Lesson 29
Inverse Operators
Non-Homogeneous Equations

3. Inverse Operator

Consider the equation,

$$F(D)\, y_P = R(x) \quad \text{———} \quad (1)$$

then,

$$y_p = \frac{1}{f(D)} R(x)$$

Where $\quad \dfrac{1}{f(D)}$ = inverse operator

Note: y_P depends on the type of function represented by R(x).

R(x) may be:

1. Exponential Function
2. Trigonometric Function
3. Polynomial Function
4. Composite Function
   - 4.1 Exponential Shift
   - 4.2 X Shift

Properties of Inverse Operators:

Let $\dfrac{1}{F(D)}$ and $\dfrac{1}{G(D)}$ be inverse operators.

1. $\dfrac{1}{F(D)} af(x) = a\dfrac{1}{F(D)} f(x)$

2. $\dfrac{1}{F(D)}\left[f(x)+g(x)\right] = \dfrac{1}{F(D)} f(x) + \dfrac{1}{F(D)} g(x)$

3. $\left\{\dfrac{1}{F(D)}\dfrac{1}{G(D)}\right\} f(x) = \dfrac{1}{F(D)}\left\{\dfrac{1}{G(D)} f(x)\right\} = \dfrac{1}{G(D)}\left\{\dfrac{1}{F(D)} f(x)\right\}$

3.1 R(x) is an Exponential Function

3.1.1 Given: f(D) $y_P = e^{ax}$

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \qquad \text{Such that } f(a) \neq 0$$

if f(a)=0, and f(D) contains the factor (D-a) n times, that is,
f(D) = Ø(D)(D-a)^n, then

3.1.2 $\phi(D)(D-a)^n y_p = e^{ax}$

$$y_p = \frac{1}{\phi(D)(D-a)^n} e^{ax}$$
$$= \frac{x^n e^{ax}}{n!\phi(a)}, \phi(a) \neq 0$$

Example: Find the particular solution of the differential equation.

1. $(D^2 - D - 2)y_p = 5$

SOLUTION:

$$\left[(D^2 - D - 2)y_p = 5\right]\frac{1}{D^2 - D - 2}$$
$$y_p = 5\left[\frac{1}{D^2 - D - 2} e^{0x}\right]; a = 0$$
$$y_p = 5\left[\frac{1}{0^2 - 0 - 2} e^{0x}\right]$$
$$y_p = 5\left[-\frac{1}{2}\right]$$
$$\underline{\underline{y_p = -\frac{5}{2}}}$$

2. $\left(D^2 + 4D + 4\right)y_p = 2e^{2x}$

SOLUTION:

$$[\left(D^2+4D+4\right)y_p = 2e^{2x}]\frac{1}{D^2+4D+4}$$

$$y_p = \left[\frac{1}{D^2+4D+4}2e^{2x}\right]$$

$$y_p = 2\left[\frac{1}{D^2+4D+4}e^{2x}\right] \text{ ; a = 2}$$

$$y_p = 2\left[\frac{1}{2^2+4(2)+4}e^{2x}\right]$$

$$\underline{\underline{y_p = \frac{1}{8}e^{2x}}}$$

Seatwork: Find the particular solution of the differential equation.

1. $(D^3 - 2D^2 - 5D + 6)\ y_p = (e^{2x} + 3)^2$
   ans: $y_p = 1/18\ (e^{4x}) - 3/2\ (e^{2x}) + 3/2$

2. $(D^3 - 5D^2 + 8D - 4)\ y_p = e^{2x} + 2e^x + 3e^{-x}$
   ans: $y_p = x^2/2\ (e^{2x}) + 2xe^x - 1/6\ (e^{-x})$

Homework: Find the particular solution of the differential equation.

1. $(D^2 - 2D - 3)\ y_p = e^{-2x} + e^x + 6$
2. $(D^2 + 2D + 1)\ y_p = 2 \sinh 2x$
3. $(D^2 - D - 2)\ y_p = e^{2x} + e^{-x}$

3.2 R(x) is a Trigonometric Function

3.2.1 $\left(D^2 + a^2\right)y_p = \sin bx, a \neq b$

$$y_p = \frac{1}{D^2+a^2}\sin bx = \frac{\sin bx}{a^2-b^2}$$

3.2.2 $\left(D^2 + a^2\right)y_p = \cos bx, a \neq b$

$$y_p = \frac{1}{D^2+a^2}\cos bx = \frac{\cos bx}{a^2-b^2}$$

3.2.3 $\left(D^2+a^2\right)y_p=\sin ax$

$$y_p=\frac{1}{D^2+a^2}\sin ax=-\frac{x}{2a}\cos ax$$

3.2.4 $\left(D^2+a^2\right)y_p=\cos ax$

$$y_p=\frac{1}{D^2+a^2}\cos ax=\frac{x}{2a}\sin ax$$

3.2.5 $F(D)^2\,y_p=\sin ax$

$$yp=\frac{1}{f(D)^2}\sin ax=\frac{1}{F\left(-a^2\right)}\sin ax$$

3.2.5 $F(D)^2\,y_p=\cos ax$

$$yp=\frac{1}{f(D)^2}\cos ax=\frac{1}{F\left(-a^2\right)}\cos ax, F\left(-a^2\right)\neq 0$$

Example: Find the particular solution of the differential equation.

1. $\left(D^2+9\right)y_p=10\sin 2x+5\cos 2x$

SOLUTION:

$$[\left(D^2+9\right)y_p=10\sin 2x+5\cos 2x\,]\ \frac{1}{D^2+9}$$

$$y_p=10\left[\frac{1}{D^2+9}\sin 2x\right]+5\left[\frac{1}{D^2+9}\cos 2x\right]$$

where $a=2, a^2=4, -a^2=-4$

$$y_p=10\left[\frac{1}{-4+9}\sin 2x\right]+5\left[\frac{1}{-4+9}\cos 2x\right]$$

$$y_p=10\left[\frac{1}{5}\sin 2x\right]+5\left[\frac{1}{5}\cos 2x\right]$$

$$\underline{y_p=2\sin 2x+\cos 2x}$$

2. $\left(D^2+3D-4\right)y_p=\sin 2x$

SOLUTION:

$$[\left(D^2+3D-4\right)y_p=\sin 2x\,]\ \frac{1}{D^2+3D-4}$$

$$y_p=\left[\frac{1}{D^2+3D-4}\sin 2x\right]$$

where $a=2, a^2=4, -a^2=-4$

$$y_p = \left[\frac{1}{-4+3D-4}\sin 2x\right]$$

$$y_p = \left[\frac{1}{3D-8}\sin 2x\right]$$

$$y_p = \left[\frac{1}{3D-8} \bullet \frac{3D+8}{3D+8}\sin 2x\right]$$

$$y_p = (3D+8)\left[\frac{1}{9D^2-64}\sin 2x\right] \quad \text{Recall: } -a^2 = -4$$

$$y_p = (3D+8)\left[\frac{1}{9(-4)-64}\sin 2x\right]$$

$$y_p = \frac{-1}{100}(3D+8)\sin 2x$$

$$y_p = \frac{-1}{100}\left[3(2\cos 2x)+8\sin 2x\right]$$

$$y_p = \frac{-2}{100}\left[3\cos 2x+4\sin 2x\right]$$

$$\underline{y_p = \frac{-1}{50}\left[3\cos 2x+4\sin 2x\right]}$$

3. $(D^2+4)y_p = \cos 2x + \cos 4x$

SOLUTION:

$$[(D^2+4)y_p = \cos 2x + \cos 4x\,]\frac{1}{D^2+4}$$

$$y_p = \left[\frac{1}{D^2+4}\cos 2x\right] + \left[\frac{1}{D^2+4}\cos 4x\right]$$

where $a = 2, a^2 = 4, -a^2 = -4$ where $a = 4, a^2 = 16, -a^2 = -16$

$$y_p = \left[\frac{1}{-4+4}\cos 2x\right] + \left[\frac{1}{-16+4}\cos 4x\right]$$

$$y_p = \frac{x}{2(2)}\sin 2x - \frac{1}{12}\cos 4x$$

$$\underline{y_p = \frac{1}{4}(x\sin 2x) - \frac{1}{12}(\cos 4x)}$$

Seatwork: Find the particular solution of the differential equation.

1. $(D^2 + 4)\ y_p = \sin 3x$ ans: $y_P = -\frac{1}{5}\sin 3x$

2. $(D^4 + 10D^2 + 9)\, y_p = \cos(2x + 3)$ ans: $y_P = -\frac{1}{15}\cos(2x+3)$

Homework: Find the particular solution of the differential equation.

1. $(D^2 + 16)\, y_p = \sin 2x + \cos 3x$ ans: $y_p = 1/12(\sin 2x) + 1/7(\cos 3x)$
2. $(D^2 + 9)\, y_p = 2\sin^2 x$ ans: $y_p = 1/9 - 1/5(\cos 2x)$
3. $(D - 6)\, y_p = \sin 2x$ ans: $y_p = -1/20(\cos 2x) - 3/20(\sin 2x)$
4. $(D^2 + D - 2)\, y_p = \sin 2x$ ans: $y_p = -1/20(\cos 2x + 3\sin 2x)$

3.3 R(x) is a Polynomial Function

Given: $F(D)\, y_P = x^n$

$$y_p = \left\{\frac{1}{F(D)}\right\} x^n = \left[1 + D + D^2 + \ldots + D^m\right] x^n; (m \geq n)$$

Note: $\left[1 + D + D^2 + \ldots + D^m\right]$ is obtained by expanding $\left\{\frac{1}{F(D)}\right\}$ in ascending powers of D and suppressing all terms beyond D^m since $D^m(x^n) = 0$ when $m > n$.

Example: Find the particular solution of the differential equation.

1. $(D^2 + 4)y_p = 4x + 3$

SOLUTION:

$$[(D^2 + 4)Y_P = 4x + 3]\ \frac{1}{D^2 + 4}$$

$$y_p = \left[\frac{1}{D^2 + 4}(4x + 3)\right]$$

$$\begin{array}{r|l} & \quad 1/4 \\ \hline 4 + 0D + D^2 & 1 + 0D + 0D^2 \\ & 1 + 0D + \tfrac{1}{4}D^2 \\ \hline & \qquad\quad -\tfrac{1}{4}D^2 \end{array}$$

$$y_p = \frac{1}{4}(4x + 3)$$

$$\underline{y_p = x + \tfrac{3}{4}}$$

2. $\left(D^2 + 2D + 2\right)y_p = 2x^2 + 3x + 8$

SOLUTION:

$$\left[\left(D^2 + 2D + 2\right)y_p = 2x^2 + 3x + 8\right] \frac{1}{D^2 + 2D + 2}$$

$$y_p = \left[\frac{1}{D^2 + 2D + 2}\left(2x^2 + 3x + 8\right)\right]$$

$$\begin{array}{r|l} & \tfrac{1}{2} - \tfrac{1}{2}D + \tfrac{1}{4}D^2 \\ \hline 2 + 2D + D^2 & 1 \\ & \underline{1 + D + \tfrac{1}{2}D^2} \\ & -D - \tfrac{1}{2}D^2 \\ & \underline{-D - D^2 - \tfrac{1}{2}D^3} \\ & \tfrac{1}{2}D^2 + \tfrac{1}{2}D^3 \end{array}$$

$$y_p = \left(\frac{1}{2} - \frac{1}{2}D + \frac{1}{4}D^2\right)\left(2x^2 + 3x + 8\right)$$

$$y_p = \frac{1}{2}\left(2x^2 + 3x + 8\right) - \frac{1}{2}D\left(2x^2 + 3x + 8\right) + \frac{1}{4}D^2\left(2x^2 + 3x + 8\right)$$

$$y_p = x^2 + \frac{3}{2}x + 4 - \frac{1}{2}(4x + 3) + \frac{1}{4}(4)$$

$$y_p = x^2 + \frac{3}{2}x + 4 - 2x - \frac{3}{2} + 1$$

$$\underline{y_p = x^2 - \tfrac{1}{2}x + \tfrac{7}{2}}$$

Seatwork: Find the particular solution of the differential equation.

1. $(2D^2 + 2D + 3)\, y_p = x^2 + 2x - 1$ ans: $y_p = \dfrac{1}{3}x^2 + \dfrac{2}{9}x - \dfrac{25}{27}$

2. $(D^3 - 4D^2 + 3D)\, y_p = x^2$ ans: $y_p = \dfrac{1}{9}x^3 + \dfrac{4}{9}x^2 + \dfrac{26}{27}x$

Homework: Find the particular solution of the differential equation.

1. $(D^2 + 1)\, y_p = x^6$ ans: $y_p = x^6 - 30x^4 + 360x^2 - 720$
2. $(D^2 + D - 2)\, y_p = x^2$ ans: $y_p = -\frac{1}{2}\left(x^2 + x + 3/2\right)$

3.4 R(x) is a Composite Function

3.4.1 Given: $F(D)y_p = e^{ax}Q(x)$

$$y_p = \left\{\frac{1}{F(D)}\right\}e^{ax}Q(x)$$

Using "exponential shift"

$$y_p = e^{ax}\left[\left\{\frac{1}{F(D+a)}\right\}Q(x)\right]$$

3.4.2 Given: $F(D)y_p = xQ(x)$

Using "x–shift"

$$y_p = x\left[\left\{\frac{1}{F(D)}\right\}Q(x)\right] - \left[\left\{\frac{F'(D)}{[F(D)]^2}\right\}Q(x)\right]$$

Example: Find the particular solution of the differential equation.

1. $(D^2 - 2D)y_p = e^x \sin x$

SOLUTION:

$$[(D^2 - 2D)y_p = e^x \sin x\,]\frac{1}{D^2 - 2D}$$

$$y_p = e^x\left[\frac{1}{D^2 - 2D}\sin x\right] \quad ; \quad D \to D+1$$

$$y_p = e^x\left[\frac{1}{(D+1)^2 - 2(D+1)}\sin x\right]$$

$$y_p = e^x\left[\frac{1}{D^2 - 1}\sin x\right]$$

where $a=1, a^2=1, -a^2=-1$

$$y_p = e^x\left[\frac{1}{-1-1}\sin x\right]$$

$$y_p = e^x\left[-\frac{1}{2}\sin x\right]$$

$$\underline{y_p = -\frac{1}{2}\left(e^x \sin x\right)}$$

2. $\left(D^2+2D+4\right)y_p = e^x \sin 2x$

SOLUTION:

$$[\left(D^2+2D+4\right)y_p = e^x \sin 2x\,]\ \frac{1}{D^2+2D+4}$$

$$y_p = \left(\frac{1}{D^2+2D+4}e^x \sin 2x\right) \quad ; \quad D \to D+1$$

$$y_p = e^x\left(\frac{1}{[D+1]^2+2[D+1]+4}\sin 2x\right)$$

$$y_p = e^x\left(\frac{1}{D^2+2D+1+2D+2+4}\sin 2x\right)$$

$$y_p = e^x\left(\frac{1}{D^2+4D+7}\sin 2x\right)$$

where $a=2, a^2=4, -a^2=-4$

$$y_p = e^x\left(\frac{1}{-4+4D+7}\sin 2x\right)$$

$$y_p = e^x\left(\frac{1}{4D+3}\bullet\frac{4D-3}{4D-3}\sin 2x\right)$$

$$y_p = e^x\left(4D-3\right)\left(\frac{1}{16D^2-9}\sin 2x\right)$$

Recall: $-a^2=-4$

$$y_p = e^x\left(4D-3\right)\left(\frac{1}{16(-4)-9}\sin 2x\right)$$

$$y_p = \frac{-1}{73}e^x\left(4D-3\right)\sin 2x$$

$$y_p = \frac{-1}{73}e^x\left[4D(\sin 2x)-3\sin 2x\right]$$

$$y_p = \frac{-1}{73}e^x\left[8(\cos 2x)-3\sin 2x\right]$$

$$\underline{y_p = -\frac{1}{73}\left(e^x\right)\left(8\cos 2x-3\sin 2x\right)}$$

3. $\left(D^2+3D+2\right)y_p = x\sin 2x$

SOLUTION:

$$[\left(D^2+3D+2\right)y_p = x\sin 2x\,]\ \frac{1}{D^2+3D+2}$$

$$y_p = \left[\frac{1}{D^2+3D+2}\left(x\sin 2x\right)\right]$$

$$y_p = x\left[\frac{1}{D^2+3D+2}\sin 2x\right] - \left[\frac{2D+3}{\left(D^2+3D+2\right)^2}\sin 2x\right]$$

$$y_{p1} = x\left[\frac{1}{D^2+3D+2}\sin 2x\right]$$

where $a=2, a^2=4, -a^2=-4$

$$y_{p1} = x\left[\frac{1}{-4+3D+2}\sin 2x\right]$$

$$y_{p1} = x\left[\frac{1}{3D-2}\bullet\frac{3D+2}{3D+2}\sin 2x\right]$$

$$y_{p1} = x\left(3D+2\right)\left[\frac{1}{9D^2-4}\sin 2x\right]$$

Recall: $-a^2=-4$

$$y_{p1} = x\left(3D+2\right)\left[\frac{1}{9(-4)-4}\sin 2x\right]$$

$$y_{p1} = -\frac{x}{40}\left(3D+2\right)\sin 2x$$

$$y_{p1} = -\frac{x}{40}\left[3D\sin 2x + 2\sin 2x\right]$$

$$y_{p1} = -\frac{x}{40}\left[3(2\cos 2x) + 2\sin 2x\right]$$

$$y_{p1} = -\frac{2x}{40}\left[3\cos 2x + 2\sin 2x\right]$$

$$\underline{y_{p1} = -\frac{1}{20}\left(x\right)\left(3\cos 2x+\sin 2x\right)}$$

$$y_{p2} = \left(2D+3\right)\left[\frac{1}{\left(D^2+3D+2\right)^2}\sin 2x\right]$$

$$y_{p2} = \left(2D+3\right)\left[\frac{1}{\left(3D-2\right)^2}\sin 2x\right]$$

$$y_{p2} = (2D+3)\left[\frac{1}{9D^2 - 12D + 4}\sin 2x\right]$$

Recall: $-a^2 = -4$

$$y_{p2} = (2D+3)\left[\frac{1}{9(-4) - 12D + 4}\sin 2x\right]$$

$$y_{p2} = (2D+3)\left[\frac{1}{-36 - 12D + 4}\sin 2x\right]$$

$$y_{p2} = (2D+3)\left[\frac{1}{-12D - 32}\sin 2x\right]$$

$$y_{p2} = \frac{-1}{4}(2D+3)\left[\frac{1}{3D+8} \bullet \frac{3D-8}{3D-8}\sin 2x\right]$$

$$y_{p2} = \frac{-1}{4}(2D+3)(3D-8)\left[\frac{1}{9D^2 - 64}\sin 2x\right]$$

Recall: $-a^2 = -4$

$$y_{p2} = -\frac{1}{4}\left(6D^2 - 7D - 24\right)\left[\frac{1}{9(-4) - 64}\sin 2x\right]$$

$$y_{p2} = \frac{1}{400}\left(6D^2 - 7D - 24\right)\sin 2x$$

$$y_{p2} = \frac{1}{400}\left(6D^2 \sin 2x - 7D\sin 2x - 24\sin 2x\right)$$

$$y_{p2} = \frac{1}{400}\left[6(-4\sin 2x) - 7(2\cos 2x) - 24\sin 2x\right]$$

$$y_{p2} = \frac{1}{400}(-24\sin 2x - 14\cos 2x - 24\sin 2x)$$

$$y_{p2} = \frac{1}{400}(-48\sin 2x - 14\cos 2x)$$

$$y_{p2} = -\frac{2}{400}(24\sin 2x + 7\cos 2x)$$

$$\underline{y_{p2} = -\frac{1}{200}(24\sin 2x + 7\cos 2x)}$$

Seatwork: Find the particular solution of the differential equation.

1. $(D^2 - 4)\ y_p = x^2 e^{3x}$

   ans: $y_p = e^{3x}\left[\frac{1}{5}x^2 - \frac{12}{25}x + \frac{62}{125}\right]$

2. $(D^3 - 3D^2 - 6D + 8)\ y = xe^{-3x}$

   ans: $y_p = -\frac{1}{28}xe^{-3x} - \frac{39}{784}e^{-3x}$

Homework: Find the particular solution of the differential equation.

1. $(D - 2)^2\ y_p = x\ e^{2x}$ ans: $y_p = 1/6\ (x^3e^{2x})$
2. $(D^2 - 4D + 13)\ y_p = e^{2x}\ \sin x$ ans: $y_p = 1/8\ (e^{2x}\ \sin x)$
3. $(D^2 + 9)\ y_p = x\ \sin 2x$ ans: $y_p = 1/25(5x\sin 2x - 4\cos 2x)$
4. $(D + 1)\ y_p = x\ \cos x$ ans: $y_p = x/2(\sin x + \cos x) - ½(\sin x)$