4

Two-dimensional crystallography

Mathematical crystallography provides one of the most important applications of elementary geometry to physics. The three-dimensional theory is complicated, but its analog in two dimensions is easy to visualize without being trivial. Patterns covering the plane arise naturally as an extension of the strip patterns considered in § 3.7. However, in spite of the restriction to two dimensions, a complete account of the enumeration of infinite symmetry groups is beyond the scope of this book.

4.1 LATTICES AND THEIR DIRICHLET REGIONS

For some minutes Alice stood without speaking, looking out in all directions over the country . . . "I declare it's marked out just like a large chessboard . . . all over the world—if this is the world at all."

Lewis Carroll

[Dodgson **2,** Chap. 2]

Infinite two-dimensional groups (the symmetry groups of repeating patterns such as those commonly used on wallpaper or on tiled floors) are distinguished from infinite "one-dimensional" groups by the presence of *independent* translations, that is, translations whose directions are neither parallel nor opposite. The crystallographer E. S. Fedorov showed that there are just seventeen such two-dimensional groups of isometries. They were rediscovered in our own century by Pólya and Niggli.* The symbols by which we denote them are taken from the International Tables for X-ray Crystallography.

The simplest instance is the group **p1**, generated by two independent

* E. S. Fedorov, *Zapiski Imperatorskogo S. Peterburgskogo Mineralogicheskogo Obshchestva* (2), **28** (1891), pp. 345–390; G. Pólya and P. Niggli, *Zeitschrift für Kristallographie und Mineralogie,* **60** (1924), pp. 278–298. [See also Fricke and Klein **1,** pp. 227–233.] Fedorov's table shows that 16 of the 17 groups had been described by C. Jordan in 1869. The remaining one was recognized by L. Sohncke in 1874; but he missed three others.

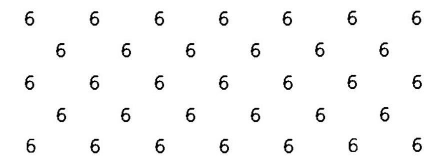


Figure 4.1a

translations X, Y. Since the inverse of a translation is a translation, and the product of two translations is a translation (3.21), this group consists entirely of translations. Since XY = YX, these translations are simply X^xY^y for all integers x, y. Abstractly, this is the "direct product" $C_\infty \times C_\infty$, which has the single defining relation

$$XY = YX$$

[Coxeter and Moser **1,** p. 40]. Any object, such as the numeral 6 in Figure 4.1*a*, is transformed by the group **p1** into an infinite array of such objects, forming a pattern. Conversely, **p1** is the complete symmetry group of the pattern, provided the object has no intrinsic symmetry. If the object is a single point, the pattern is an array of points called a two-dimensional *lattice,* which may be pictured as the plan of an infinite orchard. Each lattice point is naturally associated with the symbol for the translation by which it is derived from the original point 1 (Figure 4.1*b*).

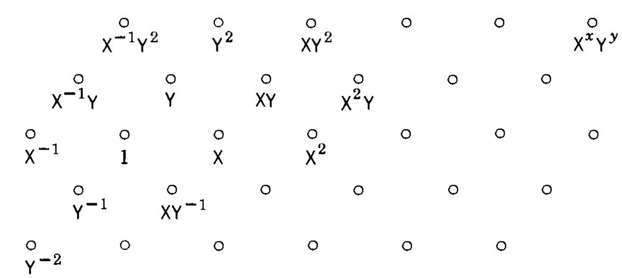


Figure 4.1b

Anyone standing in an orchard observes the alignment of trees in rows in many directions. This exhibits a characteristic property of a lattice: the line joining any two of the points contains infinitely many of them, evenly spaced, that is, a "one-dimensional lattice." In fact, the line joining the points 1 and X^xY^y contains also the points

$$X^{nx/d}Y^{ny/d} = (X^{x/d}Y^{y/d})^n$$

where d is the greatest common divisor of x and y, and n runs over all the integers. In particular, the powers of X all lie on one line, the powers of Y on another, and lines parallel to these through the remaining lattice points form a tessellation of congruent parallelograms filling the plane without interstices (Figure 4.1*c*). (We use the term *tessellation* for any arrangement of polygons fitting together so as to cover the whole plane without overlapping.)

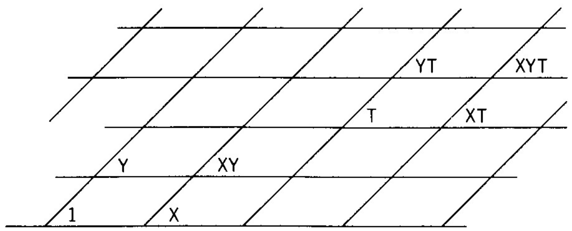


Figure 4.1c

A typical parallelogram is formed by the four points 1, X, XY, Y. The translation $T = X^xY^y$ transforms this parallelogram into another one having the point T (instead of 1) at its "first" corner. There is thus a one-to-one correspondence between the cells or tiles of the tessellation and the transformations in the group, with the property that each transformation takes any point inside the original cell to a point similarly situated in the new cell. For this reason, the typical parallelogram is called a *fundamental region.*

The shape of the fundamental region is far from unique. Any parallelogram will serve, provided it has four lattice points for its vertices but no others on its boundary or inside [Hardy and Wright **1**, p. 28]. This is the geometrical counterpart of the algebraic statement that the group generated by X, Y is equally well generated by X^aY^b, X^cY^d, provided

$$ad - bc = \pm 1.$$

To express the old generators in terms of the new, we observe that

$$(X^aY^b)^d(X^cY^d)^{-b} = X^{ad-bc}, \quad (X^aY^b)^{-c}(X^cY^d)^a = Y^{ad-bc}.$$

But there is no need for the fundamental region to be a parallelogram at all; for example, we may replace each pair of opposite sides by a pair of congruent curves, as in Figure 4.1*d*.

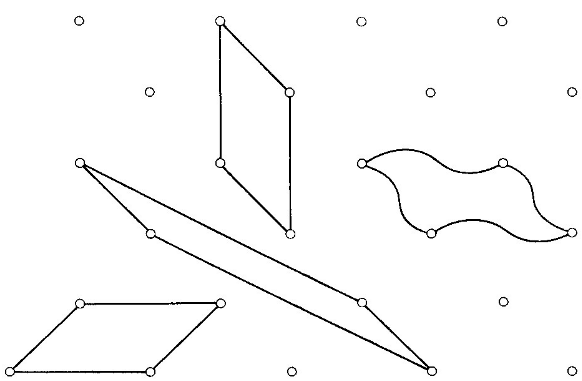


Figure 4.1d

Every possible fundamental region, whether we choose a parallelogram or any other shape, has *the same area* as the typical parallelogram of Figure 4.1*c*. For, inside a sufficiently large circle, the number of lattice points is equal to the number of replicas of any fundamental region (with an insignificant error due to mutilated regions at the circumference); thus every possible shape has for its area the same fraction of the area of the large circle.* It is an interesting fact that any *convex* fundamental region for the translation group is a centrally symmetrical polygon (namely, a parallelogram or a centrally symmetrical hexagon).†

Among the various possible parallelograms, we can select a standard or *reduced* parallelogram by taking the generator Y to be the shortest translation (or one of the shortest) in the group, and X to be an equal or next shortest translation in another direction. If the angle between X and Y then happens to be obtuse, we reverse the direction of Y. Thus, among all the parallelograms that can serve as a fundamental region, the reduced parallelogram has the shortest possible sides. The translations along these sides are naturally called reduced generators.

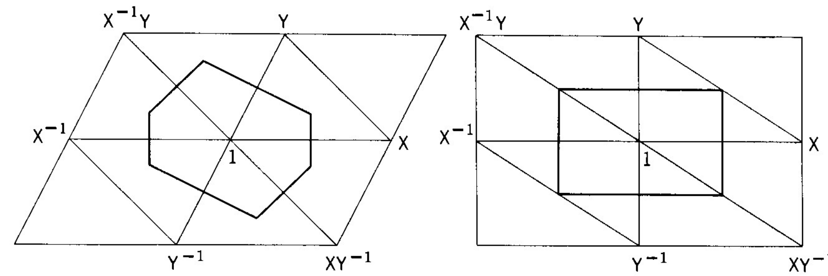


Figure 4.1e

By joining the vertices X, Y of the reduced parallelogram, and the corresponding pair of vertices of each replica, we obtain a tessellation of congruent triangles whose vertices are lattice points and whose angles are nonobtuse. Each lattice point belongs to six of the triangles; for example, the triangles surrounding the point 1 join it to pairs of adjacent points in the cycle

$$X, \quad Y, \quad X^{-1}Y, \quad X^{-1}, \quad Y^{-1}, \quad XY^{-1}$$

(Figure 4.1*e*). By joining the circumcenters of these six triangles, we obtain the *Dirichlet region* (or "Voronoi polygon") of the lattice: a polygon whose interior consists of all the points in the plane which are nearer to a particu-

* Gauss used this idea as a means of estimating π [Hilbert and Cohn-Vossen **1**, pp. 33–34].
† A. M. Macbeath. *Canadian Journal of Mathematics,* **13** (1961), p. 177.

lar lattice point (such as the point 1) than to any other lattice point.* Such regions, each surrounding a lattice point, evidently fit together to fill the whole plane; in fact, the Dirichlet region is a particular kind of fundamental region.

The lattice is symmetrical by the half-turn about the point 1 (or any other lattice point). For this half-turn interchanges the pairs of lattice points X^xY^y, $X^{-x}Y^{-y}$. (In technical language, the group **p1** has an automorphism of period 2 which replaces X and Y by their inverses.) Hence *the Dirichlet region is symmetrical by a half-turn.* Its precise shape depends on the relative lengths of the generating translations X, Y and the angle between them. If this angle is a right angle, the Dirichlet region is a rectangle (or a square), since the circumcenter of a right-angled triangle is the midpoint of the hypotenuse. In all other cases it is a hexagon (not necessarily a regular hexagon; but since it is centrally symmetrical, its pairs of opposite sides are equal and parallel).

Varying the lattice by letting the angle between the translations X and Y increase gradually to 90°, we see that two opposite sides of the hexagon shrink till they become single vertices, and then the remaining four sides form a rectangle (or square).

Reflections in the four or six sides of the Dirichlet region transform the central lattice point 1 into four or six other lattice points which we naturally call the *neighbors* of the point 1.

EXERCISES

1. Any two opposite sides of a Dirichlet region are perpendicular to the line joining their midpoints.

2. Sketch the various types of lattice that can arise if X and Y are subject to the following restrictions: they may have the same length, and the angle between them may be 90° or 60°. Indicate the Dirichlet region in each case, and state whether the symmetry group of this region is C_2, D_2, D_4, or D_6.

4.2 THE SYMMETRY GROUP OF THE GENERAL LATTICE

The investigation of the symmetries of a given mathematical structure has always yielded the most powerful results.

E. Artin (1898–1962)
[Artin **1**, p. 54]

Any given lattice is easily seen to be symmetrical by the half-turn about the midpoint of the segment joining any two lattice points [Hilbert and Cohn-Vossen **1**, p. 73]. Such midpoints form a lattice of finer mesh, whose generating translations are half as long as X and Y (see the "open" points in Figure 4.2*a*).

* G. L. Dirichlet, *Journal für die reine und angewandte Mathematik*, **40** (1850), pp. 216–219.

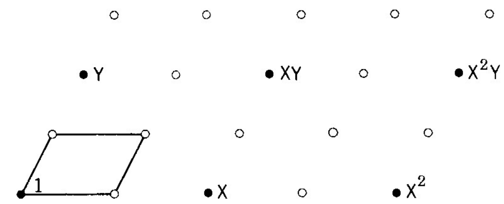


Figure 4.2a

The "general" lattice occurs when the reduced generators differ in length and the angle between them is neither 90° nor 60°. In such a case, the translations X^xY^y and the above-mentioned half-turns are its only symmetry operations. In other words, the symmetry group of the general lattice is derived from **p1** by adding an extra transformation H, which is the half-turn about the point 1. This group is denoted by **p2** [Coxeter and Moser **1**, pp. 41–42]. It is generated by the half-turn H and the translations X, Y, in terms of which the half-turn that interchanges the points 1 and $T = X^xY^y$ is HT. (Note that T itself is the product of H and HT.) The group is equally well generated by the three half-turns HX, H, HY, or (redundantly) by these three and their product

$$HX \cdot H \cdot HY = HXY,$$

which are half-turns about the four vertices of the parallelogram shown in Figure 4.2*a*.

It is remarkable that any triangle or any simple quadrangle (not necessarily convex) will serve as a fundamental region for **p2**. Half-turns about the midpoints of the three or four sides may be identified with HX, H, HY (Figure 4.2*b*), or HX, H, HY, HXY (Figure 4.2*c*).

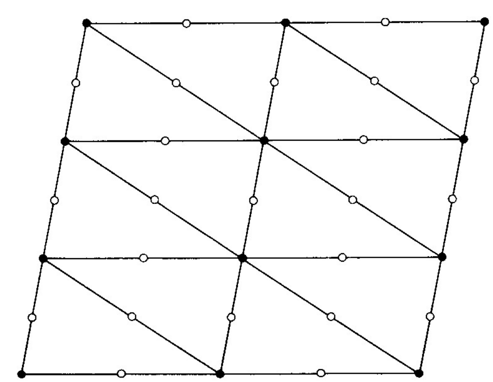


Figure 4.2b

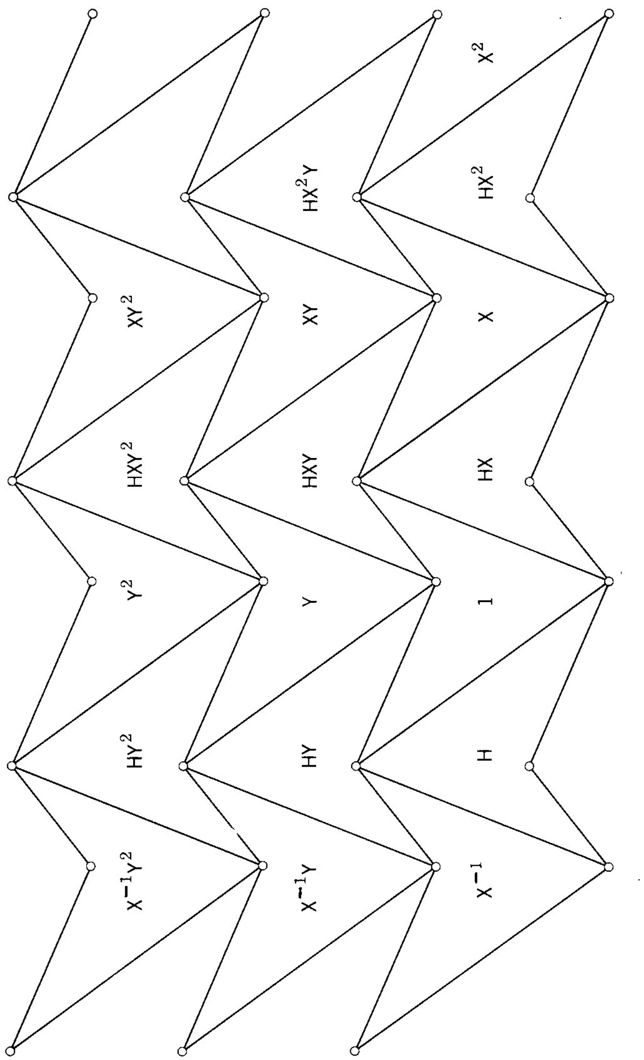


Figure 4.2c

EXERCISES

1. Why do the vertices of the quadrangles in Figure 4.2c form two superposed lattices?

2. Draw the tessellation of Dirichlet regions for a given lattice. Divide each region into two halves by means of a diagonal. The resulting tessellation is a special case of the tessellation of scalene triangles (Figure 4.2b) or of irregular quadrangles (Figure 4.2c) according as the Dirichlet region is rectangular or hexagonal.

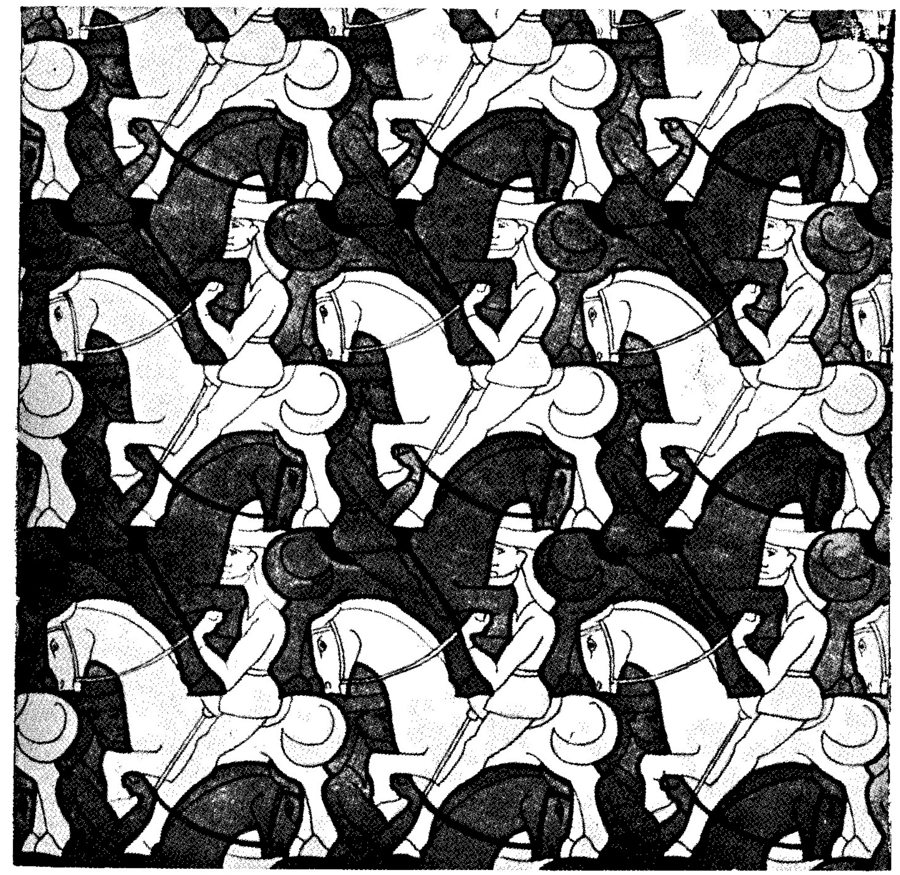


Plate I

4.3 THE ART OF M. C. ESCHER

The groups **p1** and **p2** are two of the simplest of the seventeen discrete groups of isometries involving two independent translations. Several others will be mentioned in this section and the next. Convenient generators for all of them are listed in Table I on p. 413.

The art of filling a plane with a repeating pattern reached its highest development in thirteenth-century Spain, where the Moors used all the seventeen groups in their intricate decoration of the Alhambra [Jones **1**]. Their preference for abstract patterns was due to their strict observance of the Second Commandment. The Dutch artist M. C. Escher, free from such scruples, makes an ingenious application of these groups by using animal shapes for their fundamental regions. For instance, the symmetry group of his pattern of knights on horseback (Plate I) seems at first sight to be **p1**, generated by a horizontal translation and a vertical translation. But by ignoring the distinction between the dark and light specimens we obtain the more interesting group **pg**, which is generated by two parallel glide reflections, say G and G′. We observe that the vertical translation can be expressed equally well as G^2 or G'^2. It is remarkable that the single relation

$$G^2 = G'^2$$

provides a complete abstract definition for this group [Coxeter and Moser **1**, p. 43]. Clearly, the knight and his steed (of either color) constitute a fundamental region for **pg**. But we must combine two such regions, one dark and one light, in order to obtain a fundamental region for **p1**.

Similarly, the symmetry group of Escher's pattern of beetles (Plate II) seems at first sight to be **pm**, generated by two vertical reflections and a vertical translation. But on looking more closely we see that there are both dark and light beetles, and that the colors are again interchanged by glide reflections. The complete symmetry group **cm**, whose fundamental region is the right or left half of a beetle of either color, is generated by any such vertical glide reflection along with a vertical reflection. To obtain a fundamental region for the "smaller" group **pm**, we combine the right half of a beetle of either color with the left half of an adjacent beetle of the other color.

A whole beetle (of either color) provides a fundamental region for the group **p1** (with one of its generating translations oblique) or equally well for **pg**.

EXERCISES

1. Locate the axes of two glide reflections which generate **pg** in Plates I and II.

2. Any two parallelograms whose sides are in the same two directions can together be repeated by translations to fill the plane.

4.4 SIX PATTERNS OF BRICKS

Figure 4.4*a* shows how six of the seventeen two-dimensional space groups arise as the symmetry groups of familiar patterns of rectangles, which we may think of as bricks or tiles. The generators are indicated as follows: a broken line denotes a mirror, a "lens" denotes a half-turn, a small square denotes a quarter-turn (i.e., rotation through 90°), and a "half arrow" denotes a glide reflection.

In each case, a convenient fundamental region is indicated by shading. This region is to some extent arbitrary except in the case of **pmm**, where it is entirely bounded by mirrors.

The procedure for analysing such a pattern is as follows. We observe that the symmetry group of a single brick is D_2 (of order 4), which has subgroups C_2 and D_1. If all the symmetry operations of the brick are also symmetry operations of the whole pattern, as in **cmm** and **pmm**, the fundamental region is a quarter of the brick, two of the generators are the reflections that generate D_2, and any other generator transforms the original brick into a neighboring brick. If only the subgroup C_2 or D_1 belongs to the whole pat-

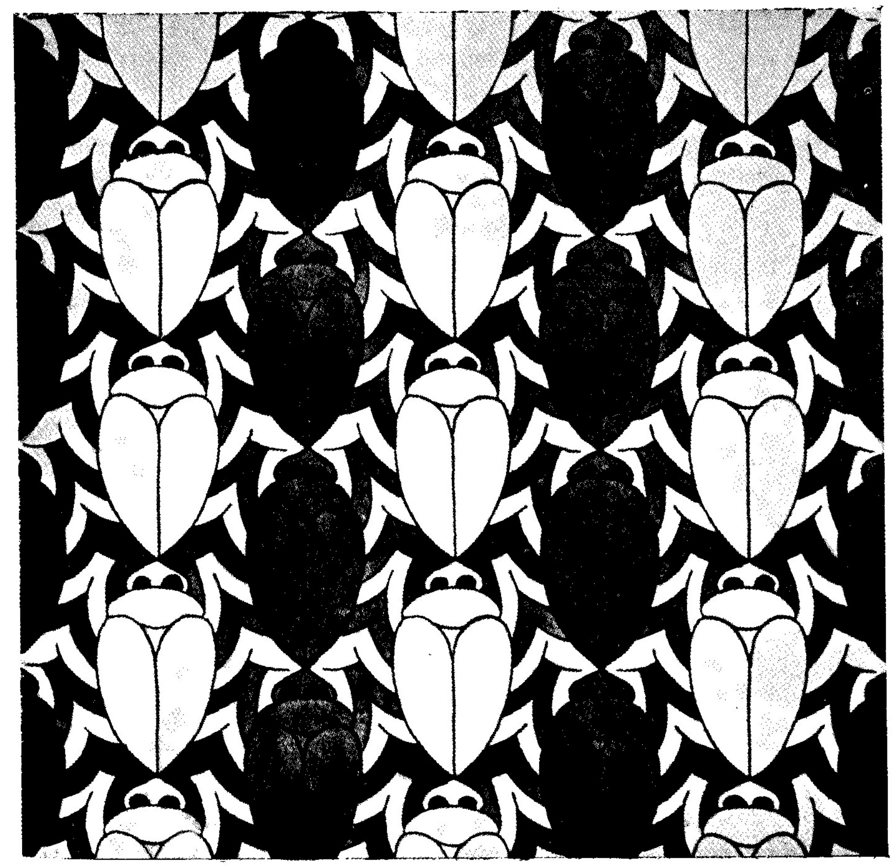


Plate II

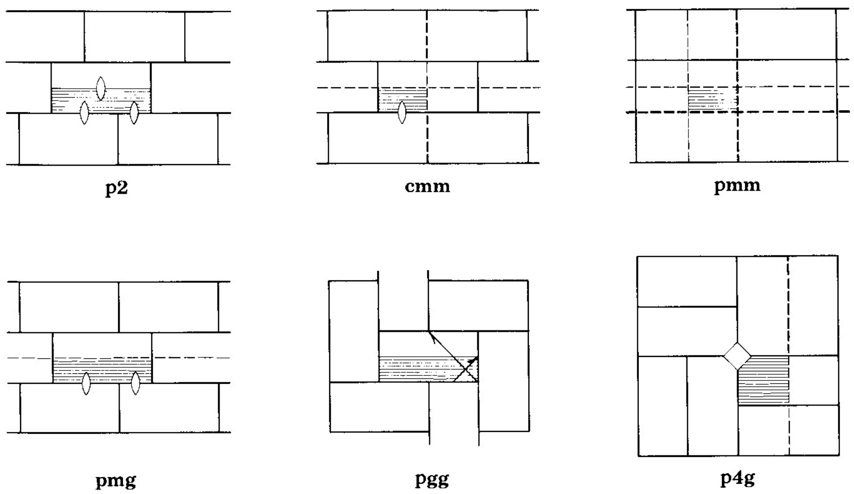


Figure 4.4a

tern (the way C_2 belongs to **p2** or **pgg**, and D_1 to **pmg** or **p4g**), the fundamental region is half a brick, and the generators are not quite so obvious.

EXERCISE

In all these patterns it is understood that a "brick" is a rectangle in which one side is twice as long as another. In each case, any brick is related to the whole pattern in the same way as any other. (In technical language, the symmetry group is *transitive* on the bricks.) Are these six the *only* transitive patterns of bricks?

4.5 THE CRYSTALLOGRAPHIC RESTRICTION

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

G. H. Hardy [**2**, p. 24]

A complete account of the enumeration of the seventeen two-dimensional space-groups would occupy too much space. But it seems worthwhile to give Barlow's elegant proof* that the only possible cyclic subgroups are C_2, C_3, C_4, and C_6. In other words:

The only possible periods for a rotational symmetry operation of a lattice are 2, 3, 4, 6.

Let P be any center of rotation of period n. The remaining symmetry

* W. Barlow, *Philosophical Magazine* (6), **1** (1901), p. 17.

operations of the lattice transform P into infinitely many other centers of rotation of the same period. Let Q be one of these other centers (Figure 4.5*a*) at the least possible distance from P. A third center, P', is derived from P by rotation through $2\pi/n$ about Q; and a fourth, Q', is derived from Q by rotation through $2\pi/n$ about P'. Of course, the segments PQ, QP', $P'Q'$, are all equal. It may happen that P and Q' coincide; then $n = 6$. In all other cases, since Q was chosen at the least possible distance from P, we must have $PQ' \geqslant PQ$; therefore $n \leqslant 4$. (If $n = 4$, $PQP'Q'$ is a square. If $n = 5$, PQ' is obviously shorter than PQ. If $n > 6$, PQ crosses $P'Q'$, but it is no longer necessary to use Q': we already have $PP' < PQ$, which is sufficiently absurd.)

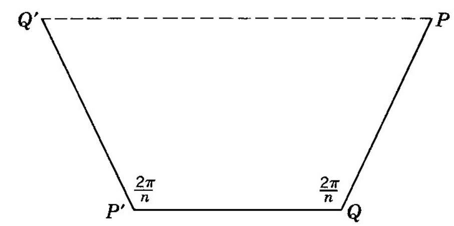


Figure 4.5a

EXERCISES

1. If S and T are rotations through $2\pi/n$ about P and Q, what is $T^{-1}ST$?
2. If a discrete group of isometries includes two rotations about distinct centers, it includes two such rotations having the same period, and therefore also a translation. If this period is greater than 2, it includes two independent translations.

4.6 REGULAR TESSELLATIONS

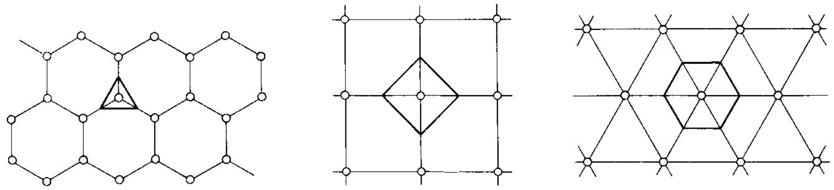
The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G. H. Hardy [**2**, p. 25]

It was probably Kepler (1571–1630) who first investigated the possible ways of filling the plane with equal regular polygons. We shall find it convenient to use the Schläfli symbol $\{p, q\}$ for the tessellation of regular p-gons, q surrounding each vertex [Schläfli **1**, p. 213]. The cases

$$\{6, 3\}, \quad \{4, 4\}, \quad \{3, 6\}$$

are illustrated in Figure 4.6*a*, where in each case the polygon drawn in heavy lines is the *vertex figure:* the q-gon whose vertices are the midpoints of the q edges at a vertex. (Since tessellations are somewhat analogous to polyhedra, it is natural to use the word *edges* for the common sides of adjacent polygons, and *faces* for the polygons themselves.)



{6, 3} {4, 4} {3, 6}

Figure 4.6a

For a formal definition, we may say that a tessellation is *regular* if it has regular faces and a regular vertex figure at each vertex.

The tessellation $\{6, 3\}$ is often used for tiled floors in bathrooms. It can also be seen in any beehive. $\{4, 4\}$ is familiar in the form of squared paper; in terms of Cartesian coordinates, its vertices are just the points for which both x and y are integers. $\{3, 6\}$ is the dual of $\{6, 3\}$ in the following sense. The *dual* of $\{p, q\}$ is the tessellation whose edges are the perpendicular bisectors of the edges of $\{p, q\}$ (see Figure 4.6*b*). Thus the dual of $\{p, q\}$ is $\{q, p\}$, and vice versa; the vertices of either are the centers of the faces of the other. In particular, the dual of $\{4, 4\}$ is an equal $\{4, 4\}$.

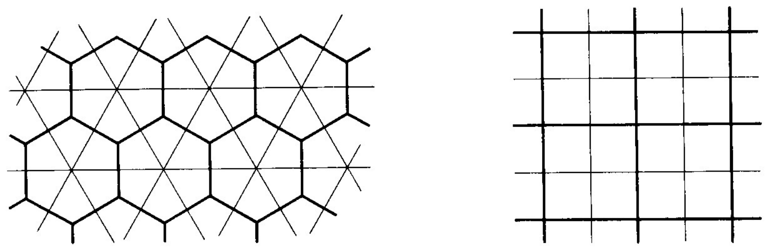


Figure 4.6b

The possible values of p and q are easily obtained by equating the angle of a p-gon, namely $(1 - 2/p)\pi$, to the value it must have if q such polygons come together at a vertex:

$$\left(1 - \frac{2}{p}\right)\pi = \frac{2\pi}{q}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$
$$(p - 2)(q - 2) = 4.$$

The three possible ways of factorizing 4, namely

$$4 \cdot 1, \quad 2 \cdot 2, \quad 1 \cdot 4,$$

yield the three tessellations already described. However, before declaring that these are the *only* regular tessellations, we should investigate the fractional solutions of our equation; for there might conceivably be a regular "star" tessellation $\{p, q\}$ whose face $\{p\}$ and vertex figure $\{q\}$ are regular polygons of the kind considered in §2.8. For instance, Figure 4.6*c* shows ten pentagons placed together at a common vertex. Although they overlap, we might expect to be able to add further pentagons so as to form a tessellation $\{5, \frac{10}{3}\}$ (whose vertex figure is a decagram), covering the plane a number of times. But in fact this number is infinite, as we shall see.

Consider the general regular tessellation $\{p, q\}$, where $p = n/d$. If it covers the plane only a finite number of times, there must be a minimum distance between the centers of pairs of faces. Let P, Q be two such centers at this minimum distance apart. Since they are centers of rotation of period n, the argument used in §4.5 proves that the only possible values of n are 3, 4, 6. Thus $d = 1$, and these are also the only possible values of p. Hence *there are no regular star tessellations* [Coxeter **1**, p. 112].

It is actually possible to cover a *sphere* three times by using twelve "pentagons" whose sides are arcs of great circles [Coxeter **1**, p. 111].

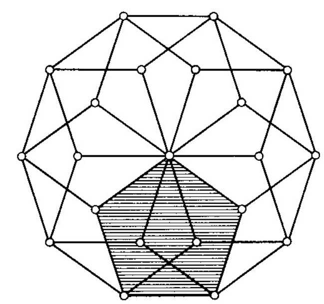
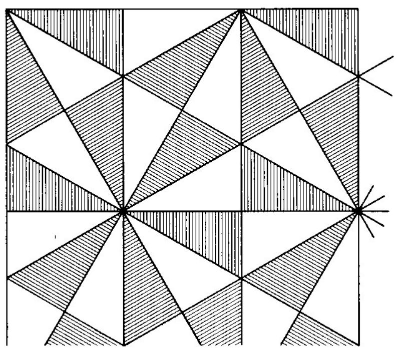


Figure 4.6c

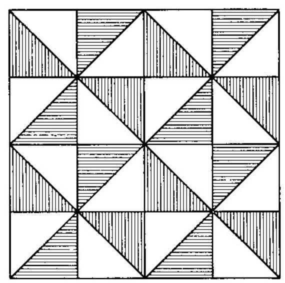
To find the symmetry group of a regular tessellation, we treat its face the way we treated one of the bricks in §4.4. Clearly, the symmetry group of $\{p, q\}$ is derived from the symmetry group D_p of one face by adding the reflection in a side of that face. Thus it is generated by reflections in the sides of a triangle whose angles are π/p (at the center of the face), $\pi/2$ (at the midpoint of an edge), and π/q (at a vertex). This triangle is a fundamental region, since it is transformed into neighboring triangles by the three generating reflections. Since each generator leaves invariant all the points on one side, the fundamental region is unique: it cannot be modified by addition and subtraction the way Escher modified the fundamental regions of some other groups.

The network of such triangles, filling the plane, is cut out by all the lines of symmetry of the regular tessellation. The lines of symmetry include the lines of the edges of both $\{p, q\}$ and its dual $\{q, p\}$. In the case of $\{6, 3\}$ and $\{3, 6\}$ (Figure 4.6*b*), these edge lines suffice; in the case of the two dual $\{4, 4\}$'s we need also the diagonals of the squares. In Figure 4.6*d*, alternate regions have been shaded so as to exhibit both the complete symmetry groups **p6m**, **p4m** and the "direct" subgroups **p6**, **p4** (consisting of rotations and translations) which preserve the colors and the direction of the shading [Brewster **1**, p. 94; Burnside **1**, pp. 416, 417].

Instead of deriving the network of triangles from the regular tessellation, we may conversely derive the tessellation from the network. For this purpose, we pick out a point in the network where the angles are π/p, that is, where p shaded and p white triangles come together. These $2p$ triangles combine to form a face of $\{p, q\}$.



p6 and p6m



p4 and p4m

Figure 4.6d

EXERCISES

1. Justify the formal definition of "regular" on page 62. (It implies that the faces are all alike and that the vertices are all surrounded alike.)

2. Give a general argument to prove that the midpoints of the edges of a regular tessellation belong to a lattice. (*Hint*: Consider the group **p2** generated by half-turns about three such midpoints.)

3. Pick out the midpoints of the edges of $\{6, 3\}$. Verify that they belong to a lattice. Do they constitute the whole lattice?

4. Draw portions of lattices whose symmetry groups are **p2**, **pmm**, **cmm**, **p4m**, **p6m**.

4.7 SYLVESTER'S PROBLEM OF COLLINEAR POINTS

Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy [**2,** p. 34]

As we saw in § 4.1, a lattice is a discrete set of points having the property that the line joining any two of them contains not only these two but infinitely many. Figure 4.7*a* shows a *finite* "orchard" in which nine points are arranged in ten rows of three [Ball **1**, p. 105]. It was probably the investigation of such configurations that led Sylvester* to propose his problem of 1893:

Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.

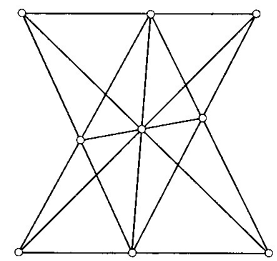


Figure 4.7a

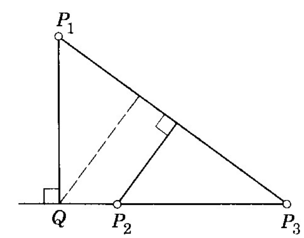


Figure 4.7b

Neither Sylvester nor any of his contemporaries were able to think of a satisfactory proof. The question was forgotten till 1933, when Karamata and Erdös revived it, and T. Gallai (*alias* Grünwald) finally succeeded, using a rather complicated argument. Sylvester's "negative" statement was rephrased "positively" by Motzkin:

If n points in the real plane are not on one straight line, then there exists a straight line containing exactly two of the points.

The following proof, which somewhat resembles Barlow's proof of the crystallographic restriction (§ 4.5), is due to L. M. Kelly.

* J. J. Sylvester, *Mathematical Questions and Solutions from the Educational Times,* **59** (1893), p. 98 (Question 11851). See also R. Steinberg, *American Mathematical Monthly,* **51** (1944), p. 170; L. M. Kelly, *ibid.,* **55** (1948), p. 28; T. Motzkin, *Transactions of the American Mathematical Society,* **70** (1951), p. 452; L. M. Kelly and W. O. J. Moser, *Canadian Journal of Mathematics,* **10** (1958), p. 213.

The n points $P_1, \ldots, P_n$ are joined by at most $\frac{1}{2}n(n-1)$ lines P_1P_2, P_1P_3, etc. Consider the pairs P_i, P_jP_k, consisting of a point and a joining line which are not incident. Since there are at most $\frac{1}{2}n(n-1)(n-2)$ such pairs, there must be at least one, say P_1, P_2P_3, for which the distance P_1Q from the point to the line is the smallest such distance that occurs.

Then the line P_2P_3 contains no other point of the set. For if it contained P_4, at least two of the points P_2, P_3, P_4 would lie on one side of the perpendicular P_1Q (or possibly one of the P's would coincide with Q). Let the points be so named that these two are P_2, P_3, with P_2 nearer to Q (or coincident with Q). Then P_2, P_3P_1 (Figure 4.7*b*) is another pair having a smaller distance than P_1Q, which is absurd.

This completes the proof that there is always a line containing exactly two of the points. Of course, there may be more than one such line; in fact, Kelly and Moser proved that the number of such lines is at least $3n/7$.

EXERCISES

1. The above proof yields a line P_2P_3 containing only these two of the P's. The point Q actually lies *between* P_2 and P_3.
2. If n points are not all on one line, they have at least n distinct joins [Coxeter **2**, p. 31].
3. Draw a configuration of n points for which the lower limit of $3n/7$ "ordinary" joins is attained. (*Hint*: $n = 7$.)

5

Similarity in the Euclidean plane

In later chapters we shall see that Euclidean geometry is by no means the only possible geometry: other kinds are just as logical, almost as useful, and in some respects simpler. According to the famous *Erlangen program* (Klein's inaugural address at the University of Erlangen in 1872), the criterion that distinguishes one geometry from another is the group of transformations under which the propositions remain true. In the case of Euclidean geometry, we might at first expect this to be the continuous group of all isometries. But since the propositions remain valid when the scale of measurement is altered, as in a photographic enlargement, the "principal group" for Euclidean geometry [Klein **2**, p. 133] includes also "similarities" (which may change distances although of course they preserve angles). In the present chapter we classify such transformations of the Euclidean plane. In particular, "dilatations" will be seen to play a useful role in the theory of the nine-point center of a triangle. These and other "direct" similarities are treated in the standard textbooks, but "opposite" similarities (§ 5.6) seem to have been sadly neglected.

5.1 DILATATION

"If I eat one of these cakes," she thought, "it's sure to make some change in my size." . . . So she swallowed one . . . and was delighted to find that she began shrinking directly.

Lewis Carroll

[Dodgson **1**, Chap. 4]

It is convenient to extend the usual definition of *parallel* by declaring that two (infinite straight) lines are parallel if they have either no common point or two common points. (In the latter case they coincide.) This convention enables us to assert that, without any exception,