10

The five Platonic solids

We saw, in §4.6, that the Euclidean plane can be filled with squares, four at each vertex. If we try to fit squares together with only three at each vertex, we find that the figure closes as soon as we have used six squares, and we have a cube $\{4, 3\}$. Similarly, we can fill the plane with equilateral triangles, six at each vertex, and it is interesting to see what happens if we use three, four, or five instead of six. Another possibility is to use pentagons, three at each vertex, in accordance with the symbol $\{5, 3\}$.

With the possible exception of spheres, such *polyhedra* are the simplest solid figures. They provide an easy approach to the subject of topology as well as an interesting exercise in trigonometry. They can be defined and generalized in various ways [see, e.g., Hilbert and Cohn-Vossen **1**, p. 290].

10.1 PYRAMIDS, PRISMS, AND ANTIPRISMS

Although a Discourse of Solid Bodies be an uncommon and neglected Part of Geometry, yet that it is no inconsiderable or unprofitable Improvement of the Science will (no doubt) be readily granted by such, whose Genius tends as well to the Practical as Speculative Parts of it, for whom this is chiefly intended.

Abraham Shorp (1651-1742)
(*Geometry Improv'd*, London, 1717, p. 65)

A *convex polygon* (such as $\{n\}$, where n is an integer) may be described as a finite region of a plane "enclosed" by a finite number of lines, in the sense that its interior lies entirely on one side of each line. Analogously, a *convex polyhedron* is a finite region of space enclosed by a finite number of planes [Coxeter **1**, p. 4]. The part of each plane that is cut off by other planes is a polygon that we call a *face*. Any common side of two faces is an *edge*.

The most familiar polyhedra are *pyramids* and *prisms*. We shall be concerned solely with "right regular" pyramids whose faces consist of a regular n-gon and n isosceles triangles, and with "right regular" prisms whose

faces consist of two regular n-gons connected by n rectangles (so that there are two rectangles and one n-gon at each vertex). The height of such a prism can always be adjusted so that the rectangles become squares, and then we have an instance of a *uniform* polyhedron: all the faces are regular polygons and all the vertices are surrounded alike [Ball **1**, p. 135]. When $n = 4$, the prism is a *cube*, which is not merely uniform but *regular*: the faces are all alike, the edges are all alike, and the vertices are all alike. (The phrase "all alike" can be made precise with the aid of the theory of groups. We mean that there is a symmetry operation that will transform any face, edge, or vertex into any other face, edge, or vertex.)

The height of an n-gonal pyramid can sometimes be adjusted so that the isosceles triangles become equilateral. In fact, this can be done when $n < 6$; but six equilateral triangles fall flat into a plane instead of forming a solid angle. A triangular pyramid is called a *tetrahedron*. If three, and therefore all four, faces are equilateral, the tetrahedron is *regular*.

By slightly distorting an n-gonal prism we obtain an n-gonal *antiprism* (or "prismatoid," or "prismoid"), whose faces consist of two regular n-gons connected by $2n$ isosceles triangles. The height of such an antiprism can always be adjusted so that the isosceles triangles become equilateral, and then we have a uniform polyhedron with three triangles and an n-gon at each vertex. When $n = 3$, the antiprism is the regular *octahedron*. When $n = 5$, we can combine it with two pentagonal pyramids, one on each "base," to form the regular *icosahedron* [Coxeter **1**, p. 5]. A pair of icosahedral dice of the Ptolemaic dynasty can be seen in one of the Egyptian rooms of the British Museum in London.

We have now constructed four of the five convex regular polyhedra, namely those regarded by Plato as symbolizing the four elements: earth, fire, air, and water. The discrepancy between four elements and five solids did not upset Plato's scheme. He described the fifth as a shape that envelops the whole universe. Later it became the *quintessence* of the medieval alchemists. A model of this regular *dodecahedron* can be made by fitting together two "bowls," each consisting of a pentagon surrounded by five other pentagons. The two bowls will actually fit together because their free edges form a skew decagon like that formed by the lateral edges of a pentagonal antiprism (with isosceles lateral faces). Steinhaus [**2**, pp. 161–163] described a very neat method for building up such a model. From a sheet of cardboard, cut out two *nets* like Figure 10.1*a*, one for each bowl. Run a blunt knife along the five sides of the central pentagon so as to make them into hinged edges. Place one net crosswise on the other, with the scored edges outward, and bind them by running an elastic band alternately above and below the corners of the double star, holding the model flat with one hand. Removing the hand so as to allow the central pentagons to move away from each other, we see the dodecahedron rising as a perfect model (Figure 10.1*b*).

The most elementary properties of the five Platonic solids are collected

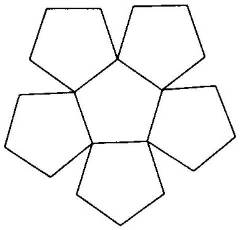

Figure 10.1a

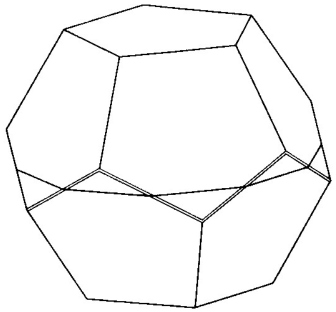

Figure 10.1b

in Table II on p. 413. Each polyhedron is characterized by a Schläfli symbol $\{p, q\}$, which means that it has p-gonal faces, q at each vertex. The numbers of vertices, edges, and faces are denoted by V, E, and F. They can easily be counted in each case, but their significance will become clearer when we have expressed them as functions of p and q. We shall also obtain an expression for the *dihedral angle,* which is the angle between the planes of two adjacent faces.

EXERCISES

1. Give an alternative description of the octahedron (as a dipyramid).

2. Describe a solid having five vertices and six triangular faces.

3. Describe the following sections: (i) of a regular tetrahedron by the plane midway between two opposite edges, (ii) of a cube by the plane midway between two opposite vertices, (iii) of a dodecahedron by the plane midway between two opposite faces.

4. Six congruent rhombi, with angles 60° and 120°, will fit together to form a *rhombohedron* ("distorted cube"). From the two opposite "acute" corners of this solid, regular tetrahedra can be cut off in such a way that what remains is an octahedron. In other words, two tetrahedra and an octahedron can be fitted together to form a rhombohedron. Deduce that the tetrahedron and the octahedron have supplementary dihedral angles, and that infinitely many specimens of these two solids can be fitted together to fill the whole Euclidean space [Ball **1**, p. 147].

10.2 DRAWINGS AND MODELS

You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view—
To preserve its symmetrical shape.

Lewis Carroll
[Dodgson **2a**, Fit 5]

Leonardo da Vinci made skeletal models of polyhedra, using strips of wood for their edges and leaving the faces to be imagined [Pacioli **1**]. When

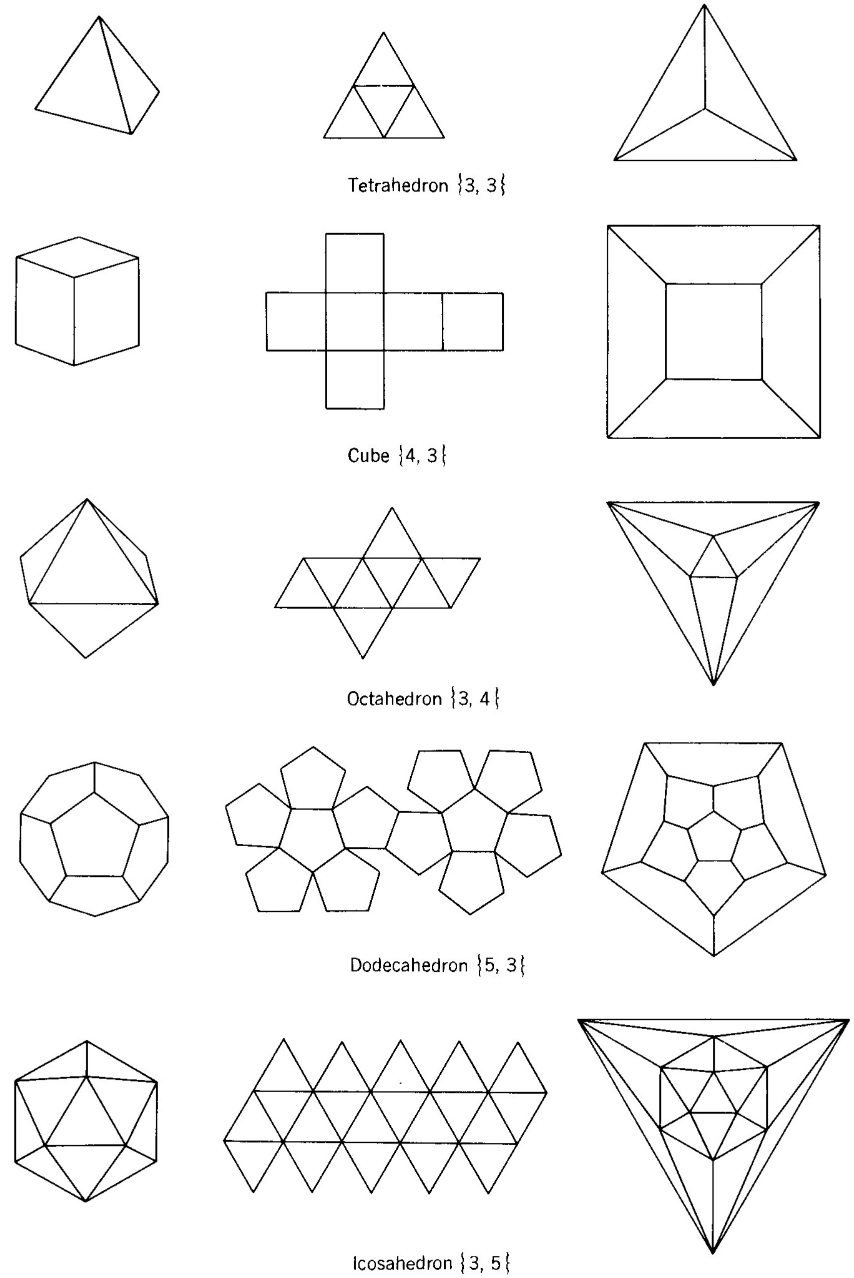

Figure 10.2a

such a model is seen in perspective from a position just outside the center of one face, this face appears as a large polygon with all the remaining faces filling its interior. Such a drawing of the solid is called a *Schlegel diagram* [Hilbert and Cohn-Vossen **1**, pp. 145–146].

Figure 10.2*a* shows each of the Platonic solids in three aspects: an ordinary perspective view, a *net* which can be folded to make a cardboard model, and a Schlegel diagram. Each can be checked by observing the nature of a face and the arrangement of faces at a vertex.

EXERCISES

1. Sketch a Schlegel diagram for a pentagonal antiprism.
2. What is the smallest number of acute-angled triangles into which a given obtuse-angled triangle can be dissected? (F. W. Levi.†)
3. What is the smallest number of acute-angled triangles into which a square can be dissected? (Martin Gardner.*)

10.3 EULER'S FORMULA

Euler . . . overlooked nothing in the mathematics of his age, totally blind though he was for the last seventeen years of his life.

E. T. Bell [**2**, p. 330]

The Schlegel diagram for a polyhedron shows at a glance which vertices belong to which edges and faces. Each face appears as a region bounded by edges, except the "initial" face, which encloses all the others. To ensure a one-to-one correspondence between faces and regions we merely have to associate the initial face with the infinite exterior region.

Any polyhedron that can be represented by a Schlegel diagram is said to be *simply connected* or "Eulerian," because its numerical properties satisfy Euler's formula

$$V - E + F = 2$$

[Hilbert and Cohn-Vossen **1**, p. 290], which is valid not only for the Schlegel diagram of such a polyhedron, but for any connected "map" formed by a finite number of points and line segments decomposing a plane into non-overlapping regions: the only restriction is that there must be at least one vertex!

A proof resembling Euler's may be expressed as follows. Any connected map can be built up, edge by edge, from the primitive map that consists of a single isolated vertex. At each stage, the new edge either joins an old vertex to a new vertex, as in Figure 10.3*a*, or joins two old vertices, as in Figure 10.3*b*. In the former case, V and E are each increased by 1 while

† *Mathematics Student*, **14** (1946).
* *Scientific American*, **202** (1960), p. 178.

F is unchanged; in the latter, V is unchanged while E and F are each increased by 1. In either case, the combination $V - E + F$ is unchanged. At the beginning, when there is only one vertex and one region (namely, all the rest of the plane), we have

$$V - E + F = 1 - 0 + 1 = 2.$$

This value 2 is maintained throughout the whole construction. Thus, Euler's formula holds for every plane map. In particular, it holds for every Schlegel diagram, and so for every simply connected polyhedron. (For another proof, due to von Staudt, see Rademacher and Toeplitz [**1**, pp. 75–76].)

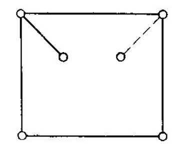

Figure 10.3a

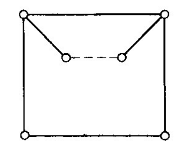

Figure 10.3b

In the case of the regular polyhedron $\{p, q\}$, the numerical properties satisfy the further relations

10.31 $$qV = 2E = pF.$$

In fact, if we count the q edges at each of the V vertices, we have counted every edge twice: once from each end. A similar situation arises if we count the p sides of each of the F faces, since every edge belongs to two faces.

We now have enough information to deduce expressions for V, E, F as functions of p and q. In fact,

$$\frac{V}{\frac{1}{q}} = \frac{E}{\frac{1}{2}} = \frac{F}{\frac{1}{p}} = \frac{V - E + F}{\frac{1}{q} - \frac{1}{2} + \frac{1}{p}} = \frac{2}{\frac{1}{q} + \frac{1}{p} - \frac{1}{2}} = \frac{4pq}{2p + 2q - pq},$$

whence

10.32 $$V = \frac{4p}{2p + 2q - pq}, \quad E = \frac{2pq}{2p + 2q - pq}, \quad F = \frac{4q}{2p + 2q - pq}.$$

Since these numbers must be positive, the possible values of p and q are restricted by the inequality $2p + 2q - pq > 0$ or

10.33 $$(p - 2)(q - 2) < 4.$$

Thus $p - 2$ and $q - 2$ are two positive integers whose product is less than 4, namely,

$$1 \cdot 1 \quad \text{or} \quad 2 \cdot 1 \quad \text{or} \quad 1 \cdot 2 \quad \text{or} \quad 3 \cdot 1 \quad \text{or} \quad 1 \cdot 3.$$

These five possibilities provide a simple proof of Euclid's assertion [Rademacher and Toeplitz **1**, pp. 84–87]:

There are just five convex regular polyhedra:

$$\{3, 3\}, \quad \{4, 3\}, \quad \{3, 4\}, \quad \{5, 3\}, \quad \{3, 5\}.$$

The inequality 10.33 is not merely a necessary condition for the existence of $\{p, q\}$ but also a sufficient condition; for in § 10.1 we saw how to construct a solid corresponding to each solution.

The same inequality arises in a more elementary manner when we construct a model of the polyhedron from its net. At a vertex we have q p-gons, each contributing an angle

$$\left(1 - \frac{2}{p}\right)\pi.$$

In order to form a solid angle, these q face angles must make a total less than 2π. Thus

$$q\left(1 - \frac{2}{p}\right)\pi < 2\pi$$

whence, as before, $(p - 2)(q - 2) < 4$.

Any maker of models soon observes that the amount by which the sum of the face angles at a vertex falls short of 2π is smaller for a complicated solid like the dodecahedron than for a simple one like the tetrahedron. Descartes proved that if this amount, say δ, is the same at every vertex, it is actually equal to $4\pi/V$ [Brückner **1**, p. 60]. In the case of $\{p, q\}$, this is an immediate consequence of the formula 10.32 for V, which yields

$$\frac{4\pi}{V} = (2p + 2q - pq)\frac{\pi}{p} = 2\pi - q\left(1 - \frac{2}{p}\right)\pi.$$

EXERCISES

1. The number of edges of $\{p, q\}$ is given by

$$E^{-1} = p^{-1} + q^{-1} - \tfrac{1}{2}.$$

2. Consider an arbitrary polyhedron having p-gonal faces for various values of p, and q faces at a vertex for various values of q. Generalize the equations 10.31 in the form

$$\Sigma q = 2E = \Sigma p,$$

where the first summation is taken over all the vertices and the last over all the faces. Deduce that every polyhedron has either a face with $p = 3$ or a vertex with $q = 3$ (or both). (*Hint*: If not, we would have $\Sigma q \geqslant 4V$ and $\Sigma p \geqslant 4F$.)

3. If the faces are all alike and the edges are all alike and the vertices are all alike, the faces are regular. Show by an example that this result for polyhedra is not valid for tessellations.

10.4 RADII AND ANGLES

A solid model of $\{p, q\}$ can evidently be built from F p-gonal pyramids of suitable altitude, placed together at their common apex, which is the center O_3 of the polyhedron. This point O_3 is the common center of three spheres: the *circumsphere* which passes through all the vertices, the *midsphere* which touches all the edges at their midpoints, and the *insphere* which touches all the faces at their centers. The *circumradius* ${}_0R$ appears as a lateral edge of any one of the pyramids (Figure 10.4*a*), the *midradius* ${}_1R$ as the altitude of a lateral face, and the *inradius* ${}_2R$ as the altitude of the whole pyramid.

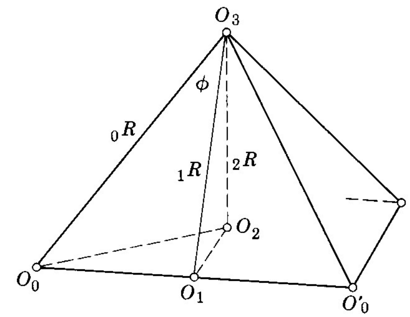

Figure 10.4a

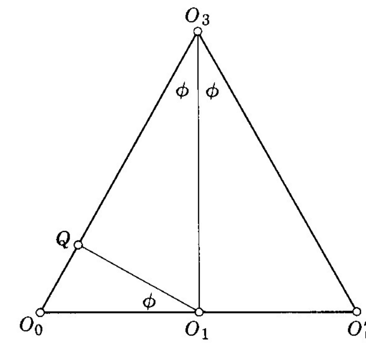

Figure 10.4b

Such a p-gonal pyramid has p planes of symmetry (or "mirrors") which join its apex O_3 to the p lines of symmetry of its base. By means of these p planes, the solid pyramid is dissected into $2p$ congruent (irregular) tetrahedra of a very special kind. Let $O_0O_1O_2O_3$ (Figure 10.4*c*) be such a tetrahedron, so that O_0 is a vertex of the polyhedron, O_1 the midpoint of an edge $O_0O'_0$, O_2 the center of a face, and O_3 the center of the whole solid. (The

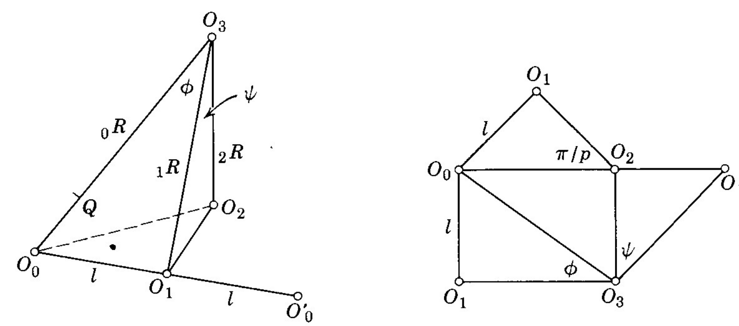

Figure 10.4c

net has been drawn to scale for the case of the cube $\{4, 3\}$, in which $O_0O_1 = O_1O_2 = O_2O_3$.) Since the plane $O_1O_2O_3$ perpendicularly bisects the edge $O_0O'_0$, O_0O_1 is perpendicular to both O_1O_2 and O_1O_3. Since $O_0O_1O_2$ is the plane of a face, the inradius O_3O_2 is perpendicular to both O_0O_2 and O_1O_2. Thus the three lines O_0O_1, O_1O_2, O_2O_3 are mutually perpendicular and the tetrahedron is "quadrirectangular": all four faces are right-angled triangles. Schläfli called such a tetrahedron an *orthoscheme* [Coxeter **1**, p. 137].

Many relations involving the radii

$$_0R = O_0O_3, \quad _1R = O_1O_3, \quad _2R = O_2O_3$$

can be derived from the four right-angled triangles, in which $O_0O_1 = l$ and $\angle O_0O_2O_1 = \pi/p$. But the whole story cannot be told till we have found the angle

$$\phi = \angle O_0O_3O_1$$

which is half the angle subtended at the center by an edge [Coxeter **1**, pp. 21–22].

Another significant angle is

$$\psi = \angle O_1O_3O_2,$$

whose complement, $\angle O_2O_1O_3$, is half the dihedral angle of the polyhedron. In other words, the dihedral angle is $\pi - 2\psi$.

In seeking these angles, it is useful to define the *vertex figure* of $\{p, q\}$: the polygon formed by the midpoints of the q edges at a vertex O_0. This is indeed a plane polygon, since its vertices lie on the circle of intersection of two spheres: the midsphere (with center O_3 and radius $_1R = O_3O_1$), and the sphere with center O_0 and radius $l = O_0O_1$. We see from 2.84 that the vertex figure of $\{p, q\}$ is a $\{q\}$ of side

$$2l\cos\frac{\pi}{p}.$$

Since its plane is perpendicular to O_3O_0, its center Q is the foot of the perpendicular from O_1 to O_3O_0 (Figure 10.4*b*), and its circumradius is

$$QO_1 = l\cos\phi.$$

By 2.81 (with $l\cos\pi/p$ for l), this circumradius is

$$l\cos\frac{\pi}{p}\csc\frac{\pi}{q}.$$

Hence

10.41 $$\cos\phi = \cos\frac{\pi}{p}\csc\frac{\pi}{q} = \cos\frac{\pi}{p}\Big/\sin\frac{\pi}{q}$$

[Coxeter **1**, p. 21].

The right-angled triangles in Figure 10.4*c* now yield

$$_0R = l\csc\phi, \quad _1R = l\cot\phi,$$

$$_2R^2 = {_1R^2} - \left(l\cot\frac{\pi}{p}\right)^2, \quad \cos\psi = \frac{_2R}{_1R}.$$

In order to eliminate ϕ, it is convenient to introduce the temporary abbreviation

$$k^2 = \sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} = \sin^2\frac{\pi}{p} - \cos^2\frac{\pi}{q},$$

so that $\sin\phi = k\csc\pi/q$. Then

$$_0R = \frac{l}{k}\sin\frac{\pi}{q}, \qquad _1R = \frac{l}{k}\cos\frac{\pi}{p},$$

10.42 $$_2R = \frac{l}{k}\cot\frac{\pi}{p}\cos\frac{\pi}{q},$$

10.43 $$\cos\psi = \cos\frac{\pi}{q}\Big/\sin\frac{\pi}{p}.$$

This last result enables us to compute the dihedral angle

$$\pi - 2\psi = 2\arcsin\left(\cos\frac{\pi}{q}\Big/\sin\frac{\pi}{p}\right).$$

EXERCISES

1. Verify that $k = \sin\pi/2c$, where $c = (2 + p + q)/(10 - p - q)$ [Coxeter **4**, p. 753], and that $E = c(c + 1)$.
2. Check the values of the dihedral angle given in Table II on p. 413. (Your calculations should agree with the observation that the dihedral angles of the tetrahedron and octahedron are supplementary. See Ex. 4 at the end of § 10.1.)
3. If a polyhedron has a circumsphere and a midsphere and an insphere, and if these three spheres are concentric, then the polyhedron is regular.

10.5 RECIPROCAL POLYHEDRA

The Platonic solid $\{p, q\}$ has a *reciprocal,* which may be defined as the polyhedron enclosed by a certain set of V planes, namely, the planes of the vertex figures at the V vertices of $\{p, q\}$. Clearly, its edges bisect the edges of $\{p, q\}$ at right angles. Among these E edges, those which bisect the p sides of a face of $\{p, q\}$ all pass through a vertex of the reciprocal, and those which bisect the q edges at a vertex $\{p, q\}$ form a face of the reciprocal. Thus

The reciprocal of $\{p, q\}$ *is* $\{q, p\}$, and vice versa. There is a vertex of either for each face of the other; in fact, the centers of the faces of $\{p, q\}$ are the vertices of a smaller version of the reciprocal $\{q, p\}$ [Steinhaus **1**, pp. 72–79].

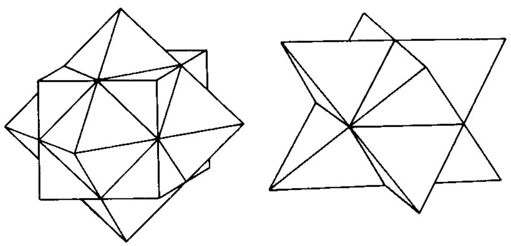

Figure 10.5a

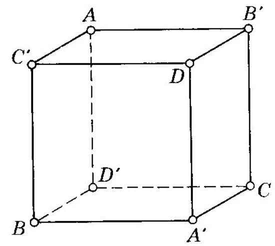

Figure 10.5b

Figure 10.5*a* shows how the octahedron $\{3, 4\}$ arises as the reciprocal of the cube $\{4, 3\}$ (or vice versa), and how the reciprocal of a regular tetrahedron $\{3, 3\}$ is an equal tetrahedron. The combination of two reciprocal tetrahedra, $ABCD$ and $A'B'C'D'$, occurs in nature as a crystal twin. Pacioli [**1**, Plates XIX, XX] named it *octaedron elevatum*. Kepler, rediscovering it a hundred years later, called it simply *stella octangula*. The twelve edges of the two tetrahedra are the diagonals of the six faces of a cube (Figure 10.5*b*).

By interchanging p and q in the formula 10.32 for V (or F) we obtain the formula for F (or V). Similarly, since Q, on O_3O_0 (Figure 10.4*b*), is the center of a face of $\{q, p\}$, the angular property ϕ of $\{p, q\}$ is equal to the angular property ψ of $\{q, p\}$, and therefore the expression 10.43 for the property ψ of $\{p, q\}$ could have been derived from 10.41 by the simple device of interchanging p and q.

EXERCISES

1. A cube of edge 1, with one vertex at the origin and three edges along the Cartesian axes, has the eight vertices (x, y, z), where each of the three coordinates is either 0 or 1, independently.

2. A cube of edge 2, with its center at the origin and its edges parallel to the Cartesian axes, has the eight vertices

$$(\pm 1, \pm 1, \pm 1).$$

3. Where is the center of the dilatation that relates the cubes described in the two preceding exercises?

4. Obtain coordinates for the vertices of a regular tetrahedron by selecting alternate vertices of a cube. Find the equations of the face planes and compute the angle between two of them.

5. Obtain coordinates for the vertices of an octahedron by locating the face centers of the cube in Ex. 2. Find the equations of the face planes and compute the angle between two that contain a common edge.

6. The points (x, y, z) that belong to the solid octahedron are given by the inequality

$$|x| + |y| + |z| \leqslant 1.$$

7. If each edge of a regular tetrahedron is projected into an arc of a great circle on the circumsphere, at what angles do these arcs intersect? Find the equations of the planes of the six great circles.