Lesson 28
Variation of Parameters
Non-Homogeneous Equations

2. *Variation of Parameters*

Step 1. Change the constants in the y_C to a function of the independent variable (parameters), say A, B, etc. where A, B are functions of x.

Step 2. Differentiate y_P according to the order of the given equation setting the sum of all functions of the derivatives of the independent variable to zero and the last nth derivative of the independent variable to R(x).

Step 3. Solve the equations simultaneously to find the parameters.

Example: Find the general solution of the differential equation.

1. $\left(D^2 - D - 2\right)y = 5$

SOLUTION:

$a)\, y_c:$

$(D^2 - D - 2) = 0$

$(m^2 - m - 2) = 0$

$m = 2, -1$

$y_c = C_1 e^{2x} + C_2 e^{-x}$

$b)\, y_p : MVP$

$y_p = Ae^{2x} + Be^{-x}$

where A and B are parameters

$Dy_p = 2Ae^{2x} + e^{2x}A' - Be^{-x} + e^{-x}B'$

Let: $e^{2x}A' + e^{-x}B' = 0 \longrightarrow (1)$

$D^2 y_p = 4Ae^{2x} + 2e^{2x}A' + e^{-x}B - e^{-x}B'$

Let: $2e^{2x}A' - e^{-x}B' = 5 \longrightarrow (2)$

Add (1) and (2)

$e^{2x}A' + e^{-x}B' = 0$

$\underline{2e^{2x}A' - e^{-x}B' = 5}$

$3e^{2x}A' = 5$

$\int A' = \int \frac{5}{3} e^{-2x}$

$$A = -\frac{5}{6}e^{-2x}$$

Multiply (1) by 2 then subtract (2)

$$2e^{2x}A' + 2e^{-x}B' = 0$$

$$\underline{2e^{2x}A' - e^{-x}B' = 5}$$

$$3e^{-x}A' = -5$$

$$\int B' = -\int \frac{5}{3}e^{x}$$

$$B = -\frac{5}{3}e^{x}$$

$$y_p = -\frac{5}{6}e^{-2x}(e^{2x}) + \left(-\frac{5}{3}e^{x}\right)e^{-x}$$

$$y_p = -\frac{5}{6} - \frac{5}{3}$$

$$y_p = -\frac{5}{2}$$

$y = y_c + y_p$

$$\underline{\underline{y = C_1e^{2x} + e^{-x} - \frac{5}{2}}}$$

2. $\left(D^3 + D\right)y = \csc x$

$a) y_c$:

$$\left(D^3 + D\right)y_c = 0$$

$$m^3 + m = 0$$

$$m\left(m^2 + 1\right) = 0$$

m=0 , $m^2 = -1$

$$m = 0 \pm i$$

$$y_c = C_1e^{0x} + \left(C_2 \cos x + C_3 \sin x\right)e^{0x}$$

$$y_c = C_1 + C_2 \cos x + C_3 \sin x$$

b) $y_p : MVP$

$$y_p = A + B\cos x + C\sin x$$

where A, B and C are parameters

$$Dy_p = A' + [B(-\sin x) + B'\cos x] + [C(\cos x) + C'\sin x]$$

Let: $A' + B'\cos x + C'\sin x = 0 \longrightarrow (1)$

$$D^2y_p = -\left[B(\cos x) + B'\sin x\right] + \left[C(-\sin x) + C'\cos x\right]$$

Let: $-B'\sin x + C'\cos x = 0 \longrightarrow (2)$

$$D^3 y_p = -[B(-\sin x) + B'\cos x] - [C(\cos x) + C'\sin x]$$

Let: $-B'\cos x - C'\sin x = \csc x \longrightarrow (3)$

add (1) and (3)
A'=cscx
integrating
A=ln(cscx-cotx)

from(2), $B' = \dfrac{C'\cos x}{\sin x} \longrightarrow (2)'$
subsitute (2)' in (3)

$$\left[-\frac{C'\cos x}{\sin x} \bullet \cos x - C'\sin x \ = \ \csc x\right]\sin x$$

$$-C'\cos^2 x - C'\sin^2 x = 1$$
$$-C'(\cos^2 x + \sin^2 x) = 1$$
$$-C' = 1$$
$$C' = -1$$

integrating
C=-x
substitute C'=-1 in (2)'

$$B' = \frac{-\cos x}{\sin x}$$

integrating
B=-ln (sinx)

y_p = ln(cscx-cotx)-cosx ln(sinx)-xsinx

y = $y_c + y_p$
$y = C_1 + C_2\cos x + C_3\sin x +$ ln(cscx-cotx)-cosx ln(sinx)-xsinx

Seatwork: Find the general solution of the differential equation.

1. $(D^2 - 6D + 9)\, y = \dfrac{e^{3x}}{x^2}$

   ans: $y = e^{3x}[C_1 + (C_2 x - 1)]$

Homework: Find the general solution of the differential equation.

1. $(D^2 - 4)y = 4x - 3e^x$ ans: $y = e^x - x + c_1e^{2x} + c_2e^{-2x}$

2. $(D^3 - 2D^2 + d)y = x$ ans: $x^2/2 + 2x + 3 + c_1xe^x + c_2e^x + c_3$