Distance geometry on $\mathbf{P}^2$ 6

Distance and the triangle inequality

So far we have discussed only the incidence structure of the projective plane. We now introduce a distance function.

Definition. *For P and Q in $\mathbf{P}^2$ define*

$$d(P, Q) = \cos^{-1}|\langle x, y\rangle|,$$

where x and y are points of $\mathbf{S}^2$ and $\pi x = P$, $\pi y = Q$.

Remark: Because of the absolute value sign, the distance is well-defined. The distance between $\{x, -x\}$ and $\{y, -y\}$ is the spherical distance between the closest representatives. Also, all distances are $\leqslant \pi/2$.

Theorem 1. *If P, Q, and R are points of $\mathbf{P}^2$, then*

$$d(P, Q) + d(Q, R) \geqslant d(P, R).$$

Proof: Let r, p, and q be the respective distances. Choose representatives P', Q', and R' so that $\langle P', R'\rangle \geqslant 0$ and $\langle Q', R'\rangle \geqslant 0$. As in Theorem 4.8,

$$|P' \times Q'| = \sin r, \quad |R' \times Q'| = \sin p,$$

$$\begin{aligned} |P' \times Q'||R' \times Q'| &\geqslant \langle P' \times Q', Q' \times R'\rangle \\ &= -\langle P', R'\rangle + \langle Q', R'\rangle\langle Q', P'\rangle. \end{aligned}$$

If $\langle P', Q'\rangle \geqslant 0$, then we get the inequalities

$$\sin r \sin p \geqslant \cos p \cos r - \cos q,$$

$$\cos q \geqslant \cos p \cos r - \sin r \sin p,$$

$$\cos q \geqslant \cos (p + r),$$

$$q \leqslant p + r.$$

Equality in this case would imply that

$$[Q' \times R'] = [P' \times Q'].$$

This means that P', Q', and R' are collinear on $\mathbf{S}^2$ and, hence, that P, Q, and R are collinear on $\mathbf{P}^2$.

If $\langle P', Q' \rangle \leqslant 0$, then use the Cauchy–Schwarz inequality in the form

$$|P' \times Q'||R' \times Q'| \geqslant \langle P', R' \rangle - \langle Q', R' \rangle \langle Q', P' \rangle, \tag{6.1}$$

which yields

$$\sin r \sin p \geqslant \cos q + \cos p \cos r,$$
$$\cos(\pi - q) \geqslant \cos(p + r),$$
$$\pi - q \leqslant p + r.$$

But $q \leqslant \pi/2$, so that $q \leqslant \pi - q \leqslant p + r$. Equality can occur in this case only if $q = \pi/2 \leqslant p + r$. As before, P', Q', and R' will be collinear. □

Remark: In this proof we have shown not only that the triangle inequality holds on $\mathbf{P}^2$ but also the familiar notion that equality cannot occur unless the three points in question are collinear. However, something unfamiliar also pops up here. Not every triple of collinear points satisfies the equality. The situation is as follows.

Theorem 2. *Three points of $\mathbf{P}^2$ are collinear if and only if they can be named P, Q, and R in such a way that either*

i. $d(P, Q) + d(Q, R) = d(P, R)$

or

ii. $d(P, Q) + d(Q, R) + d(P, R) = \pi$.

Proof: Suppose that we are given three collinear points for which (i) does not hold. Let e_3 be a pole of the line of $\mathbf{S}^2$ determined by the three given points, and let e_1 be a representative of one of the points, say P. Then we may choose θ and ϕ with $0 \leqslant \theta, \phi \leqslant \pi/2$ such that the other two points are

$$Q = \pi((\cos \phi)e_1 + (\sin \phi)e_2)$$

and

$$R = \pi((\cos \theta)e_1 \pm (\sin \theta)e_2),$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis.

Now

$$d(P, Q) = \cos^{-1}(\cos \phi) = \phi,$$
$$d(P, R) = \cos^{-1}(\cos \theta) = \theta,$$

and

$$d(Q, R) = \cos^{-1}|\cos(\phi \pm \theta)|.$$

The minus sign cannot occur because it would imply that $d(Q, R) = |\phi - \theta|$, and, hence, an equation of the form (i) would be satisfied. Similarly, if $\phi + \theta \leq \pi/2$, we would have $d(Q, R) = \phi + \theta$, another version of (i). Thus, we must conclude that $\pi/2 < \phi + \theta < \pi$, so that

$$\begin{aligned} d(Q, R) &= \cos^{-1}(-\cos(\phi + \theta)) = \cos^{-1}(\cos(\pi - (\theta + \phi))) \\ &= \pi - (\theta + \phi) = \pi - d(P, Q) - d(P, R). \end{aligned}$$

Conversely, if (i) holds, we showed in the proof of Theorem 1 that the points must be collinear. If (ii) holds, we have $p + r = \pi - q$, so that $\cos(\pi - q) = \cos(p + r)$. By the same algebra as in Theorem 1, (6.1) becomes an equality, and the three points are collinear. □

Isometries

Definition. *A map T: $\mathbf{P}^2 \to \mathbf{P}^2$ is called an* isometry *if $d(P, Q) = d(TP, TQ)$ for all P and Q in $\mathbf{P}^2$.*

Theorem 3. *Let T be an isometry. If P, Q, and R are collinear, then TP, TQ, and TR are collinear.*

Proof: Let P, Q, and R be collinear points. Let P' be the unique point on this line such that $d(P, P') = \pi/2$. Then

$$d(P, Q) + d(Q, P') = d(P, P') = \frac{\pi}{2}.$$

Hence,

$$d(TP, TQ) + d(TQ, TP') = d(TP, TP') = \frac{\pi}{2}.$$

By the previous theorem, TQ must lie on the line determined by TP and TP'. Similarly, TR must lie on this line. □

The isometries of $\mathbf{P}^2$ are closely related to the isometries of $\mathbf{S}^2$.

Theorem 4. *Let T: $\mathbf{P}^2 \to \mathbf{P}^2$ be an isometry. Then there exists a unique $A \in \mathbf{SO(3)}$ such that $T = \tilde{A}$.*

Proof: Choose e_1, e_2, and e_3 on $\mathbf{S}^2$ such that

$$T\pi\varepsilon_i = \pi e_i \quad \text{for each } i.$$

Then

$$d(T\pi\varepsilon_i, T\pi\varepsilon_j) = d(\pi\varepsilon_i, \pi\varepsilon_j) = \cos^{-1}|\langle \varepsilon_i, \varepsilon_j \rangle|.$$

But $d(\pi e_i, \pi e_j) = \cos^{-1}|\langle e_i, e_j\rangle|$, and so $|\langle \varepsilon_i, \varepsilon_j\rangle| = |\langle e_i, e_j\rangle|$. If $i \neq j$, then $\langle e_i, e_j\rangle = 0$. Otherwise, $\langle e_i, e_j\rangle = 1$. Thus, $\{e_i\}$ is an orthonormal basis of $\mathbf{R}^3$. Let A be the orthogonal matrix such that $A\varepsilon_i = e_i$ for each i. Then $\tilde{A}$ is an isometry of $\mathbf{P}^2$, and $\tilde{A}^{-1}T$ leaves $\pi\varepsilon_1$, $\pi\varepsilon_2$, and $\pi\varepsilon_3$ fixed. Let $M = \pi(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and write

$$\tilde{A}^{-1}TM = \pi(k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3),$$

where k_1, k_2, and k_3 are some numbers, with $k_1^2 + k_2^2 + k_3^2 = 1$. We claim that the $|k_i|$ are all equal. To see this, note that

$$d(\tilde{A}^{-1}TM, \tilde{A}^{-1}T\pi\varepsilon_i) = \cos^{-1}|k_i|.$$

But $d(M, \pi\varepsilon_i) = \cos^{-1}(1/\sqrt{3})$ and, hence, $|k_i| = 1/\sqrt{3}$ for all i. Let

$$B = \sqrt{3}\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

Then

$$\tilde{B}\tilde{A}^{-1}T\pi\varepsilon_i = \pi\varepsilon_i \quad \text{for each } i,$$

and since each k_i^2 is 1/3, $\tilde{B}\tilde{A}^{-1}TM = M$. □

We will next prove that any isometry that leaves each ε_i and $\pi(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ fixed must be the identity. Assuming this for the moment, we get

$$\tilde{B}\tilde{A}^{-1}T = I;$$

that is,

$$T = (\tilde{A}^{-1})^{-1}\tilde{B}^{-1} = \tilde{A}\tilde{B}^{-1} = \widetilde{AB}^{-1}.$$

But $B^{-1} = B$. Hence, $T = \widetilde{AB}$. Because AB is orthogonal, there is a unique member of **SO(3)** that determines the same isometry.

Theorem 5. *If T is an isometry of $\mathbf{P}^2$ that leaves fixed each $\pi\varepsilon_i$ and $\pi(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, then T is the identity.*

Proof: Let us work in the homogeneous coordinate system determined by ε_1, ε_2, and ε_3. Then $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ are fixed points. We first check that all points on the line joining $(1, 0, 0)$ and $(0, 1, 0)$ are fixed. A typical such point is $x = (\cos\alpha, \sin\alpha, 0)$, where $0 < \alpha < \pi$. Write $Tx = (\cos\beta, \sin\beta, 0)$, where $0 < \beta < \pi$. Then

$$d(T\pi\varepsilon_1, Tx) = d(\pi\varepsilon_1, x),$$

$$\cos^{-1}|\cos\beta| = \cos^{-1}|\cos\alpha|;$$

that is, $|\cos \beta| = |\cos \alpha|$. Thus, $\beta = \alpha$ or $\beta = \pi - \alpha$.

If $\beta = \alpha$, we are finished. If $\beta = \pi - \alpha$, then

$$Tx = \pi(-\cos \alpha, \sin \alpha, 0).$$

Let $M = (1, 1, 1)$. Then

$$d(x, M) = \cos^{-1}\left|\frac{1}{\sqrt{3}}(\cos \alpha + \sin \alpha)\right|,$$

$$d(Tx, M) = \cos^{-1}\left|\frac{1}{\sqrt{3}}(-\cos \alpha + \sin \alpha)\right|.$$

This is impossible unless $\alpha = \pi/2$, in which case $\alpha = \beta$ anyway. Similarly, we can show that all points on the sides of the triangle of reference Δ are fixed. Now each line of $\mathbf{P}^2$ contains at least two fixed points because it intersects Δ at least twice. Therefore, every line is fixed, and, hence, every point is a fixed point. □

Motions

Let ℓ be a line of $\mathbf{S}^2$. Then the *reflection* in the line $\pi\ell$ is the isometry of $\mathbf{P}^2$ defined by

$$\Omega_{\pi\ell} = \tilde{\Omega}_\ell.$$

Theorem 6. *$\Omega_{\pi\ell}$ leaves fixed every point on $\pi\ell$ and the pole of $\pi\ell$. No other points are fixed.*

Proof: Choose an orthonormal basis of $\mathbf{E}^3$ with respect to which

$$\Omega_\ell = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $\Omega_\ell x = x$ if and only if x lies on the line joining $(0, 1, 0)$ and $(0, 0, 1)$. Also $\Omega_\ell x = -x$ if and only if $x = (1, 0, 0)$ or $(-1, 0, 0)$. Thus, the fixed points of Ω_ℓ are as claimed. □

Let ℓ be a line of $\mathbf{P}^2$, and let ξ be its pole. Then the product of two reflections $\Omega_m\Omega_n$, where m and n pass through ξ, is called a *rotation about* ξ. Because line goes through ξ if and only if it is perpendicular to ℓ, we also call $\Omega_m\Omega_n$ a *translation along* ℓ, and we call $\Omega_m\Omega_n\Omega_\ell$ a *glide reflection.* If $m \perp n$, then $\Omega_m\Omega_n$ is called a *half-turn.*

Theorem 7. *The fixed lines of a reflection are the line of fixed points and all lines perpendicular to this line.*

Proof: A line is fixed if and only if its pole is a fixed point. □

Theorem 8. *A rotation other than a half-turn or the identity has a unique fixed point and a unique fixed line. The point is the pole of the line.*

Proof: Let $\tilde{A}$ be a typical rotation of $\mathbf{P}^2$, where $A \in \mathbf{SO(3)}$. Then we will have solved the problem if we can find all those points $x \in \mathbf{S}^2$ such that $Ax = \pm x$. But this calculation was done in finding the fixed lines of a rotation of $\mathbf{S}^2$. If x is a fixed point of A, then πx is the unique fixed point of $\tilde{A}$, and the line whose pole is πx is the unique fixed line of $\tilde{A}$. □

Theorem 9.
i. *Every reflection is a half-turn, and every half-turn is a reflection.*
ii. *Every glide reflection is a rotation.*

Corollary. *Every isometry of* $\mathbf{P}^2$ *is a rotation.*

Remark: It is easy to show that the three reflections theorem and the representation theorems for rotations and translations (Theorems 4.15, 16, 19, and 20) hold in $\mathbf{P}^2$ (Exercise 6).

Elliptic geometry

The geometry of $\mathbf{P}^2$ is traditionally called elliptic geometry. So far, we have discussed its incidence properties, defined the notion of distance, and classified the isometries. We have seen that elliptic geometry is a simplification of spherical geometry.

Definition. *A* segment *in* $\mathbf{P}^2$ *is a set of the form* $\pi\mathscr{s}$, *where* $\mathscr{s}$ *is a minor segment in* $\mathbf{S}^2$. *The length of* $\pi\mathscr{s}$ *is the length of* $\mathscr{s}$. *The end points of* $\pi\mathscr{s}$ *are the images by* π *of the end points of* $\mathscr{s}$.

Theorem 10.
i. *Each pair* $\{A, B\}$ *of points in* $\mathbf{P}^2$ *is the end point set of two segments. The union of these segments is the line* $\overleftrightarrow{AB}$, *and their intersection is* $\{A, B\}$.
ii. *For a segment of length* L *with end points* A *and* B, *we have* $d(A, B) = L$ *if* $L \leq \pi/2$. *Otherwise,* $d(A, B) = \pi - L$.

Definition. *A* ray *is a segment of length* $\pi/2$ *with one end point removed. The remaining endpoint is called the* origin *of the ray.*

Remark: The definitions of ray in Euclidean, spherical, and elliptic geometry may seem at first to have little in common. There is a unifying idea, however. Starting at the origin of the ray, we move in a particular direction as long as the path we have traced out is the shortest path to this origin. If this continues forever, as in the Euclidean case, the ray continues forever. On the sphere, however, once we reach the point antipodal to the ray's origin, we lose this uniqueness. In differential geometry the point where this happens is called a *cut point*. In $\mathbf{P}^2$ we reach a cut point at distance $\pi/2$.

Theorem 11. *Let P and Q be points with $d(P, Q) < \pi/2$. Then there is a unique ray with origin P that contains Q. We denote this ray by $\overrightarrow{PQ}$.*

The definition of angle in elliptic geometry is the same as our previous definitions. The radian measure of an angle $\measuredangle PQR$ is determined by choosing a representative for Q, choosing the representatives for P and R closest to Q, and computing the radian measure of the spherical angle so determined.

The notion of half-plane does not occur in $\mathbf{P}^2$. One can, however, define the interior of an angle.

A *triangle* in elliptic geometry is a figure of the form $\pi\Delta$, where Δ is a spherical triangle.

Theorem 12. *If P, Q, and R are three noncollinear points of $\mathbf{P}^2$, there is a triangle having P, Q, and R vertices. The triangle is the union of three segments.*

Remark: Our treatment of elliptic geometry has been brief. Most of the questions we have studied in Euclidean and spherical geometry have analogues that can be studied in the elliptic setting. Some of these are explored in the exercises.

EXERCISES

1. Find the distance $d(P, Q)$, where $P = (-1, 0, 1)$ and $Q = (1, 1, 0)$ in homogeneous coordinates with respect to $\{\epsilon_1, \epsilon_2, \epsilon_3\}$.
2. i. Prove that every pair of distinct lines in $\mathbf{P}^2$ has a common perpendicular. Only a slight modification of your proof of Theorem 4.10 is required.
   ii. Find the common perpendicular to the lines $x_1 + 2x_2 = 0$ and $2x_2 - x_3 = 0$.

3. i. Prove the projective version of Theorem 4.11 concerning erecting and dropping perpendiculars.
   ii. Is the foot of the perpendicular the point on ℓ closest to P?
4. Let A: $\mathbf{R}^3 \to \mathbf{R}^3$ be linear. Then define $\tilde{A}\pi x = \pi A x$ so that $\tilde{A}$ maps $\mathbf{P}^2 \to \mathbf{P}^2$. Under what conditions on A will $\tilde{A}$ be an isometry? Illustrate using the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

5. Prove Theorem 9 and its corollary.
6. Verify the remark following Theorem 9 (that the three reflections and representation theorems hold in $\mathbf{P}^2$).
7. i. Given three nonconcurrent lines α, β, and γ, show how to find a point P and a line ℓ such that $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\ell H_P$.
   ii. If $\Omega_\ell H_P = \Omega_m H_Q$, what relationships must hold among ℓ, m, P, and Q?
8. Let P and Q be distinct points. Find $\mathcal{S}(\{P, Q\})$.
9. Under what conditions will two rotations about distinct points commute?
10. Let P, Q, and R be mutually perpendicular points of $\mathbf{S}^2$. Show that there are four isometries T of $\mathbf{P}^2$ that leave πP, πQ, and πR fixed. (*Hint*: Choose an appropriate orthonormal basis and compute the possible forms of the matrix of T.)
11. Find the symmetry group of the figure in $\mathbf{P}^2$ formed by two perpendicular lines.
12. Suppose that an isometry T of $\mathbf{P}^2$ has three concurrent fixed lines. Show that T must be a half-turn.
13. Prove Donkin's theorem: Let PQR be a triangle. Let α, β, and γ be rotations (translations) that take P to Q, Q to R, and R to P, respectively. Then $\gamma\beta\alpha$ is the identity.
14. Prove Theorem 10.
15. Prove Theorem 11.
16. Let P and Q be points with $d(P, Q) < \pi/2$. Prove that $\overrightarrow{PQ} \cap \overrightarrow{QP}$ is the segment with end points P and Q and length $d(P, Q)$.
17. Prove that the notion of radian measure for angles in $\mathbf{P}^2$ is well-defined.
18. What happens to Theorem 4.41 in $\mathbf{P}^2$?
19. i. Propose a definition for the perpendicular bisector of a segment in $\mathbf{P}^2$.

ii. Define the midpoint of a segment in such a way that each segment has a unique midpoint.

20. Prove that there are exactly two reflections that interchange a given pair of lines in $\mathbf{P}^2$.

21. Let ℓ be a line of $\mathbf{P}^2$, and let P and Q be any points not on ℓ. Show that there is a segment joining P and Q that does not meet ℓ. (This is why we do not attempt to define the notion of half-plane in $\mathbf{P}^2$.)

22. Define the interior of an angle in $\mathbf{P}^2$. Does the crossbar theorem hold?

23. Define the interior of a triangle in $\mathbf{P}^2$. Show that $\mathbf{P}^2$ may be regarded as the union of four equilateral triangles (and their interiors). Each triangle should have three right angles.

24. Prove Theorem 12. Is the triangle unique?

25. Prove that if X is a point in the interior of a triangle Δ, there is a segment containing X whose end points are on Δ.

26. Prove that the perpendicular bisectors of the three sides of a triangle are concurrent. In light of Exercise 18, what further results can be obtained?

27. Prove that the congruence theorems for spherical triangles are valid in $\mathbf{P}^2$ as well (Theorems 55–57 of Chapter 4).

28. i. Show that the finite groups of isometries of $\mathbf{P}^2$ may be identified with those listed in Theorem 4.58.

    ii. For each such group find a figure in $\mathbf{P}^2$ of which it is the symmetry group.