The n points $P_1, \ldots, P_n$ are joined by at most $\frac{1}{2}n(n-1)$ lines P_1P_2, P_1P_3, etc. Consider the pairs P_i, P_jP_k, consisting of a point and a joining line which are not incident. Since there are at most $\frac{1}{2}n(n-1)(n-2)$ such pairs, there must be at least one, say P_1, P_2P_3, for which the distance P_1Q from the point to the line is the smallest such distance that occurs.

Then the line P_2P_3 contains no other point of the set. For if it contained P_4, at least two of the points P_2, P_3, P_4 would lie on one side of the perpendicular P_1Q (or possibly one of the P's would coincide with Q). Let the points be so named that these two are P_2, P_3, with P_2 nearer to Q (or coincident with Q). Then P_2, P_3P_1 (Figure 4.7*b*) is another pair having a smaller distance than P_1Q, which is absurd.

This completes the proof that there is always a line containing exactly two of the points. Of course, there may be more than one such line; in fact, Kelly and Moser proved that the number of such lines is at least $3n/7$.

EXERCISES

1. The above proof yields a line P_2P_3 containing only these two of the P's. The point Q actually lies *between* P_2 and P_3.
2. If n points are not all on one line, they have at least n distinct joins [Coxeter **2**, p. 31].
3. Draw a configuration of n points for which the lower limit of $3n/7$ "ordinary" joins is attained. (*Hint*: $n = 7$.)

5

Similarity in the Euclidean plane

In later chapters we shall see that Euclidean geometry is by no means the only possible geometry: other kinds are just as logical, almost as useful, and in some respects simpler. According to the famous *Erlangen program* (Klein's inaugural address at the University of Erlangen in 1872), the criterion that distinguishes one geometry from another is the group of transformations under which the propositions remain true. In the case of Euclidean geometry, we might at first expect this to be the continuous group of all isometries. But since the propositions remain valid when the scale of measurement is altered, as in a photographic enlargement, the "principal group" for Euclidean geometry [Klein **2**, p. 133] includes also "similarities" (which may change distances although of course they preserve angles). In the present chapter we classify such transformations of the Euclidean plane. In particular, "dilatations" will be seen to play a useful role in the theory of the nine-point center of a triangle. These and other "direct" similarities are treated in the standard textbooks, but "opposite" similarities (§ 5.6) seem to have been sadly neglected.

5.1 DILATATION

"If I eat one of these cakes," she thought, "it's sure to make some change in my size." . . . So she swallowed one . . . and was delighted to find that she began shrinking directly.

Lewis Carroll

[Dodgson **1**, Chap. 4]

It is convenient to extend the usual definition of *parallel* by declaring that two (infinite straight) lines are parallel if they have either no common point or two common points. (In the latter case they coincide.) This convention enables us to assert that, without any exception,

5.11 *For each point A and line r, there is just one line through A parallel to r.*

Two figures are said to be *homothetic* if they are similar and similarly placed, that is, if they are related by a dilatation (or "homothecy"), which may be defined as follows [Artin **1**, p. 54]:

A *dilatation* is a transformation which preserves (or reverses) direction: that is, it *transforms each line into a parallel line.*

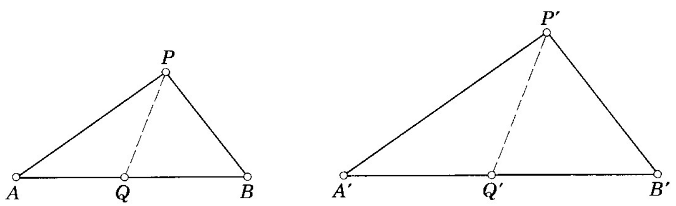

Figure 5.1a

5.12 *Two given parallel line segments AB, $A'B'$ are related by a unique dilatation $AB \to A'B'$.*

For, any point P not on AB is transformed into the point P' in which the line through A' parallel to AP meets the line through B' parallel to BP (Figure 5.1*a*); and any point Q on AB is transformed into the point Q' in which $A'B'$ meets the line through P' parallel to PQ.

In other words, a dilatation is completely determined by its effect on any two given points [Coxeter **2,** 8.51].

Clearly, the inverse of the dilatation $AB \to A'B'$ is the dilatation $A'B' \to AB$. Also $AB \to AB$ is the identity, $AB \to BA$ is a half-turn (about the midpoint of AB), and if $ABB'A'$ is a parallelogram, $AB \to A'B'$ is a translation.

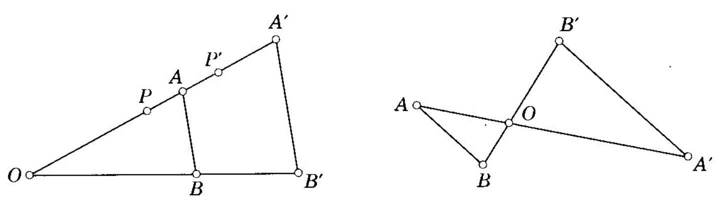

Figure 5.1b

For any dilatation which is not the identity, the two points A and B may be so chosen that A is not an invariant point and AB is not an invariant line. Such a dilatation $AB \to A'B'$ (Figure 5.1*b*) transforms any point P on AA' into a point P' on the parallel line through A', which is AA' itself.

Similarly, it transforms any point Q on BB' into a point Q' on BB'. If AA' and BB' are not parallel, these two invariant lines intersect in an invariant point O. Hence

5.13 *Any dilatation that is not a translation has an invariant point.*

This invariant point O is *unique*. For, a dilatation that has two invariant points O_1 and O_2 can only be the identity, which may reasonably be regarded as a translation, namely a translation through distance zero [Weyl **1,** p. 69].

Clearly, any point P is transformed into a point P' on OP. Let us write

$$OP' = \lambda OP,$$

with the convention that the number λ is positive or negative according as P and P' are on the same side of O or on opposite sides. With the help of some homothetic triangles (as in Figure 5.1*b*), we see that λ is a constant, that is, independent of the position of P. Moreover, any segment PQ is transformed into a segment $|\lambda|$ times as long, and oppositely directed if $\lambda < 0$. We shall use the symbol $O(\lambda)$ for the dilatation with center O and ratio λ. (Court [**2,** p. 40] prefers "(O, λ)".)

In particular, $O(1)$ is the identity and $O(-1)$ is a half-turn. Clearly, the only dilatations which are also isometries are half-turns and translations. In the case of a translation, such a symbol as $O(\lambda)$ is no longer available.

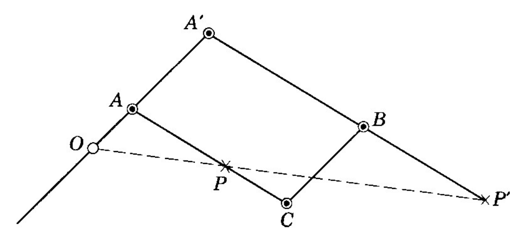

Figure 5.1c

EXERCISES

1. What is the inverse of the dilatation $O(\lambda)$?
2. If the product $O_1(\lambda_1)$ and $O_2(\lambda_2)$ is $O(\lambda_1\lambda_2)$, where is O?
3. Express the dilatation $O(\lambda)$ in terms of (a) polar coordinates, (b) Cartesian coordinates.
4. Explain the action of the *pantograph* (Figure 5.1*c*), an instrument invented by Christoph Scheiner about 1630 for the purpose of making a copy, reduced or enlarged, of any given figure. It is formed by four rods, hinged at the corners of a parallelogram $AA'BC$ whose angles are allowed to vary. The three collinear points O, P, P', on the respective rods AA', AC, $A'B$, remain collinear when the shape of the parallelogram is changed. The instrument is pivoted at O. When a pencil point is inserted at

P' and a tracing point at P (or vice versa), and the latter is traced over the lines of a given figure, the pencil point draws a homothetic copy. The positions of O and P are adjustable on their respective rods so as to allow various choices of the ratio $OA : OA'$. (Care must, of course, be taken to keep O and P collinear with P'.)

5. How could the pantograph be modified so as to yield a dilatation $O(\lambda)$ with λ negative?

5.2 CENTERS OF SIMILITUDE

I have often wondered why "similitude" ever got into elementary geometry. . . . I'm sure youngsters would be much more at ease with a pair of circles if they just had centers of "similarity" instead of being made to imagine that some new idea was insinuating itself.

E. H. Neville (1889-1961)

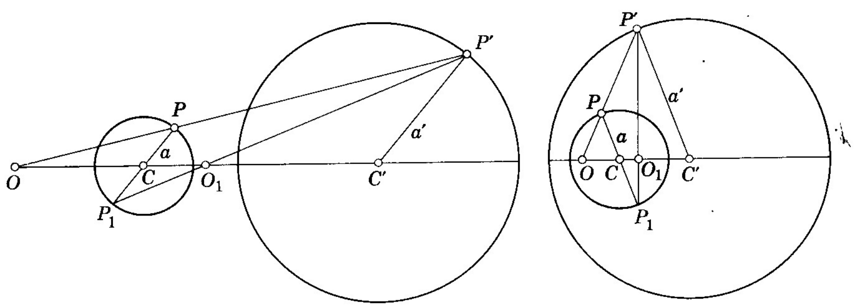

Figure 5.2a

A dilatation $O(\lambda)$, transforming C into C', transforms a circle with center C and radius r into a circle with center C' and radius $|\lambda|r$. Conversely, as we see in Figure 5.2*a*, if two circles have distinct centers C, C' and unequal radii a, a', they are related by two dilatations, $O(a'/a)$ and $O_1(-a'/a)$, whose centers O and O_1 divide the segment CC' externally and internally in the ratio $a : a'$ [Court **2**, p. 184]. These points O and O_1 are called the *centers of similitude* of the two circles. To construct them, we draw an arbitrary diameter PCP_1 of the first circle and a parallel radius $C'P'$ of the second (with P' on the same side of CC' as P); then O lies on PP', and O_1 on P_1P'.

If two circles are concentric or equal, they are still related by two dilatations, but there is only one center of similitude. In the case of concentric circles this is because the two dilatations have the same center. In the case of equal circles it is because one of the dilatations is a translation, which has no center. (The other is the half-turn about O_1, which is now the midpoint of CC'.)

A. Vandeghen and G. R. Veldkamp (*American Mathematical Monthly*, **71** (1964), p. 178) found that, for the triangle considered in Exercise 10 of § 1.5 (page 16), the centers of similitude of the two "Soddy circles" are the incenter and the *Gergonne point*: the point of concurrence of the lines joining the vertices to the points of contact of the respectively opposite sides with the incircle.

EXERCISES

1. If two equal circles have no common point, they have two parallel common tangents and two other common tangents through O_1 (midway between the centers). If they touch they have only three common tangents. If they intersect they have only the two parallel common tangents.

2. Any common tangent of two unequal circles passes through a center of similitude. Sketch the positions of the centers of similitude, and record the number of common tangents, in the five essentially different instances of two such circles. (Two of the five are shown in Figure 5.2*a*.)

3. Given two dilatations $O(\lambda)$, $O_1(\lambda_1)$, with $\lambda \neq \lambda_1$, describe the position of the unique point C on which both have the same effect.

5.3 THE NINE-POINT CENTER

Consider an arbitrary triangle ABC, with circumcenter O, centroid G, and orthocenter H. Let A', B', C' be the midpoints of the sides, and A'', B'', C'' the midpoints of the segments HA, HB, HC, as in Figure 1.7*a*. Clearly, both the triangles $A'B'C'$, $A''B''C''$ are homothetic to ABC, being derived from ABC by the respective dilatations $G(-\frac{1}{2})$, $H(\frac{1}{2})$. The former provides a new proof that the medians are concurrent and trisect one another.

Since $G(-\frac{1}{2})$ and $H(\frac{1}{2})$ are the two dilatations by which the nine-point circle can be derived from the circumcircle [Court **2**, p. 104], the points G, H are the centers of similitude of these two circles, and the Euler line GH contains the centers of both circles: not only the circumcenter O, as we know already, but also the *nine-point center* N. Since the values of μ for the dilatations are $\pm\frac{1}{2}$, the nine-point radius is half the circumradius, and the centers of similitude H, G divide the segment ON externally and internally in the ratio 2 : 1 (Figure 5.3*a*). Thus N is the midpoint of OH.

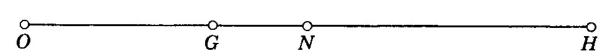

Figure 5.3a

EXERCISES

1. Using Cartesian coordinates, find the ordinates y of the centers O, G, N, H of the isosceles triangle whose vertices are $(0, 10)$, $(\pm 6, -8)$.

2. If $ABCH$ is an orthocentric quadrangle (see 1.72), the four Euler lines of the triangles BCH, CAH, ABH, ABC are concurrent.

5.4 THE INVARIANT POINT OF A SIMILARITY

When a figure is enlarged so as to remain still of the same shape, every straight line in it remains a straight line, and every angle remains congruent to itself. All the parts of the figure are equally enlarged. When one figure is an enlarged copy of another, the two are said to be similar. The degree of enlargement necessary to make one figure equal to the other is called their ratio of similitude. The ratio of two lines in the one figure is equal to the ratio of the two corresponding lines in the other.

W. K. Clifford (1845-1879)
(*Mathematical Papers*, p. 631)

A *similarity* (or "similarity transformation," or "similitude") is a transformation which takes each segment AB into a segment $A'B'$ whose length is given by

$$\frac{A'B'}{AB} = \mu,$$

where μ is a constant positive number (the same for all segments) called the *ratio of magnification* (Clifford's "ratio of similitude"). It follows that any triangle is transformed into a similar triangle, and any angle into an equal (or opposite) angle. When $\mu = 1$, the similarity is an isometry. Other special cases are the dilatations $O(\pm\mu)$.

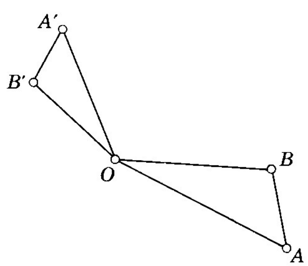

Figure 5.4a

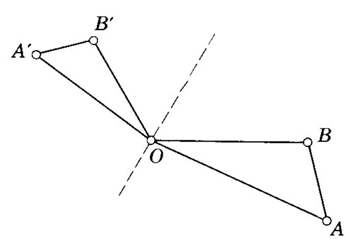

Figure 5.4b

A less familiar instance is the *dilative rotation* (or "spiral similarity", Figure 5.4*a*), which is the product of a dilatation $O(\mu)$ and a rotation about O. Another is the *dilative reflection* (Figure 5.4*b*), which is the product of a dilatation $O(\mu)$ and the reflection in a line through O. We would not obtain anything new (in either case) if we replaced this dilatation $O(\mu)$ by $O(-\mu)$. For, since $O(-\mu) = O(-1) \cdot O(\mu)$, and $O(-1)$ is a half-turn, the product of $O(\mu)$ and a rotation through α about O is the same as the product of $O(-\mu)$ and a rotation through $\alpha + \pi$; and since $O(-1)$ is the product of two perpendicular reflections, the product of $O(\mu)$ and the reflection in a line m

through O is the same as the product of $O(-\mu)$ and the reflection in the line through O perpendicular to m. In fact, a dilative reflection has two perpendicular invariant lines (its *axes*), which are the internal and external bisectors of $\angle\, AOA'$ (and of $\angle\, BOB'$).

Clearly (cf. 3.11),

5.41 *Any two similar triangles ABC, $A'B'C'$ are related by a unique similarity $ABC \to A'B'C'$, which is direct or opposite according as the sense of $A'B'C'$ agrees or disagrees with that of ABC.*

In other words, a similarity is completely determined by its effect on any three given non-collinear points. For instance, the two triangles CBF, ACF used in proving Pythagoras's theorem (Figure 1.3*d*) are related by a dilative rotation, the product of the dilatation $F(AC/CB)$ and a quarter-turn; and the two triangles ABC, ACF (in the same figure) are related by a dilative reflection whose axes are the bisectors of the angle A.

Here is another way of expressing the same idea:

Any two line segments AB, $A'B'$, are related by just two similarities: one direct and one opposite.

For instance, the segment AB can be completed to make a square $ABCD$ on either side of the line AB, and similarly there are two ways to place a square $A'B'C'D'$ on $A'B'$. The similarity

$$ABCD \to A'B'C'D'$$

is direct or opposite according as the senses round these two squares agree or disagree.

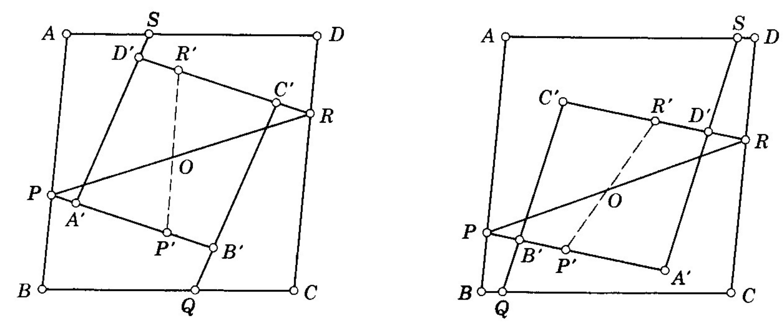

Figure 5.4c

Theorem 5.13 suggests the possibility that every similarity with $\mu \neq 1$ may have an invariant point.

If a given similarity is not a dilatation, there must be at least one line transformed into a nonparallel line. Let AB and $A'B'$ be corresponding segments on such a pair of lines, and let the given similarity (direct or opposite) be determined by similar, but not congruent, parallelograms $ABCD$ and $A'B'C'D'$ (for example, by squares, as above).

Let P, Q, R, S denote the points of intersection of the pairs of corresponding lines AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, DA and $D'A'$, as in Figure 5.4*c*. Suppose the given similarity transforms P (on AB) into P' (on $A'B'$), and R (on CD) into R' (on $C'D'$). Let O be the common point of the lines PR, $P'R'$ (which cannot be parallel, for, if they were, $PRR'P'$ would be a parallelogram, and the segments PR, $P'R'$ would be congruent, contradicting $P'R' = \mu PR$). Since the point pairs PP' and RR' lie on parallel lines $A'B'$ and $C'D'$,

$$\frac{OP}{OR} = \frac{OP'}{OR'}.$$

Therefore, the similarity leaves O invariant. Moreover, O is its *only* invariant point. For, if a similarity with $\mu \neq 1$ had two invariant points O_1 and O_2, the distance O_1O_2 would be left unchanged instead of being multiplied by μ. Hence

5.42 *Any similarity that is not an isometry has just one invariant point.*

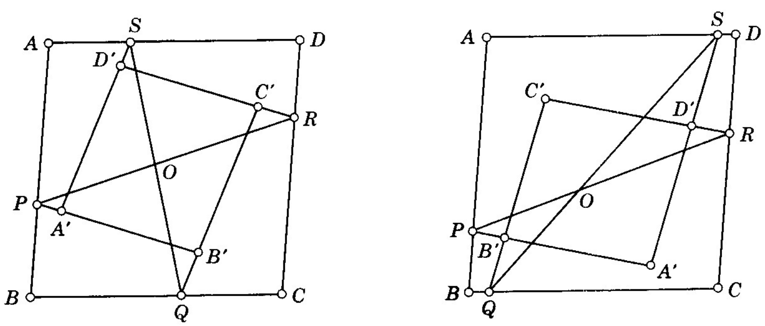

Figure 5.4d

Moreover, given two similar parallelograms $ABCD$ and $A'B'C'D'$, we can use the method indicated in Figure 5.4*d* to *construct* the center of the similarity that relates them. For, having seen that O lies on the line PR, we can apply the same reasoning (using BC and DA instead of AB and CD) to show that O lies on the line QS. This is a different line, for, if P, Q, R, S were all collinear we would have

$$\frac{PA}{PB} = \frac{PS}{PQ} = \frac{PA'}{PB'} \quad \text{and} \quad \frac{RC}{RD} = \frac{RQ}{RS} = \frac{RC'}{RD'},$$

making both P and R invariant. Hence, *O can be constructed as the point of intersection of the lines PR and QS.*

EXERCISE

How can the idea of continuity be used for a different proof of Theorem 5.42?

5.5 DIRECT SIMILARITY

Consider a given direct similarity whose ratio of magnification μ is not 1. Since there is an invariant point, say O, this similarity may be expressed as the product of the dilatation $O(\mu)$ and a direct *isometry* leaving O invariant. By Theorem 3.14, such an isometry is simply a rotation about O. Hence,

5.51 *Any direct similarity that is not an isometry is a dilatation or a dilative rotation.*

EXERCISES

1. What is the product of two dilative rotations?
2. How can two circles be used to locate the invariant point of the direct similarity that relates two given incongruent segments on nonparallel lines? [Casey **1,** p. 186.]

5.6 OPPOSITE SIMILARITY

Consider a given opposite similarity whose ratio of magnification μ is not 1. Since there is an invariant point, say O, this similarity may be expressed as the product of the dilatation $O(\mu)$ and an opposite isometry leaving O invariant. By Theorem 3.14, such an isometry is simply the reflection in a line through O. Hence

5.61 *Any opposite similarity that is not an isometry is a dilative reflection.*

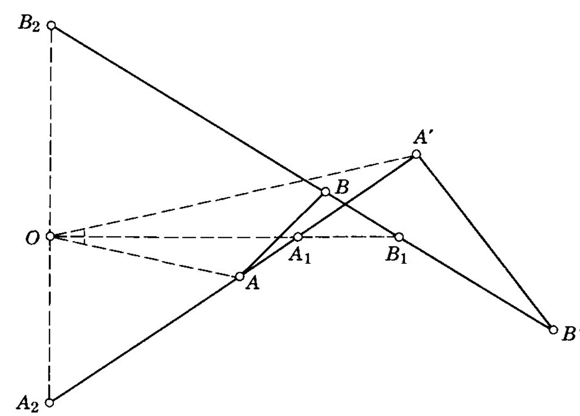

Figure 5.6a

EXERCISES

1. If two maps of the same country on different scales are drawn on tracing paper and superposed, there is just one place that is represented by the same spot on both maps. (It is understood that one of the maps may be turned over before it is superposed on the other.) [Lachlan **1**, pp. 137, 139.]

2. When all the points P on AB are related by a similarity to all the points P' on $A'B'$, the points dividing the segments PP' in the ratio $AB : A'B'$ (internally or externally) are distinct and collinear or else they all coincide.

3. If S is an opposite similarity, S^2 is a dilatation.

4. What is the product (a) of two dilative reflections? (b) of a dilative rotation and a dilative reflection?

5. Let AB and $A'B'$ be two given segments of different lengths. Let A_1 and A_2 divide AA' internally and externally in the ratio $AB{:}A'B'$ (as in Figure 5.6*a*). Let B_1 and B_2 divide BB' in the same manner. Then the lines A_1B_1 and A_2B_2 are at right angles, and are the axes of the dilative reflection that transforms AB into $A'B'$. [Lachlan **1**, p. 134; Johnson **1**, p. 27.] (It has been tacitly assumed that $A_1 \neq B_1$ and $A_2 \neq B_2$. However, if A_2 and B_2 coincide, the axes are A_1B_1 and the perpendicular line through A_2.)

6. Describe the transformation

$$(r, \theta) \rightarrow (\mu r, \theta + \alpha)$$

of polar coordinates, and the transformation

$$(x, y) \rightarrow (\mu x, -\mu y)$$

of Cartesian coordinates.

6 Circles and spheres

The present chapter shows how Euclidean geometry, in which lines and planes play a fundamental role, can be extended to *inversive* geometry, in which this role is taken over by circles and spheres. We shall see how the obvious statement, that lines and planes are circles and spheres of infinite radius, can be replaced by the sophisticated statement that lines and planes are those circles and spheres which pass through an "ideal" point, called "the point at infinity." In § 6.9 we shall briefly discuss a still more unusual geometry, called *elliptic,* which is one of the celebrated "non-Euclidean" geometries.

6.1 INVERSION IN A CIRCLE

Can it be that all the great scientists of the past were really playing a game, a game in which the rules are written not by man but by God? . . . When we play, we do not ask why we are playing—we just play. Play serves no moral code except that strange code which, for some unknown reason, imposes itself on the play. . . . You will search in vain through scientific literature for hints of motivation. And as for the strange moral code observed by scientists, what could be stranger than an abstract regard for truth in a world which is full of concealment, deception, and taboos? . . . In submitting to your consideration the idea that the human mind is at its best when playing, I am myself playing, and that makes me feel that what I am saying may have in it an element of truth.

J. L. Synge (1897 -)*

All the transformations so far discussed have been similarities, which transform straight lines into straight lines and angles into equal angles. The transformation called *inversion,* which was invented by L. J. Magnus in 1831, is new in one respect but familiar in another: it transforms some

* *Hermathena,* **19** (1958), p. 40; quoted with the editor's permission.