Finite groups of isometries of $\mathbf{E}^2$

3

Introduction

So far we have been studying particular figures or transformations with only secondary emphasis on the group structures involved. We shall now turn our attention to the study of groups of transformations and apply our results to get geometrical conclusions. Our main result is to determine precisely the finite groups of isometries of $\mathbf{E}^2$.

Cyclic and dihedral groups

Let m be any positive integer, and let $\alpha = \text{rot }(2\pi/m)$. The smallest subgroup of $\mathbf{O(2)}$ containing α is denoted by $\mathbf{C}_m$. We observe that $\mathbf{C}_1 = \{I\}$, but if $m > 1$ then $\mathbf{C}_m$ consists of the distinct elements $I, \alpha, \alpha^2, \ldots, \alpha^{m-2}, \alpha^{m-1}$, because $\alpha^m = I$. Any group isomorphic to $\mathbf{C}_m$ is called a *cyclic* group of order m.

Now let $\beta = \text{ref } 0$. The smallest subgroup of $\mathbf{O(2)}$ containing both α and β is denoted by $\mathbf{D}_m$.

Theorem 1. *In the group* $\mathbf{D}_m$ *the identity* $\beta\alpha = \alpha^{-1}\beta$ *holds.*

Proof: By Theorem 1.30 we have, for any θ, ϕ,

$$\text{ref } \theta \text{ rot } \phi = \text{ref}\left(\theta - \frac{\phi}{2}\right) = \text{ref}\left(\theta + \left(\frac{-\phi}{2}\right)\right) = \text{rot}(-\phi)(\text{ref } \theta).$$

Setting $\phi = 2\pi/m$ and $\theta = 0$ yields the desired result. □

Theorem 2. *The index* $[\mathbf{D}_m:\mathbf{C}_m]$ *of* $\mathbf{C}_m$ *in* $\mathbf{D}_m$ *is equal to* 2.

Proof: The group $\mathbf{C}_m$ consists of m distinct rotations. The coset $\beta\mathbf{C}_m$ consists of m distinct reflections. Because the identity

$$\beta\alpha^j\beta\alpha^k = \alpha^{-j}\beta^2\alpha^k = \alpha^{k-j}$$

holds, the union of the cosets $\beta\mathbf{C}_m$ and $\mathbf{C}_m$ is the group $\mathbf{D}_m$. □

Any group isomorphic to $\mathbf{D}_m$ is called a *dihedral* group.

Remark:
i. The symmetry group of an angle (other than a straight angle) is isomorphic to $\mathbf{D}_1$. It consists of the identity and a reflection.
ii. The Klein four-group (which we obtained in Theorem 2.28 as the symmetry group of a segment) is isomorphic to $\mathbf{D}_2$. It consists of the identity, a half-turn, and two reflections.
iii. The symmetry group of an equilateral triangle (Theorem 2.37) is isomorphic to $\mathbf{D}_3$.

Conjugate subgroups

Let G be a group. Two subgroups H and K are said to be *conjugate in G* if there exists an element $g \in G$ such that $K = g^{-1}Hg$.

Theorem 3. *Let g and T be isometries of $\mathbf{E}^2$. Then*
i. *If T is a reflection, so is $g^{-1}Tg$.*
ii. *If T is a rotation, so is $g^{-1}Tg$.*
iii. *If T is a translation, so is $g^{-1}Tg$.*

Proof:
i. Let $T = \Omega_\ell$ be a reflection. For any $x \in g^{-1}\ell$,

$$g^{-1}\Omega_\ell gx = g^{-1}gx = x,$$

so that $g^{-1}\ell$ is pointwise fixed. On the other hand, $g^{-1}\Omega_\ell g$ cannot be equal to the identity. Because it has this particular fixed point behavior, $g^{-1}\Omega_\ell g$ must in fact be the reflection in the line $g^{-1}\ell$ (Theorem 1.39).
ii. Suppose that $T = \Omega_\ell\Omega_m$ is a rotation with center P. Then

$$g^{-1}Tg = (g^{-1}\Omega_\ell g)(g^{-1}\Omega_m g),$$

a rotation about $g^{-1}P$.
iii. Suppose that $T = \Omega_\ell\Omega_m$ is a translation. Then $g^{-1}Tg$ is the product of reflections in the parallel lines $g^{-1}\ell$ and $g^{-1}m$. □

Groups of isometries that are conjugate in $\mathscr{I}(\mathbf{E}^2)$ are said to be *geometrically equivalent*. This definition is motivated by observations of the type made in Theorem 3. Conjugate elements perform the "same" isometries with respect to "different" positions. The group of all rotations about a point P is conjugate to the group of all rotations about any other point Q.

An example of two groups that are algebraically equivalent (i.e., isomorphic) but not geometrically equivalent (conjugate) is given by $\mathbf{D}_1$ and $\mathbf{C}_2$. Because both have order 2, they are isomorphic. But $\mathbf{C}_2$ is generated by a half-turn α, and $\mathbf{D}_1$ by a reflection β. Any equation of the form $g^{-1}\alpha g = \beta$ would be impossible because β has a line of fixed points, but α (and hence $g^{-1}\alpha g$) has only one fixed point.

The groups $\mathbf{C}_4$ and $\mathbf{D}_2$ have the same order but are not isomorphic. This can be seen by observing that every element $g \in \mathbf{D}_2$ satisfies the relation $g^2 = I$, which is not satisfied by $\alpha \in \mathbf{C}_4$.

The groups $\mathbf{C}_{2m}$ and $\mathbf{D}_m$ have the same number of elements for each m. If $m > 2$, $\mathbf{C}_{2m}$ is abelian, but $\mathbf{D}_m$ is not. Thus, $\mathbf{D}_m$ and $\mathbf{C}_{2m}$ are never geometrically equivalent and are isomorphic only if $m = 1$.

Theorem 4. *Let $\alpha = \operatorname{rot}(2\pi/m)$, $\beta = \operatorname{ref} 0$, and $\gamma = \operatorname{ref} \theta$. Then the group $\langle\{\alpha, \gamma\}\rangle$ generated by α and γ is conjugate to the dihedral group $\mathbf{D}_m = \langle\{\alpha, \beta\}\rangle$.*

Proof: Intuitively speaking, we can realize a reflection in the mirror of γ by first rotating by $-\theta$, then reflecting in the x_1-axis, and finally rotating back by θ. Algebraically, by Theorem 1.30

$$\begin{aligned} \operatorname{rot}\theta \operatorname{ref} 0 \operatorname{rot}(-\theta) &= \operatorname{rot}\theta \operatorname{ref}\frac{\theta}{2} \\ &= \operatorname{ref}\left(\frac{\theta}{2} + \frac{\theta}{2}\right) = \operatorname{ref}\theta. \end{aligned}$$ □

It is not surprising that congruent figures have geometrically equivalent symmetry groups.

Theorem 5. *Let $\mathscr{F}$ be a figure, and let g be an isometry of $\mathbf{E}^2$. Then $\mathscr{S}(\mathscr{F})$ and $\mathscr{S}(g\mathscr{F})$ are conjugate in $\mathscr{I}(\mathbf{E}^2)$.*

Proof: Let h be a symmetry of $\mathscr{F}$. Then

$$ghg^{-1}g\,\mathscr{F} = gh\,\mathscr{F} = g\,\mathscr{F},$$

and, hence, ghg^{-1} is a symmetry of $g\mathscr{F}$. Conversely, if h is a symmetry of $g\,\mathscr{F}$, we have

$$g^{-1}hg\,\mathscr{F} = g^{-1}g\,\mathscr{F} = \mathscr{F}.$$

Thus, $g^{-1}\mathscr{S}(g\mathscr{F})g = \mathscr{S}(\mathscr{F})$. □

Remark: For economy of language we will sometimes substitute "is" for "is geometrically equivalent to" when speaking of groups of isometries. For instance, we say "the symmetry group of the equilateral triangle is $\mathbf{D}_3$," even though this is strictly true only for an equilateral triangle with centroid at the origin and with the x_1-axis as a line of symmetry.

Orbits and stabilizers

Let X be a set, and let G be a group of transformations of X. Let x be a member of X. Then

$$Gx = \{gx | g \in G\}$$

is called the *orbit* of x by G. When G is understood from the context, we may write Orbit(x) for Gx. The set

$$G_x = \{g \in G | gx = x\}$$

is called the *stabilizer* of x in G. The stabilizer is also sometimes written Stab(x).

Theorem 6. *For any $x \in X$ the stabilizer of x in G is a group. There is a natural bijection of the set of cosets determined by* Stab(x) *onto* Orbit(x).

Proof: The fact that G_x is a group is easy. Now consider the mapping $\tau : G \to \text{Orbit}(x)$ defined by $\tau g = gx$. Then τ is clearly surjective. Let $\pi : G \to G/G_x$ be the natural homomorphism. We now set $\bar{\tau}\pi g = \tau g$ and note that $\bar{\tau}$ is well-defined as a map from G/G_x to Gx. To check this, suppose that $\pi g = \pi\tilde{g}$ are two representations of an element of G/G_x. Because g and $\tilde{g}$ belong to the same coset, they satisfy $\tilde{g}^{-1}gx = x$. But then

$$\tau g = gx = (\tilde{g}\tilde{g}^{-1})gx = \tilde{g}(\tilde{g}^{-1}g)x = \tilde{g}x = \tau\tilde{g}.$$

Hence, $\bar{\tau}$ is well-defined. Surjectivity is clear. Furthermore, $\bar{\tau}$ is injective because $\bar{\tau}\pi g = \bar{\tau}\pi\tilde{g}$ means that $\tau g = \tau\tilde{g}$. In other words,

$$gx = \tilde{g}x, \quad \tilde{g}^{-1}gx = x, \quad \tilde{g}^{-1}g \in G_x, \quad \text{and} \quad \pi g = \pi\tilde{g}. \qquad \square$$

Theorem 6 allows us to complete our earlier assertions about symmetry groups of rectilinear figures.

Theorem 7. *If $\mathscr{F}$ is a rectilinear figure having just one vertex, then $\mathscr{S}(\mathscr{F})$ is a finite group.*

Proof: Let P be the only vertex of $\mathscr{F}$. Then $\hat{\mathscr{F}}$ contains a finite number of lines (say m) through P. Note that $m \geqslant 2$. Let Q be some point of $\mathscr{F}$ other than P. Then the stabilizer of Q, consisting of isometries leaving P and Q fixed (Theorem 1.39) has at most two elements: the identity and possibly the reflection in $\overleftrightarrow{PQ}$. The orbit of Q by $\mathscr{S}(\mathscr{F})$ consists of points on $\mathscr{F}$ whose distance from P is $d(P, Q)$. Because $\hat{\mathscr{F}}$ has only m lines through P, there can be at most $2m$ such points. Thus

$$\#\mathscr{S}(\mathscr{F}) \leqslant \#\mathrm{Stab}(Q)\cdot\#\mathrm{Orbit}(Q) \leqslant 2(2m) = 4m.$$

This estimate is the best possible because a regular polygon with $2m$ vertices determines m lines through its center and has symmetry group of order $4m$. The figure consisting of these m lines alone has the same symmetry group. (Regular polygons are discussed in the next section.) □

Remark: Let $\mathscr{F}$ be a figure consisting of m rays with a common origin P. Then $\mathscr{S}(\mathscr{F})$ has at most $2m$ elements.

Definition. *Let G be a group of transformations of a set X. If there is a point $x_0 \in X$ that is a fixed point of every transformation in G, we call x_0 a fixed point of G.*

Remark: The orbit of the fixed point x_0 is $\{x_0\}$. The stabilizer of x_0 is the whole group G

Theorem 8. *Let $\mathscr{F}$ be a rectilinear figure with at least one vertex. Then $\mathscr{S}(\mathscr{F})$ has a fixed point.*

Leonardo's theorem

We now turn to the question, what finite groups can occur as symmetry groups of figures? The answer was known to Leonardo da Vinci ([31], p. 99). Although groups had not been invented in his day, he was aware that the only symmetries of a finite figure were rotations about a certain point and reflections in lines through that point.

We noted that symmetry groups have the fixed point property. We now show that all finite isometry groups have that property.

Theorem 9. *Let G be a finite subgroup of* $\mathbf{AF(2)}$. *Then G has a fixed point.*

Proof: Choose a point $x \in \mathbf{E}^2$. Let $n = \#G$, and $C = (1/n)\Sigma_{g\in G}gx$. That is, C is the centroid of the orbit of x. Any $T \in G$ permutes the elements of the orbit of x. A calculation similar to that of Theorem 2.27 gives us the fact that $TC = C$. □

Corollary. *Every finite subgroup of $\mathscr{I}(\mathbf{E}^2)$ consists of rotations about a certain point and reflections in lines through that point.*

Theorem 10. *Every finite subgroup G of $\mathscr{I}(\mathbf{E}^2)$ is cyclic or dihedral. If C is a fixed point of G, then G is generated by a rotation about C (possibly trivial) and/or a reflection in a line through C.*

Proof: First consider the case $C = 0$. Then G is a subgroup of $\mathbf{O(2)}$. If $G \cap \mathbf{SO(2)} = \{I\}$, then either $G = \{I\}$ or $G = \{I, \beta\}$ (where β is some reflection), because if G contained more than one reflection, it would have to contain a nontrivial rotation as well.

Suppose now that G contains a nontrivial rotation. Let ϕ be the smallest positive number such that rot $\phi \in G$. If rot ψ is another element of G, we may choose an integer ℓ so that

$$\ell\phi \leq \psi < (\ell + 1)\phi;$$

that is,

$$0 \leq \psi - \ell\phi < \phi.$$

Now $\operatorname{rot}(\psi - \ell\phi) = (\operatorname{rot}\psi)(\operatorname{rot}\phi)^{-\ell}$ is a member of G. Hence, $\psi - \ell\phi = 0$, and we conclude that all rotations in G are powers of rot ϕ. The same calculation with $\psi = 2\pi$ shows that there is a positive integer m such that $m\phi = 2\pi$; that is, $\phi = 2\pi/m$. Thus, we have shown that $G \cap \mathbf{SO(2)} = \mathbf{C}_m$.

Now if G contains a reflection β, we see that the coset $\mathbf{C}_m\beta$ contains m distinct reflections. Every reflection in G lies in $\mathbf{C}_m\beta$ because if γ is such a reflection then $\gamma\beta$ is a rotation in $\mathbf{C}_m$ and $\gamma = (\gamma\beta)\beta \in \mathbf{C}_m\beta$. Thus, every element of G may be written in the form $\alpha^j\beta^k$, where $\alpha = \operatorname{rot}(2\pi/m)$, $0 \leq j < m$, and $0 \leq k < 2$.

The identity $\beta\alpha = \alpha^{-1}\beta$ can be easily verified as in Theorem 1, and thus G is either $\mathbf{C}_m$ or $\mathbf{D}_m$.

Finally, if C is a point other than the origin, G is conjugate to the group $\tau_{-C}G\tau_C$, which does leave the origin fixed, and the first part of the proof applies. □

Regular polygons

As we have just discovered, only the cyclic and dihedral groups can occur as symmetry groups of figures. We will now describe a family of figures having precisely these symmetry groups. These are based on the familiar notion of regular polygons. Intuitively, we may imagine tracing out a figure by moving a unit distance, then turning through an angle of $2\pi/m$, and repeating this process m times. (This is the "turtle geometry" approach [1].)

We now make the formal definition.

Definition. *Let m be a positive integer greater than 2. Let P and Q be distinct points of $\mathbf{E}^2$. For each integer k let*

$$Q_k = \tau_P\left(\text{rot}\,\frac{2\pi k}{m}\right)\tau_{-P}Q,$$

and let q_k be the segment Q_kQ_{k+1}. The union of all q_k is called a regular polygon. *See Figures 3.1–3.2.*

Observing that $Q_{k+m} = Q_k$ and $q_{k+m} = q_k$ for all k, we see that there are m distinct Q_k (called vertices of the polygon) and m distinct q_k (called edges or sides of the polygon). The expression "polygon with m sides" is sometimes abbreviated "m-gon." But a regular polygon also is a rectilinear figure, and the concept of vertex has already been defined in Chapter 2. We must show that the two definitions are consistent. As a first step, we show that the centroid of the Q_k is P.

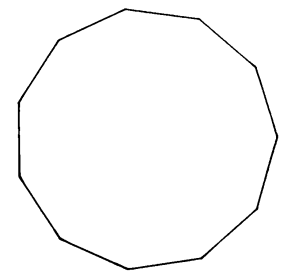

Figure 3.1 A regular 11-gon – symmetry group is dihedral.

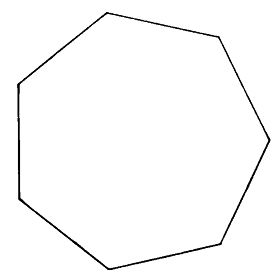

Figure 3.2 A regular 7-gon – symmetry group is dihedral.

Theorem 11. *Let $\{Q_k\}$ be the vertices of a regular m-gon. Then*

$$\frac{1}{m}\sum_{k=1}^{m} Q_k = P.$$

Proof: First note that

$$Q_k = P + \alpha^k(Q - P),$$

where $\alpha = \text{rot}(2\pi/m)$. Thus,

$$\Sigma Q_k = mP + (\Sigma\alpha^k)(Q - P).$$

The proof normally used to derive the formula for the sum of a geometric series applies here to show that the matrix equation

$$\sum_{k=1}^{m} \alpha^k = 0$$

holds. This completes the proof. □

Theorem 12. *Let $\{Q_k\}$ be the vertices of a regular m-gon. Let $\ell_k = \overleftrightarrow{Q_kQ_{k+1}}$ be a line determined by consecutive vertices. Then the whole polygon lies on the same side of ℓ_k.*

Proof: Let $v = Q_k - P$. Then, we may write

$$Q_{k+1} = P + (\text{rot}\,\theta)v, \quad Q_j = P + (\text{rot}\,\delta)v,$$

where $\theta = 2\pi/m$ and $\delta = (j - k)\theta$. Now

$$Q_{k+1} - Q_k = (\text{rot}\,\theta - I)v = 2\left(\sin\frac{\theta}{2}\right)J\left(\text{rot}\,\frac{\theta}{2}\right)v$$

and

$$Q_j - Q_k = 2\left(\sin\frac{\delta}{2}\right)J\left(\operatorname{rot}\frac{\delta}{2}\right)v.$$

The equation of the line ℓ_k is

$$\langle x - Q_k, N\rangle = 0,$$

where $N = J(Q_{k+1} - Q_k)$. We compute

$$\begin{aligned}\langle Q_j - Q_k, N\rangle &= 4\sin\frac{\theta}{2}\sin\frac{\delta}{2}\left\langle J\left(\operatorname{rot}\frac{\delta}{2}\right)v, -\left(\operatorname{rot}\frac{\theta}{2}\right)v\right\rangle\\ &= 4\sin\frac{\theta}{2}\sin\frac{\delta}{2}\left\langle\left(\operatorname{rot}\frac{\delta-\theta}{2}\right)v, Jv\right\rangle\end{aligned}$$

because $J^2 = -I$ and $J = \operatorname{rot}(\pi/2)$ commutes with the other rotations. But

$$\left(\operatorname{rot}\frac{\delta-\theta}{2}\right)v = \left(\left(\cos\frac{\delta-\theta}{2}\right)I + \left(\sin\frac{\delta-\theta}{2}\right)J\right)v,$$

therefore,

$$\langle Q_j - Q_k, N\rangle = 4\left(\sin\frac{\theta}{2}\right)\left(\sin\frac{\delta}{2}\right)\left(\sin\frac{\delta-\theta}{2}\right)|Jv|^2.$$

At this stage we remark that $\sin(\delta/2)$ and $\sin((\delta - \theta)/2)$ cannot have different signs. At worst, one can be zero, and the other nonzero. We conclude that $\langle Q_j - Q_k, N\rangle \geq 0$ for all j, and that all the Q_j lie on the same side of ℓ_k. Furthermore, equality occurs if and only if $\delta = 0$ (i.e., $Q_j = Q_k$) or $\delta = \theta$ (i.e., $Q_j = Q_{k+1}$). □

Corollary. *The line ℓ does not intersect any segment of the form Q_jQ_{j+1} except for the following cases:*

i. $Q_j = Q_k$, $Q_{j+1} = Q_{k+1}$,
ii. $Q_j = Q_{k+1}$,
iii. $Q_k = Q_{j+1}$.

Corollary. *The vertices of the polygon are precisely the m points $\{Q_j\}$, $j = 1, 2, \ldots, m$.*

Similarity of regular polygons

When computing the symmetry group of a regular polygon, it is legitimate to assume that its center is at the origin and one of its vertices is at $\varepsilon_1 = (1, 0)$. The reason is that if $\mathscr{P}$ and $\mathscr{P}'$ are two regular m-gons, then $\mathscr{S}(\mathscr{P})$ and $\mathscr{S}(\mathscr{P}')$ are conjugate in $\mathscr{I}(\mathbf{E}^2)$. We will now justify this statement.

Theorem 13. *Any two regular m-gons are similar.*

Similarity of regular polygons

Proof: Let P, Q and P', Q' be, respectively, the center and a vertex of two regular m-gons $\mathscr{P}$ and $\mathscr{P}'$. Then $\tau_{-P}\mathscr{P}$ and $\tau_{-P'}\mathscr{P}'$ are regular m-gons congruent to $\mathscr{P}$ and $\mathscr{P}'$. Let ϕ and ϕ' be chosen so that $(\text{rot}\ \phi)\tau_{-P}\mathscr{P}$ and $(\text{rot}\ \phi')\tau_{-P'}\mathscr{P}'$ are regular m-gons centered at the origin with one vertex on the positive x_1-axis (i.e., the ray with origin 0 and direction vector ε_1). Call the new polygons $\mathscr{P}_0$ and $\mathscr{P}'_0$.

Now if $Q = (q, 0)$ and $Q' = (q', 0)$, let S be the central dilatation given by

$$Sx = \frac{q'}{q}x.$$

We claim that $S\mathscr{P}_0 = \mathscr{P}'_0$. First note that if Q_j and Q'_j are the vertices of $\mathscr{P}_0$ and $\mathscr{P}'_0$, respectively, then

$$\begin{aligned} SQ_j &= S(\text{rot}\ j\theta)Q = (\text{rot}\ j\theta)\frac{q'}{q}Q \\ &= (\text{rot}\ j\theta)Q' = Q'_j, \end{aligned}$$

where $\theta = 2\pi/m$. Thus if q_j and q'_j are the edges of $\mathscr{P}_0$ and $\mathscr{P}'_0$, respectively, then

$$Sq_j = q'_j.$$

We conclude that $S\mathscr{P}_0 = \mathscr{P}'_0$. Putting all this together we see that

$$S(\text{rot}\ \phi)\tau_{-P}\mathscr{P} = (\text{rot}\ \phi')\tau_{-P'}\mathscr{P}';$$

that is,

$$\tau_{P'}(\text{rot}(-\phi')S(\text{rot}\ \phi)\tau_{-P}\mathscr{P} = \mathscr{P}'. \qquad \square$$

Theorem 14. *The symmetry groups of any two regular m-gons are conjugate in* $\mathscr{S}(\mathbf{E}^2)$.

Proof: Let T be the similarity constructed in the previous theorem. Then $\mathscr{S}(\mathscr{P})$ and $\mathscr{S}(\mathscr{P}')$ are conjugate in the group of similarities. This is not enough. However, if $g \in \mathscr{S}(\mathscr{P})$, then

$$g = \tau_P g_0 \tau_{-P},$$

where $g_0 \in \mathbf{O(2)}$. This is because g leaves P fixed. Now

$$\begin{aligned} TgT^{-1} &= \tau_{P'}(\text{rot}(\phi - \phi'))S\tau_{-P}\tau_P g_0 \tau_{-P}\tau_P S^{-1}(\text{rot}(\phi' - \phi))\tau_{-P'} \\ &= \tau_{P'}(\text{rot}(\phi - \phi'))Sg_0S^{-1}(\text{rot}(\phi' - \phi))\tau_{-P'}. \end{aligned}$$

We observe that S commutes with every member of $\mathbf{O(2)}$. Hence,

$$TgT^{-1} = \tau_{P'}(\text{rot}(\phi - \phi'))\tau_{-P}g\tau_P(\text{rot}(\phi' - \phi))\tau_{-P'} = \tilde{T}g\tilde{T}^{-1},$$

where $\bar{T}$ is an isometry. This shows that $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are in fact conjugate in $\mathcal{I}(\mathbf{E}^2)$.

Symmetry of regular polygons

Theorem 15. *Let $\mathcal{P}$ be a regular m-gon. Then $\mathcal{S}(\mathcal{P}) = \mathbf{D}_m$.*

Proof: By Theorems 12 and 13 we may assume that the vertices of $\mathcal{P}$ are of the form

$$\left(\text{rot}\frac{2\pi k}{m}\right)\varepsilon_1, \quad k = 0, 1, 2, \ldots, m - 1.$$

Clearly, $\text{rot}(2\pi/m)$ permutes the vertices. In fact,

$$\left(\text{rot}\ \frac{2\pi}{m}\right)Q_j = Q_{j+1}.$$

Secondly, ref 0 permutes the vertices. Specifically,

$$(\text{ref } 0)Q_j = Q_{m-j}.$$

Adjacent vertices remain adjacent under both of these transformations. In particular,

$$\left(\text{rot}\ \frac{2\pi}{m}\right)q_j = q_{j+1}, \quad (\text{ref } 0)\ q_j = q_{m-j-1}.$$

Thus, the edges are permuted by rot $(2\pi/m)$ and ref 0. Because each of these transformations is in $\mathcal{S}(\mathcal{P})$, we must conclude that $\mathbf{D}_m \subset \mathcal{S}(\mathcal{P})$. But Leonardo's theorem shows that $\mathcal{S}(\mathcal{P})$ is cyclic or dihedral because the centroid must be left fixed. If $\mathcal{S}(\mathcal{P})$ contains a rotation other than those in $\mathbf{C}_m$ (call it rot θ), then rot θ must permute the vertices. In particular, $(\text{rot } \theta)Q$ must be a vertex and so must be equal to $\text{rot}(2\pi j/m)\varepsilon_1$ for some j. Thus, rot $\theta \in \mathbf{C}_m$, and the proof is complete. □

Leonardo's theorem together with Theorem 7 shows that the only groups that can occur as symmetry groups of rectilinear figures (having at least one vertex) are $\mathbf{D}_m$ and $\mathbf{C}_m$.

The work on regular polygons shows that every dihedral group can be obtained as the symmetry group of some figure.

We now ask whether every cyclic group $\mathbf{C}_m$ can occur as the symmetry group of a figure. The answer can be seen as follows. A regular m-gon has $2m$ symmetries, m rotations, and m reflections. We change the figure in such a way as to destroy the bilateral symmetry while retaining the rotational symmetry. This can be done by attaching a tail to one end of each of the edges.

Let Q_j be defined as before, but let q_j be the ray with origin Q_j and direction vector $Q_{j+1} - Q_j$. One can check that no new vertices are introduced by this procedure. See Figures 3.3 and 3.4. However,

$$\left(\text{rot}\ \frac{2\pi}{m}\right)q_j = q_{j+1} \quad \text{and} \quad \left(\text{rot}\ \frac{2\pi}{m}\right)Q_j = Q_{j+1}.$$

Thus $\mathbf{C}_m \subset \mathcal{S}(\mathcal{F})$. As before, these are the only rotations in $\mathcal{S}(\mathcal{F})$.

If any reflection ref ϕ were a symmetry of $\mathcal{F}$, it would have to permute the vertices of $\mathcal{F}$ as well as leaving the centroid (the origin in this case) fixed. At most, two vertices could be fixed by ref ϕ. Thus, for some $j \neq k$, we would have

$$(\text{ref}\ \phi)Q_j = Q_k.$$

Then

$$\begin{aligned}(\text{ref}\ \phi)Q_{j+1} &= \text{ref}\ \phi(\text{rot}\ \theta)Q_j = \text{rot}(-\theta)(\text{ref}\ \phi)Q_j \\ &= (\text{rot}(-\theta))Q_k = Q_{k-1}.\end{aligned}$$

Thus, ref ϕ sends the ray $\overrightarrow{Q_jQ_{j+1}}$ to the ray $\overrightarrow{Q_kQ_{k-1}}$, which is not in the figure $\mathcal{P}$. We conclude that $\mathcal{S}(\mathcal{P})$ contains no reflections, and $\mathcal{S}(\mathcal{P}) = \mathbf{C}_m$. We can now state

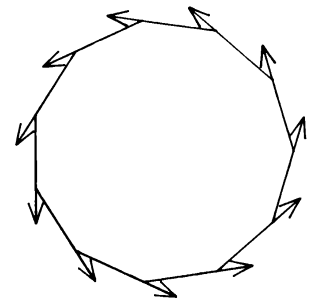

Figure 3.3 A modified regular 11-gon – symmetry group is cyclic.

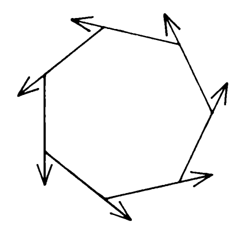

Figure 3.4 A modified regular 7-gon – symmetry group is cyclic.

Theorem 16. *Every finite cyclic or dihedral group is the symmetry group of a rectilinear figure.*

Figures with no vertices

A complete rectilinear figure with no vertices must consist of a finite number of parallel lines. Let $\mathcal{F}$ be such a figure, and let $[v]$ be the direction of the lines.

Because τ_v is a symmetry of $\mathcal{F}$, we have an example of a rectilinear figure with an infinite symmetry group.

Theorem 17. *Let $\mathcal{F}$ be a figure consisting of a finite number of parallel lines with direction $[v]$. Let $\mathcal{T}$ be the set of translations in $\mathcal{S}(\mathcal{F})$. Then*

$$\mathcal{T} = \{\tau_w | w \in [v]\}.$$

Proof: $\mathcal{F}$ is the union of lines of the form $P + [v]$. Now

$$\tau_w(P + [v]) = P + [v] + w = P + [v].$$

On the other hand, if $\tau_w \in \mathcal{T}$, then $\tau_w P = P + w$ must be in $P + [v]$. Thus $w \in [v]$. □

Theorem 18. *Let $\mathscr{F}$ be the figure of Theorem 17. Then $\mathscr{S}(\mathscr{F}) \cap \mathbf{O(2)}$ has at most four elements.*

Proof: Let A be a member of the group in question. Because A permutes the lines of $\mathscr{F}$, we must have $Av = \pm v$. Thus, A is a symmetry of the segment joining v and $-v$. By Theorem 2.28 there are at most four possibilities: two reflections, a half-turn, and the identity. In our case the identity and the reflection that interchanges v and $-v$ necessarily belong to $\mathscr{S}(\mathscr{F})$. The other two transformations will belong only if the lines are placed in a certain way.

Theorem 19. *Let $\mathscr{F}$ be a complete rectilinear figure with no vertices, and let $\mathscr{T}$ be the set of translations in $\mathscr{S}(\mathscr{F})$. Then*
i. *$\mathscr{T}$ is a normal subgroup of $\mathscr{S}(\mathscr{F})$,*
ii. *$\mathscr{S}(\mathscr{F})/\mathscr{T}$ has at most four elements.*

Proof: The homomorphism that sends each symmetry onto its linear part has $\mathscr{T}$ as its kernel and $\mathscr{S}(\mathscr{F}) \cap \mathbf{O(2)}$ as its range. Thus, $\mathscr{T}$ is normal, and the quotient group is isomorphic to $\mathscr{S}(\mathscr{F}) \cap \mathbf{O(2)}$.

EXERCISES

1. A *parallelogram* is a rectilinear figure consisting of four segments (sides) AB, BC, CD, and DA, where $AB \| CD$ and $BC \| DA$. Prove that there is an affine transformation relating any two parallelograms.
2. Find the affine symmetry group of the parallelogram.
3. A *rhombus* is a parallelogram in which all four sides have equal lengths. A *rectangle* is a parallelogram in which adjacent sides are perpendicular. A parallelogram that is both a rhombus and a rectangle is called a *square*. Find the symmetry groups of all types of parallelograms.
4. Let ℓ be a line. Show that $\mathrm{TRANS}(\ell) \cup \{H_P | P \in \ell\}$ is a group, and write down a multiplication table for it. Show that $\mathrm{TRANS}(\ell)$ is a normal subgroup, and describe the quotient group.
5. Let P and Q be distinct points. Describe the group G generated by $\{H_P, H_Q\}$. Show that the set of translations in G is a normal subgroup and describe the quotient group.
6. Let $\mathscr{F}$ be the union of three segments AB, BC, and CD. Given that $d(A, B) = d(C, D)$, $AB \perp BC$, and $BC \perp CD$, what can you say about $\mathscr{S}(\mathscr{F})$?
7. Let $\mathscr{F}$ be a complete rectilinear figure having two or more vertices, all of which are collinear. Show that $\mathscr{S}(\mathscr{F})$ is a finite group having one,

two, or four elements. Describe the configuration in each of these cases.

8. Verify the equation $\sum_{k=1}^{m} \alpha^k = 0$ in Theorem 11.