7

Isometry and similarity in Euclidean space

This chapter is the three-dimensional counterpart of Chapters 3 and 5. In § 7.5 we find a proof (independent of Euclid's Fifth Postulate) for the theorem (discovered by Michel Chasles in 1830) that every motion is a *twist*. In § 7.6 we see that every similarity (except the twist and the glide reflection, which are isometries) is a three-dimensional *dilative rotation*.

Most isometries are familiar in everyday life. When you walk straight forward you are undergoing a translation. When you turn a corner, it is a rotation; when you ascend a spiral staircase, a twist. The transformation that interchanges yourself and your image in an ordinary mirror is a reflection, and it is easy to see how you could combine this with a rotation or a translation to obtain a rotatory reflection or a glide reflection, respectively.

7.1 DIRECT AND OPPOSITE ISOMETRIES

A congruence is either proper, carrying a left screw into a left and a right one into a right, or it is improper or reflexive, changing a left screw into a right one and vice versa. The proper congruences are those transformations which . . . connect the positions of points of a rigid body before and after a motion.

H. Weyl [**1**, pp. 43-44]

The axioms of congruence, a sample of which was given in 1.26, can be extended in a natural manner from plane geometry to solid geometry. In space, an *isometry* (Weyl's "congruence") is still any transformation that preserves length, so that a line segment PQ is transformed into a congruent seg-

ment $P'Q'$. The most familiar examples are the *rotation* about a given line through a given angle and the *translation* in a given direction through a given distance. In the former case the axis of rotation has all its points invariant; in the latter there is no invariant point, except when the distance is zero so that the translation is the identity. A *reflection* is the special kind of isometry which has a whole plane of invariant points: the mirror. By a simple argument involving three spheres instead of two circles, we can easily prove the following analogue of Theorem 2.31:

7.11 *If an isometry has three non-collinear invariant points, it must be either the identity or a reflection.*

When two tetrahedra $ABCP$, $ABCP'$ are images of each other by reflection in their common face, we may regard the "broken line" formed by the three edges AB, BC, CP as a kind of rudimentary screw, and the image formed by AB, BC, CP' as an oppositely oriented screw: if one is right-handed the other is left-handed. A model is easily made from two pieces of stiff wire, with right-angled bends at B and C. In this manner the idea of *sense* can be extended from two dimensions to three: we can say whether two given congruent tetrahedra agree or disagree in sense. In the former case we shall find that either tetrahedron can be *moved* (like a screw in its nut) to the position previously occupied by the other; such a motion is called a *twist*.

This distinction arises in analytic geometry when we make a coordinate transformation. If O is the origin and X, Y, Z are at unit distances along the positive coordinate axes, the sense of the tetrahedron $OXYZ$ determines whether the system of axes is right-handed or left-handed. (A coordinate transformation determines an isometry transforming each point (x, y, z) into the point that has the *same* coordinates in the new system.)

Since an isometry is determined by its effect on a tetrahedron,

7.12 *Any two congruent tetrahedra $ABCD$, $A'B'C'D'$ are related by a unique isometry $ABCD \to A'B'C'D'$, which is direct or opposite according as the sense of $A'B'C'D'$ agrees or disagrees with that of $ABCD$.*

(Some authors, such as Weyl, say "proper or improper" instead of "direct or opposite.")

The solid analogue of Theorem 3.12 is easily seen to be:

7.13 *Two given congruent triangles are related by just two isometries: one direct and one opposite.*

As a counterpart for 3.13 we have [Coxeter **1**, p. 36]:

7.14 *Every isometry is the product of at most four reflections. If there is an invariant point, "four" can be replaced by "three."*

Since a reflection reverses sense, an isometry is direct or opposite according as it is the product of an even or odd number of reflections: 2 or 4 in the former case, 1 or 3 in the latter. In particular, a direct isometry with an invariant point is the product of just two reflections, and since the two mirrors have a common point they have a common line. Hence

7.15 *Every direct isometry with an invariant point is a rotation.*

Also, as Euler observed in 1776,

7.16 *The product of two rotations about lines through a point O is another such rotation.*

EXERCISE

The product of rotations through π about two intersecting lines that form an angle α is a rotation through 2α.

7.2 THE CENTRAL INVERSION

One of the most important opposite isometries is the *central inversion* (or "reflection in a point"), which transforms each point P into the point P' for which the midpoint of PP' is a fixed point O. This can be described as the product of reflections in any three mutually perpendicular planes through O. Taking these three mirrors to be the coordinate planes $x = 0, y = 0, z = 0$, we see that the central inversion in the origin transforms each point (x, y, z) into $(-x, -y, -z)$.

The name "central inversion," though well established in the literature of crystallography, is perhaps unfortunate: we must be careful to distinguish it from inversion in a sphere.

For most purposes the central inversion plays the same role in three dimensions as the half-turn in two. But we must remember that, since 3 is an odd number, the central inversion is an opposite isometry whereas the half-turn is direct. In space, the name *half-turn* is naturally used for the rotation through π about a line (or the "reflection in a line"), which is still direct [Lamb **1**, p. 9].

EXERCISE

What is the product of half-turns about three mutually perpendicular lines through a point?

7.3 ROTATION AND TRANSLATION

The treatment of translation in § 3.2 can be adapted to three dimensions by defining a translation as the product of two central inversions. We soon see that either the first center or the second may be arbitrarily assigned, and that the two inversions may be replaced by two half-turns about parallel axes or by two reflections in parallel mirrors.

Thus the product of two reflections is either a translation or a rotation.

The latter arises when the two mirrors intersect in a line, the axis of the rotation. In particular, the product of reflections in two perpendicular mirrors is a half-turn.

The product of reflections in two planes through a line l, being a rotation about l, is the same as the product of reflections in two other planes through l making the same dihedral angle as the given planes (in the same sense). Similarly, the product of reflections in two parallel planes, being a translation, is the same as the product of reflections in two other planes parallel to the given planes and having the same distance apart.

EXERCISE

What is the product of reflections in three parallel planes?

7.4 THE PRODUCT OF THREE REFLECTIONS

The three simplest kinds of isometry, namely rotation, translation and reflection, combine in commutative pairs to form the *twist* (or "screw displacement"), *glide reflection* and *rotatory reflection.* A twist is the product of a rotation with a translation along the direction of the axis. A glide reflection is the product of a reflection with a translation along the direction of a line lying in the mirror, that is, the product of reflections in three planes of which two are parallel while the third is perpendicular to both. A rotatory reflection is the product of a reflection with a rotation whose axis is perpendicular to the mirror. When this rotation is a half-turn, the rotatory reflection reduces to a central inversion.

Any rotatory reflection can be analysed into a central inversion and a residual rotation. For, if the rotation involved in the rotatory reflection is a rotation through θ, we may regard it as the product of a half-turn and a rotation through $\theta + \pi$ (or $\theta - \pi$). Thus a rotatory reflection can just as well be called a *rotatory inversion:* the product of a central inversion and a rotation whose axis passes through the center.

Any opposite isometry T that has an invariant point O is either a single reflection or the product of reflections in three planes through O. Its product TI with the central inversion in O, being a direct isometry with an invariant point, is simply a rotation S about a line through O. Hence the given opposite isometry is the rotatory inversion

$$\mathrm{T} = \mathrm{SI}^{-1} = \mathrm{SI}:$$

7.41 *Every opposite isometry with an invariant point is a rotatory inversion.*

Since three planes that have no common point are all perpendicular to one plane α, the reflections in them (as applied to a point in α) behave like the reflections in the lines that are their sections by α. Thus we can make use of Theorem 3.31 and conclude that

7.42 *Every opposite isometry with no invariant point is a glide reflection.*

EXERCISES

1. What is the product of reflections in three planes through a line?
2. Let ABC and $A'B'C'$ be two congruent triangles in distinct planes. Consider the perpendicular bisectors of AA', BB', CC'. If these three planes have just one common point O, the two triangles are related by a rotatory inversion with center O. (*Hint:* If they were related by a rotation, the three planes would intersect in a line.)
3. Every opposite isometry is expressible as the product of a reflection and a half-turn.

7.5 TWIST

The only remaining possibility is a direct isometry with no invariant point. Let S be any direct isometry (with or without an invariant point), transforming an arbitrary point A into A'. Let R_1 be the reflection that interchanges A and A'. Then the product R_1S is an opposite isometry leaving A' invariant. By 7.41, this is a rotatory inversion or rotatory reflection $R_2R_3R_4$, the product of a rotation R_2R_3 and a reflection R_4, the mirror for R_4 being perpendicular to the axis for R_2R_3. Since this rotation may be expressed as the product of two reflections in various ways (§7.3), we can adjust the mirrors for R_2 and R_3 so as to make the former perpendicular to the mirror for R_1. Since both these planes remain perpendicular to the mirror for R_4, we now have

$$S = R_1R_2R_3R_4,$$

the product of the two rotations R_1R_2, R_3R_4, both of which are half-turns [Veblen and Young **2,** p. 318]:

7.51 *Every direct isometry is expressible as the product of two half-turns.*

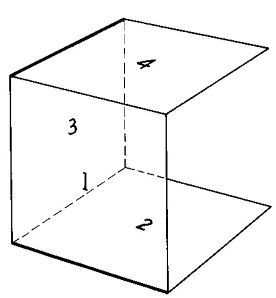

Figure 7.5a

If the isometry has an invariant point, it is a rotation, which may be expressed in various ways as the product of half-turns about two intersecting

lines. When there is no invariant point, the axes of the two half-turns are either parallel, in which case the product is a translation, or *skew,* like two opposite edges of a tetrahedron. Two skew lines always lie in a pair of parallel planes, namely, the plane through each line parallel to the other.

Since a half-turn is the product of reflections in *any* two perpendicular planes through its axis, the two half-turns R_1R_2, R_3R_4 with skew axes are respectively equal to $R'_1R'_2$, $R'_3R'_4$, where the mirrors for R'_2 and R'_4 are parallel while the other two are perpendicular to them (Figure 7.5*a*). Hence

$$R_1R_2R_3R_4 = R'_1R'_2R'_3R'_4 = R'_1R'_3R'_2R'_4,$$

where the interchange of the middle reflections is possible since the half-turn $R'_2R'_3$ may be equally well expressed as $R'_3R'_2$. We have now fulfilled our purpose of expressing the general direct isometry as a twist: the product of the rotation $R'_1R'_3$ and the translation $R'_2R'_4$ along the axis of the rotation. (This axis meets both the skew lines at right angles, and therefore measures the shortest distance between them.) In other words,

7.52 *Every displacement is either a rotation or a translation or a twist.*

(For an alternative treatment see Thomson and Tait [**1,** § 102].)

EXERCISES

1. What kind of isometry transforms the point (x, y, z) into
(a) $(x, y, -z)$, (b) $(-y, x, z)$, (c) $(x, y, z + 1)$,
(d) $(-y, x, z + 1)$, (e) $(-x, y, z + 1)$, (f) $(-y, x, -z)$?
2. The product of half-turns about two skew lines at right angles is a twist, namely, the product of a half-turn about the line of shortest distance and a translation through twice this shortest distance. (Veblen and Young [**2,** p. 324] named this a *half twist.*)

7.6 DILATIVE ROTATION

It can be proved by elementary methods that every Euclidean similarity other than a rigid motion has a fixed point.

Hilbert and Cohn-Vossen [**1,** p. 331]

In Euclidean space, the definition of *dilatation* is exactly the same as in the plane. In fact, § 5.1 can be applied, word for word, to three dimensions, except that the special dilatation $AB \to BA$ or $O(-1)$ is not a half-turn but a central inversion (§ 7.2). Likewise, § 5.2 applies to spheres just as well as to circles: Figure 5.2*a* may be regarded as a plane section of two unequal spheres with their centers C, C' and their centers of similitude O, O_1. Two equal spheres are related by a translation and by a central inversion.

However, an important difference appears when we consider questions of sense. In the plane, every dilatation is direct, but in space the dilatation $O(\lambda)$ is direct or opposite according as λ is positive or negative; for example, the central inversion $O(-1)$ is opposite, as we have seen.

In space, as in the plane, two similar figures are related by a *similarity*, which in special cases may be an isometry or a dilatation. By a natural extension of the terminology we now take a *dilative rotation* to mean the product of a rotation about a line l (the *axis*) and a dilatation whose center O lies on l. The plane through O perpendicular to l is invariant, being transformed according to the two-dimensional "dilative rotation" of § 5.5. In the special case when the rotation about l is a half-turn, there are infinitely many other invariant planes, namely all the planes through l. Any such plane is transformed according to a dilative reflection.

Suppose a dilative rotation is the product of a rotation through angle α and a dilatation $O(\lambda)$ (where O lies on the axis). The following values of α and λ yield special cases which are familiar:

α	λ	Similarity
0	1	Identity
π	1	Half-turn
α	1	Rotation
π	-1	Reflection
0	-1	Central inversion
α	-1	Rotatory inversion
0	λ	Dilatation

We observe that this table includes all kinds of isometry, both direct and opposite, except the translation, twist and glide reflection (which have no invariant points). Still more surprisingly, we shall find that, with these same three exceptions, *every similarity is a dilative rotation.*

The role of similar triangles is now taken over by similar tetrahedra. Evidently

7.61 *Two given similar tetrahedra $ABCD$, $A'B'C'D'$ are related by a unique similarity $ABCD \to A'B'C'D'$, which is direct or opposite according as the sense of $A'B'C'D'$ agrees or disagrees with that of $ABCD$.*

In other words, a similarity is completely determined by its effect on any four given non-coplanar points, and we have the following generalization of Theorem 7.13:

7.62 *Two given similar triangles ABC, $A'B'C'$ are related by just two similarities: one direct and one opposite.*

As a step towards proving that every similarity which is not an isometry is a dilative rotation, let us first prove

7.63 *Every similarity which is not an isometry has just one invariant point.*

Consider any given similarity S, whose ratio of magnification is $\mu \neq 1$. Let S transform an arbitrary point A into A'. If A' coincides with A, we have the desired invariant point. If not, let Q be the point that divides the segment AA' in the ratio $1:\mu$, externally or internally according as S is direct or opposite: that is, construct Q so that $QA' = \pm\mu QA$. Let D denote the direct or opposite dilatation $Q(\pm\mu^{-1})$. Then SD, having ratio of magnification 1, is a direct isometry which leaves A invariant. By Theorem 7.15, SD is a rotation about some line l through A. The plane through Q perpendicular to l is transformed into itself by both SD and D^{-1} and, therefore, also by their product S. On this invariant plane, S induces a two-dimensional similarity which, by Theorem 5.42, has an invariant point. Finally, this invariant point is unique, for if there were two distinct invariant points, the segment formed by them would be invariant instead of being multiplied by μ.

Having found the invariant point (or center) O, we can carry out a simplified version of the above procedure, with O for A. Since A' and Q both coincide with O, S is the product of a rotation about a line through O and the dilatation $O(\pm\mu)$, that is to say, S is a dilative rotation:

7.64 *Every similarity is either an isometry or a dilative rotation.*

In other words, every similarity is either a translation, a twist, a glide reflection, or a dilative rotation, provided we regard the last possibility as including all the special cases tabulated in the middle of page 102.

By Theorem 7.62, there are two dilative rotations, one direct and one opposite, which will transform a given triangle ABC into a similar (but not congruent) triangle $A'B'C'$. The ratio of magnification, $\mu \neq 1$, is given by the equation $A'B' = \mu AB$. Let A_1 and A_2 divide AA' internally and externally in the ratio $1 : \mu$. Let B_1 and B_2, C_1 and C_2 divide BB', CC' in the same manner. Consider the three spheres whose diameters are A_1A_2, B_1B_2, C_1C_2. These are "spheres of Appollonius" (Theorem 6.81); for example, the first is the locus of points whose distances from A and A' are in the ratio $1 : \mu$. Any point O for which

$$OA' = \mu OA, \qquad OB' = \mu OB, \qquad OC' = \mu OC$$

must lie on all three spheres. We have already established the existence of two such points. Hence the centers of the two dilative rotations may be constructed as the points of intersection of these three spheres.

In § 3.7 we used a translation to generate a geometric representation of the infinite cyclic group C_∞ (which is the free group with one generator). We see now that the same abstract group has a more interesting representation in which the generator is a dilative rotation. Some thirty elements of this group can be seen in the *Nautilus* shell [Thompson **2**, p. 843, Figure 418].

EXERCISES

1. How is the point (x, y, z) transformed by the general dilative rotation whose center and axis are the origin and the z-axis?
2. Find the axis and angle for the dilative rotation

$$(x, y, z) \rightarrow (\mu z, \mu x, \mu y).$$

3. How does the above classification of similarities deal with the three-dimensional "dilative reflection": the product of a dilatation $O(\lambda)$ and the reflection in a plane through O?
4. Could Theorem 7.63 be proved in the manner of the exercise at the end of §5.4 (page 73)?

7.7 SPHERE-PRESERVING TRANSFORMATIONS

The reasoning used in § 6.7 extends readily from two to three dimensions, yielding the following analog* of Theorem 6.71:

7.71 *Every sphere-preserving transformation of inversive space is either a similarity or the product of an inversion (in a sphere) and an isometry.*

EXERCISE

Every sphere-preserving transformation can be expressed as the product of r reflections and s inversions, where

$$r \leqslant 4, \quad s \leqslant 2, \quad r + s \leqslant 5.$$

* René Lagrange, *Produits d'inversions et métrique conforme,* Cahiers scientifiques, 23 (Gauthier-Villars, Paris, 1957), p. 7. See also Coxeter, *Annali di Matematica pura ed applicata,* **53** (1961), pp. 165–172.

Part II