TRIGONOMETRY

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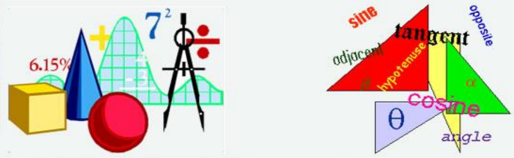

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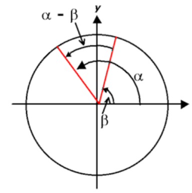
Trigonometric Identities



- Fundamental Trigonometric Identities
  - Proving
  - Addition & Subtraction
  - Double, Half-Angle and Product-Sum

Trigonometry

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Trigonometric Identities

DEFINITION

Two functions f and g are said to be **identically equal** if

$$\boxed{f(x) = g(x)}$$

for every value of x for which both functions are defined. Such an equation is referred to as an **identity.** An equation that is not an identity is called a **conditional equation.**

For example, the following are identities:

$$(x+1)^2 = x^2 + 2x + 1 \qquad \sin^2 x + \cos^2 x = 1 \qquad \csc x = \frac{1}{\sin x}$$

The following are conditional equations:

$$2x + 5 = 0 \qquad \text{True only if } x = -\frac{5}{2}$$

$$\sin x = 0 \qquad \text{True only if } x = k\pi,\ k \text{ an integer}$$

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Trigonometric Identities

The following boxes summarize the trigonometric identities that we have established thus far.

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

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Trigonometric Identities

Guidelines for Establishing Identities

1. It is almost always preferable to start with the side containing the more complicated expression.
2. Rewrite sums or differences of quotients as a single quotient.
3. Sometimes rewriting one side in terms of sine and cosine functions only will help.
4. Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

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Establish the identity: $\csc\theta \cdot \tan\theta = \sec\theta$

Establish the identity: $\sin^2(-\theta) + \cos^2(-\theta) = 1$

Trigonometric Identities

Establish the identity: $\dfrac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$

Establish the identity: $\dfrac{1 + \tan u}{1 + \cot u} = \tan u$

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Establish the identity: $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = 2\csc\theta$

Establish the identity: $\dfrac{\tan v + \cot v}{\sec v \csc v} = 1$

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Establish the identity: $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$

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Trigonometric Identities

Definition 6.2 Sum and Difference Identities

Sum and Difference Identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

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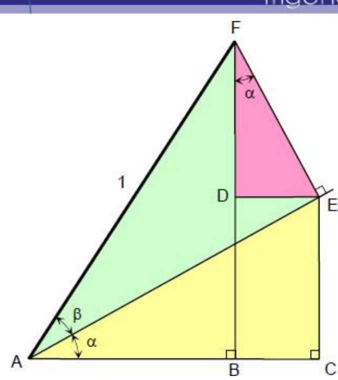

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Consider triangle *AEF*:

$\cos\beta = \dfrac{\overline{AE}}{1}$; $\overline{AE} = \cos\beta$

$\sin\beta = \dfrac{\overline{EF}}{1}$; $\overline{EF} = \sin\beta$

From triangle *EDF*:

$\sin\alpha = \dfrac{\overline{DE}}{\overline{EF}}$

$\sin\alpha = \dfrac{\overline{DE}}{\sin\beta}$

$\overline{DE} = \sin\alpha\,\sin\beta$

$\cos\alpha = \dfrac{\overline{DF}}{\overline{EF}}$

$\cos\alpha = \dfrac{\overline{DF}}{\sin\beta}$

$\overline{DF} = \cos\alpha\,\sin\beta$

$\overline{BC} = \overline{DE} = \sin\alpha\,\sin\beta$

From Triangle *ACE*:

$\sin\alpha = \dfrac{\overline{CE}}{\overline{AE}}$

$\sin\alpha = \dfrac{\overline{CE}}{\cos\beta}$

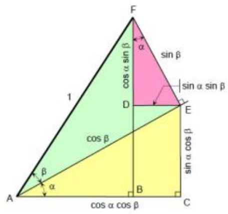
$\overline{CE} = \sin\alpha\,\cos\beta$

$\cos\alpha = \dfrac{\overline{AC}}{\overline{AE}}$

$\cos\alpha = \dfrac{\overline{AC}}{\cos\beta}$

$\overline{AC} = \cos\alpha\,\cos\beta$

$\overline{BD} = \overline{CE} = \sin\alpha\,\cos\beta$



$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$\tan(\alpha+\beta)=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$\tan(\alpha+\beta)=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\tan(\alpha+\beta)=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Sum of two angles

From triangle *ABF*
$\sin(\alpha+\beta)=\overline{BD}+\overline{DF}$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$\cos(\alpha+\beta)=\overline{AC}-\overline{BC}$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

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Trigonometric Identities

Example 6.2.1 Use sum or difference identities to find the exact value of the given function.

a) $\cos 105°$ b) $\sin 15°$ c) $\tan 375°$

Example 6.2.2 Given that $\sin A=\frac{3}{5}$, $\tan B=\frac{12}{5}$, $0<A<\frac{\pi}{2}$, and $\pi<B<\frac{3\pi}{2}$, find the following:

a) $\sin(A+B)$ c) $\cos(A-B)$

b) $\tan(A-B)$ d) Quadrant of $A-B$

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Example 6.2.3 Write each expression in terms of a trigonometric function of one angle.

a) $\sin 80°\cos 25° + \cos 80°\sin 25°$

b) $\cos 34°\cos 48° - \sin 34°\sin 48°$

c) $\cos\alpha\cos 3\alpha+\sin\alpha\sin 3\alpha$

d) $\dfrac{\tan\frac{3\pi}{4}+\tan\frac{\pi}{4}}{1-\tan\frac{3\pi}{4}\tan\frac{\pi}{4}}$

Example 6.2.4 Prove each identity.

a) $\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$

b) $\tan(\beta+45°)+\tan(\beta-45°)=2\tan 2\beta$

c) $\sin(270°+\beta)=-\cos\beta$

d) $\tan(360°-\beta)=-\tan\beta$

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Definition 6.3 Double Angle Identities

Double Angle Identities

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A$$
$$\cos 2A = 2\cos^2 A-1$$
$$\cos 2A = 1-2\sin^2 A$$
$$\tan 2A=\frac{2\tan A}{1-\tan^2 A},\qquad \tan A\neq 1$$

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Example 6.3.1 Write each expression in terms of a trigonometric function of one angle.

a) $2\sin 35°\cos 35°$ c) $1-2\sin^2 40°$

b) $\dfrac{2\tan 22.5°}{1-\tan^2 22.5°}$ d) $2\sin 67.5°\cos 67.5°$

Example 6.3.2 Use double angle identities to find the exact value of each trigonometric function.

a) If $\sin\alpha=\frac{3}{5}$ and $0°<\alpha<90°$, find $\cos 2\alpha$

b) If $\tan\theta=-\frac{4}{3}$ and $90°<\theta<180°$, find $\sin 2\theta$

c) If $\cos\beta=\frac{5}{13}$ and $270°<\beta<360°$, find $\tan 2\beta$

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Definition 6.4 Half-Angle Identities

Double Angle Identities

$$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}},\qquad \cos A\neq -1$$
$$\tan\frac{A}{2}=\frac{1-\cos A}{\sin A},\qquad \sin A\neq 0$$
$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A},\qquad \cos A\neq -1$$

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Trigonometric Identities

Example 6.4.1 Use half angle identities to find the exact values of the following.

a) $\tan 105°$

b) $\cos \frac{7\pi}{8}$

c) $\sin 22.5°$

d) $\sin \frac{\pi}{12}$

Example 6.4.2 Find the exact value of each trigonometric function. Assume $0 < \theta < 360°$.

a) $\cos \frac{\theta}{2}$ if $\cos \theta = \frac{4}{5}$ and θ lie in quadrant 1.

b) $\cos \frac{\theta}{2}$ if $\cos 2\theta = -\frac{12}{13}$ and θ lies in quadrant 3.

c) $\tan \frac{\theta}{2}$ if $\tan \theta = -2$ and θ lies in quadrant 2.

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Trigonometric Identities

Definition 6.5 Product/Sum Identities

Product/Sum Identities

$$2\cos A\cos B = \cos(A-B)+\cos(A+B)$$
$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$
$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$
$$2\cos A\sin B = \sin(A+B)-\sin(A-B)$$

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Example 6.5.1 Express each product as a sum or difference.

a) $2 \sin 50° \cos 20°$
b) $2 \cos 40° \cos 10°$

Example 6.5.2 Express each sum or difference as a product.

a) $\cos 70° + \cos 10°$
b) $\cos 4x - \cos 8x$
c) $\cos 12x + \cos 4x$
d) $\sin 10x - \sin 6x$