

Figure 2.8b

2.82

$$plr = pl^2 \cot \frac{\pi}{n}.$$

When  $d = 1$ , this is simply  $pl^2 \cot \pi/p$ ; in other cases our definition of area has the effect that every part of the interior is counted a number of times equal to the “local density” of that part; for example, the pentagonal region in the middle of the pentagram  $\{\frac{5}{2}\}$  is counted twice.

The angle  $P_0P_1P_2$  between two adjacent sides of  $\{n\}$ , being the sum of the base angles of the isosceles triangle, is the supplement of  $2\pi/n$ , namely

2.83

$$\left(1 - \frac{2}{n}\right)\pi.$$

The line segment joining the midpoints of two adjacent sides is called the *vertex figure* of  $\{n\}$ . Its length is clearly

2.84

$$2l \cos \frac{\pi}{n}$$

[Coxeter 1, pp. 16, 94].

### EXERCISES

1. If the sides of a polygon inscribed in a circle are all equal, the polygon is regular.
2. If a polygon inscribed in a circle has an odd number of vertices, and all its angles are equal, the polygon is regular. (Marcel Riesz.)
3. Find the angles of the polygons  
 $\{5\}, \{ \frac{5}{2} \}, \{9\}, \{ \frac{9}{2} \}, \{ \frac{9}{4} \}$ .
4. Find the radii and vertex figures of the polygons  
 $\{8\}, \{ \frac{8}{3} \}, \{12\}, \{ \frac{12}{5} \}$ .
5. Give polar coordinates for the  $k$ th vertex  $P_k$  of a polygon  $\{n\}$  of circumradius 1 with its center at the pole, taking  $P_0$  to be  $(1, 0)$ .
6. Can a square cake be cut into nine slices so that everyone gets the same amount of cake and the same amount of icing?

## Isometry in the Euclidean plane

Having made some use of reflections, rotations, and translations, we naturally ask why a rotation or a translation can be achieved as a continuous displacement (or “motion”) while a reflection cannot. It is also reasonable to ask whether there is any other kind of isometry that resembles a reflection in this respect. After answering these questions in terms of “sense,” we shall use the information to prove a remarkable theorem (§ 3.6) and to describe the seven possible ways to repeat a pattern on an endless strip (§ 3.7).

### 3.1 DIRECT AND OPPOSITE ISOMETRIES

“Take care of the sense, and the sounds will take care of themselves.”

Lewis Carroll

[Dodgson 1, Chap. 9]

By several applications of Axiom 1.26, it can be proved that any point  $P$  in the plane of two congruent triangles  $ABC, A'B'C'$  determines a corresponding point  $P'$  such that  $AP = A'P', BP = B'P', CP = C'P'$ . Likewise another point  $Q$  yields  $Q'$ , and  $PQ = P'Q'$ . Hence

**3.11** *Any two congruent triangles are related by a unique isometry.*

In § 1.3, we saw that Pappus’s proof of *Pons asinorum* involved the comparison of two coincident triangles  $ABC, ACB$ . We see intuitively that this is a distinction of *sense*: if one is counterclockwise the other is clockwise. It is a “topological” property of the Euclidean plane that this distinction can be extended from coincident triangles to distinct triangles: any two “directed” triangles,  $ABC$  and  $A'B'C'$ , either agree or disagree in sense. (For a deeper investigation of this intuitive idea, see Veblen and Young [2, pp. 61–62] or Denk and Hofmann [1, p. 56].)

If  $ABC$  and  $A'B'C'$  are congruent, the isometry that relates them is said to be *direct* or *opposite* according as it preserves or reverses sense, that is,

according as  $ABC$  and  $A'B'C'$  agree or disagree. It is easily seen that this property of the isometry is independent of the chosen triangle  $ABC$ : if the same isometry relates  $DEF$  to  $D'E'F'$ , where  $DEF$  agrees with  $ABC$ , then also  $D'E'F'$  agrees with  $A'B'C'$ . Clearly, direct and opposite isometries combine like positive and negative numbers (e.g., the product of two opposite isometries is direct). Since a reflection is opposite, a rotation (which is the product of two reflections) is direct. In particular, the identity is direct. Some authors call direct and opposite isometries “displacements and reversals” or “proper and improper congruences.”

Theorem 2.31 can be extended as follows:

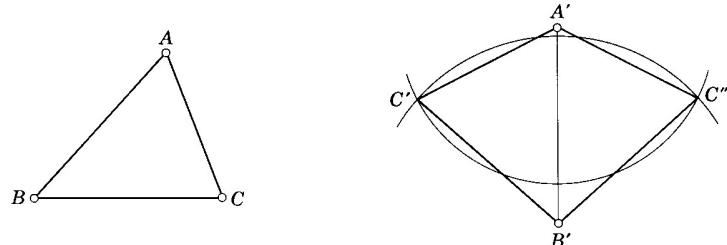


Figure 3.1a

**3.12** Two given congruent line segments (or point pairs)  $AB$ ,  $A'B'$  are related by just two isometries: one direct and one opposite.

To prove this, take any point  $C$  outside the line  $AB$ , and construct  $C'$  so that the triangle  $A'B'C'$  is congruent to  $ABC$ . The two possible positions of  $C'$  (marked  $C'$ ,  $C''$  in Figure 3.1a) provide the two isometries. Since either can be derived from the other by reflecting in  $A'B'$ , one of the isometries is direct and the other opposite.

For a complete discussion we need the following theorem [Bachmann 1, p. 3]:

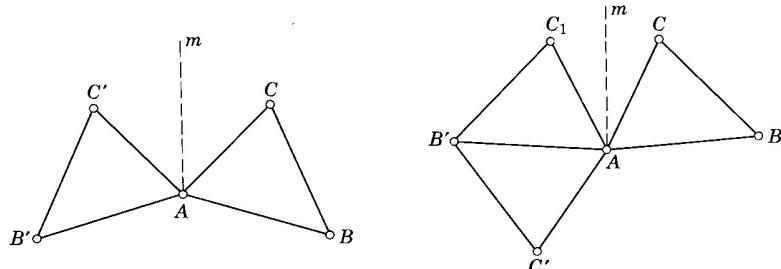


Figure 3.1b

**3.13** Every isometry of the plane is the product of at most three reflections. If there is an invariant point, “three” can be replaced by “two.”

We prove this in four stages, using 3.11. Trivially, if the triangles  $ABC$ ,  $A'B'C'$  coincide, the isometry is the identity (which is the product of a reflection with itself). If  $A$  coincides with  $A'$ , and  $B$  with  $B'$ , while  $C$  and  $C'$  are distinct, the triangles are related by the reflection in  $AB$ . The case when only  $A$  coincides with  $A'$  can be reduced to one of the previous cases by reflecting  $ABC$  in  $m$ , the perpendicular bisector of  $BB'$  (see Figure 3.1b). Finally, the general case can be reduced to one of the first three cases by reflecting  $ABC$  in the perpendicular bisector of  $AA'$  [Coxeter 1, p. 35].

Since a reflection reverses sense, an isometry is direct or opposite according as it is the product of an even or odd number of reflections.

Since the identity is the product of two reflections (namely of any reflection with itself), we may say simply that any isometry is the product of two or three reflections, according as it is direct or opposite. In particular,

**3.14** Any isometry with an invariant point is a rotation or a reflection according as it is direct or opposite.

### EXERCISES

1. Name two direct isometries.
2. Name one opposite isometry. Is there any other kind?
3. If  $AB$  and  $A'B'$  are related by a rotation, how can the center of rotation be constructed? (Hint: The perpendicular bisectors of  $AA'$  and  $BB'$  are not necessarily distinct.)
4. The product of reflections in three lines through a point is the reflection in another line through the same point [Bachmann 1, p. 5].

### 3.2 TRANSLATION

Enoch walked with God; and he was not, for God took him.  
Genesis V, 24

The particular isometries so far considered, namely reflections (which are opposite) and rotations (which are direct), have each at least one invariant point. A familiar isometry that leaves no point invariant is a *translation* [Bachmann 1, p. 7], which may be described as the product of half-turns about two distinct points  $O$ ,  $O'$  (Figure 3.2a). The first half-turn transforms an arbitrary point  $P$  into  $P^H$ , and the second transforms this into  $P^T$ , with the final result that  $PP^T$  is parallel to  $OO'$  and twice as long. Thus the length and direction of  $PP^T$  are constant: independent of the position of  $P$ . Since a translation is completely determined by its length and direction, the product of half-turns about  $O$  and  $O'$  is the same as the product of half-turns about  $Q$  and  $Q'$ , provided  $QQ'$  is equal and parallel to  $OO'$ . (This

means that  $OO'Q'Q$  is a parallelogram, possibly collapsing to form four collinear points, as in Figure 3.2a.) Thus, for a given translation, the center of one of the two half-turns may be arbitrarily assigned.

### 3.21 The product of two translations is a translation.

For, we may arrange the centers so that the first translation is the product of half-turns about  $O_1$  and  $O_2$ , while the second is the product of half-turns about  $O_2$  and  $O_3$ . When they are combined, the two half-turns about  $O_2$  cancel, and we are left with the product of half-turns about  $O_1$  and  $O_3$ .

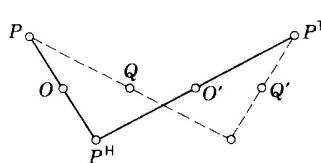


Figure 3.2a

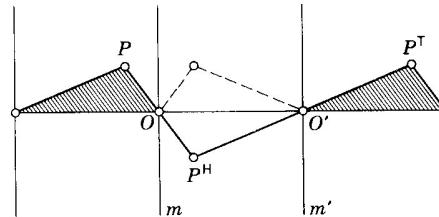


Figure 3.2b

Similarly, if  $m$  and  $m'$  (Figure 3.2b) are the lines through  $O$  and  $O'$  perpendicular to  $OO'$ , the half-turns about  $O$  and  $O'$  are the products of reflections in  $m$  and  $OO'$ ,  $OO'$  and  $m'$ . When they are combined, the two reflections in  $OO'$  cancel, and we are left with the product of reflections in  $m$  and  $m'$ . Hence

### 3.22 The product of reflections in two parallel mirrors is a translation through twice the distance between the mirrors.

If a translation  $T$  takes  $P$  to  $P^T$  and  $Q$  to  $Q^T$ , the segment  $QQ^T$  is equal and parallel to  $PP^T$ ; therefore  $PPQ^TP^T$  is a parallelogram. Similarly, if another translation  $U$  takes  $P$  to  $Q$ , it also takes  $P^T$  to  $Q^T$ ; therefore

$$TU = UT.$$

(In detail, if  $Q$  is  $P^U$ ,  $Q^T$  is  $P^{UT}$ . But  $U$  takes  $P^T$  to  $P^{TU}$ . Therefore  $P^{TU}$  and  $P^{UT}$  coincide, for all positions of  $P$ .) In other words,

### 3.23 Translations are commutative.

The product of a half-turn  $H$  and a translation  $T$  is another half-turn; for we can express the translation as the product of two half-turns, one of which is  $H$ , say  $T = HH'$ , and then we have

$$HT = H^2H' = H'.$$

### 3.24 The product of a half-turn and a translation is a half-turn.

#### EXERCISES

- If  $T$  is the product of half-turns about  $O$  and  $O'$ , what is the product of half-turns about  $O'$  and  $O$ ?
- When a translation is expressed as the product of two reflections, to what extent can one of the two mirrors be arbitrarily assigned?

3. What is the product of rotations through opposite angles ( $\alpha$  and  $-\alpha$ ) about two distinct points?

4. The product of reflections in three parallel lines is the reflection in another line belonging to the same pencil of parallels.

5. Every product of three half-turns is a half-turn [Bachmann 1, p. 7].

6. If  $H_1, H_2, H_3$  are half-turns,  $H_1H_2H_3 = H_3H_2H_1$ .

7. Express the translation through distance  $a$  along the  $x$ -axis as a transformation of Cartesian coordinates. If  $f(x, y) = 0$  is the equation for a curve, what is the equation for the transformed curve? Consider, for instance, the circle  $x^2 + y^2 - 1 = 0$ .

### 3.3 GLIDE REFLECTION

We are now familiar with three kinds of isometry: reflection, rotation, and translation. Another kind is the *glide reflection* (or simply “glide”), which is the product of the reflection in a line  $a$  and a translation along the same line. Picture this line as a straight path through snow; then, consecutive footprints are related by a glide. Such an isometry is determined by its axis  $a$  and the extent of the component translation. Since a reflection is opposite whereas a translation is direct, their product is opposite. Thus a glide reflection is an opposite isometry having no invariant point [Coxeter 1, p. 36].

If a glide reflection  $G$  transforms an arbitrary point  $P$  into  $P^G$  (Figure 3.3a),  $P$  and  $P^G$  are equidistant from the axis  $a$  on opposite sides. Hence

*The midpoint of the line segment  $PP^G$  lies on the axis for all positions of  $P$ .*

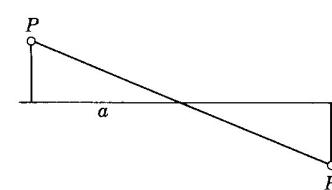


Figure 3.3a

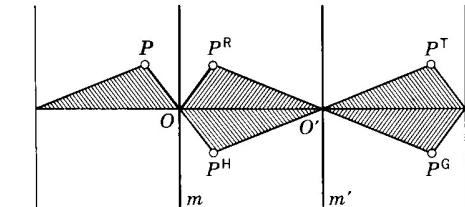


Figure 3.3b

Let  $R_1$  and  $T$  denote the component reflection and translation. They evidently commute, so that

$$G = R_1T = TR_1.$$

We have seen (Figure 3.2b) that the translation  $T$  may be expressed as the product of two half-turns or of two parallel reflections. Identifying the line  $a$  in Figure 3.3a with the line  $OO'$  in Figure 3.3b, let  $R, R'$  denote the reflections in  $m, m'$ . Then the product of the two half-turns

$$H = RR_1 = R_1R, \quad H' = R'R_1 = R_1R'$$

is

$$T = HH' = RR_1R_1R' = RR',$$

and the glide reflection is

$$\begin{aligned} G &= R_1T = R_1RR' = HR' \\ &= TR_1 = RR'R_1 = RH'. \end{aligned}$$

Thus a glide reflection may be expressed as the product of three reflections (two perpendicular to the third), or of a half-turn and a reflection, or of a reflection and a half-turn. Conversely, the product of any half-turn and any reflection (or vice versa) is a glide reflection, provided the center of the half-turn does not lie on the mirror. [Bachmann 1, p. 6.]

We saw in 3.13 that any direct isometry in the plane is the product of two reflections, that is, a translation or a rotation according as the two mirrors are parallel or intersecting; also that any opposite isometry with an invariant point is a reflection. To complete the catalog of isometries, the only remaining possibility is an opposite isometry with no invariant point. If such an isometry  $S$  transforms an arbitrary point  $A$  into  $A'$ , consider the half-turn  $H$  that interchanges these two points. The product  $HS$ , being an opposite isometry which leaves the point  $A'$  invariant, can only be a reflection  $R$ . Hence the given opposite isometry is the glide reflection

$$S = H^{-1}R = HR.$$

*Every opposite isometry with no invariant point is a glide reflection.*

In other words,

**3.31** *Every product of three reflections is either a single reflection or a glide reflection.*

In particular, the product  $RT$  of any reflection and any translation is a glide reflection, degenerating to a pure reflection when the mirror for  $R$  is perpendicular to the direction of the translation  $T$  (in which case the reflections  $R$  and  $RT$  may be used as the two parallel reflections whose product is  $T$ ). But since a given glide reflection  $G$  has a definite axis (the locus of midpoints of segments  $PP^G$ ), its decomposition into a reflection and a translation *along the mirror* is unique (unlike its decomposition into a reflection and a half-turn, where we may either take the mirror to be any line perpendicular to the axis or equivalently take the center of the half-turn to be any point on the axis).

#### EXERCISES

1. If  $B$  is the midpoint of  $AC$ , what kinds of isometry will transform (i)  $AB$  into  $CB$ , (ii)  $AB$  into  $BC$ ?
2. Every direct isometry is the product of two reflections. Every opposite isometry is the product of a reflection and a half-turn.
3. Describe the product of the reflection in  $OO'$  and the half-turn about  $O$ .
4. Describe the product of two glide reflections whose axes are perpendicular.

5. Every product of three glide reflections is a reflection or a glide reflection.
6. The product of three reflections is a reflection if and only if the three mirrors are either concurrent or parallel.
7. If  $R_1, R_2, R_3$  are three reflections,  $(R_1R_2R_3)^2$  is a translation [Rademacher and Toeplitz 1, p. 29].
8. Describe the transformation

$$(x, y) \longrightarrow (x + a, -y).$$

Justify the statement that this transforms the curve  $f(x, y) = 0$  into  $f(x - a, -y) = 0$ .

#### 3.4 REFLECTIONS AND HALF-TURNS

Thomsen\* has developed a very beautiful theory in which geometrical properties of points  $O, O_1, O_2, \dots$  and lines  $m, m_1, m_2, \dots$  (understood to be all distinct) are expressed as relations among the corresponding half-turns  $H, H_1, H_2, \dots$  and reflections  $R, R_1, R_2, \dots$ . The reader can soon convince himself that the following pairs of statements are logically equivalent:

- |                         |                       |  |
|-------------------------|-----------------------|--|
| $RR_1 = R_1R$           | $\longleftrightarrow$ | $m$ and $m_1$ are perpendicular.                   |
| $HR = RH$               | $\longleftrightarrow$ | $O$ lies on $m$ .                                  |
| $R_1R_2R_3 = R_3R_2R_1$ | $\longleftrightarrow$ | $m_1, m_2, m_3$ are either concurrent or parallel. |
| $H_1H = HH_2$           | $\longleftrightarrow$ | $O$ is the midpoint of $O_1O_2$ .                  |
| $H_1R = RH_2$           | $\longleftrightarrow$ | $m$ is the perpendicular bisector of $O_1O_2$ .    |

#### EXERCISE

Interpret the relations (a)  $H_1H_2H_3H_4 = 1$ ; (b)  $R_1R = RR_2$ .

#### 3.5 SUMMARY OF RESULTS ON ISOMETRIES

*And thick and fast they came at last,  
And more, and more, and more.*

Lewis Carroll

[Dodgson 2, Chap. 4]

Some readers may have become confused with the abundance of technical terms, many of which are familiar words to which unusually precise meanings have been attached. Accordingly, let us repeat some of the definitions, stressing both their analogies and their differences.

\* G. Thomsen, The treatment of elementary geometry by a group-calculus, *Mathematical Gazette*, 17 (1933), p. 232. Bachmann [1] devotes a whole book to the development of this idea.

In all the contexts that concern us here, a *transformation* is a one-to-one correspondence of the whole plane (or space) with itself. An *isometry* is a special kind of transformation, namely, the kind that preserves length. A *symmetry operation* belongs to a given figure rather than to the whole plane: it is an isometry that transforms the figure into itself.

In the plane, a *direct* (sense-preserving) isometry, being the product of two reflections, is a rotation or a translation according as it does or does not have an invariant point, that is, according as the two mirrors are intersecting or parallel. In the latter case the length of the translation is twice the distance between the mirrors; in the former, the angle of the rotation is twice the angle between the mirrors. In particular, the product of reflections in two perpendicular mirrors is a half-turn, that is, a rotation through two right angles. Moreover, the product of two half-turns is a translation.

An *opposite* (sense-reversing) isometry, being the product of three reflections, is, in general, a *glide reflection*: the product of a reflection and a translation. In the special case when the translation is the identity (i.e., a translation through zero distance), the glide reflection reduces to a single reflection, which has a whole line of invariant points, namely, all the points on the mirror.

To sum up:

**3.51** Any direct isometry is either a translation or a rotation. Any opposite isometry is either a reflection or a glide reflection.

#### EXERCISES

- If  $S$  is an opposite isometry,  $S^2$  is a translation.
- If  $R_1, R_2, R_3$  are three reflections,  $(R_2R_3R_1R_2R_3)^2$  is a translation along the first mirror. (*Hint:* Since  $R_1R_2R_3$  and  $R_2R_3R_1$  are glide reflections, their squares are commutative, by 3.23; thus

$$(R_1R_2R_3)^2(R_2R_3R_1)^2 = (R_2R_3R_1)^2(R_1R_2R_3)^2,$$

that is,  $R_1$  and  $(R_2R_3R_1R_2R_3)^2$  are commutative [cf. Bachmann 1, p. 13].)

#### 3.6 HJELMSLEV'S THEOREM

... a very high degree of unexpectedness, combined with inevitability and economy.

G. H. Hardy [2, p. 53]

We saw, in 3.12, that two congruent line segments  $AB, A'B'$ , are related by just two isometries: one direct and one opposite. Both isometries have the same effect on every point collinear with  $A$  and  $B$ , that is, every point on the infinite straight line  $AB$  (for instance, the midpoint of  $AB$  is transformed into the midpoint of  $A'B'$ ). The opposite isometry is a reflection or glide reflection whose mirror or axis contains all the midpoints of segments joining pairs of corresponding points. If two of these midpoints coincide, the

direct isometry is a half-turn, and they all coincide [Coxeter 3, p. 267]. Hence

**HJELMSLEV'S THEOREM.** When all the points  $P$  on one line are related by an isometry to all the points  $P'$  on another, the midpoints of the segments  $PP'$  are distinct and collinear or else they all coincide.

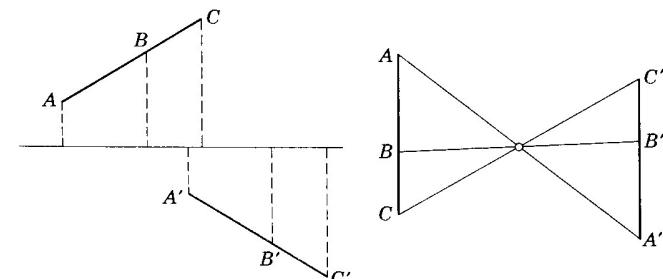


Figure 3.6a

In particular, if  $A, B, C$  are on one line and  $A', B', C'$  on another, with  
**3.61**  $AB = A'B', \quad BC = B'C'$

(Figure 3.6a), then the midpoints of  $AA', BB', CC'$  are either collinear or coincident (J. T. Hjelmslev, 1873–1950).

#### 3.7 PATTERNS ON A STRIP

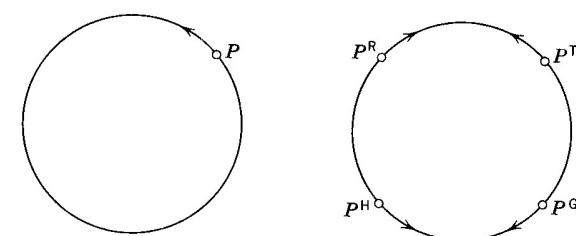


Figure 3.7a

Any kind of isometry may be used to relate two equal circles. For instance, the point  $P$  on the first circle of Figure 3.7a is transformed into  $P^T$  on the second circle by a translation, into  $P^R$  by a reflection, into  $P^H$  by a half-turn, and into  $P^G$  by a glide reflection. (Arrows have been inserted to indicate what happens to the positive sense of rotation round the first circle.) These four isometries have one important property in common: they leave invariant (as a whole) one infinite straight line, namely, the line joining the centers of the two circles. (In the fourth case this is the *only* invariant line.)

We have seen (Figure 3.2b) that the product of reflections in two parallel mirrors  $m, m'$  is a translation. This may be regarded as the limiting case of a rotation whose center is very far away; for the two parallel mirrors are the limiting case of two mirrors intersecting at a very small angle. Accordingly, the infinite group generated by a single translation is denoted by  $C_\infty$ , and the infinite group generated by two parallel reflections is denoted by  $D_\infty$ . Abstractly,  $C_\infty$  is the “free group with one generator.” If  $T$  is the generating translation, the group consists of the translations

$$\dots, T^{-2}, T^{-1}, 1, T, T^2, \dots$$

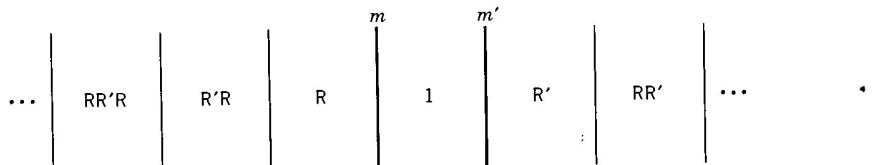


Figure 3.7b

Similarly,  $D_\infty$ , generated by the reflections  $R, R'$  in parallel mirrors  $m, m'$  (Figure 3.7b), consists of the reflections and translations

$$\dots, RR'R, R'R, R, 1, R', RR', R'RR', \dots$$

[Coxeter 1, p. 76]; its abstract definition is simply

$$R^2 = R'^2 = 1.$$

This group can be observed when we sit in a barber's chair between two parallel mirrors (cf. the *New Yorker*, Feb. 23, 1957, p. 39, where somehow the reflection  $RR'R'R$  yields a demon).

A different geometrical representation for the same abstract group  $D_\infty$  is obtained by interpreting the generators  $R$  and  $R'$  as half-turns. There is also an intermediate representation in which one of them is a reflection and the other a half-turn; but in this case their product is no longer a translation but a glide reflection.

Continuing in this manner, we could soon obtain the complete list of the seven infinite “one-dimensional” symmetry groups: the seven essentially distinct ways to repeat a pattern on a strip or ribbon [Speiser 1, pp. 81–82]:

<i>Typical pattern</i>	<i>Generators</i>	<i>Abstract Group</i>
(i) ... L L L L ...	1 translation	$C_\infty$
(ii) ... L Γ L Γ ...	1 glide reflection	
(iii) ... V V V V ...	2 reflections	
(iv) ... N N N N ...	2 half-turns	$D_\infty$
(v) ... V Λ V Λ ...	1 reflection and 1 half-turn	
(vi) ... D D D D ...	1 translation and 1 reflection	$C_\infty \times D_1$
(vii) ... H H H H ...	3 reflections	$D_\infty \times D_1$

In (iii), the two mirrors are both vertical, one in the middle of a V, reflecting it into itself, while the other reflects this V into one of its neighbors; thus one half of the V, placed between the two mirrors, yields the whole pattern. In (vi) and (vii) there is a horizontal mirror, and the symbols in the last column indicate “direct products” [Coxeter 1, p. 42]. For all these groups, except (i) and (ii), there is some freedom in choosing the generators; for example, in (iii) or (iv) one of the two generators could be replaced by a translation.

Strictly speaking, these seven groups are not “1-dimensional” but “1½-dimensional;” that is, they are 2-dimensional symmetry groups involving translation in one direction. In a purely one-dimensional world there are only two infinite symmetry groups:  $C_\infty$ , generated by one translation, and  $D_\infty$ , generated by two reflections (in point mirrors).

### EXERCISES

1. Identify the symmetry groups of the following patterns:

$$\begin{aligned} &\dots b b b b \dots, \\ &\dots b p b p \dots, \\ &\dots b d b d \dots, \\ &\dots b q b q \dots, \\ &\dots b d p q b d p q \dots. \end{aligned}$$

2. Which are the symmetry groups of (a) a cycloid, (b) a sine curve?