Lesson 26
Homogeneous Linear Equations with Constant Coefficient

Specific Objectives:

At the end of the lesson, the students are expected to:

- Identify under which case the differential equation belongs to with respect to the roots of the auxiliary equation
- determine the general solution to the homogeneous linear equations with constant coefficients

Recall:

$$a_0\left(\frac{d^n y}{dx^n}\right)+a_1\left(\frac{d^{n-1}y}{dx^{n-1}}\right)+a_2\left(\frac{d^{n-2}y}{dx^{n-2}}\right)+\dots+a_{n-1}\frac{dy}{dx}+a_n y=0$$

Using D–operators,

$$a_0D^n y+a_1D^{n-1}y+a_2D^{n-2}y+\dots+a_{n-1}Dy+a_n y=0$$

Standard Form:

$$\left(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_{n-1}D+a_n\right)y=0 \text{ ——— } (1)$$

Or $\qquad f(D)y = 0$

Suppose $y = e^{mx}$ is a solution of (1). Substitute it in (1),

$$\left(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_{n-1}D+a_n\right)e^{mx}=0$$

$$a_0D^n e^{mx}+a_1D^{n-1}e^{mx}+a_2D^{n-2}e^{mx}+\dots+a_{n-1}De^{mx}+a_n e^{mx}=0$$

Recall: $De^{mx}=me^{mx}, D^2e^{mx}=m^2e^{mx}, D^3e^{mx}=m^3e^{mx} \therefore D^n e^{mx}=m^n e^{mx}$

$$a_0m^n e^{mx}+a_1m^{n-1}e^{mx}+a_2m^{n-2}e^{mx}+\dots+a_{n-1}me^{mx}+a_n e^{mx}=0$$

$$\left(a_0m^n+a_1m^{n-1}+a_2m^{n-2}+\dots+a_{n-1}m+a_n\right)e^{mx}=0$$

$$f(m)e^{mx}=0$$

For e^{mx} to be a particular solution of (1), it is necessary and sufficient that f (m) = 0. This relation is called the characteristic or auxiliary equation of (1). The roots of the auxiliary equation, f(m) = 0 will determine the general solution of the homogeneous equation with constant coefficients.

1. The Auxiliary Equation with Real Roots

1.1 Distinct Roots

Suppose the auxiliary equation, f(m) = 0, has m_1, m_2, and m_3 as its roots, then $e^{m_1 x}$, $e^{m_2 x}$ and $e^{m_3 x}$ are the solution of (1) and the general solution is given by:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

Example: Find the general solution of the differential equation

1. $\left(D^2 - 2D - 3\right)y = 0$

SOLUTION:

$$\left(D^2 - 2D - 3\right)y = 0$$
$$m^2 - 2m - 3 = 0$$
$$(m-3)(m+1) = 0$$
$$m = 3, -1$$
$$\underline{\underline{y = C_1 e^{3x} + C_2 e^{-x}}}$$

2. $\left(D^2 + 5D + 4\right)y = 0$

SOLUTION:

$$\left(D^2 + 5D + 4\right)y = 0$$
$$m^2 + 5m + 4 = 0$$
$$(m+4)(m+1) = 0$$
$$m = -4, -1$$
$$\underline{\underline{y = C_1 e^{-4x} + C_2 e^{-x}}}$$

Seatwork: Find the general solution of the differential equation

1. $(D^3 - 7D + 6)\, y = 0$ ans: $y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-3x}$

2. $(D^5 - 9D^4 + 13D^3 + 57D^2 - 86D - 120)\ y = 0$

ans: $y = C_1e^{-x} + C_2e^{-2x} + C_3e^{3x} + C_4e^{4x} + C_5e^{5x}$

1.2 Repeated Roots

If m is an n-fold root of the auxiliary equation,(m is repeated n times), then the general solution corresponding to this repeated root will be

$$y = c_1e^{mx} + c_2xe^{mx} + c_3x^2e^{mx} + \ldots + c_nx^{n-1}e^{mx}$$

Note: add all terms arising from the application of case 1

Example: Find the general solution of the differential equation.

1. $(D^2 + 4D + 4)y = 0$

SOLUTION:

$$(D^2 + 4D + 4)y = 0$$
$$m^2 + 4m + 4 = 0$$
$$(m+2)^2 = 0$$
$$m + 2 = 0$$
$$m = -2, \textit{double root}$$
$$\underline{\underline{y = (C_1 + C_2x)e^{-2x}}}$$

2. $(D^3 + 3D^2 + 3D + 1)y = 0$

SOLUTION:

$$(D^3 + 3D^2 + 3D + 1)y = 0$$
$$m^3 + 3m^2 + 3m + 1 = 0$$
$$(m+1)^3 = 0$$
$$m + 1 = 0$$
$$m = -1, \textit{triple root}$$
$$\underline{\underline{y = (C_1 + C_2x + C_3x^2)e^{-x}}}$$

Seatwork: Find the general solution of the differential equation

1. $(D^3 - 4D^2 + 4D)\, y = 0$ ans: $y = C_1 + (C_2 + C_3x)\, e^{2x}$

2. $(D^4 - 6D^3 - 24D^2 + 224D - 384)\, y = 0$
ans: $y = C_1e^{-6x} + (C_2 + C_3x + C_4x^2)\, e^{4x}$

2. The Auxiliary Equation with Complex/Imaginary Roots

<u>2.1 Distinct Roots</u>

Suppose the roots of the auxiliary equation, $f(m) = 0$, are $m_1 = a + bi$ and $m_2 = a - bi$, then the general solution is given by:

$$y = \left(C_1 \cos bx + C_2 \sin bx\right) e^{ax}$$

Example: Find the general solution of the differential equation.

1. $\left(D^2 + 6D + 13\right)y = 0$

SOLUTION:

$$\left(D^2 + 6D + 13\right)y = 0$$
$$m^2 + 6m + 13 = 0$$

By Quadratic Formula

$$m = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)}$$
$$m = \frac{-6 \pm \sqrt{-16}}{2}$$
$$m = \frac{-6 \pm 4i}{2} = \frac{-6}{2} \pm \frac{4i}{2}$$
$$m = -3 \pm 2i$$
$$\underline{\underline{y = \left(C_1 \cos 2x + C_2 \sin 2x\right)e^{-3x}}}$$

2. $\left(D^2+4D+5\right)y=0$

SOLUTION:

$$\left(D^2+4D+5\right)y=0$$
$$m^2+4m+5=0$$

by Quadratic Formula

$$m=\frac{-(4)\pm\sqrt{(4)^2-4(1)(5)}}{2(1)}$$
$$m=\frac{-4\pm\sqrt{-4}}{2}$$
$$m=\frac{-4\pm 2i}{2}=\frac{-4}{2}\pm\frac{2i}{2}$$
$$m=-2\pm i$$
$$y=\left(C_1\cos x+C_2\sin x\right)e^{-2x}$$

Seatwork: Find the general solution of the differential equation

1. $(D^4 + 10D^2 + 16)\ y = 0$
   ans: $y=(C_1\cos 2\sqrt{2}\ x + C_2\sin 2\sqrt{2}\ x)+(C_3\cos\sqrt{2}\ x + C_4\sin\sqrt{2}\ x)$

2.2 Repeated Roots

Suppose the roots of the auxiliary equation, f(m) = 0, are $m_1 = a + bi$ (taken twice as a root) and $m_2 = a - bi$ (taken twice as a root), then the general solution is given by:

$$y=\left[\left(C_1+C_2x\right)\cos bx+\left(C_3+C_4x\right)\sin bx\right]e^{ax}$$

Example: Find the general solution of the differential equation.

1. $\left(D^2-2D+5\right)^2y=0$

SOLUTION:

$$\left(m^2-2m+5\right)^2=0$$

by Quadratic Formula

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$m = \frac{2 \pm \sqrt{-16}}{2}$$

$$m = \frac{2 \pm 4i}{2} = \frac{2}{2} \pm \frac{4i}{2}$$

$$m = 1 \pm 2i, \textit{double root}$$

$$\underline{\underline{y = [(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x]e^x}}$$

2. $(D^4 + 18D^2 + 81)y = 0$

SOLUTION:

$$m^4 + 18m + 81 = 0$$

$$(m^2 + 9)^2 = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$\sqrt{m^2} = \sqrt{-9}$$

$$m = 0 \pm 3i, \textit{double root}$$

$$\underline{\underline{y = [(C_1 + C_2 x)\cos 3x + (C_3 + C_4 x)\sin 3x]e^{0x}}}$$

Seatwork: Find the general solution of the differential equation

1. $(D^4 - 4D^3 + 8D^2 - 8D + 4)\ y = 0$
   ans: $y = [(C_1 + C_2x) \cos x + (C_3 + C_4x) \sin x]\ e^x$

2. $(D^6 + 3D^4 + 3D^2 + 1)\ y = 0$
   ans: $y = [(C_1 + C_2x + C_3x^2) \cos x + (C_4 + C_5x + C_6x^2) \sin x]$