12

Ordered geometry

During the last 2000 years, the two most widely read books have undoubtedly been the Bible and the Elements. Scholars find it an interesting task to disentangle the various accounts of the Creation that are woven together in the Book of Genesis. Similarly, as Euclid collected his material from various sources, it is not surprising that we can extract from the Elements two self-contained geometries that differ in their logical foundation, their primitive concepts and axioms. They are known as *absolute* geometry and *affine* geometry. After describing them briefly in § 12.1, we shall devote the rest of this chapter to those propositions which belong to both: propositions so fundamental and "obvious" that Euclid never troubled to mention them.

12.1 THE EXTRACTION OF TWO DISTINCT GEOMETRIES FROM EUCLID

The pursuit of an idea is as exciting as the pursuit of a whale.

Henry Norris Russell (1877-1957)

Absolute geometry, first recognized by Bolyai (1802–1860), is the part of Euclidean geometry that depends on the first four Postulates without the fifth. Thus it includes the propositions I.1–28, III.1–19, 25, 28–30; IV.4–9 (with a suitably modified definition of "square"). The study of absolute geometry is motivated by the fact that these propositions hold not only in Euclidean geometry but also in hyperbolic geometry, which we shall study in Chapter 16. In brief, absolute geometry is geometry without the assumption of a unique parallel (through a given point) to a given line.

On the other hand, in affine geometry, first recognized by Euler (1707–1783), the unique parallel plays a leading role. Euclid's third and fourth postulates become meaningless, as circles are never mentioned and angles are never measured. In fact, the only admissible isometries are half-turns and translations. The affine propositions in Euclid are those which are preserved by parallel projection from one plane to another [Yaglom **2**, p. 17]:

for example, I. 30, 33–45, and VI. 1, 2, 4, 9, 10, 24–26. The importance of affine geometry has lately been enhanced by the observation that these propositions hold not only in Euclidean geometry but also in Minkowski's geometry of time and space, which Einstein used in his special theory of relativity.

Since each of Euclid's propositions is affine or absolute or neither, we might at first imagine that the two geometries (which we shall discuss in Chapters 13 and 15, respectively) had nothing in common except Postulates I and II. However, we shall see in the present chapter that there is a quite impressive nucleus of propositions belonging properly to both. The essential idea in this nucleus is *intermediacy* (or "betweenness"), which Euclid used in his famous definition:

A line (segment) is that which lies evenly between its ends.

This suggests the possibility of regarding intermediacy as a primitive concept and using it to define a line segment as the set of all points between two given points. In the same spirit we can extend the segment to a whole infinite line. Then, if B lies between A and C, we can say that the three points A, B, C lie in *order* on their line. This relation of order can be extended from three points to four or more.

Euclid himself made no explicit use of order, except in connection with measurement: saying that one magnitude is greater or less than another. It was Pasch, in 1882, who first pointed out that a geometry of order could be developed without reference to measurement. His system of axioms was gradually improved by Peano (1889), Hilbert (1899), and Veblen (1904).

Etymologically, "geometry without measurement" looks like a contradiction in terms. But we shall find that the passage from axioms and simple theorems to "interesting" theorems resembles Euclid's work in spirit, though not in detail.

This basic geometry, the common foundation for the affine and absolute geometries, is sufficiently important to have a name. The name *descriptive* geometry, used by Bertrand Russell [**1**, p. 382], was not well chosen, because it already had a different meaning. Accordingly, we shall follow Artin [**1**, p. 73] and say *ordered* geometry.

We shall pursue this rigorous development far enough to give the reader its flavor without boring him. The whole story is a long one, adequately told by Veblen [**1**] and Forder [**1**, Chapter II, and the *Canadian Journal of Mathematics,* **19** (1967), pp. 997–1000].

It is important to remember that, in this kind of work, we must define all the concepts used (except the primitive concepts) and prove all the statements (except the axioms), however "obvious" they may seem.

EXERCISES

1. Is the ratio of two lengths along one line a concept belonging to absolute geometry or to affine geometry or to both? (*Hint:* In "one dimension," i.e., when we consider only the points on a single line, the distinction between *absolute* and *affine* disappears.)

2. Name a Euclidean theorem that belongs neither to absolute geometry nor to affine geometry.

3. The concurrence of the medians of a triangle (1.41) is a theorem belonging to both absolute geometry and affine geometry. To which geometry does the rest of § 1.4 belong?

4. Which geometry deals (a) with parallelograms? (b) with regular polygons? (c) with Fagnano's problem (§ 1.8)?

12.2 INTERMEDIACY

A discussion of order . . . has become essential to any understanding of the foundation of mathematics.

Bertrand Russell (1872-)
[Russell **1**, p. 199]

In Pasch's development of ordered geometry, as simplified by Veblen, the only primitive concepts are *points* $A, B, \ldots$ and the relation of *intermediacy* $[ABC]$, which says that B is between A and C. If B is not between A and C, we say simply "not $[ABC]$." There are ten axioms (12.21–12.27, 12.42, 12.43, and 12.51), which we shall introduce where they are needed among the various definitions and theorems.

AXIOM 12.21 *There are at least two points.*

AXIOM 12.22 *If A and B are two distinct points, there is at least one point C for which $[ABC]$.*

AXIOM 12.23 *If $[ABC]$, then A and C are distinct: $A \neq C$.*

AXIOM 12.24 *If $[ABC]$, then $[CBA]$ but not $[BCA]$.*

THEOREM 12.241 *If $[ABC]$ then not $[CAB]$.*

Proof. By Axiom 12.24, $[CAB]$ would imply not $[ABC]$.

THEOREM 12.242 *If $[ABC]$, then $A \neq B \neq C$* (that is, in view of Axiom 12.23, the three points are all distinct).

Proof. If $B = C$, the two conclusions of Axiom 12.24 are contradictory. Similarly, we cannot have $A = B$.

DEFINITIONS. If A and B are two distinct points, the *segment* AB is the set of points P for which $[APB]$. We say that such a point P is *on* the segment. Later we shall apply the same preposition to other sets, such as "lines."

THEOREM 12.243 *Neither A nor B is on the segment AB.*

Proof. If A or B were on the segment, we would have $[AAB]$ or $[ABB]$, contradicting 12.242.

THEOREM 12.244 *Segment AB = segment BA.*

Proof. By Axiom 12.24, $[APB]$ implies $[BPA]$.

DEFINITIONS. The *interval* $\overline{AB}$ is the segment AB plus its *end points* A and B:

$$\overline{AB} = A + AB + B.$$

The *ray* A/B ("from A, away from B") is the set of points P for which $[PAB]$. The *line* AB is the interval $\overline{AB}$ plus the two rays A/B and B/A:

$$\text{line } AB = A/B + \overline{AB} + B/A.$$

COROLLARY 12.2441 Interval $\overline{AB}$ = interval $\overline{BA}$; line AB = line BA.

AXIOM 12.25 *If C and D are distinct points on the line AB, then A is on the line CD.*

THEOREM 12.251 *If C and D are distinct points on the line AB then*

$$\text{line } AB = \text{line } CD.$$

Proof. If A, B, C, D are not all distinct, suppose $D = B$. To prove that line AB = line BC, let X be any point on BC except A or B. By 12.25, A, like X, is on BC. Therefore B is on AX, and X is on AB. Thus every point on BC is also on AB. Interchanging the roles of A and C, we see that similarly every point on AB is also on BC. Thus $AB = BC$. Finally, if A, B, C, D are all distinct, we have $AB = BC = CD$.

COROLLARY 12.2511 Two distinct points lie on just one line. Two distinct lines (if such exist) have at most one common point. (Such a common point F is called a point of intersection, and the lines are said to *meet* in F.)

COROLLARY 12.2512 Any three distinct points A, B, C on a line satisfy just one of the relations $[ABC]$, $[BCA]$, $[CAB]$.

AXIOM 12.26 *If AB is a line, there is a point C not on this line.*

THEOREM 12.261 *If C is not on the line AB, then A is not on BC, nor B on CA: the three lines BC, CA, AB are distinct.*

Proof. By 12.25, if A were on BC, C would be on AB.

DEFINITIONS. Points lying on the same line are said to be *collinear*. Three non-collinear points A, B, C determine a *triangle* ABC, which consists of these three points, called *vertices,* together with the three segments BC, CA, AB, called *sides*.

AXIOM 12.27 *If ABC is a triangle and $[BCD]$ and $[CEA]$, then there is, on the line DE, a point F for which $[AFB]$.* (See Figure 12.2*a*.)

THEOREM 12.271 *Between two distinct points there is another point.*

Proof. Let A and B be the two points. By 12.26, there is a point E not on the line AB. By 12.22, there is a point C for which $[AEC]$. By 12.251, the line AC is the same as AE. By 12.261 (applied to ABE), B is not on this line: therefore ABC is a triangle. By 12.22 again, there is a point D for which $[BCD]$. By 12.27 there is a point F between A and B.

THEOREM 12.272 *In the notation of Axiom* 12.27, $[DEF]$.

Proof. Since F lies on the line DE, there are (by 12.2512) just five possibilities: $F = D$, $F = E$, $[EFD]$, $[FDE]$, $[DEF]$. Either of the first two would make A, B, C collinear.

If $[EFD]$, we could apply 12.27 to the triangle DCE with $[CEA]$ and $[EFD]$ (Figure 12.2*b*), obtaining X on AF with $[DXC]$. Since AF and CD cannot meet more than once, we have $X = B$, so that $[DBC]$. Since $[BCD]$, this contradicts 12.24.

Similarly (Figure 12.2*c*) we cannot have $[FDE]$. The only remaining possibility is $[DEF]$.

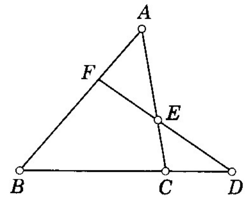

Figure 12.2a

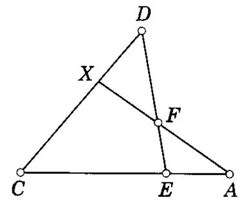

Figure 12.2b

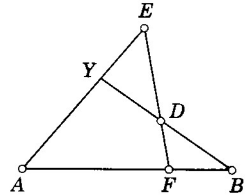

Figure 12.2c

This proof is typical; so let us be content to give the remaining theorems without proofs [Veblen **1**, pp. 9–15; Forder **1**, pp. 49–55].

12.273 *A line cannot meet all three sides of a triangle.* (Remember that the "sides" are not intervals, nor whole lines, but only segments.)

12.274 *If $[ABC]$ and $[BCD]$, then $[ABD]$.*

12.275 *If $[ABC]$ and $[ABD]$ and $C \neq D$, then $[BCD]$ or $[BDC]$, and $[ACD]$ or $[ADC]$.*

12.276 *If $[ABD]$ and $[ACD]$ and $B \neq C$, then $[ABC]$ or $[ACB]$.*

12.277 *If $[ABC]$ and $[ACD]$, then $[BCD]$ and $[ABD]$.*

DEFINITION. If $[ABC]$ and $[ACD]$, we write $[ABCD]$.

This four-point order is easily seen to have all the properties that we should expect, for example, if $[ABCD]$, then $[DCBA]$, but all the other orders are false.

Any point O on a segment AB decomposes the segment into two segments: AO and OB. (We are using the word *decomposes* in a technical sense [Veblen **1**, p. 21], meaning that every point on the segment AB except O itself is on just one of the two "smaller" segments.) Any point O on a ray from A decomposes the ray into a segment and a ray: AO and O/A. Any point O on a line decomposes the line into two "opposite" rays; if $[AOB]$, the rays are O/A and O/B. The ray O/A, containing B, is sometimes more conveniently called *the ray OB.*

For any integer $n > 1$, n distinct collinear points decompose their line into two rays and $n - 1$ segments. The points can be named $P_1, P_2, \ldots, P_n$ in such a way that the two rays are P_1/P_n, P_n/P_1, and the $n - 1$ segments are

$$P_1P_2,\ P_2P_3,\ \ldots,\ P_{n-1}P_n,$$

each containing none of the points. We say that the points are in the *order* $P_1P_2 \ldots P_n$, and write $[P_1P_2 \ldots P_n]$. Necessary and sufficient conditions for this are

$$[P_1P_2P_3],\quad [P_2P_3P_4],\quad \ldots,\quad [P_{n-2}P_{n-1}P_n].$$

Naturally, the best logical development of any subject uses the simplest or "weakest" possible set of axioms. (The worst occurs when we go to the opposite extreme and assume everything, so that there is no development at all!) In his original formulation of Axiom 12.27 [Pasch and Dehn **1**, p. 2 : "IV. Kernsatz"] Pasch made the following far stronger statement: If a line in the plane of a given triangle meets one side, it also meets another side (or else passes through a vertex). Peano's formulation, which we have adopted, excels this in two respects. The word "plane" (which we shall define in § 12.4) is not used at all, and the line DE penetrates the triangle ABC in a special manner, namely, before entering through the side CA, it comes from a point D on C/B. It might just as easily have come from a point on A/B (which is the same with C and A interchanged) or from a point on B/A or B/C (which is quite a different story). The latter possibility (with a slight change of notation) is covered by the following theorem (12.278). Axiom 12.27 is "only just strong enough"; for, although it enables us to deduce the statement 12.278 of apparently equal strength, we could not reverse the roles: if we tried instead to use 12.278 as an axiom, we would not be able to deduce 12.27 as a theorem!

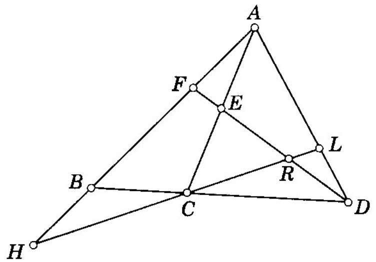

Figure 12.2d

THEOREM **12.278** *If ABC is a triangle and $[AFB]$ and $[BCD]$, then there is, on the line DF, a point E for which $[CEA]$.*

Proof. Take H on B/F (as in Figure 12.2*d*) and consider the triangle DFB with $[FBH]$ and $[BCD]$. By 12.27 and 12.272, there is a point R for which $[DRF]$ and $[HCR]$. By 12.274, $[AFB]$ and $[FBH]$ imply $[AFH]$. Thus we have a triangle DAF with $[AFH]$ and $[FRD]$. By 12.27 and 12.272 again, there is a point L for which $[DLA]$ and $[HRL]$. By 12.277, $[HCR]$ and $[HRL]$ imply $[CRL]$. Thus we have a triangle CAL with $[ALD]$ and $[LRC]$. By 12.27 a third time, there is, on the line DR $(=DF)$, a point E for which $[CEA]$.

EXERCISES

1. A line contains infinitely many points.
2. We have defined a segment as a set of points. At what stage in the above development can we assert that this set is never the *null* set? [Forder **1**, p. 50.]
3. In the proof of 12.272, we had to show that the relation $[FDE]$ leads to a contradiction. Do this by applying 12.27 to the triangle BFD (instead of EAF).
4. Given a finite set of lines, there are infinitely many points not lying on any of the lines.
5. If ABC is a triangle and $[BLC]$, $[CMA]$, $[ANB]$, then there is a point E for which $[AEL]$ and $[MEN]$. [Forder **1**, p. 56.]
6. If ABC is a triangle, the three rays B/C, A/C, A/B have a *transversal* (that is, a line meeting them all). (K. B. Leisenring.)
7. If ABC is a triangle, the three rays B/C, C/A, A/B have no transversal.

12.3 SYLVESTER'S PROBLEM OF COLLINEAR POINTS

Almost any field of mathematics offered an enchanting world for discovery to Sylvester.

E. T. Bell [**1**, p. 433]

It may seem to some readers that we have been using self-evident axioms to prove trivial results. Any such feeling of irritation is likely to evaporate when it is pointed out that the machinery so far developed is sufficiently powerful to deal effectively with Sylvester's conjecture (§ 4.7), which baffled the world's mathematicians for forty years. This matter of collinearity clearly belongs to ordered geometry. Kelly's Euclidean proof involves the extraneous concept of distance: it is like using a sledge hammer to crack an almond. The really appropriate nutcracker is provided by the following argument.

THEOREM. *If n points are not all collinear, there is at least one line containing exactly two of them.*

Proof. Let $P_1, P_2, \ldots, P_n$ be the n points, so named that the first three are not collinear (Figure 12.3*a*). Lines joining P_1 to all the other points of the set meet the line P_2P_3 in at most $n - 1$ points (including P_2 and P_3). Let Q be any *other* point on this line. Then the line P_1Q contains P_1 but no other P_i.

Lines joining pairs of P's meet the line P_1Q in at most $\binom{n-1}{2} + 1$ points (including P_1 and Q). Let P_1A be one of the segments that arise in the decomposition of this line by all these points. (Possibly $A = Q$.) Then no joining line P_iP_j can meet the "empty" segment P_1A. By its definition, A

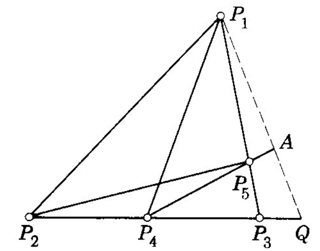

Figure 12.3a

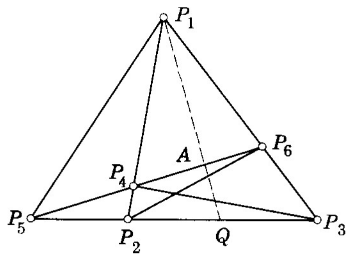

Figure 12.3b

lies on at least one joining line, say P_4P_5. If P_4 and P_5 are the only P's on this line (as in Figure 12.3*a*) our task is finished. If not, we have a joining line through A containing at least three of the P's, which we can name P_4, P_5, P_6 in such an order that the segment AP_5 contains P_4 but not P_6. (Since A decomposes the line into two opposite rays, one of which contains at least two of the three P's, this special naming is always possible. See Figure 12.3*b*.) We can now prove that the line P_1P_5 contains only these two P's.

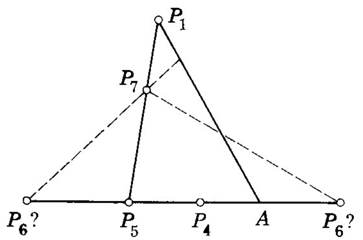

Figure 12.3c

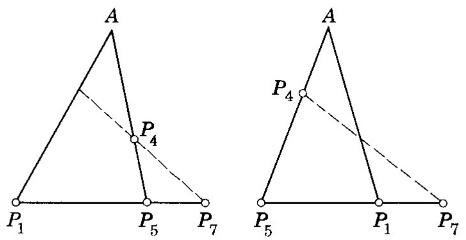

Figure 12.3d

We argue by *reductio ad absurdum*. If the line P_1P_5 contains (say) P_7, we can use 12.27 and 12.278 to deduce that the segment P_1A meets one of the joining lines, namely, P_6P_7 or P_4P_7. In fact, it meets P_6P_7 if $[P_1P_7P_5]$ (as in Figure 12.3*c*), and it meets P_4P_7 if $[P_1P_5P_7]$ or $[P_5P_1P_7]$ (as in Figure 12.3*d*). In either case our statement about the "empty" segment is contradicted.

Thus we have found, under all possible circumstances, a line (P_4P_5 or P_1P_5) containing exactly two of the P's.

EXERCISE

Justify the statement that the joining lines meet the line P_1Q in at most $\binom{n-1}{2}+1$ points. In the example shown in Figure 12.3*b*, this number (at most 11) is only 5; why? (The symbol $\binom{i}{j}$ stands for the number of combinations of i things taken j at a time; for instance, $\binom{i}{2}$ is the number of pairs, namely $\frac{1}{2}i(i-1)$.)

12.4 PLANES AND HYPERPLANES

If i hyperplanes in n dimensions are so placed that every n but no n + 1 have a common point, the number of regions into which they decompose the space is

$$\binom{i}{0}+\binom{i}{1}+\binom{i}{2}+\binom{i}{3}+\ldots+\binom{i}{n}=f(n,i).$$

Ludwig Schläfli (1814-1895)
[Schläfli **1**, p. 209]

It is remarkable that we can do so much plane geometry before defining a plane. But now, as the Walrus said, "The time has come . . . . "

DEFINITIONS. If A, B, C are three non-collinear points, the *plane ABC* is the set of all points collinear with pairs of points on one or two sides of the triangle ABC. A segment, interval, ray, or line is said to be *in* a plane if all its points are.

Axioms 12.21 to 12.27 enable us to prove all the familiar properties of incidence in a plane, including the following two which Hilbert [**1**, p. 4] took as axioms:

Any three non-collinear points in a plane α completely determine that plane.

If two distinct points of a line a lie in a plane α, then every point of a lies in α.

DEFINITIONS. An *angle* consists of a point O and two non-collinear rays going out from O. The point O is the *vertex* and the rays are the *sides* of the angle [Veblen **1**, p. 21; Forder **1**, p. 69]. If the sides are the rays OA and OB, or a_1 and b_1, the angle is denoted by $\angle AOB$ or a_1b_1 (or $\angle BOA$, or b_1a_1). The same angle a_1b_1 is determined by any points A and B on its respective sides. If C is any point between A and B, the ray OC is said to be *within* the angle.

From here till the statement of Axiom 12.41, we shall assume that all the points and lines considered are *in one plane*.

A *convex region* is a set of points, any two of which can be joined by a segment consisting entirely of points in the set, with the extra condition that each of the points is on at least two non-collinear segments consisting entirely of points in the set. In particular, an *angular* region is the set of all points on rays within an angle, and a *triangular* region is the set of all points between pairs of points on distinct sides of a triangle. An angular (or triangular) region is said to be *bounded* by the angle (or triangle).

It can be proved [Veblen **1**, p. 21] that any line containing a point of a convex region "decomposes" it into two convex regions. In particular, a line a decomposes a plane (in which it lies) into two *half planes.* Two points are said to be on the *same side* of a if they are in the same half plane, on *opposite sides* if they are in opposite half planes, that is, if the segment joining them meets a. In the latter case we also say that a *separates* the two points. (It is unfortunate that the word "side" is used with two different meanings, both well established in the literature. However, the context will always show whether we are considering the two sides of an angle, which are rays, or the two sides of a line, which are half planes.)

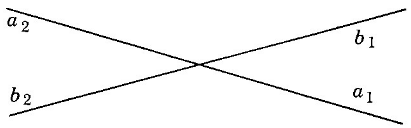

Figure 12.4a

As we remarked in § 12.2, any point O on a line a decomposes a into two rays, say a_1 and a_2. Any other line b through O is likewise decomposed by O into two rays b_1 and b_2, one in each of the half planes determined by a. Each of these rays decomposes the half plane containing it into two angular regions. Thus any two intersecting lines a and b together decompose their plane into four angular regions, bounded by the angles

$$a_1b_1, \quad b_1a_2, \quad a_2b_2, \quad b_2a_1,$$

as in Figure 12.4a. The opposite rays a_1 and a_2 are said to *separate* the rays b_1 and b_2; they likewise separate all the rays within either of the angles a_1b_1, b_1a_2 from all the rays within either of the angles a_2b_2, b_2a_1. We also say that the rays a_1 and b_1 separate all the rays between them from a_2, b_2, and from all the rays within b_1a_2, a_2b_2, or b_2a_1.

It follows from the definition of a line that two distinct points, A and B, decompose their line into three parts: the segment AB and the two rays A/B, B/A. Somewhat similarly, two nonintersecting (but coplanar) lines, a and b, decompose their plane into three regions. One of these regions lies *between* the other two, in the sense that it contains the segment AB for any A on a and B on b. Another line c is said to lie between a and b if it meets such a segment AB but does not meet a or b, and we naturally write $[acb]$.

12.401 *If ABC and $A'B'C'$ are two triads of collinear points, such that the three lines AA', BB', CC' have no intersection, and if $[ACB]$, then $[A'C'B']$.*

Analogous consideration of an angular region yields

12.402 *If ABC and $A'B'C'$ are two triads of collinear points on distinct lines, such that the three lines AA', BB', CC' have a common point O which is not between A and A', nor between B and B', nor between C and C', and if $[ACB]$, then $[A'C'B']$.*

We need one or more further axioms to determine the number of *dimensions*. If we are content to work in two dimensions we say

AXIOM 12.41 *All points are in one plane.*

If not [Forder **1**, p. 60], we say instead:

AXIOM 12.42 *If ABC is a plane, there is a point D not in this plane.*

We then define the *tetrahedron $ABCD$*, consisting of the four non-coplanar points A, B, C, D, called *vertices*, the six joining segments AD, BD, CD, BC, CA, AB, called *edges*, and the four triangular regions BCD, CDA, DAB, ABC, called *faces*. The *space* (or "3-space") $ABCD$ is the set of all points collinear with pairs of points in one or two faces of the tetrahedron $ABCD$.

We can now deduce the familiar properties of incidence of lines and planes [Forder **1**, pp. 61–65]. In particular, any four non-coplanar points of a space determine it, and the line joining any two points of a space lies entirely in the space. If Q is in the space $ABCD$ and P is in a face of the tetrahedron $ABCD$, then PQ meets the tetrahedron again in a point distinct from P.

If we are content to work in three dimensions, we say

AXIOM 12.43 *All points are in the same space.*

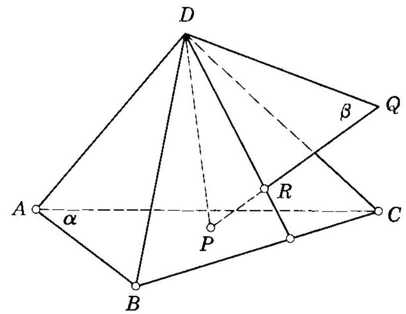

Figure 12.4b

Consequently:

THEOREM 12.431 *Two planes which meet in a point meet in another point, and so in a line.*

Proof. Let P be the common point and α one of the planes. Take A, B, C in α so that P is inside the triangle ABC. Let DPQ be a triangle in the other plane β (Figure 12.4b). If D or Q lies in α, then α and β have two common points. If not, PQ meets the tetrahedron $ABCD$ in a point R distinct from P; and DR, in β, meets the triangle ABC in a point common to α and β.

If, on the other hand, we wish to increase the number of dimensions, we replace 12.43 by

AXIOM 12.44 *If $A_0A_1A_2A_3$ is a 3-space, there is a point A_4 not in this 3-space.*

We then define the *simplex* $A_0A_1A_2A_3A_4$ which has 5 vertices A_i, 10 edges A_iA_j $(i < j)$, 10 faces $A_iA_jA_k$ $(i < j < k)$, and 5 *cells* $A_iA_jA_kA_l$ (which are tetrahedral regions.) The 4-*space* $A_0A_1A_2A_3A_4$ is the set of points collinear with pairs of points on one or two cells of the simplex.

The possible extension to n dimensions (using mathematical induction) is now clear. The n-space $A_0A_1 \ldots A_n$ is decomposed into two convex regions (half-spaces) by an $(n-1)$-dimensional subspace such as $A_0A_1 \ldots A_{n-1}$, which is called a *hyperplane* (or "prime," or "$(n-1)$-flat").

EXERCISES

1. Any 5 coplanar points, no 3 collinear, include 4 that form a convex quadrangle.

2. A ray OC within $\angle AOB$ decomposes the angular region into two angular regions, bounded by the angles AOC and COB. [Veblen **1**, p. 24.]

3. If m distinct coplanar lines meet in a point O, they decompose their plane into $2m$ angular regions [Veblen **1**, p. 26].

4. If ABC is a triangle, the three lines BC, CA, AB decompose their plane into seven convex regions, just one of which is triangular.

5. If m coplanar lines are so placed that every 2 but no 3 have a common point, they decompose their plane into a certain number of convex regions. Call this number $f(2, m)$. Then

$$f(2, m) = f(2, m-1) + m.$$

But $f(2, 0) = 1$. Therefore $f(2, 1) = 2$, $f(2, 2) = 4$, $f(2, 3) = 7$, and $f(2, m) = 1 + m + \binom{m}{2}$.

6. If m planes in a 3-space are so placed that every 3 but no 4 have a common point, they decompose their space into (say) $f(3, m)$ convex regions. Then

$$f(3, m) = f(3, m-1) + f(2, m-1).$$

But $f(3, 0) = 1$. Therefore $f(3, 1) = 2$, $f(3, 2) = 4$, $f(3, 3) = 8$, $f(3, 4) = 15$, and $f(3, m) = 1 + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$. [Steiner **1**, p. 87.]

7. Obtain the analogous result for m hyperplanes in an n-space.

12.5 CONTINUITY

Nothing but Geometry can furnish a thread for the labyrinth of the composition of the continuum . . . and no one will arrive at a truly solid metaphysic who has not passed through that labyrinth.

G. W. Leibniz (1646-1716)

[Russell **2**, pp. 108-109]

Between any two rational numbers (§ 9.1) there is another rational number, and therefore an infinity of rational numbers; but this does not mean that every real number (§ 9.2) is rational. Similarly, between any two points (12.271) there is another point, and therefore an infinity of points; but this does not mean that the axioms in § 12.2 make the line "continuous." In fact, continuity requires at least one further axiom. There are two well-recognized approaches to this subtle subject. One, due to Cantor and Weierstrass, defines a monotonic sequence of points, with an axiom stating that *every bounded monotonic sequence has a limit* [Coxeter **2,** Axiom 10.11]. The other, due to Dedekind, obtains a general point on a line as the common origin of two opposite rays [Coxeter **3**, p. 162]. Its arithmetical counterpart is illustrated by describing $\sqrt{2}$ as the "section" between rational numbers whose squares are less than 2 and rational numbers whose squares are greater than 2. Dedekind's Axiom, though formidable in appearance, is the more readily applicable; so we shall use it here:

AXIOM 12.51 *For every partition of all the points on a line into two nonempty sets, such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.*

This axiom is easily seen to imply several modified versions of the same statement. Instead of "the points on a *line*" we could say "the points on a *ray*" or "the points on a *segment*" or "the points on an *interval*." (In the last case, for instance, the rest of the line consists of two rays which can be added to the two sets in an obvious manner.) Another version [Forder **1**, p. 299] is:

THEOREM 12.52 *For every partition of all the rays within an angle into two nonempty sets, such that no ray of either lies between two rays of the other, there is a ray of one set which lies between every other ray of that set and every ray of the other set.*

To prove this for an angle $\angle AOB$, we consider the section of all the rays by the line AB, and apply the "segment" version of 12.51 to the segment AB.

12.6 PARALLELISM

In the last few weeks I have begun to put down a few of my own Meditations, which are already to some extent nearly 40 years old. These I had never put in writing, so I have been compelled three or four times to go over the whole matter afresh in my head.

C. F. Gauss (1777-1855)

(Letter to H. K. Schumacher, May 17, 1831, as translated by Bonola [**1**, p. 67])

The idea of defining, through a given point, two rays parallel to a given line (in opposite senses), was developed independently by Gauss, Bolyai, and Lobachevsky. The following treatment is closest to that of Gauss.

THEOREM 12.61 *For any point A and any line r, not through A, there are just two rays from A, in the plane Ar, which do not meet r and which separate all the rays from A that meet r from all the other rays that do not.*

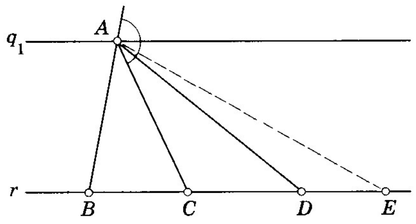

Figure 12.6a

Proof. Taking any two distinct points B and C on r, we apply 12.52 to the angle between the rays AC and A/B (marked in Figure 12.6*a*). We consider the partition of all the rays within this angle into two sets according as they do or do not meet the ray C/B. Clearly, these sets are not empty, and no ray in either set lies between two in the other. We conclude that one of the sets contains a special ray p_1 which lies between every other ray of that set and every ray of the other set.

In fact, p_1 belongs to the second set. For, if it met C/B, say in D, we would have $[BCD]$. By Axiom 12.22, we could take a point E such that $[CDE]$, with the absurd conclusion that AE belongs to both sets: to the first, because E is on C/B, and to the second, because AD lies between AC and AE.

We have thus found a ray p_1, within the chosen angle, which is the "first" ray that fails to meet the ray C/B; this means that every ray within the angle between AC and p_1 does meet C/B. Interchanging the roles of B and C, we obtain another special ray q_1, on the other side of AB, which may be described (for a counterclockwise rotation) as the "last" ray that fails to meet B/C. Since the line r consists of the two rays B/C, C/B, along with the interval $\overline{BC}$, we have now found two rays p_1, q_1, which separate all the rays from A that meet r from all the other rays (from A) that do not. [Forder **1**, p. 300.]

These special rays from A are said to be *parallel* to the line r in the two senses: p_1 parallel to C/B, and q_1 parallel to B/C. (Two rays are said to have the same sense if they lie on the same side of the line joining their initial points.)

For the sake of completeness, we define the rays parallel to r from a point A on r itself to be the two rays into which A decomposes r. The distinction between affine geometry and hyperbolic geometry depends on the question whether, for other positions of A, the two rays p_1, q_1 are still the two halves of one line. If they are, this line decomposes the plane into two half planes, one of which contains the whole of the line r. If not, the lines p and q (which

contain the rays) decompose the plane into four angular regions

$$p_1q_1, \quad q_1p_2, \quad p_2q_2, \quad q_2p_1.$$

In this case, by 12.61, r lies entirely in the region p_1q_1.

COROLLARY 12.62 *For any point A and any line r, not through A, there is at least one line through A, in the plane Ar, which does not meet r.*

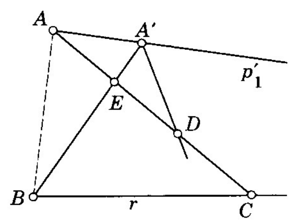

Figure 12.6b

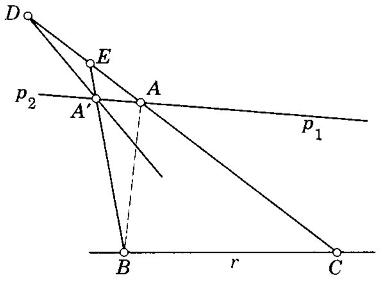

Figure 12.6c

Another familiar property of parallelism is its "transmissibility":

THEOREM 12.63 *The parallelism of a ray and a line is maintained when the beginning of the ray is changed by the subtraction or addition of a segment.*

Proof [Gauss **1**, vol. 8, p. 203]. Let p_1 be a ray from A which is parallel to a line r through B, and let A' be any point on this ray (Figure 12.6*b*) or on the opposite ray p_2 (Figure 12.6*c*). The modified ray p'_1, beginning at A', is A'/A or $A'A$, respectively; it obviously does not meet r. What remains to be proved is that every ray from A', within the angle between $A'B$ and p'_1, does meet r. Let D be any point on such a ray (Figure 12.6*b*) or on its opposite (Figure 12.6*c*). Since p_1 (from A) is parallel to r, the line AD (containing a ray within the angle between AB and p_1) meets r, say in C. The line $A'B$, separating A from D, meets the segment AD, say in E. By Axiom 2.27, applied to the triangle CBE with $[BEA']$ and $[EDC]$ (Figure 12.6*b*) or to the triangle BCE with $[CED]$ and $[EA'B]$ (Figure 12.6*c*), the line $A'D$ meets BC. Thus p'_1 is parallel to r.

This property of transmissibility enables us to say that the *line* $p = \overline{AA'}$ is parallel to the line $r = \overline{BC}$, provided we remember that this property is associated with a definite "sense" along each line.

Busemann [**1**, p. 139 (23.5)] has proved that it is not possible, within the framework of two-dimensional ordered geometry, to establish the "symmetry" of parallelism: that if p is parallel to r then r is parallel to p. To supply this important step we need either Axiom 12.42 [as in Coxeter **3**, pp. 165–177] or the affine axiom of parallelism (13.11) or the absolute axioms of congruence (§ 15.1).

THEOREM 12.64 *If two lines are both parallel to a third in the same sense, there is a line meeting all three.*

Proof. We have to show that, if lines p and s are both parallel to r in the same sense, then the three lines p, r, s have a transversal. In affine geometry this is obvious, so let us assume the geometry to be hyperbolic. Of the two lines parallel to r through a point A on p, one is p itself. Let q be the other, and let r be in the angular region $\mathrm{p}_1\mathrm{q}_1$, so that the rays p_1 and q_1 (from A) are parallel to r in opposite senses and s is parallel to r in the same sense as p_1. Let B and D be arbitrary points on r and s, respectively.

If D is in the region $\mathrm{p}_1\mathrm{q}_1$, the line AD is a transversal. If D is in $\mathrm{p}_1\mathrm{q}_2$, BD is a transversal. If D is in $\mathrm{p}_2\mathrm{q}_2$, both AD and BD are transversals. Finally, if D is in $\mathrm{p}_2\mathrm{q}_1$, AB is a transversal.

Hyperbolic geometry will be considered further in Chapters 15, 16, and 20.

EXERCISES

1. If p is parallel to s and $[prs]$, then p is parallel to r. (See Figure 15.2*c* with s for q.)
2. Consider all the points strictly inside a given circle in the Euclidean plane. Regard all other points as nonexistent. Let chords of the circle be called lines. Then all the axioms 12.21–12.27, 12.41, and 12.51 are satisfied. Locate the two rays through a given point parallel to a given line. Note that they form an angle (as in Figure 16.2*b*).

13 Affine geometry

The first three sections of this chapter contain a systematic development of the foundations of affine geometry. In particular, we shall see how length may be measured along a line, though independent units are required for lines in different directions. In §§ 13.4–7 we shall investigate such topics as area, affine transformations, lattices, vectors, barycentric coordinates, and the theorems of Ceva and Menelaus. Finally, in § 13.8 and § 13.9, we shall extend these ideas from two dimensions to three.

According to Blaschke [**1,** p. 31; **2,** p. 12], the word "affine" (German *affin*) was coined by Euler. But it was only after the launching of Klein's Erlangen program (see Chapter 5) that this geometry became recognized as a self-contained discipline. Many of the propositions may seem familiar; in fact, most readers will discover that they have often been working in the affine plane without realizing that it could be so designated.

Our treatment is somewhat more geometric and less algebraic than that of Artin's *Geometric Algebra* [Artin **1;** see especially pp. 58, 63, 71]. Incidentally, we shall find that our Axiom 13.12 (which he calls DP) implies Theorem 13.122 (his $\mathrm{D_a}$): this presumably means that his Axiom 4b implies 4a.

13.1 THE AXIOM OF PARALLELISM AND THE "DESARGUES" AXIOM

Mathematical language is difficult but imperishable. I do not believe that any Greek scholar of to-day can understand the idiomatic undertones of Plato's dialogues, or the jokes of Aristophanes, as thoroughly as mathematicians can understand every shade of meaning in Archimedes' works.

M. H. A. Newman

(*Mathematical Gazette* **43,** 1959, p. 167)

In this axiomatic treatment, we regard the real affine plane as a special case of the ordered plane. Accordingly, the primitive concepts are *point*