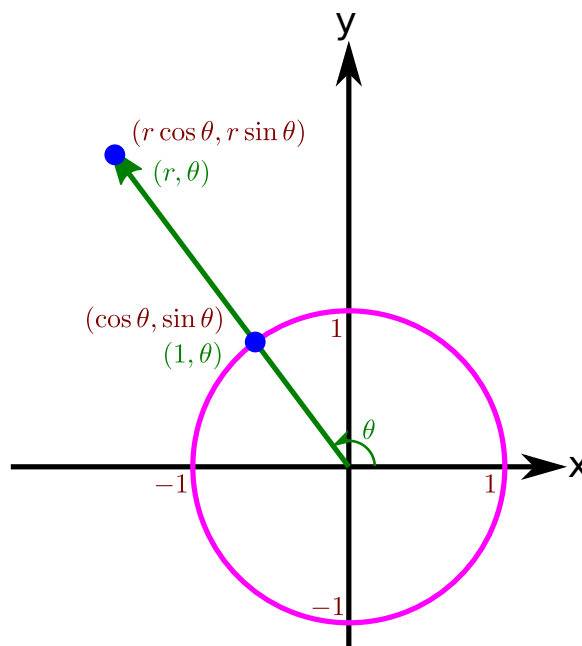
**Examples:**

- The polar coordinate $(r, \theta) = (3, \pi/5)$ converted to a Cartesian coordinate is $(x, y) = (3 \cos(\pi/5), 3 \sin(\pi/5)) \approx (2.42705, 1.76336)$

- The polar coordinate $(r, \theta) = (7, 0.77\pi)$ converted to a Cartesian coordinate is $(x, y) = (7 \cos(0.77\pi), 7 \sin(0.77\pi)) \approx (-5.25078, 4.62918)$

- The polar coordinate $(r, \theta) = (2.6, -1.67\pi)$ converted to a Cartesian coordinate is $(x, y) = (2.6 \cos(-1.67\pi), 2.6 \sin(-1.67\pi)) \approx (1.32351, 2.23793)$

- The polar coordinate $(r, \theta) = (2.6, 0.33\pi)$ converted to a Cartesian coordinate is $(x, y) = (2.6 \cos(0.33\pi), 2.6 \sin(0.33\pi)) \approx (1.32351, 2.23793)$

- The polar coordinate $(r, \theta) = (-3, \pi/5)$ converted to a Cartesian coordinate is $(x, y) = (-3 \cos(\pi/5), -3 \sin(\pi/5)) \approx (-2.42705, -1.76336)$

## Cartesian to polar conversion

Given an arbitrary Cartesian coordinate $(x, y)$, a polar coordinate $(r, \theta)$ that generates the same point as $(x, y)$ is sought.

There are infinitely many choices of polar coordinate $(r, \theta)$ for the given Cartesian coordinate $(x, y)$. To limit our choices down to one alternative, the following restrictions will be placed on $r$ and $\theta$:

$$\left\{ \begin{array}{c} r \geq 0 \\ (r = 0) \implies (\theta = 0) \\ -\pi < \theta \leq \pi \end{array} \right.$$

- Since $r \geq 0$, only non-negative choices of $r$ will be considered, so $r$ is simply the distance that the point $(x, y)$ is from the origin: $r = \sqrt{x^2 + y^2}$.

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- The condition $(r = 0) \implies (\theta = 0)$ means that $\theta = 0$ if $r = 0$, which means that the $\theta$ coordinate of the origin point, which could have been any value, is defaulted to $\theta = 0$.

- The condition $-\pi < \theta \leq \pi$ means that a counterclockwise twist is used to reach points above the $x$-axis, and a clockwise twist is used to reach points beneath the $x$-axis. Moreover, a twist of $\pi$ (which is counterclockwise) is used to reach the negative $x$-axis.

It is clear that $r$ is simply the distance that the point $(x, y)$ is from the origin: $r = \sqrt{x^2 + y^2}$. Computing $\theta$ is a more difficult task.

To find $\theta$, consider the line segment that connects the origin to point $(x, y)$. The counterclockwise angle that this line segment makes with the positive $x$-axis is $\theta$. Extrapolating this line segment to a line, this line will intersect the vertical line defined by $x = 1$ at the Cartesian point $(1, y/x)$. This is because the line segment has a slope of $y/x$, so the change in $y$ over a unit change in $x$ is $y/x$. **Use the image below for illustration.** Given a line that makes a counterclockwise angle of $\theta$ with the $x$-axis, this line intersects the line $x = 1$ at the Cartesian point $(1, \tan \theta)$ (revisit the discussions related to the unit circle). Since $(1, y/x)$ and $(1, \tan \theta)$ are the same point, $\tan \theta = y/x$.

The fact that $\tan \theta = y/x$ can also be established from $x = r \cos \theta$ and $y = r \sin \theta$:

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

In the image below, it can be seen that if $x > 0$, then $\theta = \tan^{-1}(y/x)$, whereas if $x < 0$, then a half rotation needs to be either added to or subtracted from $\tan^{-1}(y/x)$ to get $\theta$.

In the case where $x < 0$,

- If $y/x \leq 0$, then the range of values of $\tan^{-1}(y/x)$ is $(-\pi/2, 0]$, so a half rotation must be added to give values of $\theta$ from the range $(\pi/2, \pi]$ which is a valid subset of the target range $(-\pi, \pi]$. This can be seen with point $(x_3, y_3)$ in the image, where a half rotation needs to be added to $\tan^{-1}(y_3/x_3)$ to get $\theta_3$.

- If $y/x > 0$, then the range of values of $\tan^{-1}(y/x)$ is $(0, \pi/2)$, so a half rotation must be subtracted to give values of $\theta$ from the range $(-\pi, -\pi/2)$ which is a valid subset of the target range $(-\pi, \pi]$. This can be seen with point $(x_4, y_4)$ in the image, where a half rotation needs to be subtracted from $\tan^{-1}(y_4/x_4)$ to get $\theta_4$.

In summary,

- If $x > 0$, then $\theta = \tan^{-1}(y/x)$

- If $x = 0$, then

  * If $y > 0$, then $\theta = \pi/2$
  * If $y = 0$, then $\theta = 0$ (this is the origin point)
  * If $y < 0$, then $\theta = -\pi/2$

- If $x < 0$, then

  * If $y \geq 0$, then $\theta = \tan^{-1}(y/x) + \pi$
  * If $y < 0$, then $\theta = \tan^{-1}(y/x) - \pi$
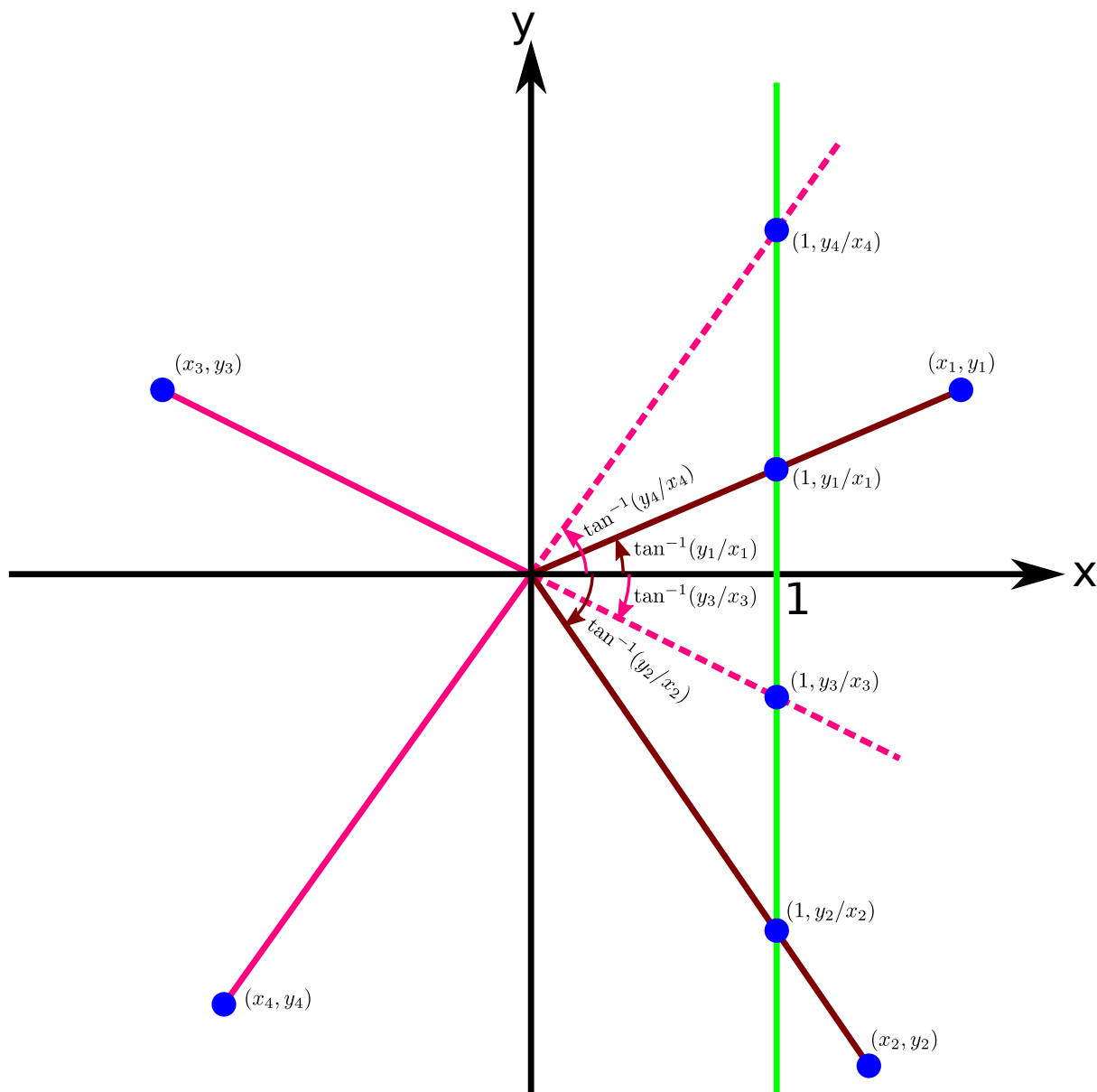
**Examples:**

- The Cartesian coordinate $(x, y) = (1, 2)$ converted to a polar coordinate gives $r = \sqrt{1^2 + 2^2} \approx 2.23607$ and $\theta = \tan^{-1}(2/1) \approx 1.10715$. Hence $(r, \theta) \approx (2.23607, 1.10715)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (4, -3)$ converted to a polar coordinate gives $r = \sqrt{4^2 + (-3)^2} = 5$ and $\theta = \tan^{-1}((-3)/4) \approx -0.643501$. Hence $(r, \theta) \approx (5, -0.643501)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (0, 6)$ converted to a polar coordinate gives $r = \sqrt{0^2 + 6^2} = 6$ and

$\theta = \pi/2 \approx 1.57080$. Hence $(r, \theta) \approx (6, 1.57080)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (0, 0)$ converted to a polar coordinate gives $r = \sqrt{0^2 + 0^2} = 0$ and $\theta = 0$. Hence $(r, \theta) = (0, 0)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (0, -3)$ converted to a polar coordinate gives $r = \sqrt{0^2 + (-3)^2} = 3$ and $\theta = -\pi/2 \approx -1.57080$. Hence $(r, \theta) \approx (3, -1.57080)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (-2, 5)$ converted to a polar coordinate gives $r = \sqrt{(-2)^2 + 5^2} \approx 5.38516$ and $\theta = \tan^{-1}(5/(-2)) + \pi \approx 1.95130$. Hence $(r, \theta) \approx (5.38516, 1.95130)$ is one possible choice of polar coordinate.

- The Cartesian coordinate $(x, y) = (-3, -1)$ converted to a polar coordinate gives $r = \sqrt{(-3)^2 + (-1)^2} \approx 3.16228$ and $\theta = \tan^{-1}((-1)/(-3)) - \pi \approx -2.81984$. Hence $(r, \theta) \approx (3.16228, -2.81984)$ is one possible choice of polar coordinate.

## Double Integrals using Polar Coordinates

2D regions in polar coordinates can be quantified by first establishing bounds on $\theta$, and then establishing bounds on $r$ that are functions of $\theta$. In the image to the right, the bounds on $\theta$ are $a$ and $b$: $a \leq \theta \leq b$. The bounds on $r$ are functions of $\theta$: $h_L(\theta) \leq r \leq h_U(\theta)$. The region itself is:

$$R = \{(r, \theta) | a \leq \theta \leq b \ \& \ h_L(\theta) \leq r \leq h_U(\theta)\}$$

**It will now always be assumed that $r \geq 0$ for all valid polar regions.**
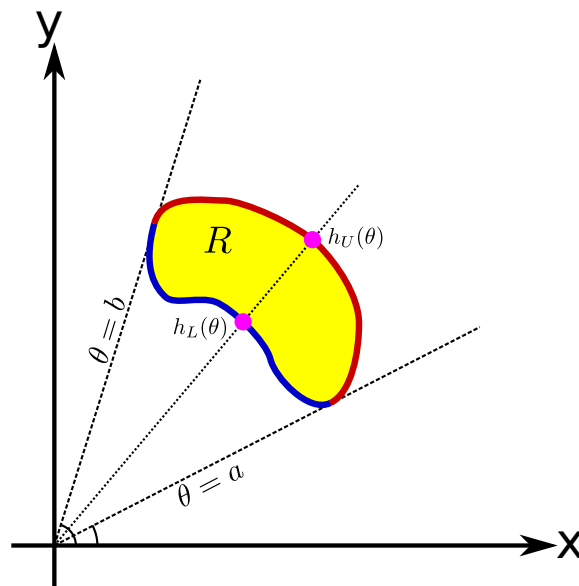


The double integral $\iint_R f(r, \theta) dA$ over a polar region:

$$R = \{(r, \theta) | a \leq \theta \leq b \ \& \ h_L(\theta) \leq r \leq h_U(\theta)\}$$

will now be evaluated using nested single variable integrals. To establish the single integral expression for $\iint_R f(r, \theta) dA$, region $R$ will be decomposed into a series of $N$ thin wedges where $N$ is a large number, as depicted in the image below. For each $i = 1, 2, ..., N$, the angle of the $i^{\text{th}}$ wedge is $\Delta\theta_i$, and $\theta_i^*$ is a "representative" $\theta$ value from the $i^{\text{th}}$ wedge. Now for each wedge, further partition the wedge into a series of $M$ "polar rectangles" where $M$ is a large number, as depicted in the image below. For each $i = 1, 2, ..., N$ and for each $j = 1, 2, ..., M$, the radial thickness of the $j^{\text{th}}$ polar rectangle in the $i^{\text{th}}$ wedge is $\Delta r_{(i,j)}$, and $r_{(i,j)}^*$ is a "representative" $r$ value from the $j^{\text{th}}$ polar rectangle in the $i^{\text{th}}$ wedge.

$$\Delta A = r^* \Delta r \Delta \theta$$
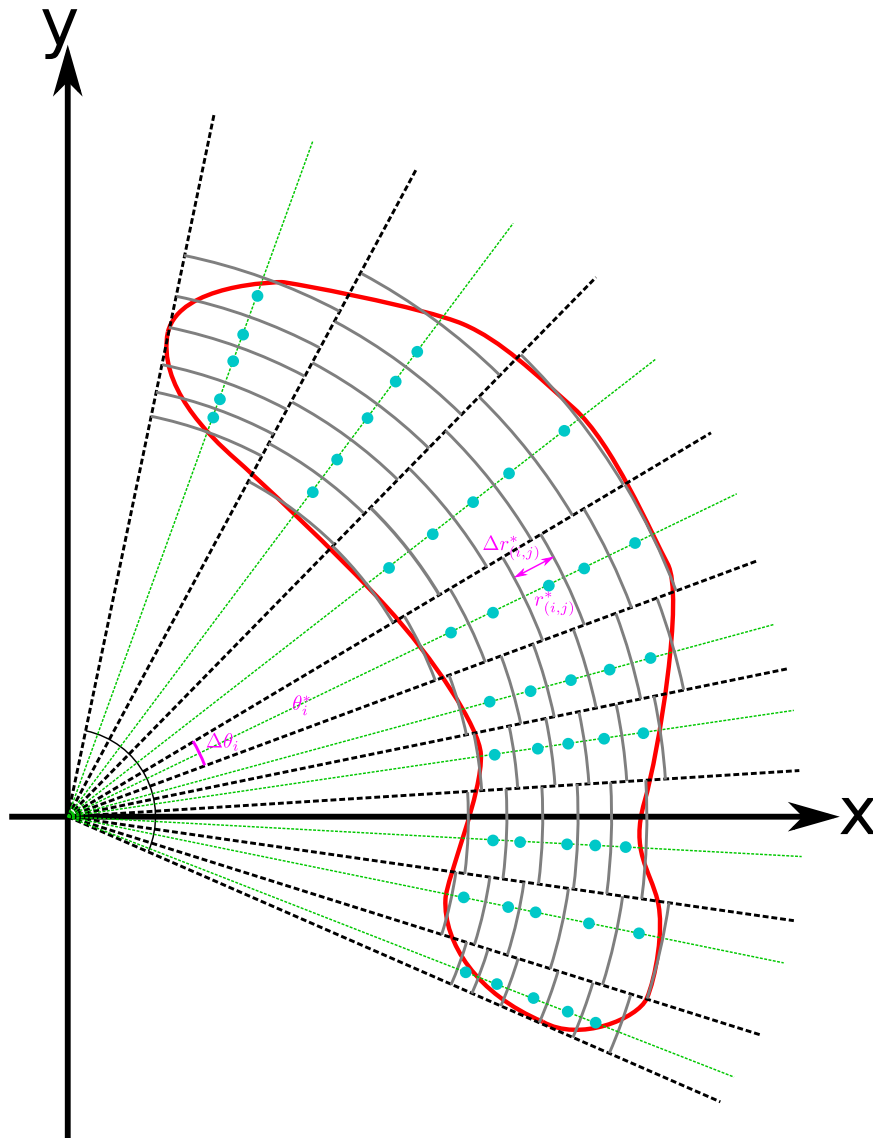
The area of the $j^{\text{th}}$ polar rectangle in the $i^{\text{th}}$ wedge is not $\Delta r_{(i,j)} \Delta \theta_i$.

To the right is shown a polar rectangle with an angular length of $\Delta \theta$ and a radial thickness of $\Delta r$. The average of the inner and outer radii is $r^*$. It can be computed that the area on this polar rectangle is $(r\Delta\theta)\Delta r = r\Delta r\Delta\theta$.

The area of the $j^{\text{th}}$ polar rectangle in the $i^{\text{th}}$ wedge is in fact:

$$\Delta A_{(i,j)} = r^*_{(i,j)} \Delta r_{(i,j)} \Delta \theta_i$$

where an additional factor of $r^*_{(i,j)}$ has been introduced.
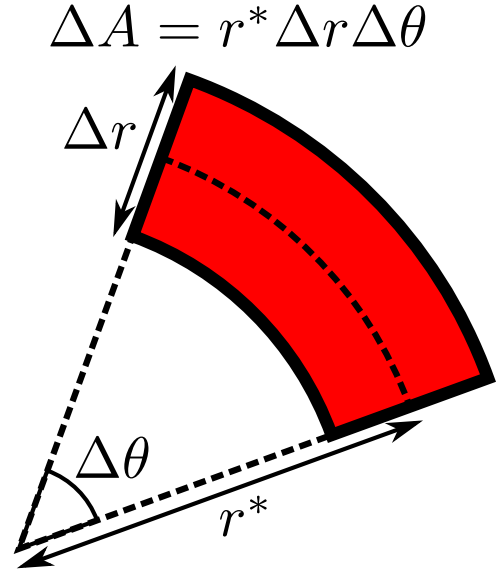
Evaluating the Riemann sum gives:

$$\iint_R f(r,\theta)dA = \lim_{N\to+\infty} \lim_{M\to+\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} f(r^*_{(i,j)},\theta_i^*)\Delta A_{(i,j)} = \lim_{N\to+\infty} \lim_{M\to+\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} f(r^*_{(i,j)},\theta_i^*)(r^*_{(i,j)}\Delta r_{(i,j)}\Delta\theta_i)$$

$$= \lim_{N\to+\infty} \sum_{i=1}^{N}\left(\lim_{M\to+\infty}\sum_{j=1}^{M} f(r^*_{(i,j)},\theta_i^*)\cdot r^*_{(i,j)}\cdot \Delta r_{(i,j)}\right)\Delta\theta_i$$

$$= \lim_{N\to+\infty} \sum_{i=1}^{N}\left(\int_{r=h_L(\theta_i^*)}^{h_U(\theta_i^*)} f(r,\theta_i^*)\cdot r\cdot dr\right)\Delta\theta_i = \int_{\theta=a}^{b}\left(\int_{r=h_L(\theta)}^{h_U(\theta)} f(r,\theta)\cdot r\cdot dr\right)d\theta$$

Therefore:

$$\iint_R f(r,\theta) = \int_{\theta=a}^{b}\left(\int_{r=h_L(\theta)}^{h_U(\theta)} f(r,\theta)\cdot r\cdot dr\right)d\theta$$

Note the introduction of an additional factor of $r$ to the integrand when the double integral is replaced with the nested integral.

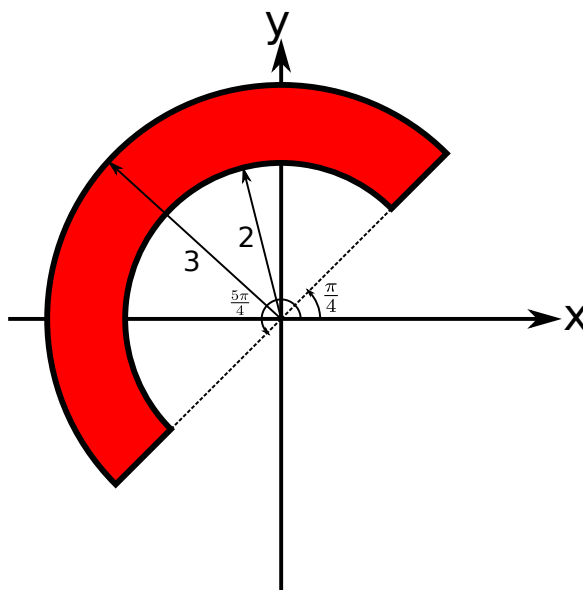**Example 1:**

Consider the polar region:

$$R = \left\{ (r,\theta) \middle| \frac{\pi}{4} \le \theta \le \frac{5\pi}{4} \ \& \ 2 \le r \le 3 \right\}$$

which is depicted on the right.

Given an arbitrary integrand $f(r,\theta)$, the double integral over the region $R$ is:

$$\iint_R f(r,\theta)dA = \int_{\theta=\pi/4}^{5\pi/4} \left( \int_{r=2}^{3} f(r,\theta) \cdot r \cdot dr \right) d\theta$$

Note the introduction of an additional factor of $r$ to the integrand when the double integral is replaced with the nested integral.

Consider the specific integrand $f(r,\theta) = \frac{1}{r}\cos^2\theta\sin^2\theta$. The double integral over $R$ is:

$$\iint_R \frac{1}{r}\cos^2\theta\sin^2\theta \cdot dA = \int_{\theta=\pi/4}^{5\pi/4} \left( \int_{r=2}^{3} \frac{1}{r}\cos^2\theta\sin^2\theta \cdot r \cdot dr \right) d\theta = \int_{\theta=\pi/4}^{5\pi/4} \left( \int_{r=2}^{3} \cos^2\theta\sin^2\theta \cdot dr \right) d\theta$$

$$= \int_{\theta=\pi/4}^{5\pi/4} r\cos^2\theta\sin^2\theta \Big|_{r=2}^{3} d\theta = \int_{\theta=\pi/4}^{5\pi/4} \cos^2\theta\sin^2\theta \, d\theta = \int_{\theta=\pi/4}^{5\pi/4} \left( \frac{1+\cos(2\theta)}{2} \right)\left( \frac{1-\cos(2\theta)}{2} \right) d\theta$$

$$= \int_{\theta=\pi/4}^{5\pi/4} \frac{1-\cos^2(2\theta)}{4} d\theta = \int_{\theta=\pi/4}^{5\pi/4} \frac{1 - \frac{1+\cos(4\theta)}{2}}{4} d\theta = \int_{\theta=\pi/4}^{5\pi/4} \frac{1-\cos(4\theta)}{8} d\theta = \frac{\theta - \frac{1}{4}\sin(4\theta)}{8} \Big|_{\theta=\pi/4}^{5\pi/4}$$

$$= \frac{4\theta - \sin(4\theta)}{32} \Big|_{\theta=\pi/4}^{5\pi/4} = \frac{5\pi - \sin(5\pi)}{32} - \frac{\pi - \sin(\pi)}{32} = \frac{5\pi}{32} - \frac{\pi}{32} = \frac{4\pi}{32} = \frac{\pi}{8}$$
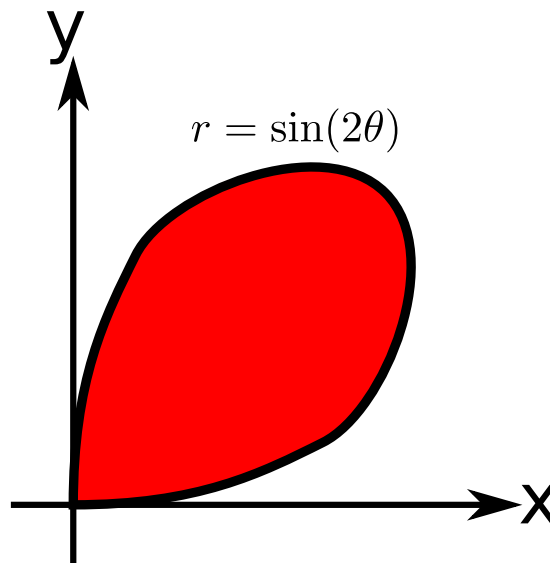
**Example 2:**

Consider the polar region:

$$R = \left\{(r,\theta)\middle|0 \le \theta \le \frac{\pi}{2} \ \& \ 0 \le r \le \sin(2\theta)\right\}$$

which is depicted on the right.

Given an arbitrary integrand $f(r,\theta)$, the double integral over the region $R$ is:

$$\iint_R f(r,\theta)dA = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\sin(2\theta)} f(r,\theta)\cdot r\cdot dr\right)d\theta$$

Note the introduction of an additional factor of $r$ to the integrand when the double integral is replaced with the nested integral.

$$r = \sin(2\theta)$$



Consider the specific integrand $f(r,\theta) = r$. The double integral over $R$ is:

$$\iint_R rdA = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\sin(2\theta)} r\cdot r\cdot dr\right)d\theta = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\sin(2\theta)} r^2\cdot dr\right)d\theta = \int_{\theta=0}^{\pi/2} \frac{1}{3}r^3\Big|_{r=0}^{\sin(2\theta)}d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{3}\sin^3(2\theta)d\theta = \int_{\theta=0}^{\pi/2} \frac{1}{3}\sin(2\theta)\cdot\sin^2(2\theta)d\theta = \int_{\theta=0}^{\pi/2} \frac{1}{3}\sin(2\theta)(1-\cos^2(2\theta))d\theta$$

$$= \int_{\theta=0}^{\pi/2} -\frac{1}{6}(1-\cos^2(2\theta))(-2\sin(2\theta)d\theta) = -\frac{1}{6}\left(\cos(2\theta)-\frac{1}{3}\cos^3(2\theta)\right)\Big|_{\theta=0}^{\pi/2}$$

$$= (-\frac{1}{6}(\cos(\pi)-\frac{1}{3}\cos^3(\pi))) - (-\frac{1}{6}(\cos(0)-\frac{1}{3}\cos^3(0))) = (-\frac{1}{6}(-1+\frac{1}{3})) - (-\frac{1}{6}(1-\frac{1}{3}))$$

$$= (\frac{1}{9}) - (-\frac{1}{9}) = \frac{2}{9}$$

# Cartesian integral to Polar integral conversions

Often, it may be necessary to convert a nested integral that uses Cartesian coordinates to a nested integral that uses polar coordinates. Firstly, the nested Cartesian integral needs to be converted to a double integral. The limits of integration define the region of integration for the double integral:

$$\int_{x=a}^{b}\left(\int_{y=h_L(x)}^{h_U(x)} f(x,y)dy\right)dx = \iint_R f(x,y)dA \quad\text{where}\quad R = \{(x,y)|a\le x\le b \ \& \ h_L(x)\le y\le h_U(x)\}$$

and

$$\int_{y=a}^{b}\left(\int_{x=h_L(y)}^{h_U(y)} f(x,y)dx\right)dy = \iint_R f(x,y)dA \quad\text{where}\quad R = \{(x,y)|a\le y\le b \ \& \ h_L(y)\le x\le h_U(y)\}$$

Once the nested integral has been converted to a double integral, the integrand needs to be rewritten to use the polar coordinates $r$ and $\theta$ instead of $x$ and $y$. This is done using the following replacements:

- Replace $x$ with $r\cos\theta$

- Replace $y$ with $r\sin\theta$

- For speed, replace $x^2 + y^2$ with $r^2$

**Examples:**

- $$\iint_R (3x - 5y)dA = \iint_R (3r\cos\theta - 5r\sin\theta)dA$$

- $$\iint_R \frac{x}{x^2 + y^2}dA = \iint_R \frac{r\cos\theta}{r^2}dA = \iint_R \frac{\cos\theta}{r}dA$$

- $$\iint_R \frac{2xy}{x^2 + y^2}dA = \iint_R \frac{2(r\cos\theta)(r\sin\theta)}{r^2}dA = \iint_R 2\cos\theta\sin\theta dA = \iint_R \sin(2\theta)dA$$

The region $R$ needs to be expressed using polar coordinates. The first thing to do is to **sketch the region $R$**. The sketch will help keep track of which curves/lines form which limits and boundaries. The equations for the curves that bound $R$ need to have $x$ and $y$ replaced so the equation now uses polar coordinates. The equation is then rewritten so $r$ is a function of $\theta$, or $\theta$ is a constant value. When simplifying the equation, the case where $r = 0$ can often be simply ignored so division of both sides by $r$ can be done safely.

In particular, lines that pass through the origin correspond to constant values of $\theta$, and often set the limits on $\theta$. The line $y = mx$ where $m$ is an arbitrary slope consists of all points where $\theta = \tan^{-1}(m) + k\cdot\pi$ where $k$ is an arbitrary integer that puts $\theta$ in the correct range.

Lastly, when converting the double integral to a nested polar integral, an extra factor of $r$ is multiplied onto the integrand.

**Example 1:**

Consider the Cartesian nested integral:

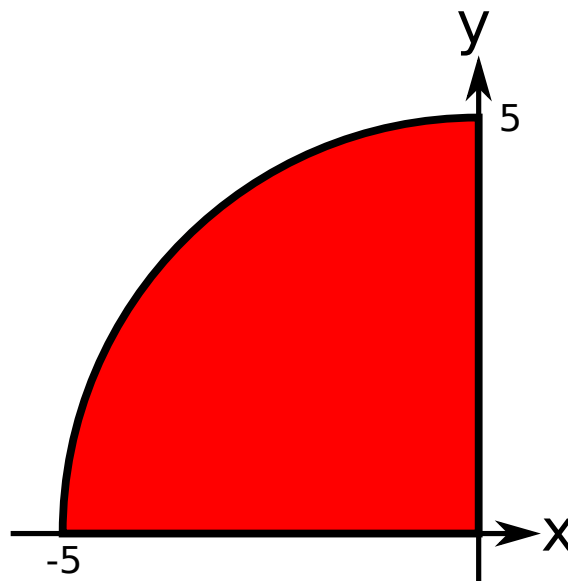$$\int_{y=0}^5 \left( \int_{x=-\sqrt{25-y^2}}^0 \sqrt{x^2 + y^2}dx \right) dy$$

This integral is difficult to evaluate, and so it will be converted to polar coordinates.

$$\int_{y=0}^5 \left( \int_{x=-\sqrt{25-y^2}}^0 \sqrt{x^2 + y^2}dx \right) dy = \iint_R \sqrt{x^2 + y^2}\cdot dA$$

where

$$R = \{(x,y) | 0 \leq y \leq 5 \ \& \ -\sqrt{25 - y^2} \leq x \leq 0\}$$

This region is sketched on the right.



The integrand is:

$$\sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

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so

$$\iint_R \sqrt{x^2 + y^2} \cdot dA = \iint_R r \cdot dA$$

Now to rewrite $R$ using polar coordinates.

The positive $y$ and negative $x$ axes form the lower and upper bounds on $\theta$ whcih are $\frac{\pi}{2}$ and $\pi$.

The equation of the circle (which forms the upper bound on $r$) is:

$$x = -\sqrt{25 - y^2} \implies r\cos\theta = -\sqrt{25 - (r\sin\theta)^2} \iff r^2\cos^2\theta = 25 - r^2\sin^2\theta$$

$$\iff r^2 = 25 \iff r = 5$$

From the sketch, the bounds on $\theta$ are $\frac{\pi}{2}$ and $\pi$, while the bounds on $r$ are 0 and 5:

$$R = \left\{ (r, \theta) \Big| \frac{\pi}{2} \le \theta \le \pi \ \& \ 0 \le r \le 5 \right\}$$

The double integral becomes the following polar nested integral. Note the inclusion of an additional factor of $r$ in the integrand of the nested integral.

$$\iint_R r \cdot dA = \int_{\theta=\pi/2}^{\pi} \left( \int_{r=0}^{5} r^2 \cdot dr \right) d\theta$$

The Cartesian nested integral has been rewritten as a polar nested integral. This integral can in fact be further evaluated:

$$\int_{\theta=\pi/2}^{\pi} \left( \int_{r=0}^{5} r^2 \cdot dr \right) d\theta = \int_{\theta=\pi/2}^{\pi} \frac{1}{3} r^3 \Big|_{r=0}^{5} d\theta = \int_{\theta=\pi/2}^{\pi} \frac{125}{3} d\theta = \frac{125\pi}{6}$$

Therefore:

$$\int_{y=0}^{5} \left( \int_{x=-\sqrt{25-y^2}}^{0} \sqrt{x^2 + y^2} dx \right) dy = \frac{125\pi}{6}$$

**Example 2:**

Consider the Cartesian nested integral:

$$\int_{x=0}^{3} \left( \int_{y=x^2}^{3x} \frac{x^4}{(x^2 + y^2)^2} dy \right) dx$$
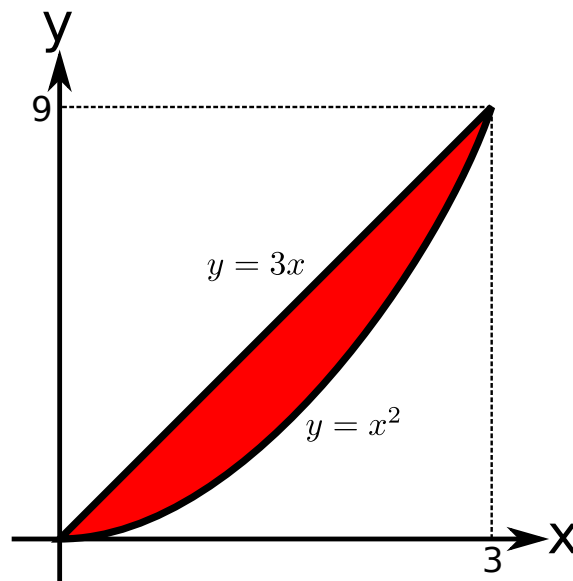
This integral is difficult to evaluate, and so it will be converted to polar coordinates.

$$\int_{x=0}^{3} \left( \int_{y=x^2}^{3x} \frac{x^4}{(x^2 + y^2)^2} dy \right) dx = \iint_R \frac{x^4}{(x^2 + y^2)^2} dA$$

where

$$R = \{ (x, y) | 0 \le x \le 3 \ \& \ x^2 \le y \le 3x \}$$

This region is sketched on the right.

The integrand is:
$$\frac{x^4}{(x^2+y^2)^2} = \frac{(r\cos\theta)^4}{(r^2)^2} = \frac{r^4\cos^4\theta}{r^4} = \cos^4\theta$$

so
$$\iint_R \frac{x^4}{(x^2+y^2)^2}dA = \iint_R \cos^4\theta\, dA$$

Now to rewrite $R$ using polar coordinates.

The line $y = 3x$ corresponds to an upper bound of $\theta = \tan^{-1}(3)$.

The equation of the parabola (which forms the upper bound on $r$) is:

$$y = x^2 \iff r\sin\theta = (r\cos\theta)^2 \iff r\sin\theta = r^2\cos^2\theta \iff (\cos^2\theta)r = \sin\theta \iff r = \frac{\sin\theta}{\cos^2\theta}$$

The parabola is tangent to the $x$-axis, so the lower bound on $\theta$ is 0.

From the sketch, the bounds on $\theta$ are 0 and $\tan^{-1}(3)$, while the bounds on $r$ are 0 and $\frac{\sin\theta}{\cos^2\theta}$:

$$R = \left\{ (r,\theta) \,\middle|\, 0 \le \theta \le \tan^{-1}(3) \ \& \ 0 \le r \le \frac{\sin\theta}{\cos^2\theta} \right\}$$

The double integral becomes the following polar nested integral. Note the inclusion of an additional factor of $r$ in the integrand of the nested integral.

$$\iint_R \cos^4\theta\, dA = \int_{\theta=0}^{\tan^{-1}(3)} \left( \int_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} \cos^4\theta \cdot r \cdot dr \right) d\theta$$

The Cartesian nested integral has been rewritten as a polar nested integral. This integral can in fact be further evaluated:

$$\int_{\theta=0}^{\tan^{-1}(3)} \left( \int_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} \cos^4\theta \cdot r \cdot dr \right) d\theta = \int_{\theta=0}^{\tan^{-1}(3)} \frac{1}{2}r^2\cos^4\theta \Big|_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} d\theta$$

$$= \int_{\theta=0}^{\tan^{-1}(3)} \frac{1}{2}\sin^2\theta\, d\theta = \int_{\theta=0}^{\tan^{-1}(3)} \frac{1-\cos(2\theta)}{4} d\theta = \frac{\theta - \frac{1}{2}\sin(2\theta)}{4} \Big|_{\theta=0}^{\tan^{-1}(3)}$$

$$= \frac{1}{4}\tan^{-1}(3) - \frac{1}{8}\sin(2\tan^{-1}(3)) = \frac{1}{4}\tan^{-1}(3) - \frac{1}{4}\sin(\tan^{-1}(3))\cos(\tan^{-1}(3))$$

$$= \frac{1}{4}\tan^{-1}(3) - \frac{1}{4}\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}} = \frac{1}{4}\tan^{-1}(3) - \frac{3}{40}$$

Therefore:
$$\int_{x=0}^{3} \left( \int_{y=x^2}^{3x} \frac{x^4}{(x^2+y^2)^2}dy \right) dx = \frac{1}{4}\tan^{-1}(3) - \frac{3}{40}$$

**Example 3:**

Consider the Cartesian nested integral:

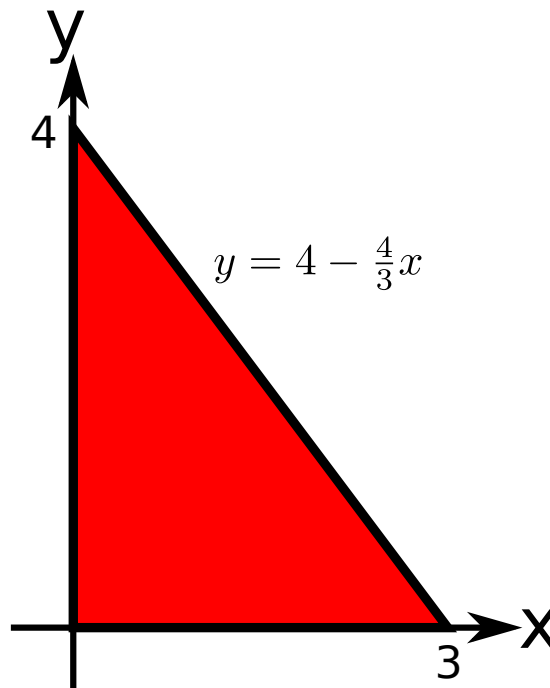$$\int_{x=0}^{3} \left( \int_{y=0}^{4-\frac{4}{3}x} \frac{4x+3y}{x^2+y^2} dy \right) dx$$

This integral is difficult to evaluate, and so it will be converted to polar coordinates.

$$\int_{x=0}^{3} \left( \int_{y=0}^{4-\frac{4}{3}x} \frac{4x+3y}{x^2+y^2} dy \right) dx = \iint_{R} \frac{4x+3y}{x^2+y^2} dA$$

where

$$R = \{(x,y)|0 \le x \le 3 \ \& \ 0 \le y \le 4 - \frac{4}{3}x\}$$

This region is sketched on the right.

The integrand is:

$$\frac{4x+3y}{x^2+y^2} = \frac{4r\cos\theta + 3r\sin\theta}{r^2} = \frac{4\cos\theta + 3\sin\theta}{r}$$

so

$$\iint_{R} \frac{4x+3y}{x^2+y^2} dA = \iint_{R} \frac{4\cos\theta + 3\sin\theta}{r} dA$$

Now to rewrite $R$ using polar coordinates.
The positive $x$ and positive $y$ axes form the lower and upper bounds on $\theta$ whcih are $0$ and $\frac{\pi}{2}$.
The equation of the line (which forms the upper bound on $r$) is:

$$y = 4 - \frac{4}{3}x \iff r\sin\theta = 4 - \frac{4}{3}r\cos\theta \iff (\frac{4}{3}\cos\theta + \sin\theta)r = 4 \iff r = \frac{12}{4\cos\theta + 3\sin\theta}$$

From the sketch, the bounds on $\theta$ are $0$ and $\frac{\pi}{2}$, while the bounds on $r$ are $0$ and $\frac{12}{4\cos\theta+3\sin\theta}$:

$$R = \left\{ (r,\theta) \middle| 0 \le \theta \le \frac{\pi}{2} \ \& \ 0 \le r \le \frac{12}{4\cos\theta + 3\sin\theta} \right\}$$

The double integral becomes the following polar nested integral. Note the inclusion of an additional factor of $r$ in the integrand of the nested integral.

$$\iint_{R} \frac{4\cos\theta + 3\sin\theta}{r} dA = \int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} \frac{4\cos\theta + 3\sin\theta}{r} \cdot r \cdot dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} (4\cos\theta + 3\sin\theta) dr \right) d\theta$$

The Cartesian nested integral has been rewritten as a polar nested integral. This integral can in fact be further evaluated:

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$$\int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} (4\cos\theta + 3\sin\theta)dr \right) d\theta = \int_{\theta=0}^{\pi/2} (4\cos\theta + 3\sin\theta)r \Big|_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} d\theta$$

$$= \int_{\theta=0}^{\pi/2} 12d\theta = 12\theta \Big|_{\theta=0}^{\pi/2} = 6\pi$$

Therefore:

$$\int_{x=0}^{3} \left( \int_{y=0}^{4-\frac{4}{3}x} \frac{4x+3y}{x^2+y^2} dy \right) dx = 6\pi$$
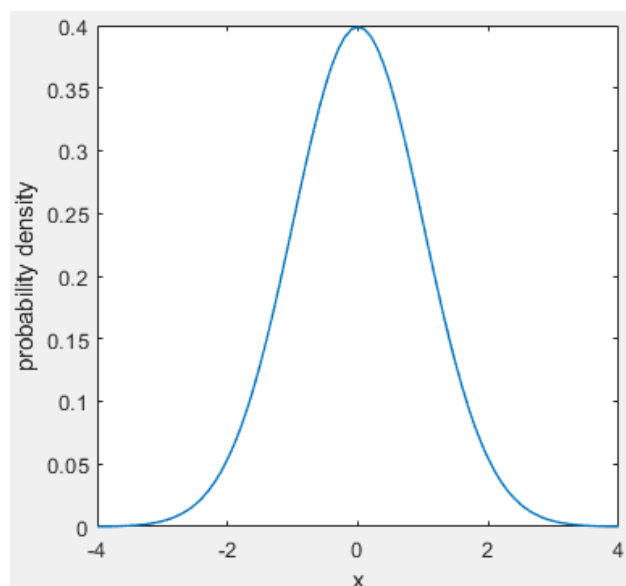
**Example 4:**

Those familiar with probability theory will have been introduced to the "normal" probability distribution:

$$\Pr(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

This probability distribution is a "bell curve" where the variance is 1.

The natural question if where does the coefficient of $\frac{1}{\sqrt{2\pi}}$ come from? The answer is that the coefficient makes the total probability equal to 1:

$$\int_{x=-\infty}^{+\infty} \Pr(x)dx = 1$$



This means that

$$\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

**How can this be proven?** An anti-derivative for the function $e^{-\frac{1}{2}x^2}$ cannot be expressed in terms of the "usual" mathematical symbols $+$, $-$, $\times$, $\div$, exp, log, cos, sin, etc. Despite this, the definite integral $\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx$ can still be computed.

Let $I = \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx$, so

$$I^2 = \left( \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \right)^2$$

$$= \left( \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \right) \left( \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \right) = \left( \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \right) \left( \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy \right)$$

$$= \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} \left( \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy \right) dx = \int_{x=-\infty}^{+\infty} \left( \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} dy \right) dx$$

$$= \int_{x=-\infty}^{+\infty} \left( \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+y^2)} dy \right) dx$$

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This Cartesian nested integral will be evaluated by converting it to a polar nested integral.

$$\int_{x=-\infty}^{+\infty} \left( \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+y^2)} dy \right) dx = \iint_R e^{-\frac{1}{2}(x^2+y^2)} dA$$

where $R$ is the set of all points in 2D space:

$$R = \{(x,y)| -\infty < x < +\infty \;\&\; -\infty < y < +\infty\}$$

The integrand is:

$$e^{-\frac{1}{2}(x^2+y^2)} = e^{-\frac{1}{2}r^2}$$

so

$$\iint_R e^{-\frac{1}{2}(x^2+y^2)} dA = \iint_R e^{-\frac{1}{2}r^2} dA$$

In terms of polar coordinates, the set of points in 2D space can be expressed as:

$$R = \{(r,\theta)|0 \le r \le +\infty \;\&\; 0 \le \theta \le 2\pi\}$$

The double integral becomes the following polar nested integral. Note the inclusion of an additional factor of $r$ in the integrand of the nested integral.

$$\iint_R e^{-\frac{1}{2}r^2} dA = \int_{\theta=0}^{2\pi} \left( \int_{r=0}^{+\infty} e^{-\frac{1}{2}r^2} \cdot r \cdot dr \right) d\theta$$

The Cartesian nested integral has been rewritten as a polar nested integral. **The additional factor of $r$ in the integrand is what makes this integral solvable using the chain rule.** This integral can in fact be further evaluated:

$$\int_{\theta=0}^{2\pi} \left( \int_{r=0}^{+\infty} e^{-\frac{1}{2}r^2} \cdot r \cdot dr \right) d\theta = \int_{\theta=0}^{2\pi} -e^{-\frac{1}{2}r^2} \Big|_{r=0}^{+\infty} d\theta = \int_{\theta=0}^{2\pi} (0-(-1)) d\theta = \int_{\theta=0}^{2\pi} d\theta = \theta \Big|_{\theta=0}^{2\pi} = 2\pi$$

Therefore $I^2 = 2\pi$, so $I = \sqrt{2\pi}$. Therefore:

$$\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

This explains why the normal distribution has the coefficient $\frac{1}{\sqrt{2\pi}}$.

## Polar integral to Cartesian integral conversions

Often, it may be necessary to convert a nested integral that uses polar coordinates to a nested integral that uses Cartesian coordinates. Firstly, the nested polar integral needs to be converted to a double integral. The limits of integration define the region of integration for the double integral:

$$\int_{\theta=a}^{b} \left( \int_{r=h_L(\theta)}^{h_U(\theta)} f(r,\theta) dr \right) d\theta = \iint_R \frac{f(r,\theta)}{r} dA \quad \text{where} \quad R = \{(r,\theta)|a \le \theta \le b \;\&\; h_L(\theta) \le r \le h_U(\theta)\}$$

When the polar nested integral is converted to a double integral, the integrand is divided by $r$ to remove the factor of $r$ that is present in the polar nested integral.

Once the nested integral has been converted to a double integral, the integrand needs to be rewritten to use Cartesian coordinates $x$ and $y$ instead of $r$ and $\theta$. This is done using the following replacements:

- Replace $\cos\theta$ with $x/r$

- Replace $\sin\theta$ with $y/r$

- Replace $\tan\theta$ with $y/x$

- Replace $\sec\theta$ with $r/x$

- Replace $\cot\theta$ with $x/y$

- Replace $\csc\theta$ with $r/y$

Next, the expression should be simplified to eliminate as many instances of $r$ as possible. After as many instances of $r$ have been eliminated, the remaining appearances of $r$ should be replaced.

- Replace $r$ with $\sqrt{x^2 + y^2}$

**Examples:**

-
$$\iint_R r\sec\theta \, dA = \iint_R r \cdot \frac{r}{x} dA = \iint_R \frac{r^2}{x} dA = \iint_R \frac{x^2 + y^2}{x} dA$$

-
$$\iint_R r^3\cos\theta \, dA = \iint_R r^3(x/r) dA = \iint_R r^2 x \, dA = \iint_R (x^2 + y^2)x \, dA$$

-
$$\iint_R r^2\cos\theta\sin\theta \, dA = \iint_R r^2(x/r)(y/r) dA = \iint_R xy \, dA$$

The region $R$ needs to be expressed using Cartesian coordinates. The first thing to do is to **sketch the region** $R$. The sketch will help keep track of which curves/lines form which limits and boundaries. The equations for the curves that bound $R$ need to have $r$ and $\theta$ replaced so the equation now uses Carteisan coordinates. The equation is then rewritten so $y$ is a function of $x$, or $x$ is a constant value (or express $x$ as a function of $y$). When simplifying the equation, the case where $r = 0$ can often be simply ignored so division of both sides by $r$ can be done safely.

In particular, the bounds on $\theta$ correspond to straight lines that pass through the origin. The condition $\theta = a$ corresponds to the line $y = \tan(a)x$.

Lastly, the double integral is converted to a Cartesian nested integral.

**Example 1:**

Consider the polar nested integral:

$$\int_{\theta=0}^{\pi} \left( \int_{r=0}^{5} \frac{r}{\sqrt{25 - r^2 \cos^2 \theta}} dr \right) d\theta$$

This integral is difficult to evaluate, and so it will be converted to Cartesian coordinates.

$$\int_{\theta=0}^{\pi} \left( \int_{r=0}^{5} \frac{r}{\sqrt{25 - r^2 \cos^2 \theta}} dr \right) d\theta$$
$$= \iint_{R} \frac{1}{\sqrt{25 - r^2 \cos^2 \theta}} dA$$

where

$$R = \left\{ (r, \theta) \Big| 0 \leq \theta \leq \pi \,\&\, 0 \leq r \leq 5 \right\}$$

This region is sketched on the right.
Note the removal of a factor of $r$ from the integrand.

The integrand is:

$$\frac{1}{\sqrt{25 - r^2 \cos^2 \theta}} = \frac{1}{\sqrt{25 - r^2 (x/r)^2}} = \frac{1}{\sqrt{25 - x^2}}$$

so

$$\iint_{R} \frac{1}{\sqrt{25 - r^2 \cos^2 \theta}} dA = \iint_{R} \frac{1}{\sqrt{25 - x^2}} dA$$

Now to rewrite $R$ using Carteisan coordinates.
The bounds on $x$ are $-5$ and $5$. The lower bound on $y$ is $0$. The curve $r = 5$ defines the upper bound on $y$:

$$r = 5 \iff \sqrt{x^2 + y^2} = 5 \iff x^2 + y^2 = 25 \iff y^2 = 25 - x^2 \iff y = \pm\sqrt{25 - x^2}$$

The upper bound on $y$ is the non-negative solution, $\sqrt{25 - x^2}$.

$$R = \{ (x, y) | -5 \leq x \leq 5 \,\&\, 0 \leq y \leq \sqrt{25 - x^2} \}$$

The double integral becomes the following Cartesian nested integral.

$$\iint_{R} \frac{1}{\sqrt{25 - x^2}} dA = \int_{x=-5}^{5} \left( \int_{y=0}^{\sqrt{25-x^2}} \frac{1}{\sqrt{25 - x^2}} dy \right) dx$$

The polar nested integral has been rewritten as a Carteisan nested integral. This integral can in fact be further evaluated:

$$\int_{x=-5}^{5} \left( \int_{y=0}^{\sqrt{25-x^2}} \frac{1}{\sqrt{25 - x^2}} dy \right) dx = \int_{x=-5}^{5} \frac{y}{\sqrt{25 - x^2}} \Big|_{y=0}^{\sqrt{25-x^2}} dx = \int_{x=-5}^{5} dx = x \Big|_{x=-5}^{5} = 10$$

Therefore:

$$\int_{\theta=0}^{\pi} \left( \int_{r=0}^{5} \frac{r}{\sqrt{25 - r^2 \cos^2 \theta}} dr \right) d\theta = 10$$

**Example 2:**

Consider the polar nested integral:

$$\int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} r^2 \cos\theta\, dr \right) d\theta$$

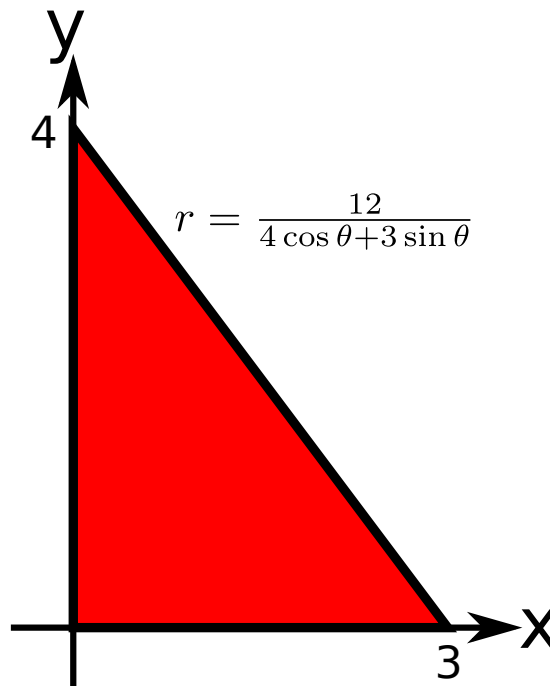This integral is difficult to evaluate, and so it will be converted to Cartesian coordinates.

$$\int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta+3\sin\theta}} r^2 \cos\theta\, dr \right) d\theta = \iint_R r\cos\theta\, dA$$

where

$$R = \left\{ (r,\theta) \middle| 0 \le \theta \le \frac{\pi}{2} \ \& \ 0 \le r \le \frac{12}{4\cos\theta + 3\sin\theta} \right\}$$

This region is sketched on the right.
Note the removal of a factor of $r$ from the integrand.

The integrand is:

$$r\cos\theta = r(x/r) = x$$

so

$$\iint_R r\cos\theta\, dA = \iint_R x\, dA$$

Now to rewrite $R$ using Carteisan coordinates.
The lower bound on $\theta$, $\theta = 0$, is the positive $x$-axis.
The upper bound on $\theta$, $\theta = \frac{\pi}{2}$, is the positive $y$-axis.
The equation of the curve $r = \frac{12}{4\cos\theta+3\sin\theta}$ (which forms the upper bound on $y$) is:

$$r = \frac{12}{4\cos\theta + 3\sin\theta} \iff r = \frac{12}{4(x/r) + 3(y/r)} \iff r = \frac{12r}{4x + 3y} \iff 1 = \frac{12}{4x + 3y}$$

$$\iff 4x + 3y = 12 \iff y = 4 - \frac{4}{3}x$$

From the sketch, the bounds on $x$ are 0 and 3, while the bounds on $y$ are 0 and $4 - \frac{4}{3}x$:

$$R = \left\{ (x,y) \middle| 0 \le x \le 3 \ \& \ 0 \le y \le 4 - \frac{4}{3}x \right\}$$

The double integral becomes the following Cartesian nested integral.

$$\iint_R x\, dA = \int_{x=0}^{3} \left( \int_{y=0}^{4-\frac{4}{3}x} x\, dy \right) dx$$

The polar nested integral has been rewritten as a Carteisan nested integral. This integral can in fact be further evaluated:

$$\int_{x=0}^{3} \left( \int_{y=0}^{4-\frac{4}{3}x} xy\, dy \right) dx = \int_{x=0}^{3} xy\Big|_{y=0}^{4-\frac{4}{3}x} dx = \int_{x=0}^{3} (4x - \frac{4}{3}x^2)dx = (2x^2 - \frac{4}{9}x^3)\Big|_{x=0}^{3}$$

$$= (18 - 12) - 0 = 6$$

Therefore:

$$\int_{\theta=0}^{\pi/2} \left( \int_{r=0}^{\frac{12}{4\cos\theta + 3\sin\theta}} r^2 \cos\theta\, dr \right) d\theta = 6$$

**Example 3:**

Consider the polar nested integral:

$$\int_{\theta=\tan^{-1}(2/5)}^{\tan^{-1}(7/5)} \left( \int_{r=\sec\theta}^{5\sec\theta} \frac{r}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2}\, dr \right) d\theta$$

This integral is difficult to evaluate, and so it will be converted to Cartesian coordinates.
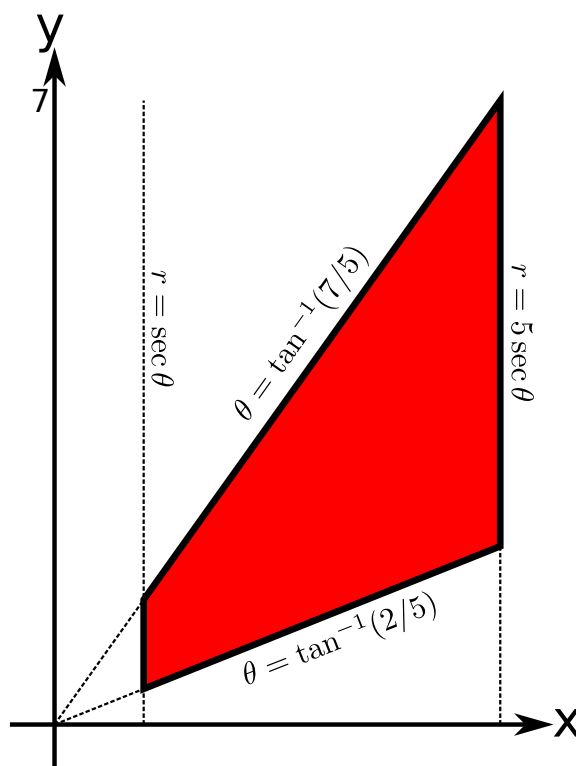
$$\int_{\theta=\tan^{-1}(2/5)}^{\tan^{-1}(7/5)} \left( \int_{r=\sec\theta}^{5\sec\theta} \frac{r}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2}\, dr \right) d\theta$$

$$= \iint_R \frac{1}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2}\, dA$$

where

$$R = \Big\{ (r,\theta)\,\Big|\, \tan^{-1}(2/5) \le \theta \le \tan^{-1}(7/5)$$

$$\&\ \sec\theta \le r \le 5\sec\theta \Big\}$$

This region is sketched on the right.
Note the removal of a factor of $r$ from the integrand.



The integrand is:

$$\frac{1}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2} = \frac{1}{(r(y/r) - \frac{2}{5}r(x/r) + 1)^2} = \frac{1}{(y - \frac{2}{5}x + 1)^2}$$

so

$$\iint_R \frac{1}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2}\, dA = \iint_R \frac{1}{(y - \frac{2}{5}x + 1)^2}\, dA$$

Now to rewrite $R$ using Carteisan coordinates.
The lower bound on $\theta$, $\theta = \tan^{-1}(2/5)$, is the line $y = \frac{2}{5}x$.
The upper bound on $\theta$, $\theta = \tan^{-1}(7/5)$, is the line $y = \frac{7}{5}x$.
The equation of the curve $r = \sec\theta$ is:

$$r = \sec\theta \iff r = \frac{1}{\cos\theta} \iff r = \frac{1}{x/r} \iff r = \frac{r}{x} \iff 1 = \frac{1}{x} \iff x = 1$$

The equation of the curve $r = 5 \sec \theta$ is:

$$r = 5 \sec \theta \iff r = \frac{5}{\cos \theta} \iff r = \frac{5}{x/r} \iff r = \frac{5r}{x} \iff 1 = \frac{5}{x} \iff x = 5$$

From the sketch, the bounds on $x$ are 1 and 5, while the bounds on $y$ are $\frac{2}{5}x$ and $\frac{7}{5}x$:

$$R = \left\{ (x,y) \,\middle|\, 1 \le x \le 5 \ \& \ \frac{2}{5}x \le y \le \frac{7}{5}x \right\}$$

The double integral becomes the following Cartesian nested integral.

$$\iint_R \frac{1}{(y - \frac{2}{5}x + 1)^2} dA = \int_{x=1}^{5} \left( \int_{y=(2/5)x}^{(7/5)x} \frac{1}{(y - \frac{2}{5}x + 1)^2} dy \right) dx$$

The polar nested integral has been rewritten as a Carteisan nested integral. This integral can in fact be further evaluated:

$$\int_{x=1}^{5} \left( \int_{y=(2/5)x}^{(7/5)x} \frac{1}{(y - \frac{2}{5}x + 1)^2} dy \right) dx = \int_{x=1}^{5} \frac{-1}{y - \frac{2}{5}x + 1} \Big|_{y=(2/5)x}^{(7/5)x} dx = \int_{x=1}^{5} \left( \frac{-1}{x+1} - (-1) \right) dx$$

$$= \int_{x=1}^{5} \left( 1 - \frac{1}{x+1} \right) dx = (x - \ln|x+1|) \Big|_{x=1}^{5} = (5 - \ln(6)) - (1 - \ln(2)) = 4 - \ln(3)$$

Therefore:

$$\int_{\theta=\tan^{-1}(2/5)}^{\tan^{-1}(7/5)} \left( \int_{r=\sec\theta}^{5\sec\theta} \frac{r}{(r\sin\theta - \frac{2}{5}r\cos\theta + 1)^2} dr \right) d\theta = 4 - \ln(3)$$

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