2
Regular Polygons

We begin this chapter by discussing (without proofs) the possibility of constructing certain regular polygons with the instruments allowed by Euclid. We then consider all these polygons, regardless of the question of constructibility, from the standpoint of symmetry. Finally, we extend the concept of a regular polygon so as to include star polygons.

2.1 CYCLOTOMY

One, two! One, two! And through and through
The vorpal blade went snicker-snack!

Lewis Carroll
[Dodgson **2**, Chap. 1]

Euclid's postulates imply a restriction on the instruments that he allowed for making constructions, namely the restriction to ruler (or straightedge) and compasses. He constructed an equilateral triangle (I.1), a square (IV.6), a regular pentagon (IV.11), a regular hexagon (IV.15), and a regular 15-gon (IV.16). The number of sides may be doubled again and again by repeated angle bisections. It is natural to ask which other regular polygons can be constructed with Euclid's instruments. This question was completely answered by Gauss (1777–1855) at the age of nineteen [see Smith **2**, pp. 301–302]. Gauss found that a regular n-gon, say $\{n\}$, can be so constructed if the odd prime factors of n are distinct "Fermat primes"

$$F_k = 2^{2^k} + 1.$$

The only known primes of this kind are

$$F_0 = 2^1 + 1 = 3, \quad F_1 = 2^2 + 1 = 5, \quad F_2 = 2^4 + 1 = 17,$$
$$F_3 = 2^8 + 1 = 257, \quad F_4 = 2^{16} + 1 = 65537.$$

To inscribe a regular pentagon in a given circle, simpler constructions than Euclid's were given by Ptolemy and Richmond.* The former has been repeated in many textbooks. The latter is as follows (Figure 2.1a).

To inscribe a regular pentagon $P_0P_1P_2P_3P_4$ in a circle with center O: draw the radius OB perpendicular to OP_0; join P_0 to D, the midpoint of OB; bisect the angle ODP_0 to obtain N_1 on OP_0; and draw N_1P_1 perpendicular to OP_0 to obtain P_1 on the circle. Then P_0P_1 is a side of the desired pentagon.

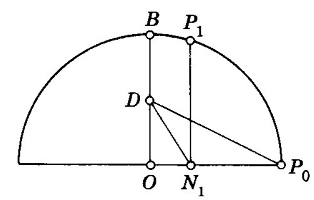


Figure 2.1a

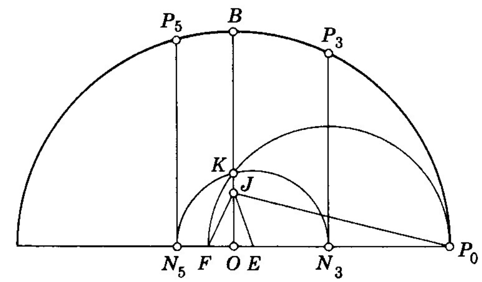


Figure 2.1b

Richmond also gave a simple construction for the $\{17\}$ $P_0P_1 \ldots P_{16}$ (Figure 2.1b). Join P_0 to J, one quarter of the way from O to B. On the diameter through P_0 take E, F, so that $\angle OJE$ is one quarter of OJP_0 and $\angle FJE$ is 45°. Let the circle on FP_0 as diameter cut OB in K, and let the circle with center E and radius EK cut OP_0 in N_3 (between O and P_0) and N_5. Draw perpendiculars to OP_0 at these two points, to cut the original circle in P_3 and P_5. Then the arc P_3P_5 (and likewise P_1P_3) is $\frac{2}{17}$ of the circumference. (The proof involves repeated application of the principle that the roots of the equation $x^2 + 2x \cot 2C - 1 = 0$ are $\tan C$ and $-\cot C$.)

Richelot and Schwendenwein constructed the regular 257-gon in 1832. J. Hermes spent ten years on the regular 65537-gon and deposited the manuscript in a large box in the University of Göttingen, where it may still be found.

The next number of the form $F_k = 2^{2^k} + 1$ is $F_5 = 4294967297$. Fermat incorrectly assumed it to be prime. G. T. Bennett gave the following neat proof† that it is composite [Hardy and Wright **1**, p. 14]: the number

$$641 = 5^4 + 2^4 = 5 \cdot 2^7 + 1,$$

dividing both $5^4 \cdot 2^{28} + 2^{32}$ and $5^4 \cdot 2^{28} - 1$, divides their difference, which is F_5.

* H. W. Richmond, *Quarterly Journal of Mathematics*, **26** (1893), pp. 296–297; see also H. E. Dudeney, *Amusements in Mathematics* (London 1917), p. 38.

† Rediscovered by P. Kanagasabapathy, *Mathematical Gazette*, **42** (1958), p. 310.

The question naturally arises whether F_k may be prime for some greater value of k. It is now known that this can happen only if F_k divides $3^{(F_k-1)/2} + 1$. Using this criterion, electronic computing machines have shown that F_k is composite for $5 \leqslant k \leqslant 16$. Therefore Hermes's construction is the last of its kind that will ever be undertaken!

EXERCISES

1. Verify the correctness of Richmond's construction for $\{5\}$ (Figure 2.1*a*).
2. Assuming Richmond's construction for $\{17\}$, how would you inscribe $\{51\}$ in the same circle?

2.2 ANGLE TRISECTION

To trisect a given angle, we may proceed to find the sine of the angle—say a—then, if x is the sine of an angle equal to one-third of the given angle, we have $4x^3 = 3x - a$.

W. W. Rouse Ball (1850-1925)

[Ball **1**, p. 327]

Gauss was almost certainly aware of the fact that his cyclotomic condition is necessary as well as sufficient, but he does not seem to have said so explicitly. The missing step was supplied by Wantzel*, who proved that, if the odd prime factors of n are *not* distinct Fermat primes, $\{n\}$ cannot be constructed with ruler and compasses. For instance, since 7 is not a Fermat prime, Euclid's instruments will not suffice for the regular heptagon $\{7\}$; and since the factors of 9 are not distinct, the same is true for the enneagon $\{9\}$.

The problem of trisecting an arbitrary angle with ruler and compasses exercised the ingenuity of professional and amateur mathematicians for two thousand years [Ball **1**, pp. 333–335]. It is, of course, easy to trisect certain particular angles, such as a right angle. But any construction for trisecting an arbitrary angle could be applied to an angle of 60°, and then we could draw a regular enneagon. In view of Wantzel's theorem, we may say that it has been known since 1837 that the classical trisection problem can never be solved.

This is probably the reason why Morley's Theorem (§1.9) was not discovered till the twentieth century: people felt uneasy about mentioning the trisectors of an angle. However, although the trisectors cannot be constructed by means of the ruler and compasses, they can be found in other ways [Cundy and Rollett **1**, pp. 208–211]. Even if these more versatile instruments had never been discovered, the theorem would still be meaningful. Most mathematicians are willing to accept the existence of things that they have not been able to construct. For instance, it was proved in 1909 that the Fermat numbers F_7 and F_8 are composite, but their smallest prime factors still remain to be computed.

EXERCISE

The number $2^n + 1$ is composite whenever n is not a power of 2.

* P. L. Wantzel, *Journal de Mathématiques pures et appliquées*, **2** (1837), pp. 366–372.

2.3 ISOMETRY

One way of describing the structure of space, preferred by both Newton and Helmholtz, is through the notion of congruence. Congruent parts of space V, V′ are such as can be occupied by the same rigid body in two of its positions. If you move the body from one into the other position the particle of the body covering a point P of V will afterwards cover a certain point P′ of V′, and thus the result of the motion is a mopping P → P′ of V upon V′. We can extend the rigid body either actually or in imagination so as to cover an arbitrarily given point P of space, and hence the congruent mopping P → P′ can be extended to the entire space.

Hermann Weyl (1885-1955)

[Weyl **1**, p. 43]

We shall find it convenient to use the word *transformation* in the special sense of a one-to-one correspondence $P \to P'$ among all the points in the plane (or in space), that is, a rule for associating pairs of points, with the understanding that each pair has a first member P and a second member P' and that every point occurs as the first member of just one pair and also as the second member of just one pair. It may happen that the members of a pair coincide, that is, that P' coincides with P; in this case P is called an *invariant* point (or "double point") of the transformation.

In particular, an *isometry* (or "congruent transformation," or "congruence") is a transformation which preserves length, so that, if (P, P') and (Q, Q') are two pairs of corresponding points, we have $PQ = P'Q'$: PQ and $P'Q'$ are congruent segments. For instance, a *rotation* of the plane about P (or about a line through P perpendicular to the plane) is an isometry having P as an invariant point, but a *translation* (or "parallel displacement") has no invariant point: every point is moved.

A *reflection* is the special kind of isometry in which the invariant points consist of all the points on a line (or plane) called the *mirror*.

A still simpler kind of transformation (so simple that it may at first seem too trivial to be worth mentioning) is the *identity*, which leaves every point unchanged. The result of applying several transformations successively is called their *product*. If the product of two transformations is the identity, each is called the *inverse* of the other, and their product in the reverse order is again the identity.

2.31 *If an isometry has more than one invariant point, it must be either the identity or a reflection.*

To prove this, let A and B be two invariant points, and P any point not on the line AB (Figure 1.3*b*). The corresponding point P', satisfying

$$AP' = AP, \qquad BP' = BP,$$

must lie on the circle with center A and radius AP, and on the circle with center B and radius BP. Since P is not on AB, these circles do not touch each other but intersect in two points, one of which is P. Hence P' is either P itself or the image of P by reflection in AB.

2.4 SYMMETRY

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Dare frame thy fearful symmetry?

William Blake (1757-1827)

When we say that a figure is "symmetrical" we mean that we can apply certain isometries, called *symmetry operations*, which leave the whole figure unchanged while permuting its parts. For example, the capital letters E and A (Figure 2.4*a*) have bilateral symmetry, the mirror being horizontal for the former, vertical for the latter. The letter N (Figure 2.4*b*) is symmetrical by a *half-turn*, or rotation through 180° (or "reflection in a point," or "central inversion"), which may be regarded as the result of reflecting horizontally and then vertically, or vice versa. The swastika (Figure 2.4*c*) is symmetrical by rotation through any number of right angles.

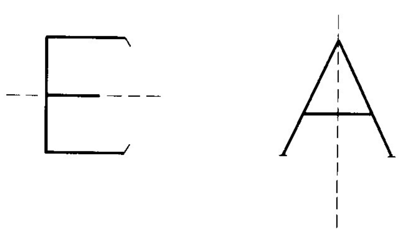


Figure 2.4a

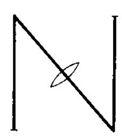


Figure 2.4b

In counting the symmetry operations of a figure, it is usual to include the identity; any figure has this trivial symmetry. Thus the swastika admits four distinct symmetry operations: rotations through 1, 2, 3, or 4 right angles. The last is the identity. The first and third are inverses of each other, since their product is the identity.

This use of the word "product" suggests an algebraic symbolism in which the transformations are denoted by capital letters while 1 denotes the identity. (Instead of 1, some authors write E.) Thus if S is the counterclockwise quarter-turn, the four symmetry operations of the swastika are

$$\mathrm{S},\quad \mathrm{S}^2,\quad \mathrm{S}^3 = \mathrm{S}^{-1} \quad\text{and}\quad \mathrm{S}^4 = 1.$$

Since the smallest power of S that is equal to the identity is the fourth power, we say that S is of *period* 4. Similarly S^2, being a half-turn, is of period 2 [see Coxeter **1**, p. 39]. The only transformation of period 1 is the identity. A translation is aperiodic (that is, it has no period), but it is conveniently said to be of infinite period.

Some figures admit both reflections and rotations as symmetry operations. The letter H (Figure 2.4*d*) has a horizontal mirror (like E) and a vertical mirror (like A), as well as a center of rotational symmetry (like N) where the two mirrors intersect. Thus it has four symmetry operations: the identity 1, the horizontal reflection R_1, the vertical reflection R_2, and the half-turn $\mathrm{R}_1\mathrm{R}_2 = \mathrm{R}_2\mathrm{R}_1$.

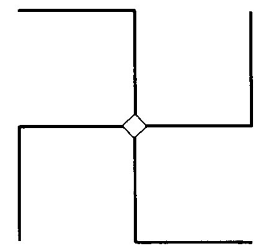


Figure 2.4c

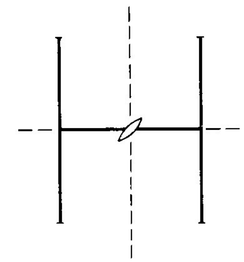


Figure 2.4d

EXERCISES

1. Every isometry of period 2 is either a reflection or a half-turn [Bachmann **1**, pp. 2-3].

2. Express (*a*) a half-turn, (*b*) a quarter-turn, as transformations of (i) Cartesian coordinates, (ii) polar coordinates. (Take the origin to be the center of rotation.)

2.5 GROUPS

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

Hermann Weyl [**1**, p. 5]

A set of transformations [Birkhoff and MacLane **1**, pp. 115-118] is said to form a *group* if it contains the inverse of each and the product of any two (including the product of one with itself or with its inverse). The number of distinct transformations is called the *order* of the group. (This may be either finite or infinite.) Clearly the symmetry operations of any figure form a group. This is called the *symmetry group* of the figure. In the extreme case when the figure is completely irregular (like the numeral 6) its symmetry group is of order one, consisting of the identity alone.

The symmetry group of the letter E or A (Figure 2.4*a*) is the so-called *dihedral* group of order 2, generated by a single reflection and denoted by D_1. (The name is easily remembered, as the Greek origin of the word "dihedral" is almost equivalent to the Latin origin of "bilateral.") The symmetry group of the letter N (Figure 2.4*b*) is likewise of order 2, but in this case the generator is a half-turn and we speak of the *cyclic* group, C_2. The two groups D_1 and C_2 are abstractly identical or *isomorphic;* they are different geometrical representations of the single abstract group of order 2, defined by the relation

2.51 $$R^2 = 1$$

or $R = R^{-1}$ [Coxeter and Moser **1**, p. 1].

The symmetry group of the swastika is C_4, the cyclic group of order 4, generated by the quarter-turn S and abstractly defined by the relation $S^4 = 1$. That of the letter H (Figure 2.4*d*) is D_2, the dihedral group of order 4, generated by the two reflections R_1, R_2 and abstractly defined by the relations

2.52 $$R_1^2 = 1, \quad R_2^2 = 1, \quad R_1R_2 = R_2R_1.$$

Although C_4 and D_2 both have order 4, they are *not* isomorphic: they have a different structure, different "multiplication tables." To see this, it suffices to observe that C_4 contains two operations of period 4, whereas all the operations in D_2 (except the identity) are of period 2: the generators obviously, and their product also, since

$$(R_1R_2)^2 = R_1R_2R_1R_2 = R_1R_2R_2R_1 = R_1R_2^2R_1 = R_1R_1 = R_1^2 = 1.$$

This last remark illustrates what we mean by saying that 2.52 is an *abstract definition* for D_2, namely that every true relation concerning the generators R_1, R_2 is an algebraic consequence of these simple relations. An alternative abstract definition for the same group is

2.53 $$R_1^2 = 1, \quad R_2^2 = 1, \quad (R_1R_2)^2 = 1,$$

from which we can easily deduce $R_1R_2 = R_2R_1$.

The general cyclic group C_n, of order n, has the abstract definition

2.54 $$S^n = 1.$$

Its single generator S, of period n, is conveniently represented by a rotation through $360°/n$. Then S^k is a rotation through k times this angle, and the n operations in C_n are given by the values of k from 1 to n, or from 0 to $n - 1$. In particular, C_5 occurs in nature as the symmetry group of the periwinkle flower.

EXERCISE

Express a rotation through angle α about the origin as a transformation of (i) polar coordinates, (ii) Cartesian coordinates. If $f(r, \theta) = 0$ is the equation for a curve in polar coordinates, what is the equation for the transformed curve?

2.6 THE PRODUCT OF TWO REFLECTIONS

Thou in thy lake dost see
Thyself.

J. M. Legoré (1823-1859)
(To a Lily)

In any group of transformations, the associative law

$$(RS)T = R(ST)$$

is automatically satisfied, but the commutative law

$$RS = SR$$

does not necessarily hold, and care must be taken in inverting a product, for example,

$$(RS)^{-1} = S^{-1}R^{-1},$$

not $R^{-1}S^{-1}$. (This becomes clear when we think of R and S as the operations of putting on our socks and shoes, respectively.)

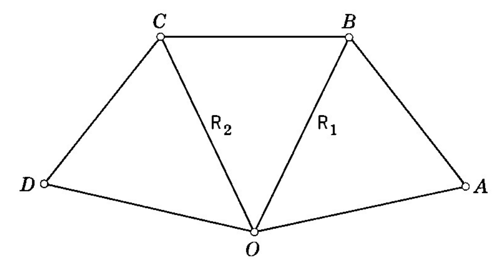


Figure 2.6a

The product of reflections in two intersecting lines (or planes) is a rotation through twice the angle between them. In fact, if *A, B, C, D, . . .* are evenly spaced on a circle with center *O*, let R_1 and R_2 be the reflections in *OB* and *OC* (Figure 2.6*a*). Then R_1 reflects the triangle *OAB* into *OCB*, which is reflected by R_2 to *OCD;* thus R_1R_2 is the rotation through $\angle AOC$ or $\angle BOD$, which is twice $\angle BOC$. Since a rotation is completely determined by its center and its angle, R_1R_2 is equal to the product of reflections in any two lines through *O* making the same angle as *OB* and *OC*. (The reflections in *OA* and *OB* are actually $R_1R_2R_1$ and R_1, whose product is $R_1R_2R_1^2 = R_1R_2$.) In particular, the half-turn about *O* is the product of reflections in any two perpendicular lines through *O*.

Since R_1R_2 is a counterclockwise rotation, R_2R_1 is the corresponding clockwise rotation; in fact,

$$R_2R_1 = R_2^{-1}R_1^{-1} = (R_1R_2)^{-1}.$$

This is the same as R_1R_2 if the two mirrors are at right angles, in which case R_1R_2 is a half-turn and $(R_1R_2)^2 = 1$.

EXERCISES

1. The product of quarter-turns (in the same sense) about C and B is the half-turn about the center of a square having BC for a side.
2. Let $ACPQ$ and $BARS$ be squares on the sides AC and BA of a triangle ABC. If B and C remain fixed while A varies freely, PS passes through a fixed point.

2.7 THE KALEIDOSCOPE

D_2 is a special case of the general dihedral group D_n, which is, for $n > 2$, the symmetry group of the regular n-gon, $\{n\}$. (See Figure 2.7a for the cases $n = 3, 4, 5$.) This is evidently a group of order $2n$, consisting of n rotations (through the n effectively distinct multiples of $360°/n$) and n reflections. When n is odd, each of the n mirrors joins a vertex to the midpoint of the opposite side; when n is even, $\frac{1}{2}n$ mirrors join pairs of opposite vertices and $\frac{1}{2}n$ bisect pairs of opposite sides [see Birkhoff and MacLane **1**, pp. 117–118, 135].

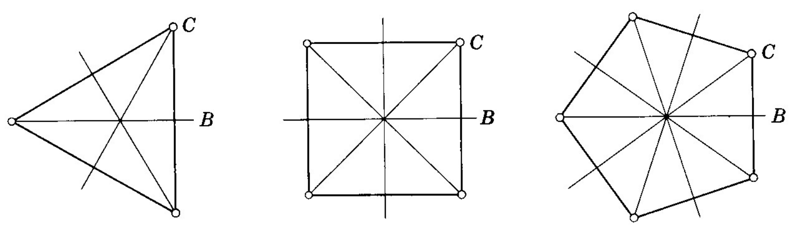


Figure 2.7a

The n rotations are just the operations of the cyclic group C_n. Thus the operations of D_n include all the operations of C_n: in technical language, C_n is a *subgroup* of D_n. The rotation through $360°/n$, which generates the subgroup, may be described as the product $S = R_1R_2$ of reflections in two adjacent mirrors (such as OB and OC in Figure 2.7a) which are inclined at $180°/n$.

Let $R_1, R_2, \ldots, R_n$ denote the n reflections in their natural order of arrangement. Then R_1R_{k+1}, being the product of reflections in two mirrors inclined at k times $180°/n$, is a rotation through k times $360°/n$:

$$R_1R_{k+1} = S^k.$$

Thus $R_{k+1} = R_1S^k$, and the n reflections may be expressed as

$$R_1, R_1S, R_1S^2, \ldots, R_1S^{n-1}.$$

In other words, D_n is generated by R_1 and S. By substituting R_1R_2 for S, we

see that the same group is equally well generated by R_1 and R_2, which satisfy the relations

2.71 $$R_1^2 = 1, \quad R_2^2 = 1, \quad (R_1R_2)^n = 1.$$

(The first two relations come from 2.51 and the third from 2.54.) These relations can be shown to suffice for an abstract definition [see Coxeter and Moser **1**, pp. 6, 36].

A practical way to make a model of D_n is to join two ordinary mirrors by a hinge and stand them on the lines OB, OC of Figure 2.7a so that they are inclined at $180°/n$. Any object placed between the mirrors yields $2n$ visible images (including the object itself). If the object is your right hand, n of the images will look like a left hand, illustrating the principle that, since a reflection reverses sense, the product of any even number of reflections preserves sense, and the product of any odd number of reflections reverses sense.

The first published account of this instrument seems to have been by Athanasius Kircher in 1646. The name *kaleidoscope* (from καλος, beautiful; ειδος, a form; and σκοπειν, to see) was coined by Sir David Brewster, who wrote a treatise on its theory and history. He complained [Brewster **1**, p. 147] that Kircher allowed the angle between the two mirrors to be any submultiple of 360° instead of restricting it to submultiples of 180°.

The case when $n = 2$ is, of course, familiar. Standing between two perpendicular mirrors (as at a corner of a room), you see your image in each and also the image of the image, which is the way other people see you.

Having decided to use the symbol D_n for the dihedral group generated by reflections in two planes making a "dihedral" angle of $180°/n$, we naturally stretch the notation so as to allow the extreme value $n = 1$. Thus D_1 is the group of order 2 generated by a single reflection, that is, the symmetry group of the letter E or A, whereas the isomorphic group C_2, generated by a half-turn, is the symmetry group of the letter N.

According to Weyl [**1**, pp. 66, 99], it was Leonardo da Vinci who discovered that the only finite groups of isometries in the plane are

$$C_1, C_2, C_3, \ldots,$$
$$D_1, D_2, D_3, \ldots.$$

His interest in them was from the standpoint of architectural plans. Of course, the prevalent groups in architecture have always been D_1 and D_2. But the pyramids of Egypt exhibit the group D_4, and Leonardo's suggestion has been followed to some extent in modern times: the Pentagon Building in Washington has the symmetry group D_5, and the Bahai Temple near Chicago has D_9. In nature, many flowers have dihedral symmetry groups such as D_6. The symmetry group of a snowflake is usually D_6 but occasionally only D_3. [Kepler **1**, pp. 259–280.]

If you cut an apple the way most people cut an orange, the core is seen to have the symmetry group D_5. Extending the five-pointed star by straight cuts in each half, you divide the whole apple into ten pieces from each of which the core can be removed in the form of two thin flakes.

EXERCISES

1. Describe the symmetry groups of
(a) a scalene triangle, (b) an isosceles triangle,

(c) a parabola,
(d) a parallelogram,
(e) a rhombus,
(f) a rectangle,
(g) an ellipse.

2. Use inverses and the associative law to prove algebraically the "cancellation rule" which says that the relation

$$\mathrm{RT} = \mathrm{ST}$$

implies $\mathrm{R} = \mathrm{S}$.

3. Show how the usual defining relations for D_3, namely 2.71 with $n = 3$, may be deduced by algebraic manipulation from the simpler relations

$$\mathrm{R}_1{}^2 = 1, \quad \mathrm{R}_1\mathrm{R}_2\mathrm{R}_1 = \mathrm{R}_2\mathrm{R}_1\mathrm{R}_2.$$

4. The cyclic group C_m is a subgroup of C_n if and only if the number m is a divisor of n. In particular, if n is prime, the only subgroups of C_n are C_n itself and C_1.

2.8 STAR POLYGONS

Instead of deriving the dihedral group D_n from the regular polygon $\{n\}$, we could have derived the polygon from the group: the vertices of the polygon are just the n images of a point P_0 (the C of Figure 2.7*a*) on one of the two mirrors of the kaleidoscope. In fact, there is no need to use the whole group D_n: its subgroup C_n will suffice. The vertex P_k of the polygon $P_0P_1 \dots P_{n-1}$ can be derived from the initial vertex P_0 by a rotation through k times $360°/n$.

More generally, rotations about a fixed point O through angles θ, 2θ, 3θ, ... transform any point P_0 (distinct from O) into other points P_1, P_2, P_3, ... on the circle with center O and radius OP_0. In general, these points become increasingly dense on the circle; but if the angle θ is commensurable with a right angle, only a finite number of them will be distinct. In particular, if $\theta = 360°/n$, where n is a positive integer greater than 2, then there will be n points P_k whose successive joins

$$P_0P_1, P_1P_2, \dots, P_{n-1}P_0$$

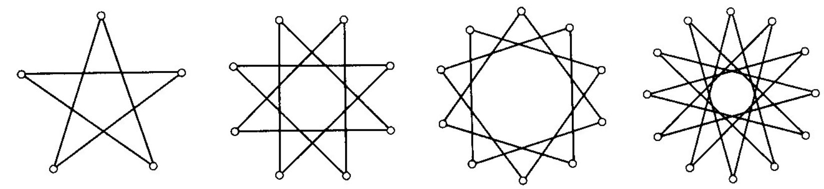
are the sides of an ordinary regular n-gon.

Let us now extend this notion by allowing n to be any rational number greater than 2, say the fraction p/d (where p and d are coprime). Accordingly, we define a (generalized) *regular polygon* $\{n\}$, where $n = p/d$. Its p vertices are derived from P_0 by repeated rotations through $360°/n$, and its p sides (enclosing the center d times) are

$$P_0P_1, P_1P_2, \dots, P_{p-1}P_0.$$

Since a ray coming out from the center without passing through a vertex will cross d of the p sides, this denominator d is called the *density* of the polygon [Coxeter **1**, pp. 93–94]. When $d = 1$, so that $n = p$, we have the ordinary regular p-gon, $\{p\}$. When $d > 1$, the sides cross one another, but the crossing points are not counted as vertices. Since d may be any positive integer relatively prime to p and less than $\frac{1}{2}p$, there is a regular polygon $\{n\}$ for each rational number $n > 2$. In fact, it is occasionally desirable to include also the *digon* $\{2\}$, although its two sides coincide.

When $p = 5$, we have the pentagon $\{5\}$ of density 1 and the *pentagram* $\{\frac{5}{2}\}$ of density 2, which was used as a special symbol by the Babylonians and by the Pythagoreans. Similarly, the *octagram* $\{\frac{8}{3}\}$ and the *decagram* $\{\frac{10}{3}\}$ have density 3, while the *dodecagram* $\{\frac{12}{5}\}$ has density 5 (Figure 2.8*a*). These particular polygons have names as well as symbols because they occur as faces of interesting polyhedra and tessellations.*



$\left\{\frac{5}{2}\right\}$ $\left\{\frac{8}{3}\right\}$ $\left\{\frac{10}{3}\right\}$ $\left\{\frac{12}{5}\right\}$

Figure 2.8a

Polygons for which $d > 1$ are known as *star* polygons. They are frequently used in decoration. The earliest mathematical discussion of them was by Thomas Bradwardine (1290–1349), who became archbishop of Canterbury for the last month of his life. They were also studied by the great German scientist Kepler (1571–1630) [see Coxeter **1**, p. 114]. It was the Swiss mathematician L. Schläfli (1814–1895) who first used a numerical symbol such as $\{p/d\}$. This notation is justified by the occurrence of formulas that hold for $\{n\}$ equally well whether n be an integer or a fraction. For instance, any side of $\{n\}$ forms with the center O an isosceles triangle OP_0P_1 (Figure 2.8*b*) whose angle at O is $2\pi/n$. (As we are introducing trigonometrical ideas, it is natural to use radian measure and write 2π instead of $360°$.) The base of this isosceles triangle, being a side of the polygon, is conveniently denoted by $2l$. The other sides of the triangle are equal to the circumradius R of the polygon. The altitude or median from O is the inradius r of the polygon. Hence

2.81 $$R = l \csc \frac{\pi}{n}, \qquad r = l \cot \frac{\pi}{n}.$$

If $n = p/d$, the area of the polygon is naturally defined to be the sum of the areas of the p isosceles triangles, namely

* H. S. M. Coxeter, M. S. Longuet-Higgins, and J. C. P. Miller, Uniform polyhedra, *Philosophical Transactions of the Royal Society*, **A, 246** (1954), pp. 401–450.

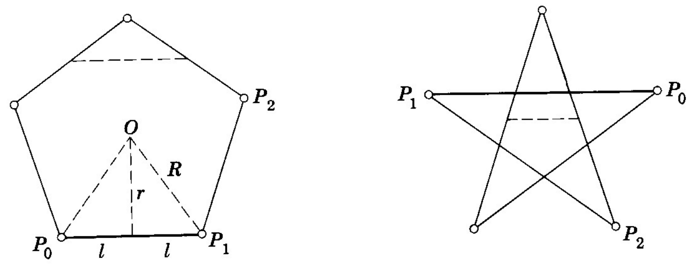


Figure 2.8b

2.82 $$plr = pl^2 \cot \frac{\pi}{n}.$$

When $d = 1$, this is simply $pl^2 \cot \pi/p$; in other cases our definition of area has the effect that every part of the interior is counted a number of times equal to the "local density" of that part; for example, the pentagonal region in the middle of the pentagram $\{\frac{5}{2}\}$ is counted twice.

The angle $P_0P_1P_2$ between two adjacent sides of $\{n\}$, being the sum of the base angles of the isosceles triangle, is the supplement of $2\pi/n$, namely

2.83 $$\left(1 - \frac{2}{n}\right)\pi .$$

The line segment joining the midpoints of two adjacent sides is called the *vertex figure* of $\{n\}$. Its length is clearly

2.84 $$2l \cos \frac{\pi}{n}$$

[Coxeter **1**, pp. 16, 94].

EXERCISES

1. If the sides of a polygon inscribed in a circle are all equal, the polygon is regular.
2. If a polygon inscribed in a circle has an odd number of vertices, and all its angles are equal, the polygon is regular. (Marcel Riesz.)
3. Find the angles of the polygons
$$\{5\},\quad \{\tfrac{5}{2}\},\quad \{9\},\quad \{\tfrac{9}{2}\},\quad \{\tfrac{9}{4}\}.$$
4. Find the radii and vertex figures of the polygons
$$\{8\},\quad \{\tfrac{8}{3}\},\quad \{12\},\quad \{\tfrac{12}{5}\}.$$
5. Give polar coordinates for the kth vertex P_k of a polygon $\{n\}$ of circumradius 1 with its center at the pole, taking P_0 to be $(1, 0)$.
6. Can a square cake be cut into nine slices so that everyone gets the same amount of cake and the same amount of icing?

3

Isometry in the Euclidean plane

Having made some use of reflections, rotations, and translations, we naturally ask why a rotation or a translation can be achieved as a continuous displacement (or "motion") while a reflection cannot. It is also reasonable to ask whether there is any other kind of isometry that resembles a reflection in this respect. After answering these questions in terms of "sense," we shall use the information to prove a remarkable theorem (§ 3.6) and to describe the seven possible ways to repeat a pattern on an endless strip (§ 3.7).

3.1 DIRECT AND OPPOSITE ISOMETRIES

"Take care of the sense, and the sounds will take core of themselves."

Lewis Carroll

[Dodgson **1**, Chap. 9]

By several applications of Axiom 1.26, it can be proved that any point P in the plane of two congruent triangles ABC, $A'B'C'$ determines a corresponding point P' such that $AP = A'P'$, $BP = B'P'$, $CP = C'P'$. Likewise another point Q yields Q', and $PQ = P'Q'$. Hence

3.11 *Any two congruent triangles are related by a unique isometry.*

In § 1.3, we saw that Pappus's proof of *Pons asinorum* involved the comparison of two coincident triangles ABC, ACB. We see intuitively that this is a distinction of *sense*: if one is counterclockwise the other is clockwise. It is a "topological" property of the Euclidean plane that this distinction can be extended from coincident triangles to distinct triangles: any two "directed" triangles, ABC and $A'B'C'$, either agree or disagree in sense. (For a deeper investigation of this intuitive idea, see Veblen and Young [**2**, pp. 61–62] or Denk and Hofmann [**1**, p. 56].)

If ABC and $A'B'C'$ are congruent, the isometry that relates them is said to be *direct* or *opposite* according as it preserves or reverses sense, that is,