1
Triangles

In this chapter we review some of the well-known propositions of elementary geometry, stressing the role of symmetry. We refer to Euclid's propositions by his own numbers, which have been used throughout the world for more than two thousand years. Since the time of F. Commandino (1509–1575), who translated the works of Archimedes, Apollonius, and Pappus, many other theorems in the same spirit have been discovered. Such results were studied in great detail during the nineteenth century. As the present tendency is to abandon them in favor of other branches of mathematics, we shall be content to mention a few that seem particularly interesting.

1.1 EUCLID

Euclid's work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity.

Sir Thomas L. Heath (1861 -1940)*

About 300 B.C., Euclid of Alexandria wrote a treatise in thirteen books called the *Elements*. Of the author (sometimes regrettably confused with the earlier philosopher, Euclid of Megara) we know very little. Proclus (410–485 A.D.) said that he "put together the Elements, collecting many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolemy, [who] once asked him if there was in geometry any shorter way than that of the Elements, and he answered that there was no royal road to geometry." Heath quotes a story by Stobaeus, to the effect that someone who had begun to read geometry with Euclid asked him "What shall I get by learning these things?" Euclid called his slave and said "Give him a dime, since he must make gain out of what he learns."

* Heath **1**, p. vi. (Such references are collected at the end of the book, pp. 415–417.)

Of the thirteen books, the first six may be very briefly described as dealing respectively with triangles, rectangles, circles, polygons, proportion, and similarity. The next four, on the theory of numbers, include two notable achievements: IX.2 and X.9, where it is proved that there are infinitely many prime numbers, and that $\sqrt{2}$ is irrational [Hardy **2**, pp. 32–36]. Book XI is an introduction to solid geometry, XII deals with pyramids, cones, and cylinders, and XIII is on the five regular solids.

According to Proclus, Euclid "set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." This notion of Euclid's purpose is supported by the Platonic theory of a mystical correspondence between the four solids

$$\left.\begin{array}{l}\text{cube,}\\ \text{tetrahedron,}\\ \text{octahedron,}\\ \text{icosahedron}\end{array}\right\}\text{ and the four "elements" }\left\{\begin{array}{l}\text{earth,}\\ \text{fire,}\\ \text{air,}\\ \text{water}\end{array}\right.$$

[cf. Coxeter **1**, p. 18]. Evidence to the contrary is supplied by the arithmetical books VII–X, which were obviously included for their intrinsic interest rather than for any application to solid geometry.

1.2 PRIMITIVE CONCEPTS AND AXIOMS

"When I use a word," Humpty-Dumpty said, "it means just what I choose it to mean—neither more nor less."

Lewis Carroll (1832-1898)
[Dodgson **2**, Chap. 6]

In the logical development of any branch of mathematics, each definition of a concept or relation involves other concepts and relations. Therefore the only way to avoid a vicious circle is to allow certain *primitive* concepts and relations (usually as few as possible) to remain undefined [Synge **1**, pp. 32–34]. Similarly, the proof of each proposition uses other propositions, and therefore certain primitive propositions, called *postulates* or *axioms*, must remain unproved. Euclid did not specify his primitive concepts and relations, but was content to give definitions in terms of ideas that would be familiar to everybody. His five Postulates are as follows:

1.21 *A straight line may be drawn from any point to any other point.*

1.22 *A finite straight line may be extended continuously in a straight line.*

1.23 *A circle may be described with any center and any radius.*

1.24 *All right angles are equal to one another.*

1.25 *If a straight line meets two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.**

It is quite natural that, after a lapse of about 2250 years, some details are now seen to be capable of improvement. (For instance, Euclid I.1 constructs an equilateral triangle by drawing two circles; but how do we know that these two circles will intersect?) The marvel is that so much of Euclid's work remains perfectly valid. In the modern treatment of his geometry [see, for instance, Coxeter **3**, pp. 161–187], it is usual to recognize the primitive concept *point* and the two primitive relations of *intermediacy* (the idea that one point may be between two others) and *congruence* (the idea that the distance between two points may be equal to the distance between two other points, or that two line segments may have the same length). There are also various versions of the axiom of *continuity*, one of which says that every convergent sequence of points has a limit.

Euclid's "principle of superposition," used in proving I.4, raises the question whether a figure can be moved without changing its internal structure. This principle is nowadays replaced by a further explicit assumption such as the axiom of "the rigidity of a triangle with a tail" (Figure 1.2*a*):

1.26 *If ABC is a triangle with D on the side BC extended, while D' is analogously related to another triangle $A'B'C'$, and if $BC = B'C'$, $CA = C'A'$, $AB = A'B'$, $BD = B'D'$, then $AD = A'D'$.*

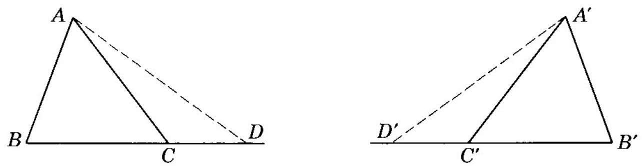

Figure 1.2a

This axiom can be used to extend the notion of congruence from line segments to more complicated figures such as angles, so that we can say precisely what we mean by the relation

$$\angle ABC = \angle A'B'C'.$$

Then we no longer need the questionable principle of superposition in order to prove Euclid I.4:

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal sides equal, they will also have their third sides equal, and their remaining angles equal respectively; in fact, they will be congruent triangles.

* In Chapter 15 we shall see how far we can go without using this unpleasantly complicated Fifth Postulate.

1.3 PONS ASINORUM

Minos: It is proposed to prove I.5 by taking up the isosceles Triangle, turning it over, and then laying it down again upon itself.

Euclid: Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?

Minos: I suppose its defenders would say that it is conceived to leave a trace of itself behind, and that the reversed Triangle is laid down upon the trace so left.

C. L. Dodgson (1832-1898)
[Dodgson **3,** p. 48]

I.5. *The angles at the base of an isosceles triangle are equal.*

The name *pons asinorum* for this famous theorem probably arose from the bridgelike appearance of Euclid's figure (with the construction lines required in his rather complicated proof) and from the notion that anyone unable to cross this bridge must be an ass. Fortunately, a far simpler proof was supplied by Pappus of Alexandria about 340 A.D. (Figure 1.3*a*):

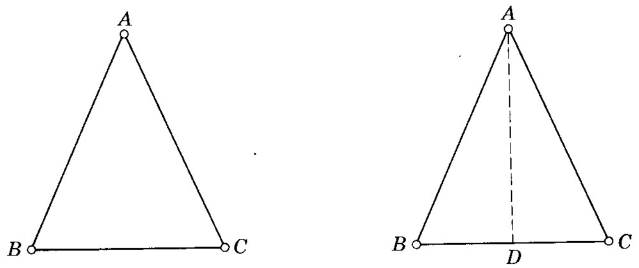

Figure 1.3a

Let ABC be an isosceles triangle with AB equal to AC. Let us conceive this triangle as two triangles and argue in this way. Since $AB = AC$ and $AC = AB$, the two sides AB, AC are equal to the two sides AC, AB. Also the angle BAC is equal to the angle CAB, for it is the same. Therefore all the corresponding parts (of the triangles ABC, ACB) are equal. In particular,

$$\angle ABC = \angle ACB.$$

The pedagogical difficulty of comparing the isosceles triangle ABC with itself is sometimes avoided by joining the apex A to D, the midpoint of the base BC. The median AD may be regarded as a *mirror* reflecting B into C. Accordingly, we say that an isosceles triangle is symmetrical by *reflection*, or that it has *bilateral symmetry*. (Of course, the idealized mirror used in geometry has no thickness and is silvered on both sides, so that it not only reflects B into C but also reflects C into B.)

Any figure, however irregular its shape may be, yields a symmetrical figure when we place it next to a mirror and waive the distinction between object and image. Such bilateral symmetry is characteristic of the external shape of most animals.

Given any point P on either side of a geometrical mirror, we can construct its reflected image P' by drawing the perpendicular from P to the mirror and extending this perpendicular line to an equal distance on the other side, so that the mirror perpendicularly bisects the line segment PP'. Working in the plane (Figure 1.3*b*) with a line AB for mirror, we draw two circles with centers A, B and radii AP, BP. The two points of intersection of these circles are P and its image P'.

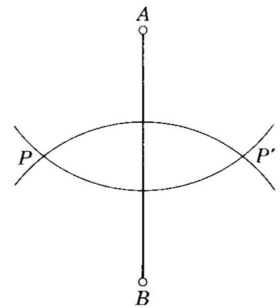

Figure 1.3b

We shall find that many geometrical proofs are shortened and made more vivid by the use of reflections. But we must remember that this procedure is merely a short cut: every such argument could have been avoided by means of a circumlocution involving congruent triangles. For instance, the above construction is valid because the triangles ABP, ABP' are congruent.

Pons asinorum has many useful consequences, such as the following five:

III.3. *If a diameter of a circle bisects a chord which does not pass through the center, it is perpendicular to it; or, if perpendicular to it, it bisects it.*

III.20. *In a circle the angle at the center is double the angle at the circumference, when the rays forming the angles meet the circumference in the same two points.*

III.21. *In a circle, a chord subtends equal angles at any two points on the same one of the two arcs determined by the chord* (e.g., in Figure 1.3*c*, $\angle PQQ' = \angle PP'Q'$).

III.22. *The opposite angles of any quadrangle inscribed in a circle are together equal to two right angles.*

III.32. *If a chord of a circle be drawn from the point of contact of a tangent, the angle made by the chord with the tangent is equal to the angle subtended by the chord at a point on that part of the circumference which lies on the far side of the chord* (e.g., in Figure 1.3*c*, $\angle OTP' = \angle TPP'$).

We shall also have occasion to use two familiar theorems on similar triangles:

VI.2. *If a straight line be drawn parallel to one side of a triangle, it will cut the other sides proportionately; and, if two sides of the triangle be cut proportionately, the line joining the points of section will be parallel to the remaining side.*

VI.4. *If corresponding angles of two triangles are equal, then corresponding sides are proportional.*

Combining this last result with III.21 and 32, we deduce two significant properties of secants of a circle (Figure 1.3*c*):

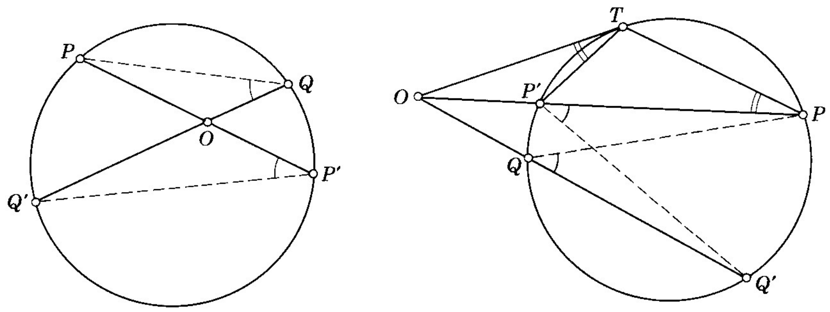

Figure 1.3c

III.35. *If in a circle two straight lines cut each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other* (i.e., $OP \times OP' = OQ \times OQ'$).

III.36. *If from a point outside a circle a secant and a tangent be drawn, the rectangle contained by the whole secant and the part outside the circle will be equal to the square on the tangent* (i.e., $OP \times OP' = OT^2$).

Book VI also contains an important property of area:

VI.19. *Similar triangles are to one another in the squared ratio of their corresponding sides* (i.e., if ABC and $A'B'C'$ are similar triangles, their areas are in the ratio $AB^2 : A'B'^2$).

This result yields the following easy proof for the theorem of Pythagoras [see Heath **1**, p. 353; **2**, pp. 210, 232, 269]:

I.47. *In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the two catheti.*

In the triangle ABC, right-angled at C, draw CF perpendicular to the hypotenuse AB, as in Figure 1.3*d*. Then we have three similar right-angled triangles ABC, ACF, CBF, with hypotenuses AB, AC, CB. By VI.19, the areas satisfy

$$\frac{ABC}{AB^2} = \frac{ACF}{AC^2} = \frac{CBF}{CB^2}.$$

Evidently, $ABC = ACF + CBF$. Therefore $AB^2 = AC^2 + CB^2$.

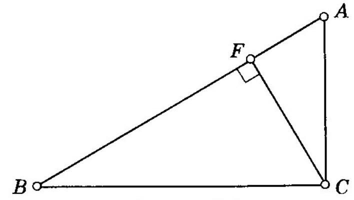

Figure 1.3d

EXERCISES

1. Using rectangular Cartesian coordinates, show that the reflection in the y-axis ($x = 0$) reverses the sign of x. What happens when we reflect in the line $x = y$?

2. Deduce I.47 from III.36 (applied to the circle with center A and radius AC).

3. Inside a square $ABDE$, take a point C so that CDE is an isosceles triangle with angles 15° at D and E. What kind of triangle is ABC?

4. Prove the Erdös-Mordell theorem: If O is any point inside a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon the respective sides BC, CA, AB, then

$$OA + OB + OC \geqslant 2(OP + OQ + OR).$$

(*Hint:** Let P_1 and P_2 be the feet of the perpendiculars from R and Q upon BC. Define analogous points Q_1 and Q_2, R_1 and R_2 on the other sides. Using the similarity of the triangles PRP_1 and OBR, express P_1P in terms of RP, OR, and OB. After substituting such expressions into

$$OA + OB + OC \geqslant OA(P_1P + PP_2)/RQ + OB(Q_1Q + QQ_2)/PR + OC(R_1R + RR_2)/QP,$$

collect the terms involving OP, OQ, OR, respectively.)

5. Under what circumstances can the sign $\geqslant$ in Ex. 4 be replaced by $=$?

6. In the notation of Ex. 4,

$$OA \times OB \times OC \geqslant (OQ + OR)(OR + OP)(OP + OQ).$$

(A. Oppenheim, *American Mathematical Monthly*, **68** (1961), p. 230. See also L. J. Mordell, *Mathematical Gazette*, **46** (1962), pp. 213–215.)

7. Prove the Steiner-Lehmus theorem: Any triangle having two equal internal angle bisectors (each measured from a vertex to the opposite side) is isosceles. (*Hint:*† If a triangle has two different angles, the smaller angle has the longer internal bisector.)

* Leon Bankoff, *American Mathematical Monthly*, **65** (1958), p. 521. For other proofs see G. R. Veldkamp and H. Brabant, *Nieuw Tijdschrift voor Wiskunde*, **45** (1958), pp. 193–196; **46** (1959), p. 87.

† Court **2**, p. 72. For Lehmus's proof of 1848, see Coxeter and Greitzer **1**, p. 15.

1.4 THE MEDIANS AND THE CENTROID

Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. . . . The Greeks, as Littlewood said to me once, are not clever schoolboys or "scholarship candidates," but "Fellows of another college." So Greek mathematics is "permanent," more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.

G. H. Hardy (1877-1947)
[Hardy **2,** p. 21]

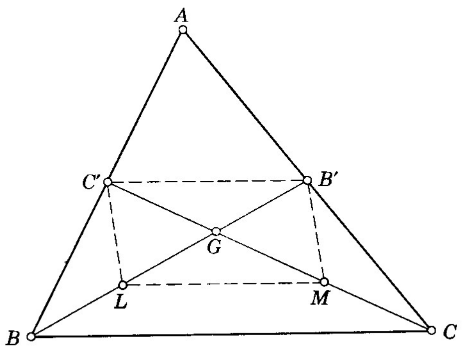

Figure 1.4a

The line joining a vertex of a triangle to the midpoint of the opposite side is called a *median*.

Let two of the three medians, say BB' and CC', meet in G (Figure 1.4*a*). Let L and M be the midpoints of GB and GC. By Euclid VI.2 and 4 (which were quoted on page 8), both $C'B'$ and LM are parallel to BC and half as long. Therefore $B'C'LM$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we have

$$B'G = GL = LB, \qquad C'G = GM = MC.$$

Thus the two medians BB', CC' trisect each other at G. In other words, this point G, which could have been defined as a point of trisection of one median, is also a point of trisection of another, and similarly of the third. We have thus proved [by the method of Court **1**, p. 58] the following theorem:

1.41 *The three medians of any triangle all pass through one point.*

This common point G of the three medians is called the *centroid* of the triangle. Archimedes (*c.* 287–212 B.C.) obtained it as the center of gravity of a triangular plate of uniform density.

EXERCISES

1. Any triangle having two equal medians is isosceles.*
2. The sum of the medians of a triangle lies between $\frac{3}{4}$p and p, where p is the sum of the sides. [Court **1**, pp. 60–61.]

1.5 THE INCIRCLE AND THE CIRCUMCIRCLE

Alone at nights,
I read my Bible more and Euclid less.

Robert Buchanan (1841-1901)
(An Old Dominie's Story)

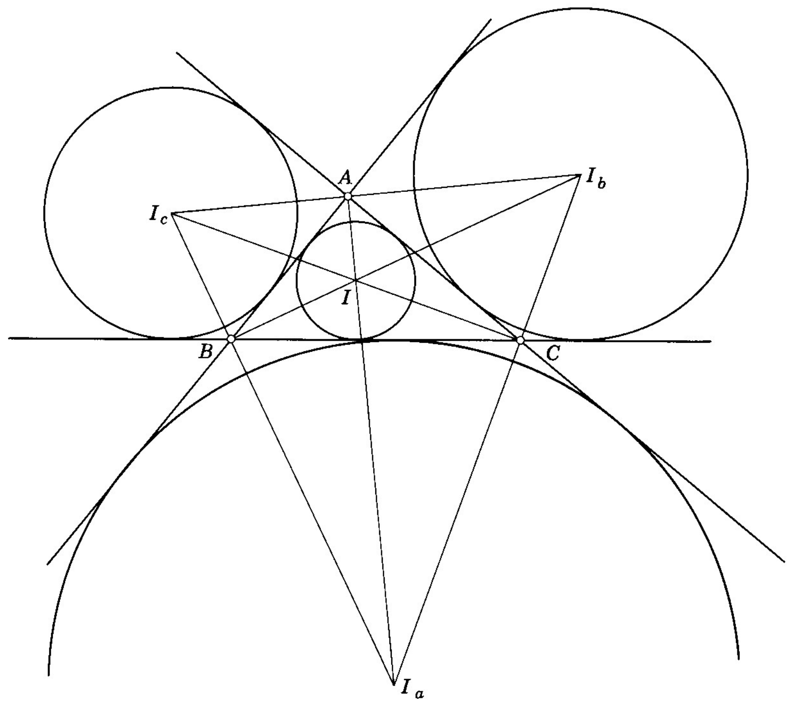

Figure 1.5a

Euclid III.3 tells us that a circle is symmetrical by reflection in any diameter (whereas an ellipse is merely symmetrical about two special diameters: the major and minor axes). It follows that the angle between two intersecting tangents is bisected by the diameter through their common point.

* It is to be understood that any exercise appearing in the form of a theorem is intended to be *proved*. It saves space to omit the words "Prove that" or "Show that."

By considering the loci of points equidistant from pairs of sides of a triangle ABC, we see that the internal and external bisectors of the three angles of the triangle meet by threes in four points I, I_a, I_b, I_c, as in Figure 1.5*a*. These points are the centers of the four circles that can be drawn to touch the three lines BC, CA, AB. One of them, the *incenter I*, being inside the triangle, is the center of the inscribed circle or *incircle* (Euclid IV.4). The other three are the *excenters* I_a, I_b, I_c: the centers of the three escribed circles or *excircles* [Court **2,** pp. 72–88]. The radii of the incircle and excircles are the *inradius* r and the *exradii* r_a, r_b, r_c.

In describing a triangle ABC, it is customary to call the sides

$$a = BC, \qquad b = CA, \qquad c = AB,$$

the semiperimeter

$$s = \tfrac{1}{2}(a + b + c),$$

the angles A, B, C, and the area Δ.

Since $A + B + C = 180°$, we have

1.51 $$\angle BIC = 90° + \tfrac{1}{2}A,$$

a result which we shall find useful in § 1.9.

Since IBC is a triangle with base a and height r, its area is $\tfrac{1}{2}ar$. Adding three such triangles we deduce

$$\Delta = \tfrac{1}{2}(a + b + c)r = sr.$$

Similarly $\Delta = \tfrac{1}{2}(b + c - a)r_a = (s - a)r_a$. Thus

1.52 $$\Delta = sr = (s - a)r_a = (s - b)r_b = (s - c)r_c.$$

From the well-known formula $\cos A = (b^2 + c^2 - a^2)/2bc$, we find also

$$\sin A = [-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2]^{\frac{1}{2}}/2bc,$$

whence

1.53
$$\begin{aligned}\Delta &= \tfrac{1}{2}bc\sin A\\ &= \tfrac{1}{4}[-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2]^{\frac{1}{2}}\\ &= \tfrac{1}{4}[(a + b + c)(-a + b + c)(a - b + c)(a + b - c)]^{\frac{1}{2}}\\ &= [s(s - a)(s - b)(s - c)]^{\frac{1}{2}}.\end{aligned}$$

This remarkable expression, which we shall use in § 18.4, is attributed to Heron of Alexandria (about 60 A.D.), but it was really discovered by Archimedes. (See B. L. van der Waerden, *Science Awakening*, Oxford University Press, New York, 1961, pp. 228, 277.) Combining Heron's formula with 1.52, we obtain

1.531 $$r^2 = \left(\frac{\Delta}{s}\right)^2 = \frac{(s - a)(s - b)(s - c)}{s}, \quad r_a^2 = \left(\frac{\Delta}{s - a}\right)^2 = \frac{s(s - c)(s - b)}{s - a}.$$

Another consequence of the symmetry of a circle is that the perpendicular bisectors of the three sides of a triangle all pass through the *circumcenter O*,

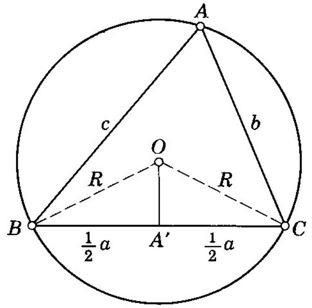

Figure 1.5b

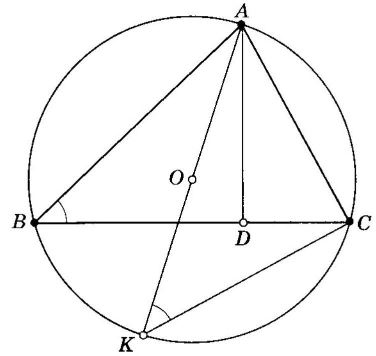

Figure 1.5c

which is the center of the circumscribed circle or *circumcircle* (Euclid IV.5). This is the only circle that can be drawn through the three vertices A, B, C. Its radius R is called the *circumradius* of the triangle. Since the "angle at the center," $\angle BOC$ (Figure 1.5*b*), is double the angle A, the congruent right-angled triangles OBA', OCA' each have an angle A at O, whence

1.54
$$R\sin A = BA' = \tfrac{1}{2}a,$$
$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Draw AD perpendicular to BC, and let AK be the diameter through A of the circumcircle, as in Figure 1.5*c*. By Euclid III.21, the right-angled triangles ABD and AKC are similar; therefore

$$\frac{AD}{AB} = \frac{AC}{AK}, \quad AD = \frac{bc}{2R}.$$

Since $\Delta = \tfrac{1}{2}BC \times AD$, it follows that

1.55
$$\begin{aligned}4\Delta R &= abc\\ &= s(s - b)(s - c) + s(s - c)(s - a) + s(s - a)(s - b)\\ &\qquad - (s - a)(s - b)(s - c)\\ &= \frac{\Delta^2}{s - a} + \frac{\Delta^2}{s - b} + \frac{\Delta^2}{s - c} - \frac{\Delta^2}{s}\\ &= \Delta(r_a + r_b + r_c - r).\end{aligned}$$

Hence the five radii are connected by the formula

1.56 $$4R = r_a + r_b + r_c - r.$$

Let us now consider four circles E_1, E_2, E_3, E_4, tangent to one another at six distinct points. Each circle E_i has a *bend* ε_i, defined as the reciprocal of its radius with a suitable sign attached, namely, if all the contacts are external (as in the case of the light circles in Figure 1.5*d*), the bends are all positive, but if one circle surrounds the other three (as in the case of the

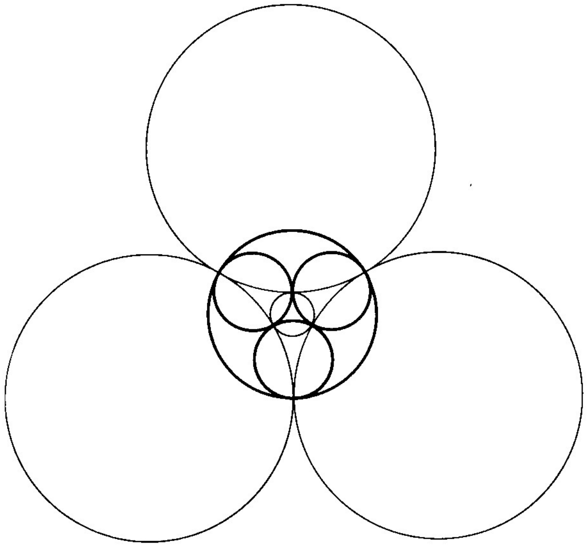

Figure 1.5d

heavy circles) the bend of this largest circle is taken to be negative; and a line counts as a circle of bend 0. In any case, the sum of all four bends is positive.

In a letter of November 1643 to Princess Elisabeth of Bohemia, René Descartes developed a formula relating the radii of four mutually tangent circles. In the "bend" notation it is

1.57 $$2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2.$$

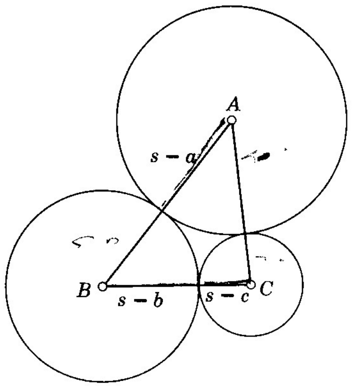

Figure 1.5e

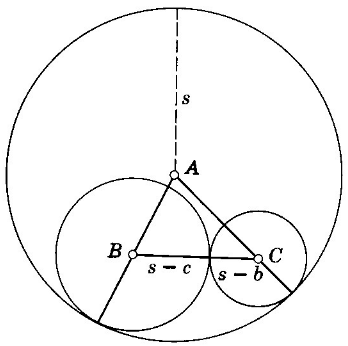

Figure 1.5f

This *Descartes circle theorem* was rediscovered in 1842 by an English amateur, Philip Beecroft, who observed that the four circles E_i determine another set of four circles H_i, mutually tangent at the same six points: H_1 through the three points of contact of E_2, E_3, E_4, and so on. Let η_i denote the bend of H_i. If the centers of E_1, E_2, E_3 form a triangle ABC, H_4 is either the incircle or an excircle. In the former case (Figure 1.5*e*),

1.58 $$\varepsilon_1 = \frac{1}{s-a},\quad \varepsilon_2 = \frac{1}{s-b},\quad \varepsilon_3 = \frac{1}{s-c},\quad \eta_4 = \mp\frac{1}{r}.$$

In the latter (Figure 1.5*f*),

$$\varepsilon_1 = -\frac{1}{s},\quad \varepsilon_2 = \frac{1}{s-c},\quad \varepsilon_3 = \frac{1}{s-b},\quad \eta_4 = \pm\frac{1}{r_a}.$$

In either case, we see from 1.531 that

$$\varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2 = \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\varepsilon_1\varepsilon_2\varepsilon_3 = \eta_4^2.$$

Similarly $\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2 = \varepsilon_4^2$, and of course we can permute the subscripts 1, 2, 3, 4. Hence

$$(\Sigma\varepsilon_i)^2 = \varepsilon_1^2 + \ldots + \varepsilon_4^2 + 2\varepsilon_1\varepsilon_2 + \ldots + 2\varepsilon_3\varepsilon_4 = \Sigma\varepsilon_i^2 + \Sigma\eta_i^2.$$

Since this expression involves ε_i and η_i symmetrically, it is also equal to $(\Sigma\eta_i)^2$; thus

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = \eta_1 + \eta_2 + \eta_3 + \eta_4 > 0.$$

Also, since

$$\begin{aligned}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) &= (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 - \varepsilon_4^2 \\ &= \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 - \varepsilon_4^2 + 2\eta_4^2 \\ &= (\eta_2\eta_3 + \eta_2\eta_4 + \eta_3\eta_4) + (\eta_1\eta_3 + \ldots) + (\eta_1\eta_2 + \ldots) - (\eta_1\eta_2 + \ldots) + 2\eta_4^2 \\ &= 2(\eta_1\eta_4 + \eta_2\eta_4 + \eta_3\eta_4) + 2\eta_4^2 = 2\eta_4(\eta_1 + \eta_2 + \eta_3 + \eta_4),\end{aligned}$$

1.59 $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 = 2\eta_4.$$

Adding four such equations after squaring each side, we deduce $\Sigma\varepsilon_i^2 = \Sigma\eta_i^2$, whence

$$2\Sigma\varepsilon_i^2 = \Sigma\varepsilon_i^2 + \Sigma\eta_i^2 = (\Sigma\varepsilon_i)^2.$$

Thus 1.57 has been proved.

In 1936, this theorem was rediscovered again by Sir Frederick Soddy, who had received a Nobel prize in 1921 for his discovery of isotopes. He expressed the theorem in the form of a poem, *The Kiss Precise**, of which the middle verse runs as follows:

Four circles to the kissing come,
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb

* *Nature*, **137** (1936), p. 1021; **139** (1937), p. 62. In the next verse, Soddy announced his discovery of the analogous formula for 5 spheres in 3 dimensions. A final verse, added by Thorold Gosset (1869–1962) deals with $n+2$ spheres in n dimensions; see Coxeter, *Aequationes Mathematicae*, **1** 1968), pp. 104–121.

There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

EXERCISES

1. Find the locus of the image of a fixed point P by reflection in a variable line through another fixed point O.

2. For the general triangle ABC, establish the identities

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}, \quad r r_a r_b r_c = \Delta^2.$$

3. The lengths of the tangents from the vertex A to the incircle and to the three excircles are respectively

$$s-a, \quad s, \quad s-c, \quad s-b.$$

4. The circumcenter of an obtuse-angled triangle lies outside the triangle.

5. Where is the circumcenter of a right-angled triangle?

6. Let U, V, W be three points on the respective sides BC, CA, AB of a triangle ABC. The perpendiculars to the sides at these points are concurrent if and only if

$$AW^2 + BU^2 + CV^2 = WB^2 + UC^2 + VA^2.$$

7. A triangle is right-angled if and only if $r + 2R = s$.

8. The bends of Beecroft's eight circles satisfy

$$\varepsilon_1+\eta_1=\varepsilon_2+\eta_2=\varepsilon_3+\eta_3=\varepsilon_4+\eta_4, \quad \Sigma\varepsilon_i\eta_i=0.$$

9. For any four numbers satisfying $k+l+m+n=0$, there is a "Beecroft configuration" having bends

$$\varepsilon_1=k(k+l),\ \varepsilon_2=(k+l)l,\quad \varepsilon_3=n^2-kl,\quad \varepsilon_4=m^2-kl,$$
$$\eta_1=l^2-mn,\ \eta_2=k^2-mn,\ \eta_3=m(m+n),\ \eta_4=(m+n)n.$$

(*Hint:* Express $\varepsilon_3, \varepsilon_4, \eta_1, \eta_2$ as rational functions of $\varepsilon_1, \varepsilon_2, \eta_3, \eta_4$.)

10. If three circles, externally tangent to one another, have centers forming a triangle ABC, they are all tangent to two other circles (or possibly a circle and a line) whose bends are

$$\frac{r+4R\pm 2s}{\Delta}.$$

11. Given a point P on the circumcircle of a triangle, the feet of the perpendiculars from P to the three sides all lie on a straight line. (This line is commonly called the *Simson line* of P with respect to the triangle, although it was first mentioned by W. Wallace, thirty years after Simson's death [Johnson **1**, p. 138].)

12. Given a triangle ABC and a point P in its plane (but not on a side nor on the circumcircle), let $A_1B_1C_1$ be the derived triangle formed by the feet of the perpendiculars from P to the sides BC, CA, AB. Let $A_2B_2C_2$ be derived analogously from $A_1B_1C_1$ (using the same P), and $A_3B_3C_3$ from $A_2B_2C_2$. Then $A_3B_3C_3$ is directly similar to ABC. [Casey **1**, p. 253.] (*Hint:* $\angle PBA = \angle PA_1C_1 = \angle PC_2B_2 = \angle PB_3A_3$.) This result has been extended by B. M. Stewart from the third derived triangle of a triangle to the nth derived n-gon of an n-gon. (*American Mathematical Monthly* **47** (1940), pp. 462–466).

1.6 THE EULER LINE AND THE ORTHOCENTER

Although the Greeks worked fruitfully, not only in geometry but also in the most varied fields of mathematics, nevertheless we today have gone beyond them everywhere and certainly also in geometry.

F. Klein (1849-1925)
[Klein **2**, p. 189]

From now on, we shall have various occasions to mention the name of L. Euler (1707–1783), a Swiss who spent most of his life in Russia, making important contributions to all branches of mathematics. Some of his simplest discoveries are of such a nature that one can well imagine the ghost of Euclid saying, "Why on earth didn't I think of that?"

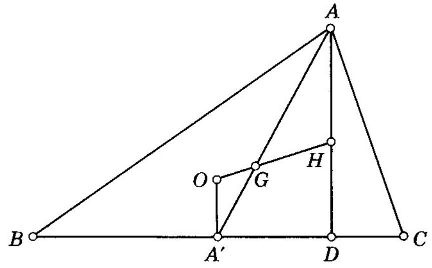

Figure 1.6a

If the circumcenter O and centroid G of a triangle coincide, each median is perpendicular to the side that it bisects, and the triangle is "isosceles three ways," that is, equilateral. Hence, if a triangle ABC is not equilateral, its circumcenter and centroid lie on a unique line OG. On this so-called *Euler line,* consider a point H such that $OH = 3OG$, that is, $GH = 2OG$ (Figure 1.6*a*). Since also $GA = 2A'G$, the latter half of Euclid VI.2 tells us that AH is parallel to $A'O$, which is the perpendicular bisector of BC. Thus AH is perpendicular to BC. Similarly BH is perpendicular to CA, and CH to AB.

The line through a vertex perpendicular to the opposite side is called an *altitude.* The above remarks [cf. Court **2**, p. 101] show that

The three altitudes of any triangle all pass through one point on the Euler line.

This common point H of the three altitudes is called the *orthocenter* of the triangle.

EXERCISES

1. Through each vertex of a given triangle ABC draw a line parallel to the opposite side. The perpendicular bisectors of the sides of the triangle so formed suggest an alternative proof that the three altitudes of ABC are concurrent.

2. The orthocenter of an obtuse-angled triangle lies outside the triangle.

3. Where is the orthocenter of a right-angled triangle?
4. Any triangle having two equal altitudes is isosceles.
5. Construct an isosceles triangle ABC (with base BC), given the median BB' and the altitude BE. (*Hint:* The centroid is two-thirds of the way from B to B'.) (H. Freudenthal.)
6. The altitude AD of any triangle ABC is of length

$$2R \sin B \sin C.$$

7. Find the perpendicular distance from the centroid G to the side BC.
8. If the Euler line passes through a vertex, the triangle is either right-angled or isosceles (or both).
9. If the Euler line is parallel to the side BC, the angles B and C satisfy

$$\tan B \tan C = 3.$$

1.7 THE NINE-POINT CIRCLE

This circle is the first really exciting one to appear in any course on elementary geometry.

Daniel Pedoe (1910 -)
[Pedoe **1**, p. 1]

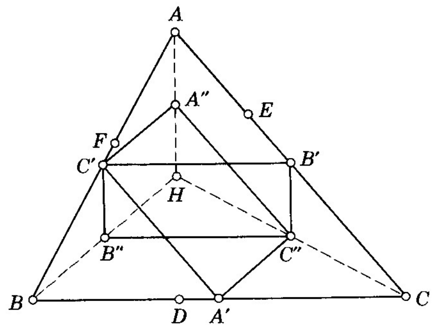

Figure 1.7a

The feet of the altitudes (that is, three points like D in Figure 1.6*a*) form the *orthic triangle* (or "pedal triangle") of ABC. The circumcircle of the orthic triangle is called the *nine-point circle* (or "Feuerbach circle") of the original triangle, because it contains not only the feet of the three altitudes but also six other significant points. In fact,

1.71 *The midpoints of the three sides, the midpoints of the lines joining the orthocenter to the three vertices, and the feet of the three altitudes, all lie on a circle.*

Proof [Coxeter **2,** 9.29]. Let $A', B', C', A'', B'', C''$ be the midpoints of BC, CA, AB, HA, HB, HC, and let D, E, F be the feet of the altitudes, as in Figure 1.7*a*. By Euclid VI.2 and 4 again, both $C'B'$ and $B''C''$ are parallel to BC while both $B'C''$ and $C'B''$ are parallel to AH. Since AH is perpendicular to BC, it follows that $B'C'B''C''$ is a rectangle. Similarly $C'A'C''A''$ is a rectangle. Hence $A'A''$, $B'B''$, $C'C''$ are three diameters of a circle. Since these diameters subtend right angles at D, E, F, respectively, the same circle passes through these points too.

If four points in a plane are joined in pairs by six distinct lines, they are called the *vertices* of a *complete quadrangle,* and the lines are its six *sides.* Two sides are said to be *opposite* if they have no common vertex. Any point of intersection of two opposite sides is called a *diagonal point.* There may be as many as three such points (see Figure 1.7*b*).

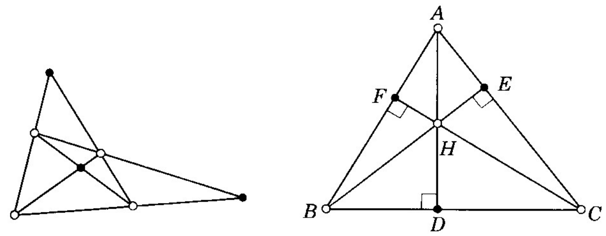

Figure 1.7b

If a triangle ABC is not right-angled, its vertices and orthocenter form a special kind of quadrangle whose opposite sides are perpendicular. In this terminology, the concurrence of the three altitudes can be expressed as follows:

1.72 *If two pairs of opposite sides of a complete quadrangle are pairs of perpendicular lines, the remaining sides are likewise perpendicular.*

Such a quadrangle $ABCH$ is called an *orthocentric quadrangle.* Its six sides

$$BC, \quad CA, \quad AB, \quad HA, \quad HB, \quad HC$$

are the sides and altitudes of the triangle ABC, and its diagonal points D, E, F are the feet of the altitudes. Among the four vertices of the quadrangle, our notation seems to give a special role to the vertex H. Clearly, however,

1.73 *Each vertex of an orthocentric quadrangle is the orthocenter of the triangle formed by the remaining three vertices.*

The four triangles (just one of which is acute-angled) all have the same orthic triangle and consequently the same nine-point circle.

It is proved in books on affine geometry [such as Coxeter **2,** 8.71] that the midpoints of the six sides of any complete quadrangle and the three diagonal points all lie on a conic. The above remarks show that, when the quadrangle is orthocentric, this "nine-point conic" reduces to a circle.

EXERCISES

1. Of the nine points described in 1.71, how many coincide when the triangle is (*a*) isosceles, (*b*) equilateral?

2. The feet of the altitudes decompose the nine-point circle into three arcs. If the triangle is scalene, the remaining six of the nine points are distributed among the three arcs as follows: One arc contains just one of the six points, another contains two, and the third contains three.

3. On the arc $A'D$ of the nine-point circle, take the point X one-third of the way from A' to D. Take points Y, Z similarly, on the arcs $B'E$, $C'F$. Then XYZ is an equilateral triangle.

4. The incenter and the excenters of any triangle form an orthocentric quadrangle. [Casey **1**, p. 274.]

5. In the notation of § 1.5, the Euler line of $I_aI_bI_c$ is IO.

6. The four triangles that occur in an orthocentric quadrangle have equal circumradii.

1.8 TWO EXTREMUM PROBLEMS

Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music.

G. H. Hardy [**2**, p. 26]

Their interest will be stimulated if only we can eliminate the aversion toward mathematics that so many have acquired from childhood experiences.

Hans Rademacher (1892-)
[Rademacher and Toeplitz **1**, p. 5]

We shall describe the problems of Fagnano and Fermat in considerable detail because of the interesting methods used in solving them. The first was proposed in 1775 by J. F. Toschi di Fagnano, who solved it by means of differential calculus. The method given here was discovered by L. Fejér while he was a student [Rademacher and Toeplitz **1**, pp. 30–32].

FAGNANO'S PROBLEM. *In a given acute-angled triangle ABC, inscribe a triangle UVW whose perimeter is as small as possible.*

Consider first an arbitrary triangle UVW with U on BC, V on CA, W on AB. Let U', U'' be the images of U by reflection in CA, AB, respectively. Then

$$UV + VW + WU = U'V + VW + WU'',$$

which is a path from U' to U'', usually a broken line with angles at V and W. Such a path from U' to U'' is minimal when it is straight, as in Figure 1.8*a*.

Hence, among all inscribed triangles with a given vertex U on BC, the one with smallest perimeter occurs when V and W lie on the straight line $U'U''$. In this way we obtain a definite triangle UVW for each choice of U on BC. The problem will be solved when we have chosen U so as to minimize $U'U''$, which is equal to the perimeter.

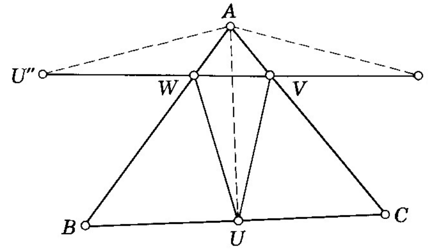

Figure 1.8a

Since AU' and AU'' are images of AU by reflection in AC and AB, they are congruent and

$$\angle U'AU'' = 2A.$$

Thus $AU'U''$ is an isosceles triangle whose angle at A is independent of the choice of U. The base $U'U''$ is minimal when the equal sides are minimal, that is, when AU is minimal. In other words, AU is the shortest distance from the given point A to the given line BC. Since the hypotenuse of a right-angled triangle is longer than either cathetus, the desired location of U is such that AU is perpendicular to BC. Thus AU is the altitude from A.

This choice of U yields a unique triangle UVW whose perimeter is smaller than that of any other inscribed triangle. Since we could equally well have begun with B or C instead of A, we see that BV and CW are the altitudes from B and C. Hence

The triangle of minimal perimeter inscribed in an acute-angled triangle ABC is the orthic triangle of ABC.

The same method can be used to prove the analogous result for spherical triangles [Steiner **2**, p. 45, No. 7].

The other problem, proposed by Pierre Fermat (1601–1665), likewise seeks to minimize the sum of three distances. The solution given here is due to J. E. Hofmann.*

FERMAT'S PROBLEM. *In a given acute-angled triangle ABC, locate a point P whose distances from A, B, C have the smallest possible sum.*

Consider first an arbitrary point P inside the triangle. Join it to A, B, C and rotate the inner triangle APB through 60° about B to obtain $C'P'B$, so that ABC' and PBP' are equilateral triangles, as in Figure 1.8*b*. Then

$$AP + BP + CP = C'P' + P'P + PC,$$

* Elementare Lösung einer Minimumsaufgabe, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, **60** (1929), pp. 22–23.

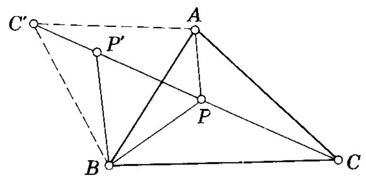

Figure 1.8b

which is a path from C' to C, usually a broken line with angles at P' and P. Such a path (joining C' to C by a sequence of three segments) is minimal when it is straight, in which case

$$\angle BPC = 180^\circ - \angle BPP' = 120^\circ$$

and $$\angle APB = \angle C'P'B = 180^\circ - \angle PP'B = 120^\circ.$$

Thus the desired point P, for which $AP + BP + CP$ is minimal, is the point from which each of the sides BC, CA, AB subtends an angle of 120°. This "Fermat point" is most simply constructed as the second intersection of the line CC' and the circle ABC' (that is, the circumcircle of the equilateral triangle ABC').

It has been pointed out [for example by Pedoe **1**, pp. 11–12] that the triangle ABC need not be assumed to be acute-angled. The above solution is valid whenever there is no angle greater than 120°.

Instead of the equilateral triangle ABC' on AB, we could just as well have drawn an equilateral triangle BCA' on BC, or CAB' on CA, as in Figure 1.8c. Thus the three lines AA', BB', CC' all pass through the Fermat point P, and any two of them provide an alternative construction for it. Moreover, the line segments AA', BB', CC' are all equal to $AP + BP + CP$. Hence

If equilateral triangles BCA', CAB', ABC' are drawn outwards on the sides of any triangle ABC, the line segments AA', BB', CC' are equal, concurrent, and inclined at 60° *to one another.*

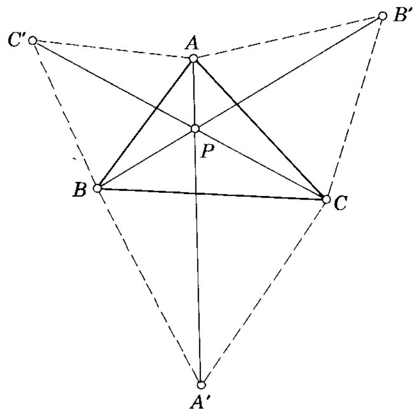

Figure 1.8c

EXERCISES

1. In Figure 1.8*a*, UV and VW make equal angles with CA. Deduce that the orthocenter of any triangle is the incenter of its orthic triangle. (In other words, if ABC is a triangular billiard table, a ball at U, hit in the direction UV, will go round the triangle UVW indefinitely, that is, until it is stopped by friction.)

2. How does Fagnano's problem collapse when we try to apply it to a triangle ABC in which the angle A is obtuse?

3. The circumcircles of the three equilateral triangles in Figure 1.8*c* all pass through P, and their centers form a fourth equilateral triangle.*

4. Three holes, at the vertices of an arbitrary triangle, are drilled through the top of a table. Through each hole a thread is passed with a weight hanging from it below the table. Above, the three threads are all tied together and then released. If the three weights are all equal, where will the knot come to rest?

5. Four villages are situated at the vertices of a square of side one mile. The inhabitants wish to connect the villages with a system of roads, but they have only enough material to make $\sqrt{3} + 1$ miles of road. How do they proceed? [Courant and Robbins **1**, p. 392.]

6. Solve Fermat's problem for a triangle ABC with $A > 120^\circ$, and for a convex quadrangle $ABCD$.

7. If two points P, P', inside a triangle ABC, are so situated that $\angle CBP = \angle PBP' = \angle P'BA$, $\angle ACP' = \angle P'CP = \angle PCB$, then $\angle BP'P = \angle PP'C$.

8. If four squares are placed externally (or internally) on the four sides of any parallelogram, their centers are the vertices of another square. [Yaglom **1**, pp. 96–97.]

9. Let X, Y, Z be the centers of squares placed externally on the sides BC, CA, AB of a triangle ABC. Then the segment AX is congruent and perpendicular to YZ (also BY to ZX and CZ to XY). (W. A. J. Luxemburg.)

10. Let Z, X, U, V be the centers of squares placed externally on the sides AB, BC, CD, DA of any simple quadrangle (or "quadrilateral") $ABCD$. Then the segment ZU (joining the centers of two "opposite" squares) is congruent and perpendicular to XV. [Forder **2**, p. 40.]

1.9 MORLEY'S THEOREM

Many of the proofs in mathematics are very long and intricate. Others, though not long, are very ingeniously constructed.

E. C. Titchmarsh (1899-1963)
[Titchmarsh **1**, p. 23]

One of the most surprising theorems in elementary geometry was discovered about 1899 by F. Morley (whose son Christopher wrote novels such as *Thunder on the Left*). He mentioned it to his friends, who spread it over

* Court [**1**, pp. 105–107]. See also *Mathesis* 1938, p. 293 (footnote, where this theorem is attributed to Napoleon); and Forder [**2**, p. 40] for some interesting generalizations.

the world in the form of mathematical gossip. At last, after ten years, a trigonometrical proof by M. Satyanarayana and an elementary proof by M. T. Naraniengar were published.*

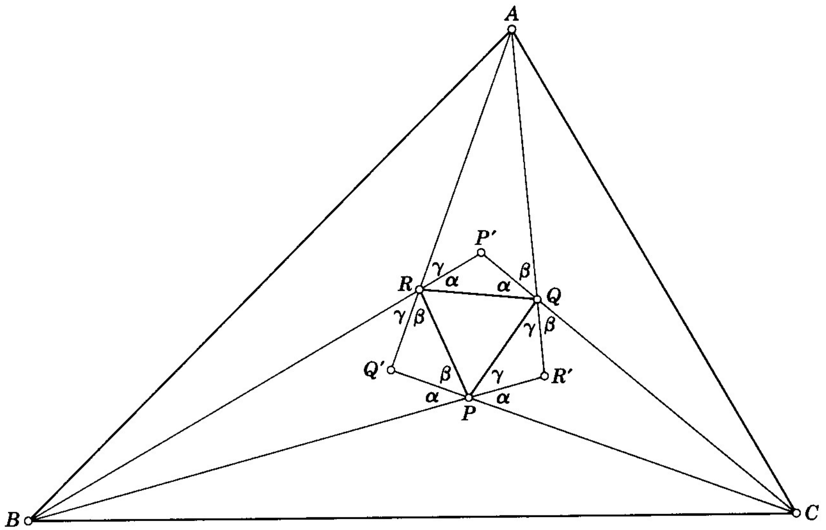

Figure 1.9a

MORLEY'S THEOREM. *The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.*

In other words, any triangle ABC yields an equilateral triangle PQR if the angles A, B, C are trisected by AQ and AR, BR and BP, CP and CQ, as in Figure 1.9*a*. (Much trouble is experienced if we try a direct approach, but the difficulties disappear if we work backwards, beginning with an equilateral triangle and building up a general triangle which is afterwards identified with the given triangle ABC.)

On the respective sides QR, RP, PQ of a given equilateral triangle PQR, erect isosceles triangles $P'QR$, $Q'RP$, $R'PQ$ whose base angles α, β, γ satisfy the equation and inequalities

$$\alpha + \beta + \gamma = 120°, \quad \alpha < 60°, \quad \beta < 60°, \quad \gamma < 60°.$$

* *Mathematical Questions and their Solutions from the Educational Times* (New Series), **15** (1909), pp. 23–24, 47. See also C. H. Chepmell and R. F. Davis, *Mathematical Gazette*, **11** (1923), pp. 85–86; F. Morley, *American Journal of Mathematics*, **51** (1929), pp. 465–472, H. D. Grossman, *American Mathematical Monthly*, **50** (1943), p. 552, and L. Bankoff, *Mathematics Magazine*, **35** (1962), pp. 223–224. The treatment given here is due to Raoul Bricard, *Nouvelles Annales de Mathématiques* (5), **1** (1922), pp. 254–258. A similar proof was devised independently by Bottema [**1**, p. 34].

Extend the sides of the isosceles triangles below their bases until they meet again in points A, B, C. Since $\alpha + \beta + \gamma + 60° = 180°$, we can immediately infer the measurement of some other angles, as marked in Figure 1.9*a*. For instance, the triangle AQR must have an angle $60° - \alpha$ at its vertex A, since its angles at Q and R are $\alpha + \beta$ and $\gamma + \alpha$.

Referring to 1.51, we see that one way to characterize the incenter I of a triangle ABC is to describe it as lying on the bisector of the angle A at such a distance that

$$\angle BIC = 90° + \tfrac{1}{2}A.$$

Applying this principle to the point P in the triangle $P'BC$, we observe that the line PP' (which is a median of both the equilateral triangle PQR and the isosceles triangle $P'QR$) bisects the angle at P'. Also the half angle at P' is $90° - \alpha$, and

$$\angle BPC = 180° - \alpha = 90° + (90° - \alpha).$$

Hence P is the incenter of the triangle $P'BC$. Likewise Q is the incenter of $Q'CA$, and R of $R'AB$. Therefore all the three small angles at C are equal; likewise at A and at B. In other words, the angles of the triangle ABC are trisected.

The three small angles at A are each $\frac{1}{3}A = 60° - \alpha$; similarly at B and C. Thus

$$\alpha = 60° - \tfrac{1}{3}A, \quad \beta = 60° - \tfrac{1}{3}B, \quad \gamma = 60° - \tfrac{1}{3}C.$$

By choosing these values for the base angles of our isosceles triangles, we can ensure that the above procedure yields a triangle ABC that is similar to any given triangle.

This completes the proof.

EXERCISES

1. The three lines PP', QQ', RR' (Figure 1.9*a*) are concurrent. In other words, the trisectors of A, B, C meet again to form another triangle $P'Q'R'$ which is perspective with the equilateral triangle PQR. (In general $P'Q'R'$ is *not* equilateral.)

2. What values of α, β, γ will make the triangle ABC (i) equilateral, (ii) right-angled isosceles? Sketch the figure in each case.

3. Let P_1 and P_2 (on CA and AB) be the images of P by reflection in CP' and BP'. Then the four points P_1, Q, R, P_2 are evenly spaced along a circle through A. In the special case when the triangle ABC is equilateral, these four points occur among the vertices of a regular enneagon (9-gon) in which A is the vertex opposite to the side QR.