Lesson 27
The Method of Undetermined Coefficients
Non–Homogeneous Equations

Specific Objectives:

At the end of the lesson, the students are expected to:

- differentiate the different methods on the determination of the particular solution
- solve non homogeneous equations using different methods

A non-homogeneous linear differential equation with constant coefficients is of the form:

$$\left(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_{n-1}D + a_n\right)y = R\left(x\right)$$

To find its general solution, $y = y_c + y_p$. The complementary solution, y_c, may be determined from the roots of the auxiliary equation, f(m) = 0, and the particular solution, y_p, may be determined by:

The particular solution may be found by any of the following methods:

1. The Method of Undetermined Coefficients
2. Variation of Parameters
3. Inverse Operators
4. By Inspection

1. The Method of Undetermined Coefficients

This method may be used if R(x) is itself a solution of some homogeneous linear differential equation with constant coefficients.

Step 1. Consider the right hand side of the equation, R(x) and determine its roots.

Case 1. If there is no repetition of roots between the auxiliary equation and R(x) such that

when R(x) is a	then y_P is
a. constant like 2, 3, etc.	A
b. linear function like 3x, 2x + 5, etc.	Ax + B

c. quadratic function like x^2, $x^2 - 3x$, etc.	$Ax^2 + Bx + C$
d. x^n	$Ax^n + Bx^{n-1} + ... + Dx + F$
e. e^{ax}	Ae^{ax}
f. sin ax, cos ax, sin ax + cos ax	A sin ax + B cos ax

Case 2. If there is a repetition of roots between the auxiliary equation and R(x), follow the principles discussed in the homogeneous linear differential equation under repetition of roots.

Step 2. Differentiate y_P according to the order of the given equation.

Step 3. Substitute these equations in the given equation. Solve for the literal constants by comparing/collecting their coefficients.

Example: Find the general solution of the differential equation.

1. $(D^2 - D - 2)y = 5$

SOLUTION:

$$y_c:\ m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2, -1$$
$$y_c = C_1 e^{2x} + C_2 e^{-x}$$

$$y_p : (D^2 - D - 2)y_p = 5$$

$$y_p = A \qquad\qquad y_p = -\tfrac{5}{2}$$
$$Dy_p = 0$$
$$D^2 y_p = 0$$
$$0 - 0 - 2A = 5$$
$$A = -\tfrac{5}{2}$$

$$\underline{\underline{y_p = C_1 e^{2x} + C_2 e^{-x} - \tfrac{5}{2}}}$$

2. $(D^2+4)y=4x+3$

SOLUTION:

$y_c: m^2+4=0$

$m^2=-4$

$m=0\pm 2i$

$y_c=[C_1\cos 2x+C_2\sin 2x]e^{0x}$

$y_p:(D^2+4)y_p=4x+3$

$y_p=Ax+B$

$Dy_p=A$

$D^2y_p=0$

$0+4(Ax+B)=4x+3$

$4Ax+4B=4x+3$

collect coefficients

$x: 4A=4; A=1$

$k: 4B=3; B=3/4$

$$\underline{\underline{y=[C_1\cos 2x+C_2\sin 2x]+x+3/4}}$$

3. (D^2+2D+2)y = 2 x^2+ 3x + 8

SOLUTION::

a) y_c:

$(D^2+2D+2)y_c=0$

$m^2+2m+2=0$

by Q.F.

$$m=\frac{-(2)\pm\sqrt{(2)^2-4(1)(2)}}{2(1)}$$

$$m=\frac{-2\pm 2i}{2}$$

$m=-1\pm i$

$y_c=(c_1\cos x+c_2\sin x)e^{-x}$

b) y_p:

$$\left(D^2 + 2D + 2\right)y_p = 2x^2 + 3x + 8 \qquad \text{equation 1}$$

by MUC

$R(x) = 2x^2 + 3x + 8, m = 0,$ triple root

$f(m) = 0, m = -1 \pm i$

$\therefore$ No repetition of roots between $R(x)$ and $f(m) = 0$

Since R(x) is a quadratic function, then

$y_p = Ax^2 + Bx + C$

$D(y_p) = A(2x) + B(1) = 2Ax + B$

$D^2(y_p) = 2A(1) = 2A$

Substitute in equation 1

$2A + 2(2Ax + B) + 2(Ax^2 + Bx + C) = 2x^2 + 3x + 8$

$2A + 4Ax + 2B + 2Ax^2 + 2Bx + 2C = 2x^2 + 3x + 8$

Compare coefficients,

$x^2 : 2A = 2, A = 1$

$x : 4A + 2B = 3$

$4(1) + 2B = 3$

$2B = 3 - 4 = -1$

$B = -\frac{1}{2}$

$k : 2A + 2B + 2C = 8$

$2(1) + 2\left(-\frac{1}{2}\right) + 2C = 8$

$2 - 1 + 2C = 8$

$1 + 2C = 8$

$2C = 8 - 1 = 7$

$C = \frac{7}{2}$

$y_p = x^2 - \frac{1}{2}x + \frac{7}{2}$

$$y = y_c + y_p$$

$$\underline{\underline{y = (c_1 \cos x + c_2 \sin x)e^{-x} + x^2 - \frac{1}{2}x + \frac{7}{2}}}$$

4. $(D^2 + 4D + 4)y \; = 2e^{2x}$

a) y_c: $(D^2 + 4D + 4)y_c = 0$

$$m^2 + 4m + 4 = 0$$
$$(m+2)^2 = 0$$
$$m + 2 = 0$$

$m = -2$, double root

$$y_c = (c_1 + c_2 x)e^{-2x}$$

b) y_p: $(D^2 + 4D + 4)y_p = 2e^{2x}$ equation 1

By MUC

$R(x) = 2e^{2x}, m = 2$

$f(m) = 0, m = -2$, double root

$\therefore$ No repetition of roots between $R(x) and f(m) = 0$

Since $R(x)$ is an exponential function,

Then $y_p = Ae^{2x}$

$$D(y_p) = A(2e^{2x}) = 2Ae^{2x}$$
$$D^2(y_p) = 2A(2e^{2x}) = 4Ae^{2x}$$

Substituting in equation 1

$$4Ae^{2x} + 4(2Ae^{2x}) + 4(Ae^{2x}) = 2e^{2x}$$
$$16Ae^{2x} = 2e^{2x}$$
$$e^{2x} : 16A = 2$$
$$A = \frac{2}{16} = \frac{1}{8}$$

$$y_p = \frac{1}{8}e^{2x}$$
$$y = y_c + y_p$$
$$y = (c_1 + c_2 x)e^{-2x} + \frac{1}{8}e^{2x}$$

5. $(D^2+9)y = 10\sin 2x + 5\cos 2x$

SOLUTION:

a) y_c:

$$(D^2+9)y_c = 0$$
$$m^2+9=0$$
$$m^2=-9$$
$$m = 0 \pm 3i$$
$$y_c = (c_1\cos 3x + c_2\sin 3x)e^{0x}$$

b) y_p:

$$(D^2+9)y_p = 10\sin 2x + 5\cos 2x \qquad \text{equation 1}$$

By MUC

$R(x) = 10\sin 2x + 5\cos 2x, m = 0 \pm 2i$

$f(m) = 0, m = 0 \pm 3i$

$\therefore$ No repetition of roots between $R(x) and f(m) = 0$

Since $R(x)$ is a trigonometric function,

Then $y_p = A\sin 2x + B\cos 2x$

$$D(y_p) = A(2\cos 2x) + B(-2\sin 2x)$$
$$D^2(y_p) = 2A(-2\sin 2x) - 2B(2\cos 2x)$$

Substituting in equation 1

$$-4A\sin 2x - 4B\cos 2x + 9A\sin 2x + 9B\cos 2x = 10\sin 2x + 5\cos 2x$$
$$5A\sin 2x + 5B\cos 2x = 10\sin 2x + 5\cos 2x$$
$$\sin 2x : 5A = 10, A = 2$$
$$\cos 2x : 5B = 5, B = 1$$
$$y_p = 2\sin 2x + \cos 2x$$

$$y = y_c + y_p$$
$$\underline{\underline{y = c_1\cos 3x + c_2\sin 3x + 2\sin 2x + \cos 2x}}$$

Seatwork: Find the general solution of the differential equation.

1. $(D^2 - 3D + 2)\ y = 2x^3 - 4x^2 + 6x + 3$

   ans. $y = C_1e^{x} + C_2e^{2x} + x^3 + \frac{5}{2}x^2 + \frac{15}{2}x + \frac{41}{4}$

2. $(D^2 - 9)\ y = x + e^{2x} - \sin 2x$

   ans. $y = C_1e^{3x} + C_2e^{-3x} - \frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$

Homework: Find the general solution of the differential equation.

1. $(D^2 - 2D - 3)\ y = x^3 + \sin x$

   ans. $y = C_1e^{3x} + C_2e^{-x} - \frac{1}{3}x^3 + \frac{2}{3}x^2 - \frac{14}{9}x + \frac{40}{27} - \frac{1}{5}\sin x + \frac{1}{10}\cos x$

2. $(D^2 - 2D)\ y = e^x \sin x$

   ans: $y = C_1e^{2x} + C_2 - \frac{1}{2}e^{x}\sin x$

3. $(D^2 - 4D + 4)\ y = x^3e^{2x} + 2xe^{2x}$

   ans. $y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{20}x^5e^{2x} + \frac{1}{3}x^3e^{2x}$

4. $(D^2 - 9)\ y = 2x + 3e^{2x} - \sin x$

5. $(D^2 + 1)\ y = -2\sin x + 4x\cos x$