Mathematical notation review

Sets

Mathematics involves more than just numbers. In the topics that are covered in this class, a basic understanding of the notation related to “sets” of entities is incredibly useful.

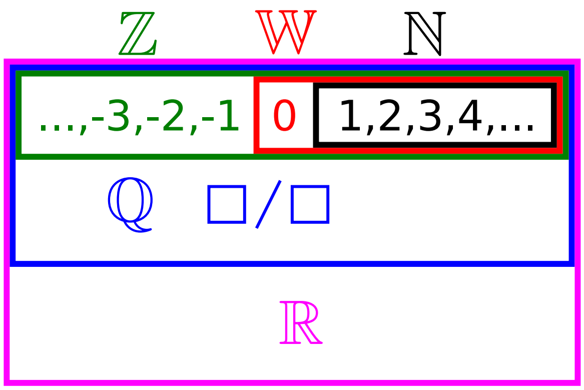
- A “set” is a collection of objects referred to as “elements”. Elements can be, but are not limited to, numbers.
- Sets **do not** contain duplicate elements (duplicate elements are ignored).
- A set will be denoted by listing the elements enclosed by $\{...\}$. The ordering of the elements does not matter. Duplicate elements are ignored.
- The empty set is denoted by $\emptyset$ or $\{\}$.
- Sets A and B are equal $A = B$ if and only if A and B have the same elements. Every element of set A can be found in set B and vice versa.
- $x \in A$ if and only if x belongs to set A. $x \notin A$ if and only if x does not belong to set A.

Examples:

- $\{1, 6, 5\} = \{5, 1, 6\} = \{5, 1, 1, 6\}$
- $2 \in \{4, 2, 3\}$
- $5 \notin \{4, 2, 3\}$

Important sets of numbers:

- “natural numbers”: $\mathbb{N} = \{1, 2, 3, 4, ...\}$
- “whole numbers”: $\mathbb{W} = \{0, 1, 2, 3, 4, ...\}$
- “integers”: $\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
- “rational numbers”: $\mathbb{Q}$ is the set of all numbers that are the division of two integers.
- “real numbers”: $\mathbb{R}$ is the set of all numbers that are distances and their negatives. The set of real numbers contains the set of rational numbers.
- “irrational numbers”: $\mathbb{I}$ is the set of all real numbers that are not also rational.



Examples:

- Classify $\sqrt{2}$: Irrational
- Classify -4: Integer, but not a whole number
- Classify $6/3$: Natural number
- Classify 0: Whole number, but not a natural number
- Classify $-4/5$: Rational number, but not an integer
- Classify 100: Natural number

How can large sets, or sets with an infinite number of elements be denoted? Here will be introduced **set builder notation**. A set can be denoted by the syntax:

$$\{\text{expression}|\text{condition}\}$$

The "expression" is an algebraic expression, most often a single variable, whose attainable values form the elements of the set. The "condition" is a condition that must be satisfied for the value of the expression to count towards the set. The symbol "|" reads as "where". To help denote conditions, the following notations will be used to shorten the description of conditions:

- Given conditions A and B, $A \wedge B$ denotes A "AND" B
- Given conditions A and B, $A \vee B$ denotes A "OR" B
- Given condition A, $\neg A$ denotes "NOT" A
- Given conditions A and B, $A \implies B$ denotes that the truth of A "implies" the truth of B (A is true only if B is also true). This notation is important when solving equations. For example, $x = 2 \implies x^2 = 4$. Also, $x \geq 3 \implies x \geq 2$.
- Given conditions A and B, $A \iff B$ denotes that A is true "if and only if" (abbreviated by "iff") B is true. A and B are either both true, or both false. This notation is important when solving equations. For example, $x + 1 = 3 \iff x = 2$. Also, $x^2 = 4 \iff ((x = 2) \vee (x = -2))$.
- Given set A and condition $B(x)$ ($B(x)$ depends on x), the notation $\forall x : B(x)$ denotes "for all values of x, $B(x)$ is true". The notation $\forall x \in A : B(x)$ denotes "for all values of x from set A, $B(x)$ is true". For example, $\forall x \in \mathbb{R} : (x + 1) - 1 = x$.

- Given set A and condition $B(x)$ ($B(x)$ depends on x), the notation $\exists x : B(x)$ denotes "there exists a value of x such that $B(x)$ is true". The notation $\exists x \in A : B(x)$ denotes "there exists a value of x from set A such that $B(x)$ is true". For example, $\exists x \in \mathbb{R} : x + 4 = 10$.

Examples

- The set of rational numbers $\mathbb{Q}$ can be denoted by $\mathbb{Q} = \{n/m | (n \in \mathbb{Z}) \wedge (m \in \mathbb{Z})\}$
- The set of even integers can be denoted by either $\{2n | n \in \mathbb{Z}\}$ or by $\{n | \exists m \in \mathbb{Z} : 2m = n\}$
- The set of odd integers can be denoted by either $\{2n + 1 | n \in \mathbb{Z}\}$ or by $\{n | \neg\exists m \in \mathbb{Z} : 2m = n\}$
- The set of square integers can be denoted by either $\{n^2 | n \in \mathbb{Z}\}$ or by $\{n | \exists m \in \mathbb{Z} : m^2 = n\}$

In many cases, when the values of the "expression" are to be limited to a set A, the notation:

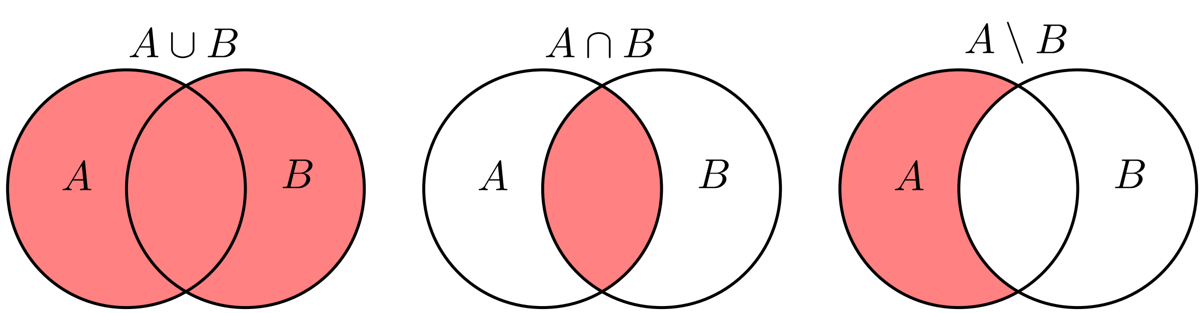
$$\{\text{expression} \in A | \text{condition}\}$$

replaces

$$\{\text{expression} | (\text{expression} \in A) \wedge \text{condition}\}$$

Set operators and relations

- Sets A and B are equal if and only if A and B have the same elements: $\forall x : (x \in A) \iff (x \in B)$
- Set B is a subset of set A, denoted by $B \subseteq A$, if and only if the elements of B can also be found in A: $\forall x : (x \in B) \implies (x \in A)$
- Set B is a proper subset of set A, denoted by $B \subset A$, if and only if $B \subseteq A$ and $B \neq A$
- The union of two sets A and B is a set that consists of the elements from both sets. Duplicate elements are not included. $A \cup B = \{x | (x \in A) \vee (x \in B)\}$
- The intersection of two sets A and B is a set that consists of the elements that are common to both sets. $A \cap B = \{x | (x \in A) \wedge (x \in B)\}$
- The "set difference" between sets A and B consists of all elements of A that do not also appear in B. $A \setminus B = \{x | (x \in A) \wedge \neg(x \in B)\}$
- The number of unique elements in set A is denoted by $|A|$.



Examples:

- $\{3, 4\} \subset \{4, 2, 3\}$
- $\{3, 4\} \subseteq \{4, 2, 3\}$

- $\{3,4,2\} \not\subset \{4,2,3\}$
- $\{3,4,2\} \subseteq \{4,2,3\}$
- $\{1,-1,3\} \cup \{4,-2\} = \{1,-1,3,4,-2\}$
- $\{1,-1,4\} \cup \{4,-2\} = \{1,-1,4,-2\}$
- $\{1,-1,4\} \cap \{4,-2\} = \{4\}$
- $\{1,-1,3\} \cup \emptyset = \{1,-1,3\}$
- $\{1,-1,3\} \cap \emptyset = \emptyset$
- $\{1,-1,3\} \setminus \{4,-2\} = \{1,-1,3\}$
- $\{1,-1,4\} \setminus \{4,-2\} = \{1,-1\}$
- $|\{7,2,1\}| = 3$
- $|\{1,2,1\}| = 2$
- $|\emptyset| = 0$

In many cases, the set of all real numbers between two bounds a and b needs to expressed in a compact manner. For this purpose interval notation is introduced. Given bounds a and b where $a < b$,

- $(a,b) = \{x \in \mathbb{R} | a < x < b\}$
- $(a,b] = \{x \in \mathbb{R} | a < x \leq b\}$
- $[a,b) = \{x \in \mathbb{R} | a \leq x < b\}$
- $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

Tuples and vectors

Given two values x and y, an **ordered pair** involving x and y is a 2 element list $\langle x,y \rangle$ where x is the 1^{st} or "left" entry, and y is the 2^{nd} or "right" entry. As the name suggests, the order of the entries in an ordered pair is important. If $x \neq y$, then $\langle x,y \rangle \neq \langle y,x \rangle$.

Given sets A and B, the "set product" $A \times B$ is the set of all possible ordered pairs that are formed by choosing the first entry from A, and the second entry from B:

$$A \times B = \{\langle x,y \rangle | x \in A \wedge y \in B\}$$

Examples:

- $\{5,6,1\} \times \{0,2\} = \{\langle 5,0 \rangle, \langle 5,2 \rangle, \langle 6,0 \rangle, \langle 6,2 \rangle, \langle 1,0 \rangle, \langle 1,2 \rangle\}$
- $\{1,3\} \times \{0,1,3\} = \{\langle 1,0 \rangle, \langle 1,1 \rangle, \langle 1,3 \rangle, \langle 3,0 \rangle, \langle 3,1 \rangle, \langle 3,3 \rangle\}$
- $\{4,2,-1\} \times \emptyset = \emptyset$
- $\{5,0,5\} \times \{0,1,1,2\} = \{5,0\} \times \{0,1,2\} = \{\langle 5,0 \rangle, \langle 5,1 \rangle, \langle 5,2 \rangle, \langle 0,0 \rangle, \langle 0,1 \rangle, \langle 0,2 \rangle\}$

Given a list of n values x_1, x_2, ..., x_n, the list $\langle x_1, x_2, ..., x_n\rangle$ is often referred to as an "n-tuple" (a generalization of the words double, triple, quadruple, etc.) or an n component vector. An ordered pair is simply a 2-tuple.

An n-tuple/vector can be denoted by either $\langle x_1, x_2, ..., x_n\rangle$ or $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Both representations are valid. The number of components n in a vector is referred to as the **dimension** of said vector.

Given the n sets $A_1, A_2, ..., A_n$, the set product $A_1 \times A_2 \times ... \times A_n$ is the set of all n component vectors where the first entry is from A_1, the second entry is from A_2, the third entry is from A_3 and so on:

$$A_1 \times A_2 \times ... \times A_n = \{\langle x_1, x_2, ..., x_n\rangle | x_1 \in A_1 \land x_2 \in A_2 \land ... \land x_n \in A_n\}$$

The set A^n is the set of all n component vectors where the entries all come from A:

$$A^n = \underbrace{A \times A \times ... \times A}_{n} = \{\langle x_1, x_2, ..., x_n\rangle | x_1 \in A \land x_2 \in A \land ... \land x_n \in A\}$$

The set of all n component vectors whose entries are real numbers is $\mathbb{R}^n$.

Examples:

- $\mathbb{R}^2$ is the set of all 2D Cartesian coordinates.
- $\mathbb{R}^3$ is the set of all 3D Cartesian coordinates.

Functions

Functions are mathematical objects that take values from a domain set X and with this value generate another value from a codomain set Y. A function f with a domain of X and a codomain of Y is introduced by the notation:

$$f : X \to Y$$

The variable x that denotes the input value from X is often referred to as the "input variable" or the "parameter" of function f. The value that is generated from the input value of x, denoted by $f(x)$, is often referred to as the "output value" or the "return value" from function f.

The domain of a function f will be denoted by $\mathbf{domain}(f)$. The "range" of function f, denoted by $\mathbf{range}(f)$, is the set of all values from the codomain Y that are actually generated by some value from the domain $\mathbf{domain}(f)$:

$$\mathbf{range}(f) = \{f(x) | x \in \mathbf{domain}(f)\}$$

If a function f is "1 to 1", then no two different input values result in the same output value:

$$\forall x_1, x_2 \in \mathbf{domain}(f) : (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$$

Given a 1-to-1 function f, then for each value y from the range of f, there is exactly one input value x from the domain of f that returns y. When a function f is "1 to 1", the function f has an inverse. The inverse function of f, denoted by f^{-1}, "reverses" function f. The domain of f^{-1} is the range of f, while the range of f^{-1} is the domain of f. Given any value x from the range of f, the value of $f^{-1}(x)$ is the unique value from the domain of f such that $f(f^{-1}(x)) = x$. For every value x from the domain of f, applying f followed by f^{-1} brings x full circle:

$$\forall x \in \mathbf{domain}(f) : f^{-1}(f(x)) = x$$

For every value x from the range of f, applying f^{-1} followed by f also brings x full circle:

$$\forall x \in \mathbf{range}(f) : f(f^{-1}(x)) = x$$

Examples:

- Let function $f : [1, 2) \to \mathbb{R}$ be defined by $f(x) = 2x + 3$. Note that even though the expression $2x + 3$ is defined for all real values of x, the domain of f has been artificially restricted to the interval $[1, 2)$. While the codomain of f is $\mathbb{R}$, the range of f is $\textbf{range}(f) = [5, 7)$. Function f is 1-to-1, and the inverse of f has a domain of $[5, 7)$ and a range of $[1, 2)$:

$$f^{-1} : [5, 7) \to [1, 2)$$

moreover, $f^{-1}(y) = \frac{y-3}{2}$.

- Let function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 5 - x^2$. The range of g is $\textbf{range}(g) = (-\infty, 5]$. Function g is not 1-to-1, and so g does not have an inverse. If the domain of g were then artificially restricted to $(-\infty, -1)$, then the range is now $(-\infty, 4)$, and g becomes 1-to-1. The inverse function $g^{-1} : (-\infty, 4) \to (-\infty, -1)$ is $g^{-1}(y) = -\sqrt{5 - y}$.

Multivariable calculus studies functions f whose domains are sets of vectors, and whose codomains are also sets of vectors:

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

When the domain of a function is a set of vectors with 2 or more components/dimensions, the function is referred to as being **multivariable**. For a multivariable function, components of the input vector can be treated as separate parameters. Consider for example a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + y - 3$$

The notation $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ will be interchangeable with the more common notation $f(x, y)$ where the components of the input vector now form separate parameter slots. In either case, when $x = 1$ and $y = 2$,

$$f(1, 2) = f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2(1) + (2) - 3 = 1$$

In general, the following notations are equivalent:

$$f(x_1, x_2, ..., x_n) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right)$$

When the range of a function is a set of vectors with 2 or more components/dimensions, the function is referred to as being **vector valued**. The components of the output vector can be treated as separate functions. Consider for example a vector valued function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\mathbf{g}(t) = \begin{bmatrix} 5 - t \\ t^2 \\ 3t \end{bmatrix}$$

For example. if $t = 3$, then $g(3) = \begin{bmatrix} 2 \\ 9 \\ 9 \end{bmatrix}$.

Define the single variable functions $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$, and $g_3 : \mathbb{R} \to \mathbb{R}$ so that:

$$\mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix}$$

The definition of $\mathbf{g}$ can be re-expressed as:

$$\begin{cases} g_1(t) = 5 - t \\ g_2(t) = t^2 \\ g_3(t) = 3t \end{cases}$$

these are referred to as **parametric equations**.