7 The hyperbolic plane

Introduction

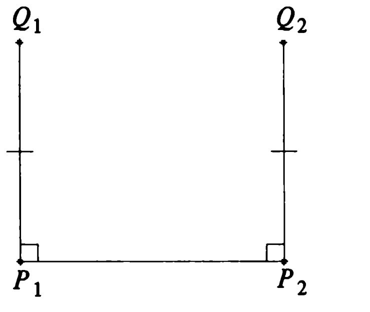

Figure 7.1 In hyperbolic geometry, $d(Q_1, Q_2) > d(P_1, P_2)$.

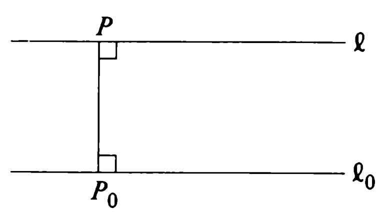

Figure 7.2 In hyperbolic geometry, ℓ_0 and ℓ do not meet.

The projective plane provides one alternative to Euclidean geometry. A second alternative is explored in this chapter.

The three geometries are contrasted in the following example: Take a segment P_1P_2 as shown in Figure 7.1. Erect equal segments P_1Q_1 and P_2Q_2 perpendicular to P_1P_2.

In $\mathbf{E}^2$ the segment Q_1Q_2 will have length equal to that of P_1P_2. However, in $\mathbf{P}^2$, the length of Q_1Q_2 will be less than that of P_1P_2. In $\mathbf{H}^2$ we shall see that Q_1Q_2 will be longer than P_1P_2.

This construction is also related to the question of parallelism. Let ℓ_0 be a line, and let P be a point not on ℓ_0. Drop a perpendicular PP_0 from P to ℓ_0, and let ℓ be the line through P perpendicular to PP_0. (See Figure 7.2.)

In $\mathbf{E}^2$, ℓ will be parallel to ℓ_0. In $\mathbf{P}^2$, ℓ will meet ℓ_0. In $\mathbf{H}^2$ it will turn out that ℓ does not meet ℓ_0.

We will now proceed to construct the geometry $\mathbf{H}^2$. It will again consist of "points" and "lines" with a "distance" function defined for each pair of points. As in the case of $\mathbf{E}^2$ and $\mathbf{P}^2$, we find that isometries of $\mathbf{H}^2$ are generated by reflections and satisfy the three reflections theorems.

Algebraic preliminaries

Our model of spherical geometry was a certain subset of $\mathbf{R}^3$, and the usual inner product of $\mathbf{R}^3$ played an important role. Our model of hyperbolic geometry will also be a subset of $\mathbf{R}^3$. However, the bilinear form on which hyperbolic geometry is based is defined by

$$b(x, y) = x_1y_1 + x_2y_2 - x_3y_3$$

(see also Chapter 6). A function of this type is used in Einstein's special theory of relativity. (See Frankel [15] or Taylor–Wheeler [29].) This explains some of the terms used in discussing its properties.

Definition. *A nonzero vector $v \in \mathbf{R}^3$ is said to be*

i. spacelike *if $b(v, v) > 0$. If $b(v, v) = 1$, it is a unit spacelike vector. An example is ε_1.*
ii. timelike *if $b(v, v) < 0$. If $b(v, v) = -1$, it is a unit timelike vector. An example is ε_3.*
iii. lightlike *if $b(v, v) = 0$. An example is $\varepsilon_1 - \varepsilon_3$.*

We use the notation $|v|$ for the "length" of a vector v (i.e., $|v| = |b(v, v)|^{1/2}$). Unit vectors satisfy $|v| = 1$.

In this chapter we use the term "orthonormal" to mean orthonormal with respect to b. Note that $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is orthonormal.

Theorem 1.

i. *Every orthonormal set of three vectors is a basis for $\mathbf{R}^3$.*
ii. *Every orthonormal basis has two spacelike vectors and one timelike vector.*
iii. *For every orthonormal pair $\{u, v\}$ of vectors, $\{u, v, u \times v\}$ is an orthonormal basis. (The cross product is taken with respect to b.)*
iv. *For every unit spacelike or unit timelike vector v, there is an orthonormal basis containing v.*

Proof:

i. We need only show that an orthonormal set is linearly independent. If an equation of the form

$$0 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

holds, where $\{e_1, e_2, e_3\}$ is orthonormal, then for each i,

$$0 = b(0, e_i) = \lambda_i b(e_i, e_i)$$

implies that $\lambda_i = 0$.

ii. First note that all three vectors cannot be spacelike. In fact, if all $b(e_i, e_i)$ are equal and

$$x = \sum_{i=1}^{3} x_i e_i,$$

we have

$$b(x, x) = \sum_{i=1}^{3} x_i^2 b(e_i, e_i).$$

This would imply that all vectors are spacelike. Similarly, if all the e_i were timelike, every vector in $\mathbf{R}^3$ would be timelike. We conclude that any orthonormal basis has at least one spacelike vector and one timelike vector.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis. Suppose that e_1 is spacelike and e_3 is timelike. Then $(e_1 \times e_3) \times e_2 = 0$, so that e_2 is a multiple of $e_1 \times e_3$. Further,

$$b(e_1 \times e_3, e_1 \times e_3) = -b(e_1, e_1)\, b(e_3, e_3) = 1,$$

so that $e_1 \times e_3$ (and hence e_2) is spacelike.

iii. We note that

$$b(u \times v, u \times v) = -b(u, u)\, b(v, v) = \pm 1,$$

and, hence, $\{u, v, u \times v\}$ is orthonormal.

iv. Suppose that v is spacelike. Let w be any unit timelike vector (e.g., $\varepsilon_3 = (0, 0, 1)$). If $b(v, w) = 0$, we can use $\{v, w, v \times w\}$ as our basis. If not, choose $\tilde{u} = v + \lambda w$, where $\lambda = -1/b(v, w)$. Then

$$\begin{aligned} b(\tilde{u}, \tilde{u}) &= 1 + 2\lambda b(v, w) - \lambda^2 \\ &= 1 - 2 - \lambda^2 = -(1 + \lambda^2). \end{aligned}$$

If we set

$$u = \frac{v + \lambda w}{\sqrt{1 + \lambda^2}},$$

then $\{u, v, u \times v\}$ is an orthonormal basis.

Suppose now that v is timelike. A similar construction, using a unit spacelike vector w, leads to an orthonormal basis $\{u, v, u \times v\}$, where $u = (v + \lambda w)/\sqrt{1 + \lambda^2}$ and $\lambda = 1/b(v, w)$. □

Theorem 2.

i. *For any* $x \in \mathbf{R}^3$,

$$x = \sum_{i=1}^{3} b(x, e_i) b(e_i, e_i) e_i \tag{7.1}$$

if $\{e_1, e_2, e_3\}$ *is an orthonormal basis.*

ii. *Let* v *be a timelike vector. Suppose that* $w \times v \neq 0$ *and* $b(v, w) = 0$. *Then* w *is spacelike.*

The Cauchy–Schwarz inequality played an important role in $\mathbf{E}^2$ and $\mathbf{S}^2$. Here is the hyperbolic version.

Theorem 3. *Let* ξ *and* η *be spacelike vectors in* $\mathbf{R}^3$ *such that* $\xi \times \eta$ *is timelike. Then*

$$b(\xi, \eta)^2 < b(\xi, \xi) b(\eta, \eta). \tag{7.2}$$

Proof: Let P be a unit timelike vector in the direction $[\xi \times \eta]$. As in the proof of Theorem 1.4, we consider the function

$$f(t) = b(\xi + t\eta, \xi + t\eta).$$

Because $b(\xi + t\eta, P) = 0$ for all real values of t and $P \times (\xi + t\eta) \neq 0$, Theorem 2 applies, and $\xi + t\eta$ is spacelike. In other words, $f(t) > 0$ for all t and

$$b(\xi, \eta)^2 < b(\xi, \xi)b(\eta, \eta). \qquad \square$$

Remark: If we weaken the hypothesis to $b(\xi \times \eta, \xi \times \eta) \leqslant 0$, the conclusion becomes

$$b(\xi, \eta)^2 \leqslant b(\xi, \xi)b(\eta, \eta).$$

However, equality can occur even if ξ and η are not proportional. (See Exercise 2.)

There is a similar result for timelike vectors.

Theorem 4. *Let v and w be timelike vectors. Then*

$$b(v, w)^2 \geqslant b(v, v)b(w, w). \tag{7.3}$$

Proof: By Theorem 2, $v \times w$ is spacelike or zero. Thus

$$b(v \times w, v \times w) \geqslant 0.$$

In other words,

$$b(v, v)b(w, w) - b(v, w)^2 \leqslant 0$$

with equality holding if and only if v and w are proportional. $\square$

Corollary. *If v and w are unit timelike vectors, then $|b(v, w)| \geqslant 1$. The "inner product" $b(v, w)$ is positive if and only if $b(v, \varepsilon_3)$ and $b(w, \varepsilon_3)$ have opposite signs.*

Proof: The first statement is immediate from the theorem. To prove the second, we introduce the following notation. Let $v = (p_1, p_2, r)$ and $w = (q_1, q_2, s)$. Consider $p = (p_1, p_2)$ and $q = (q_1, q_2)$ as vectors in $\mathbf{R}^2$. Then

$$b(v, w) = \langle p, q \rangle - rs.$$

Because $(|p| + |q|)^2 \geqslant 0$ with equality if and only if $p = q = 0$, we have

$$|p|^2 + |q|^2 \geqslant -2|p||q|.$$

Adding $1 + |p|^2|q|^2$ to each side yields

$$(1 + |p|^2)(1 + |q|^2) \geqslant (|p||q| - 1)^2.$$

But $|p|^2 - r^2 = -1$ and $|q|^2 - s^2 = -1$, so that

$$(|p||q| - 1)^2 \leqslant r^2 s^2. \tag{7.4}$$

Suppose now that r and s are both positive but $b(v, w)$ is also positive. Then $\langle p, q\rangle \geqslant 1 + rs$; that is, $\langle p, q\rangle - 1 \geqslant rs$. By the Cauchy–Schwarz inequality for $\mathbf{R}^2$, we get

$$|p||q| - 1 \geqslant rs,$$

which is incompatible with (7.4). We conclude that $b(v, w)$ must be negative when r and s are positive. The conclusion now follows from the linearity of the function b. □

Incidence geometry of $\mathbf{H}^2$

The hyperbolic plane $\mathbf{H}^2$ is defined as follows:

$$\mathbf{H}^2 = \{x \in \mathbf{R}^3 | x_3 > 0 \quad \text{and} \quad b(x, x) = -1\}.$$

Thus, as a set, $\mathbf{H}^2$ is just the upper half of a hyperboloid of two sheets.

Definition. *Let ξ be a unit spacelike vector. Then*

$$\ell = \{x \in \mathbf{H}^2 | b(\xi, x) = 0\}$$

is called the line *with unit normal (or pole) ξ.*

Remark: Like the situation in spherical geometry, a line of $\mathbf{H}^2$ is the intersection with $\mathbf{H}^2$ of a plane through the origin of $\mathbf{R}^3$. Not all planes through the origin meet $\mathbf{H}^2$. However, if ξ is timelike, it can be completed to a basis orthonormal with respect to b (Theorem 1). In particular, there are points $x \in \mathbf{H}^2$ such that $b(\xi, x) = 0$. We will now proceed to a detailed study of lines in hyperbolic geometry.

Theorem 5. *Let P and Q be distinct points of $\mathbf{H}^2$. Then there is a unique line containing P and Q, which we denote by $\overleftrightarrow{PQ}$.*

Proof: Apply Theorem 2(ii) with $v = P$ and $w = P \times Q$. The triple product formula shows that $P \times (P \times Q) \neq 0$ and, hence, that $P \times Q$ is spacelike. Let ξ be a unit vector in the direction $[P \times Q]$. Then the line whose unit normal is ξ must pass through P and Q. This is the only line through P and Q because the unit normal to any such line must be orthogonal to P and Q (with respect to b) and, hence, must be a multiple of $P \times Q$. □

Just as in spherical geometry, the cross product is used to find the point of intersection of a pair of lines. However, if ξ and η are spacelike unit vectors, $\xi \times \eta$ need not be timelike, and therefore the lines may not intersect in $\mathbf{H}^2$. In fact, all three possibilities for $\xi \times \eta$ can occur. This is what makes $\mathbf{H}^2$ a richer incidence geometry than any we have studied previously.

Definition. *Let ℓ and m be two lines with respective unit normals ξ and η. We say that ℓ and m are*

i. intersecting *lines if $\xi \times \eta$ is timelike,*
ii. parallel *lines if $\xi \times \eta$ is lightlike,*
iii. ultraparallel *lines if $\xi \times \eta$ is spacelike.*

Theorem 6. *Intersecting lines have exactly one point in common. This point is the unique point of $\mathbf{H}^2$ that is a multiple of $\xi \times \eta$.*

Proof: Clearly, the point in question lies on both lines. If P is any other point that lies on both lines, then

$$P \times (\xi \times \eta) = -b(P, \eta)\xi + b(P, \xi)\eta = 0,$$

so that P is a multiple of $\xi \times \eta$ as required. □

Remark: Neither parallel nor ultraparallel lines intersect.

Perpendicular lines

Definition. *Two lines with unit normals ξ and η are said to be* perpendicular *if $b(\xi, \eta) = 0$.*

Theorem 7. *If two lines are ultraparallel, there is a unique line γ that is perpendicular to both of them. Conversely, if two lines have a common perpendicular, they must be ultraparallel.*

Proof: Let ξ and η be unit normals of two ultraparallel lines. Let ζ be the unit (spacelike) vector that is a multiple of $\xi \times \eta$. Then $b(\xi, \zeta) = b(\eta, \zeta) = 0$, so the line with unit normal ζ is a common perpendicular to the two lines.

Conversely, if the two lines have a common perpendicular, its unit normal ζ is a spacelike vector satisfying $\zeta \times (\xi \times \eta) = 0$ and, thus, is a multiple of $\xi \times \eta$. This means that $\xi \times \eta$ is spacelike, and the lines are ultraparallel. □

Theorem 8.

i. *If ℓ and m are perpendicular lines of $\mathbf{H}^2$, then ℓ intersects m.*
ii. *Let X be a point of $\mathbf{H}^2$ and ℓ a line of $\mathbf{H}^2$. Then there is a unique line through X perpendicular to ℓ.*

Proof:

i. Let ξ and η be unit normals to ℓ and m, respectively. Then $\{\xi, \eta, \xi \times \eta\}$ is an orthonormal basis by Theorem 1. Hence, $\xi \times \eta$ is timelike.

ii. Let ξ be a unit normal to ℓ. Let η be a unit vector proportional to $\xi \times X$. This is possible because $\xi \times X$, being a nonzero vector orthogonal to X, must be spacelike.

The line m whose unit normal is η clearly passes through X but is perpendicular to ℓ. There is only one line with this property, because a unit normal to such a line must be orthogonal to ξ and X and, therefore, a multiple of their cross product. □

Definition. *The point F where m intersects ℓ is called the* foot *of the perpendicular from X to ℓ (provided X is not on ℓ).*

Remark: In the next section we define distance between two points of $\mathbf{H}^2$. As in $\mathbf{E}^2$ we can use this to define

$$d(X, \ell) = d(X, F),$$

where F is the foot of the perpendicular from X to ℓ.

Pencils

Definition. *Let ℓ and m be a pair of distinct lines with respective unit normals ξ and η. Then the set $\mathscr{P}$ of lines whose unit normals ζ are orthogonal to $\xi \times \eta$ is called a* pencil *of lines. $\mathscr{P}$ is called a pencil of intersecting lines, a pencil of parallels, or a pencil of ultraparallels according to whether $\xi \times \eta$ is timelike, lightlike, or spacelike.*

Remark: At the moment this definition may look somewhat strange. Clearly, if $\xi \times \eta$ is timelike, then lines with unit normal ζ will be the lines passing through the point of intersection, as expected. If $\xi \times \eta$ is spacelike, the pencil will consist of all lines perpendicular to a certain line. However, it is not yet evident what the pencil looks like when $\zeta \times \eta$ is lightlike. When we look at $\mathbf{H}^2$ as a subset of $\mathbf{P}^2$, we will get a more concrete interpretation for $\xi \times \eta$ and the associated pencils.

Remark:

i. The set of all lines of $\mathbf{H}^2$ perpendicular to a certain line of $\mathbf{H}^2$ is a pencil of ultraparallels.

ii. Any two lines of $\mathbf{H}^2$ determine a unique pencil.

Distance in $\mathbf{H}^2$

We parametrize lines of $\mathbf{H}^2$ much as we did in $\mathbf{S}^2$. Let e_3 be an arbitrary point of $\mathbf{H}^2$. Let e_1 and e_2 be vectors of $\mathbf{R}^3$ such that $\{e_1, e_2, e_3\}$ is an orthonormal basis.

A typical point on the plane through the origin spanned by $\{e_1, e_3\}$ is $\lambda e_3 + \mu e_1$. This point is on $\mathbf{H}^2$ if and only if $\lambda > 0$ and

$$b(\lambda e_3 + \mu e_1, \lambda e_3 + \mu e_1) = -1;$$

that is,

$$\lambda^2 = 1 + \mu^2.$$

Using Theorem 3F, we may call $\lambda = \cosh t$ and $\mu = \sinh t$. Then as t ranges through all real numbers, $(\cosh t)e_3 + (\sinh t)e_1$ runs through all the points of the line. We define distance in such a way that t measures distance along the line.

Definition. *For x, y in $\mathbf{H}^2$ define*

$$d(x, y) = \cosh^{-1}(-b(x, y)).$$

Remark: This definition is possible because $b(x, y) \leqslant -1$, as was shown in the corollary to Theorem 4.

Theorem 9. *Let $\alpha(t) = (\cosh t)e_3 + (\sinh t)e_1$. Then*

$$d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2|.$$

Proof: Exercise 6. □

Definition. *If $t_1 < t < t_2$, then $\alpha(t)$ is* between *$\alpha(t_1)$ and $\alpha(t_2)$.*

Now that we have defined distance between two points in the hyperbolic plane, it is necessary to determine which of the properties of Euclidean distance carry over to the hyperbolic case. The following is immediate from the definition.

Theorem 10. *If P and Q are points of $\mathbf{H}^2$, then*
i. $d(P, Q) \geqslant 0$.
ii. $d(P, Q) = 0$ *if and only if* $P = Q$.
iii. $d(P, Q) = d(Q, P)$.

We now address ourselves to the triangle inequality. Our proof of the triangle inequality in the spherical case relied on the cross product operation of $\mathbf{E}^3$. Here we use the hyperbolic cross product.

Theorem 11 (Triangle Inequality). *Let P, Q, and R be points of $\mathbf{H}^2$. Then $d(P, Q) + d(P, R) \geqslant d(Q, R)$ with equality if and only if P, Q, R are collinear and P lies between Q and R.*

Proof: If P, Q, and R are not collinear, then $P \times Q$ and $R \times Q$ will not be proportional. Thus, $(P \times Q) \times (R \times Q) = b(P \times Q, R)Q$ is timelike. We may apply the hyperbolic Cauchy–Schwarz inequality (Theorem 3) to get

$$b(P \times Q, R \times Q)^2 \leqslant b(P \times Q, P \times Q)b(R \times Q, R \times Q). \tag{7.5}$$

But

$$\begin{aligned} b(P \times Q, R \times Q) &= b((P \times Q) \times R, Q) \\ &= -b(P, R)b(Q, Q) + b(Q, R)b(P, Q) \\ &= b(P, R) + b(Q, R)b(P, Q) \end{aligned}$$

because $b(Q, Q) = -1$. Let $d(Q, R) = p$, $d(P, R) = q$, $d(P, Q) = r$. Then

$$\cosh p = -b(Q, R), \quad \cosh r = -b(Q, P), \quad \cosh q = -b(P, R).$$

Thus,

$$b(P \times Q, R \times Q) = \cosh p \cosh r - \cosh q.$$

Also

$$\begin{aligned} b(P \times Q, P \times Q) &= -b(P, P)b(Q, Q) + b(R, Q)^2 \\ &= -1 + \cosh^2 r = \sinh^2 r. \end{aligned}$$

and, similarly, $b(R \times Q, R \times Q) = \sinh^2 p$. Equation (7.5) now becomes

$$(\cosh p \cosh r - \cosh q)^2 \leqslant \sinh^2 r \sinh^2 p.$$

Hence,

$$\begin{aligned} \cosh p \cosh r - \cosh q &\leqslant \sinh r \sinh p, \\ \cosh q &\geqslant \cosh(p - r), \\ q &\geqslant p - r, \\ p &\leqslant q + r. \end{aligned}$$

This is what we wanted to prove. Now if $p = q + r$, we have equality in (7.5). From Theorem 3 this means that $(P \times Q) \times (R \times Q)$ is not timelike, and, hence, $b(P \times Q, R) = 0$; that is, R lies on $\overleftrightarrow{PQ}$. The fact that P lies between Q and R can be deduced easily from Theorem 9 and is left as an exercise (Exercise 7). $\square$

Remark: The properties of the hyperbolic functions used in this section may be found in Appendix F.

Isometries of $\mathbf{H}^2$

A map T: $\mathbf{H}^2 \to \mathbf{H}^2$ is called an *isometry* if for all X and Y in $\mathbf{H}^2$,

$$d(TX, TY) = d(X, Y).$$

As in the case of $\mathbf{E}^2$, $\mathbf{S}^2$, and $\mathbf{P}^2$, isometries preserve collinearity. Specifically, we have the following.

Theorem 12. *Let T be an isometry of $\mathbf{H}^2$. Then three distinct points P, Q, and R of $\mathbf{H}^2$ are collinear if and only if TP, TQ, and TR are collinear.*

Proof: Exercise 8. □

Reflections

Let α be a line of $\mathbf{H}^2$ with unit normal ξ. For $x \in \mathbf{R}^3$ let

$$\Omega_\alpha x = x - 2b(x, \xi)\xi.$$

Theorem 13.

i. $\Omega_\alpha{}^2 = I$.
ii. Ω_α *is a bijection of* $\mathbf{R}^3$ *onto* $\mathbf{R}^3$.
iii. $b(\Omega_\alpha x, \Omega_\alpha Y) = b(x, y)$ *for all* $x, y \in \mathbf{R}^3$.

Proof:

i. $$\begin{aligned}\Omega_\alpha\Omega_\alpha x &= \Omega_\alpha x - 2b(\Omega_\alpha x, \xi)\xi \\ &= x - 2b(x, \xi)\xi - 2b(x, \xi)\xi + 4b(x, \xi)b(\xi, \xi) \\ &= x.\end{aligned}$$

ii. Follows easily from (i).

iii. $$\begin{aligned}b(\Omega_\alpha x, \Omega_\alpha y) &= b(x - 2b(x, \xi)\xi, y - 2b(y, \xi)\xi) \\ &= b(x, y) - 2b(x, \xi)b(\xi, y) - 2b(y, \xi)b(x, \xi) \\ &\quad + 4b(x, \xi)b(y, \xi)b(\xi, \xi) \\ &= b(x, y).\end{aligned}$$ □

Corollary. *For any line α of $\mathbf{H}^2$ and $x \in \mathbf{R}^3$, we have the following:*

i. *If x is timelike, so is $\Omega_\alpha x$.*
ii. *If x is lightlike, so is $\Omega_\alpha x$.*
iii. *If x is spacelike, so is $\Omega_\alpha x$.*
iv. *If x is a unit vector, so is $\Omega_\alpha x$.*
v. *If $x \in \mathbf{H}^2$, so is $\Omega_\alpha x$.*

Definition. *Given a line α of $\mathbf{H}^2$, the restriction of Ω_α to $\mathbf{H}^2$ is called the* reflection *in α.*

Theorem 14. *Every reflection is an isometry of* $\mathbf{H}^2$.

Proof: For $X, Y \in \mathbf{H}^2$,

$$\begin{aligned} d(\Omega_\alpha X, \Omega_\alpha Y) &= \cosh^{-1}(-b(\Omega_\alpha X, \Omega_\alpha Y)) \\ &= \cosh^{-1}(-b(X, Y)) = d(X, Y). \end{aligned}$$ □

Theorem 15. *Let* β *be a line of* $\mathbf{H}^2$ *with unit normal* η. *Then*

$$\Omega_\alpha\beta = \{X \in \mathbf{H}^2 | b(X, \Omega_\alpha\eta) = 0\};$$

that is, if β *has unit normal* η, *then* $\Omega_\alpha\beta$ *is a line with unit normal* $\Omega_\alpha\eta$.

Proof: Let $Y \in \Omega_\alpha\beta$. Then for some $X \in \beta$, $Y = \Omega_\alpha X$ and

$$b(Y, \Omega_\alpha\eta) = b(\Omega_\alpha X, \Omega_\alpha\eta) = b(X, \eta) = 0.$$

Conversely, if $b(X, \Omega_\alpha\eta) = 0$, then $b(\Omega_\alpha X, \eta) = b(\Omega_\alpha\Omega_\alpha X, \Omega_\alpha\eta) = 0$. In other words, $\Omega_\alpha X \in \beta$ and $X \in \Omega_\alpha\beta$. □

Theorem 16.

i. *Let* x *be a point of* $\mathbf{H}^2$. *Then* $\Omega_\alpha x = x$ *if and only if* $x \in \alpha$.
ii. *Let* β *be a line of* $\mathbf{H}^2$. *Then* $\Omega_\alpha\beta = \beta$ *if and only if* $\alpha = \beta$ *or* $\alpha \perp \beta$.

Proof:

i. $x - 2b(x, \xi)\xi = x$ if and only if $b(x, \xi) = 0$.
ii. Let $\beta = \{x \in \mathbf{H}^2 | b(x, \eta) = 0\}$, where $b(\eta, \eta) = 1$. Then $\Omega_\alpha\beta = \beta$ if and only if $\eta - 2b(\eta, \xi)\xi = \pm\eta$. This holds if and only if $b(\eta, \xi) = 0$ or $\xi = \pm\eta$. The former means that $\alpha \perp \beta$, and the latter means that $\alpha = \beta$. □

Motions

As before, an isometry that is a product of reflections is called a *motion*. In addition to reflections we distinguish four special kinds of motions.

Let α and β be lines of $\mathbf{H}^2$. If α and β intersect in a point P of $\mathbf{H}^2$, then $\Omega_\alpha\Omega_\beta$ is called a *rotation about* P.

If α and β are parallel, then $\Omega_\alpha\Omega_\beta$ is called a *parallel displacement.* If α and β are ultraparallel with common perpendicular ℓ, then $\Omega_\alpha\Omega_\beta$ is called a *translation along* ℓ.

A *glide reflection* in $\mathbf{H}^2$ is the product of reflection in a line ℓ with a translation along ℓ. The line ℓ is called the *axis* of the glide reflection.

Rotations

Let P be an arbitrary point of $\mathbf{H}^2$. The set of rotations about P is denoted by ROT(P). We construct matrices representing each element of ROT(P)

and prove that ROT(P) is a group isomorphic to **SO(2)**.

Choose an orthonormal basis $\{e_1, e_2, e_3\}$ so that $e_3 = P$. If α is a line through P, we can write

$$\alpha = \{x | b(x, \xi) = 0\},$$

where

$$\xi = (-\sin\theta)e_1 + (\cos\theta)e_2.$$

Then

$$\Omega_\alpha e_1 = e_1 - 2(-\sin\theta)\xi = (\cos 2\theta)e_1 + (\sin 2\theta)e_2,$$

$$\Omega_\alpha e_2 = e_2 - (2\cos\theta)\xi = (\sin 2\theta)e_1 - (\cos 2\theta)e_1,$$

$$\Omega_\alpha e_3 = e_3.$$

Thus, the matrix of Ω_α with respect to $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{ref}\ \theta & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, if β is another line through P with pole

$$\eta = (-\sin\phi)e_1 + (\cos\phi)e_2,$$

then $\Omega_\alpha\Omega_\beta$ takes a similar form with θ replaced by ϕ. By the calculations of Chapter 1, the matrix of $\Omega_\alpha\Omega_\beta$ is

$$\begin{bmatrix} \cos 2(\theta - \phi) & -\sin 2(\theta - \phi) & 0 \\ \sin 2(\theta - \phi) & \cos 2(\theta - \phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \text{rot}\ 2(\theta - \phi) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The function $\Omega_\alpha \to \text{ref}\ \theta$ determines an isomorphism of REF(P) (the group generated by reflections of $\mathbf{H}^2$ in lines through P) onto **O(2)**. Under this isomorphism ROT(P) goes into **SO(2)**. Recalling the formulas of Chapter 1 (Theorems 33 and 34), we conclude the following:

Theorem 17 (Three reflections theorem). *Let α, β, and γ be lines through P in $\mathbf{H}^2$. Then there is a fourth line δ through P such that*

$$\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta.$$

The related representation theorem for rotations holds.

Theorem 18. *Let ρ be a rotation about P. Let ℓ be a line through P. Then there exist lines m and m' through P such that*

$$\rho = \Omega_\ell\Omega_m = \Omega_{m'}\Omega_\ell.$$

$\mathbf{H}^2$ as a subset of $\mathbf{P}^2$

As well as being an interesting subject of study in its own right, the projective plane provides a framework in which other geometries can be embedded, often allowing an approach that facilitates both computation and understanding. In Chapter 5 we saw that it was possible to regard the incidence geometry of $\mathbf{E}^2$ as a subgeometry of $\mathbf{P}^2$.

We now show that the hyperbolic plane can also be regarded as a subgeometry of $\mathbf{P}^2$. Let $\mathbf{D}^2$ be the subset of $\mathbf{P}^2$ determined by the condition $b(x, x) < 0$. This set of points may be regarded as the interior of the conic $b(x, x) = 0$. We will call the remaining points of $\mathbf{P}^2$ (those with $b(x, x) > 0$) *exterior* points.

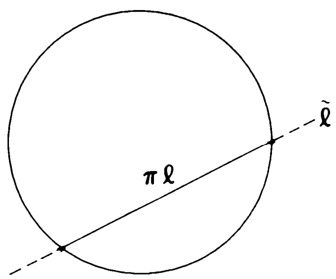

Figure 7.3 The Klein model. $\tilde{\ell} \cap \mathbf{D}^2$ represents a line of the hyperbolic plane.

Theorem 19.

i. *The usual projection π: $\mathbf{R}^3 - \{0\} \to \mathbf{P}^2$ maps $\mathbf{H}^2$ bijectively to $\mathbf{D}^2$.*
ii. *For each point X of $\mathbf{P}^2$ exterior to $\mathbf{D}^2$, there is a unique pair $\{\xi, -\xi\}$ of unit spacelike vectors such that $\pi\xi = \pi(-\xi) = X$. Conversely, each unit spacelike vector determines such an exterior point.*
iii. *A vector $v \in \mathbf{R}^3$ is lightlike if and only if πv lies on the conic.*

For most purposes we can look at $\mathbf{D}^2$ as the unit disk $x_1^2 + x_2^2 < 1$ in the plane $x_3 = 1$ of $\mathbf{E}^3$ and work in this model of $\mathbf{E}^2$ rather than in $\mathbf{P}^2$. We use the correspondence defined in Chapter 5, which relates $\mathbf{E}^2$ and $\mathbf{P}^2 - \ell_\infty$. In terms of homogeneous coordinates, ℓ_∞ is the line $x_3 = 0$.

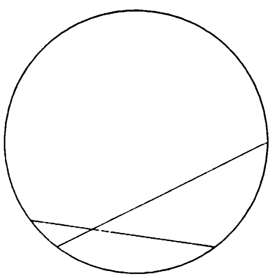

Figure 7.4 Intersecting lines.

Theorem 20. *In terms of the model described in Theorem 19, if ℓ is a line of $\mathbf{H}^2$ then $\pi\ell$ is a chord of the disk $\mathbf{D}^2$. The end points of the chord are, of course, not included in $\mathbf{D}^2$ nor in $\pi\ell$. (See Figure 7.3.)*

Remark: If ℓ is a line of $\mathbf{H}^2$, then $\pi\ell$ is contained in a unique line $\tilde{\ell}$ of $\mathbf{P}^2$. On the other hand, not all lines of $\mathbf{P}^2$ determine lines of $\mathbf{H}^2$; only those that are secants of the conic.

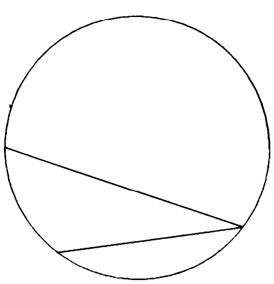

Figure 7.5 Parallel lines.

Theorem 21. *Let ℓ_1 and ℓ_2 be lines of $\mathbf{H}^2$. Then*

i. *ℓ_1 and ℓ_2 are intersecting lines if and only if $\tilde{\ell}_1$ and $\tilde{\ell}_2$ intersect in $\mathbf{D}^2$. (See Figure 7.4.)*
ii. *ℓ_1 and ℓ_2 are parallel if and only if $\tilde{\ell}_1$ and $\tilde{\ell}_2$ intersect at a point on the boundary of $\mathbf{D}^2$. (See Figure 7.5.)*
iii. *ℓ_1 and ℓ_2 are ultraparallel if and only if $\tilde{\ell}_1$ and $\tilde{\ell}_2$ intersect at a point exterior to $\mathbf{D}^2$. (See Figure 7.6.)*

Theorem 22. *Two lines ℓ_1 and ℓ_2 are perpendicular if and only if $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are conjugate.*

Theorem 23. *Let ℓ be a line of $\mathbf{H}^2$. Let P and Q be points of $\mathbf{P}^2$ where $\tilde{\ell}$ meets the conic. Then the pole of $\tilde{\ell}$ is the intersection R of the respective tangents through P and Q.*

Corollary. *A line m of $\mathbf{H}^2$ is perpendicular to ℓ if and only if $\tilde{m}$ passes through R, the pole of $\tilde{\ell}$. Figure 7.7 illustrates this and the previous two theorems.*

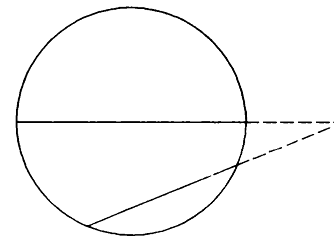

Figure 7.6 Ultraparallel lines.

Theorem 24. *Each point of $\mathbf{P}^2$ determines a unique pencil of $\mathbf{H}^2$ as follows:*

i. *Each point $\pi x = P$ of $\mathbf{D}^2$ determines the pencil of intersecting lines through $x \in \mathbf{H}^2$.*
ii. *Each point $\pi v = P$ (where v is lightlike) determines a pencil of parallels.*
iii. *Each point $\pi\xi = P$ (where ξ is a unit spacelike vector) determines a pencil of ultraparallels. The common perpendicular to this pencil corresponds to the polar line of P.*

In each case the pencil consists of all lines ℓ of $\mathbf{H}^2$ such that $\tilde{\ell}$ passes through the designated point P of $\mathbf{P}^2$.

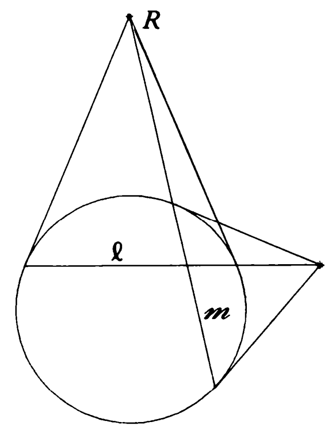

Figure 7.7 Two perpendicular lines, ℓ and m.

Remark: The pictures in Figures 7.8–7.10 give an intuitive idea of these relationships.

The discussion of this section should provide a motivation for some of the constructions we have been making in hyperbolic geometry. The incidence geometry of $\mathbf{D}^2$ is precisely that of $\mathbf{H}^2$, and this model of $\mathbf{H}^2$ is called the *Klein model.* Unfortunately, the Klein model does not represent either distance or angle faithfully, so it is unwise to rely too heavily on it. For example, a line is infinitely long, although it is represented in $\mathbf{D}^2$ by a (finite) chord.

Parallel displacements

Let $\mathscr{P}$ be a pencil of parallels determined by two lines with unit normals ξ and η. Choose an orthonormal basis by setting $e_1 = \xi$, $e_3 \in \mathbf{H}^2$, and $e_2 = e_3 \times e_1$. If we write

$$\eta = \lambda e_1 + \mu e_2 + \nu e_3, \quad \text{with} \quad \lambda \geqslant 0,$$

the conditions that $b(\eta, \eta) = 1$ and that $\xi \times \eta$ is lightlike give $\mu = \pm\nu$ and $\lambda = 1$. Hence,

$$\eta = e_1 + \mu(e_2 \pm e_3).$$

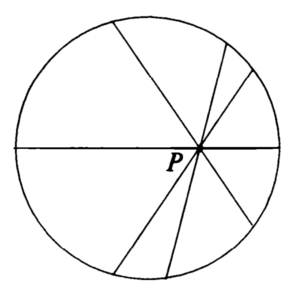

Figure 7.8 A pencil of intersecting lines.

Theorem 25. *In terms of the basis just described, suppose that $\mathscr{P}$ contains lines with unit normals* $(1, 0, 0)$ *and* $(1, \mu, -\mu)$ *for some real number* μ. *Then $\mathscr{P}$ consists precisely of those lines with unit normals of the form* $(1, r, -r)$, *where r ranges through the real numbers.*

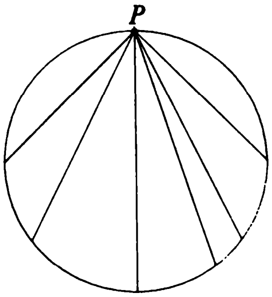

Figure 7.9 A pencil of parallels.

Proof: First note that

$$\xi \times \eta = \mu(e_1 \times e_2 - e_1 \times e_3) = -\mu(e_3 - e_2),$$

so that $\xi \times \eta$ is lightlike. Furthermore, if $\zeta = e_1 + r(e_2 - e_3)$, then

$$b(\zeta, \xi \times \eta) = \mu r b(e_2 - e_3, e_2 - e_3) = 0.$$

Conversely, if ζ is a unit spacelike vector orthogonal to $\xi \times \eta$, it is easy to check that $\pm\zeta$ must be of the form $(1, r, -r)$. □

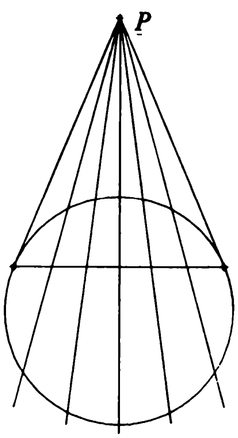

Figure 7.10 A pencil of ultraparallels.

Remark: The projective model of $\mathbf{H}^2$ provides some insight into what is going on here. The self-conjugate point P of $\mathbf{P}^2$ through which all lines of the pencil pass is $(0, 1, -1)$. A typical line of the pencil has its pole on the tangent to the conic at P. (See Figure 7.11.)

Note that a line of $\mathbf{H}^2$ belongs to two distinct pencils. In our example there is a second pencil through $Q = (0, 1, 1)$. The same basis may be used, but in this case the unit normals of lines of the pencil are $(1, r, r)$.

We have shown that a pencil of parallels is parametrized by the set of real numbers. Let α be a line of the pencil with pole $(1, r, -r)$ in homogeneous coordinates. Then

$$\begin{aligned}
\Omega_\alpha e_1 &= e_1 - 2b(\xi, e_1)\xi = -e_1 - 2re_2 + 2re_3,\\
\Omega_\alpha e_2 &= e_2 - 2r(e_1 + re_2 - re_3) = -2re_1 + (1 - 2r^2)e_2 + 2r^2e_3,\\
\Omega_\alpha e_3 &= e_3 - 2r(e_1 + re_2 - re_3) = -2re_1 - 2r^2e_2 + (1 + 2r^2)e_3.
\end{aligned}$$

The matrix of Ω_α is

$$\begin{bmatrix} -1 & -2r & -2r \\ -2r & 1 - 2r^2 & -2r^2 \\ 2r & 2r^2 & 1 + 2r^2 \end{bmatrix}. \tag{7.6}$$

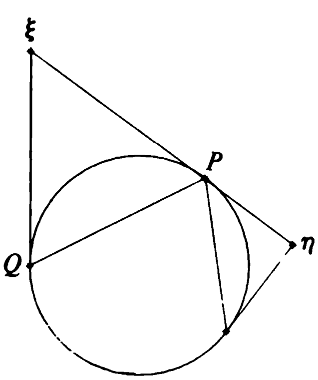

Figure 7.11 Two lines of a parallel pencil.

If β is a second line of this pencil, a calculation shows that

$$\Omega_\alpha\Omega_\beta = \begin{bmatrix} 1 & 2h & 2h \\ -2h & 1 - 2h^2 & -2h^2 \\ 2h & 2h^2 & 1 + 2h^2 \end{bmatrix} = D_h, \tag{7.7}$$

where β has pole $(1, s, -s)$ and $h = s - r$. Thus, the parallel displacement $\Omega_\alpha\Omega_\beta$ is represented with respect to this basis by the matrix D_h of (7.7). One can check that for real numbers h and k,

$$D_hD_k = D_{h+k}.$$

Theorem 26 (Three reflections theorem). *Let α, β, and γ be lines in a pencil of parallels. Then there is a fourth line δ in the pencil such that*

$$\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta.$$

Proof: With respect to an appropriate basis of $\mathbf{R}^3$, there exist real numbers r, s, and t representing α, β, and γ in the sense that $(1, r, -r)$ is a unit normal to α, and so forth. Now

$$\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta \quad \text{iff} \quad \Omega_\beta \Omega_\gamma = \Omega_\alpha \Omega_\delta;$$

that is,

$$D_{t-s} = D_{u-r},$$

where u is the real number representing δ. If we choose $u = r + t - s$, this last equation becomes true. Hence, the theorem is true, and the pole of the required line δ is represented by

$$e_1 + (r + t - s)(e_2 - e_3). \qquad \square$$

A representation theorem for parallel displacements holds also.

Theorem 27. *Let ρ be a parallel displacement arising from a pencil $\mathscr{P}$. Let ℓ be a line of $\mathscr{P}$. Then there are lines m and m' in $\mathscr{P}$ such that*

$$\rho = \Omega_\ell \Omega_m = \Omega_{m'} \Omega_\ell.$$

Proof: Exercise 18. $\square$

Let REF($\mathscr{P}$) be the group generated by all reflections in lines of the pencil $\mathscr{P}$. Let DIS($\mathscr{P}$) be the set of all parallel displacements determined by the pencil $\mathscr{P}$. In Exercise 19 you will show that DIS($\mathscr{P}$) is a group and investigate its algebraic properties.

Translations

Let $\mathscr{P}$ be an ultraparallel pencil with common perpendicular ℓ. Let e_1 be a unit normal of ℓ. Choose e_2 and e_3 spacelike and timelike, respectively, so that $\{e_1, e_2, e_3\}$ is an orthonormal basis.

Let α be an arbitrary line of the pencil. Its unit normal can be written

$$\xi = (\cosh u)e_2 + (\sinh u)e_3.$$

Then

$$\Omega_\alpha e_1 = e_1 - 2b(e_1, \xi)\xi = e_1,$$

$$\begin{aligned}\Omega_\alpha e_2 &= e_2 - (2 \cosh u)((\cosh u)e_2 + (\sinh u)e_3) \\ &= -(\cosh 2u)e_2 - (\sinh 2u)e_3,\end{aligned}$$

$$\Omega_\alpha e_3 = (\sinh 2u)e_2 + (\cosh 2u)e_3.$$

$$\Omega_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cosh 2u & \sinh 2u \\ 0 & -\sinh 2u & \cosh 2u \end{bmatrix}. \tag{7.8}$$

If β is a second line of the pencil whose pole is parametrized by v, then

$$\Omega_\alpha\Omega_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh 2k & \sinh 2k \\ 0 & \sinh 2k & \cosh 2k \end{bmatrix}, \tag{7.9}$$

where $k = u - v$.

Denote this last matrix by T_k. Then one can easily verify that

$$T_k T_m = T_{k+m}.$$

Theorem 28 (Three reflections theorem). *Let α, β, and γ be lines of a pencil of ultraparallels. Then there is a line δ in the pencil such that*

$$\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta.$$

Proof: Exercise 21. □

Theorem 29 (Representation of translations). *Let ρ be a translation along a line ℓ. Let m be any line perpendicular to ℓ. Then there exist lines α and α' perpendicular to ℓ such that*

$$\rho = \Omega_m\Omega_\alpha = \Omega_{\alpha'}\Omega_m.$$

Let $\mathscr{P}$ be a pencil of ultraparallels, and let ℓ be the common perpendicular. The group generated by all reflections in lines of $\mathscr{P}$ is denoted by REF($\mathscr{P}$). Let TRANS(ℓ) be the set of translations along ℓ. Properties of TRANS(ℓ) will be left to the exercises (Exercise 22).

Glide reflections

With respect to the basis used in the previous section, we construct the matrix of the glide reflection $\Omega_\ell T_k$. One can easily check that

$$\Omega_\ell e_1 = -e_1, \quad \Omega_\ell e_2 = e_2, \quad \Omega_\ell e_3 = e_3.$$

Thus,

$$\Omega_\ell T_k = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cosh 2k & \sinh 2k \\ 0 & \sinh 2k & \cosh 2k \end{bmatrix}. \tag{7.10}$$

Products of more than three reflections

In each of the geometries studied so far, any motion can be realized as the product of two or three reflections. The same is true in $\mathbf{H}^2$. However, the incidence structure of $\mathbf{H}^2$ is more complicated. More cases must be considered in the proof.

Our approach is to show that any product $\Omega_\alpha\Omega_\beta\Omega_\gamma\Omega_\delta$ of four reflections can be reduced to a product of two reflections (as in Theorem 1.36). As a first observation, if the pencil determined by α and β has a line in common with the pencil determined by γ and δ, our representation theorems may be applied to rewrite our product of four reflections in such a way that the second and third reflections are the same.

We begin with

Theorem 30. *Let P be a point of $\mathbf{H}^2$, and let $\mathscr{P}$ be a pencil. Then there is a line through P belonging to the pencil $\mathscr{P}$. Except in the case of the pencil of all lines through P, this line is unique.*

Proof: If $\mathscr{P}$ is a pencil of intersecting lines or a pencil of ultraparallels, the conclusion is given by Theorem 4 and Theorem 8, part (ii), respectively. Now let $\mathscr{P}$ be a pencil of parallels determined by lines with unit normal ξ and η. Then $\xi \times \eta$ is lightlike, and $P \times (\xi \times \eta)$ is nonzero. By Theorem 2, part (ii), $P \times (\xi \times \eta)$ is spacelike. The line whose unit normal is in this direction belongs to $\mathscr{P}$ and passes through P and is the only line satisfying these conditions. □

Theorem 31. *Let $\mathscr{P}_1$ be a pencil of parallels. Let $\mathscr{P}_2$ be the pencil consisting of all lines perpendicular to a line γ. If $\gamma \notin \mathscr{P}_1$, there is a unique line belonging to both pencils.*

Proof: Choose an orthonormal basis as follows. Let e_2 be a unit normal to γ. Let w be a lightlike vector such that the unit normals ξ to lines of $\mathscr{P}_1$ are precisely those unit spacelike vectors satisfying $b(\xi, w) = 0$. See Figure 7.12.

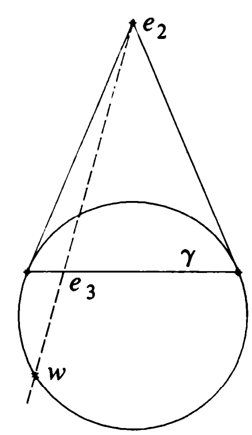

Figure 7.12 Construction of a line common to a parallel pencil and an ultraparallel pencil.

We wish to choose $e_3 \in \mathbf{H}^2$ so that it lies in $[w, e_2]$. To do this, note that

$$b(w \times e_2, w \times e_2) = b(w, e_2)^2 > 0$$

because $\gamma \notin \mathscr{P}_1$. Choose e_1 to be a unit vector in the direction $[w \times e_2]$, and $e_3 = e_1 \times e_2$. There are two choices for e_1, but only one that will ensure that e_3 lies in $\mathbf{H}^2$.

Now that we have this basis, it is easy to see that the line with unit normal e_1 is the unique line belonging to both pencils. □

Remark: If the two pencils are related in such a way that $\gamma \in \mathscr{P}_1$, then they can have no line in common. (See Exercise 33.)

Theorem 32. *Two distinct pencils of parallels have a unique line in common.*

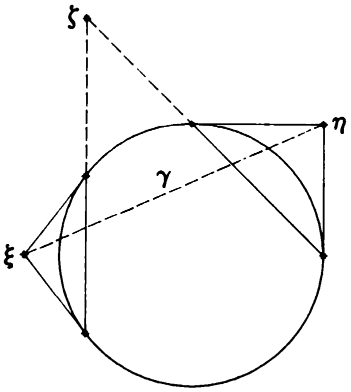

Figure 7.13 Theorem 7. Two ultraparallel pencils with a common line γ.

Proof: Let v and w be lightlike vectors determining distinct pencils. Then Theorem 5.26 gives

$$b(v \times w,\, v \times w) = b(v,\, w)^2,$$

which is positive (Exercise 34). The line whose unit normal is a multiple of $v \times w$ is the unique line common to both pencils. □

Remark: Two ultraparallel pencils have a line in common if and only if the common perpendiculars to the two pencils are themselves ultraparallel. See Figure 7.13 and Theorem 7. Thus, we have completed our analysis of the question of when two pencils have a line in common. We now have enough ammunition to attempt the task set out at the beginning of this section.

Theorem 33. *Let* α, β, γ, *and* δ *be lines. Then there exist lines u and* v *such that*

$$\Omega_\alpha \Omega_\beta \Omega_\gamma \Omega_\delta = \Omega_u \Omega_v.$$

Proof: If $\alpha = \beta$ or $\gamma = \delta$, there is nothing to prove. Assume that α and β determine a pencil $\mathscr{P}_1$, whereas γ and δ determine a pencil $\mathscr{P}_2$. As remarked at the beginning of this section, the result clearly holds if $\mathscr{P}_1$ and $\mathscr{P}_2$ have a line in common. In view of Theorems 30–32, we have still to consider the following cases:

1. α and β have γ as a common perpendicular, and δ is parallel to γ. In this case Ω_β commutes with Ω_γ, and Theorem 30 applies.
2. α and β have a common perpendicular ℓ, γ and δ have a common perpendicular m, and ℓ intersects m in a point P. Using Theorem 29, we may replace the given representation by $\Omega_{\alpha'}\Omega_{\beta'}\Omega_{\gamma'}\Omega_{\delta'}$, where β' and γ' pass through P. Then $\Omega_{\beta'}\Omega_{\gamma'}$ may be replaced by $\Omega_\ell\Omega_n$ for some line n through P. Because α' is perpendicular to ℓ, Theorem 30 now applies.
3. α and β have a common perpendicular ℓ, γ and δ have a common perpendicular m, but ℓ is parallel to m. In this case we let Q be the point where m intersects γ, and we let β' be the line through Q perpendicular to ℓ. Then the motion can be written $\Omega_{\alpha'}\Omega_{\beta'}\Omega_\gamma\Omega_\delta$ for some line $\alpha' \perp \ell$. As in case (2) we may now write

$$\Omega_{\beta'}\Omega_\gamma = \Omega_n\Omega_m$$

for some line n. Again apply Theorem 30. □

Remark: Because

$$(\Omega_\alpha\Omega_\beta\Omega_\gamma\Omega_\delta)^{-1} = \Omega_\delta\Omega_\gamma\Omega_\beta\Omega_\alpha,$$

the foregoing set of cases is exhaustive. For example, it is not necessary to consider the case where α and β are parallel while γ and δ have a common perpendicular.

Theorem 34. *Let α, β, and γ be lines not belonging to any pencil. Then $\Omega_\alpha\Omega_\beta\Omega_\gamma$ is a nontrivial glide reflection.*

The proof of Theorem 34 uses techniques similar to those we have been using in Theorem 33. It is left as an exercise (Exercise 35).

We can now assert the following classification of motions of $\mathbf{H}^2$.

Theorem 35. *The group of motions of $\mathbf{H}^2$ consists of all reflections, rotations, translations, parallel displacements, and glide reflections. Every motion is the product of two or three suitably chosen reflections.*

Fixed points of isometries

Consider the isometry $\rho = \Omega_\alpha\Omega_\beta$. Fixed points of ρ are found by solving for $X \in \mathbf{H}^2$ the equation $\rho X = X$; that is, $\Omega_\alpha X = \Omega_\beta X$. Any solution must satisfy $b(X, \xi)\xi = b(X, \eta)\eta$. If $b(X, \xi) = b(X, \eta) = 0$, then X is a multiple of $\xi \times \eta$. This means that X is the point of intersection of α and β in $\mathbf{H}^2$. On the other hand, if $b(X, \xi) \neq 0$ or $b(X, \eta) \neq 0$, ξ must be a multiple of η, and so $\alpha = \beta$. Thus, we can state

Theorem 36.

i. *A nontrivial translation has no fixed points.*
ii. *A nontrivial rotation has exactly one fixed point, the center of rotation.*
iii. *A nontrivial parallel displacement has no fixed points.*
iv. *A reflection has a line of fixed points, the axis of reflection.*
v. *A nontrivial glide reflection has no fixed points.*

This result may be compared with the Euclidean analogue, Theorem 1.39. For the proof see Exercise 37.

Fixed lines of isometries

If Ω_α is a reflection whose axis has unit normal ξ, then Ω_α will leave fixed the lines whose unit normals ζ satisfy $\Omega_\alpha\zeta = \pm\zeta$; that is, ζ must be orthogonal to ξ or $\zeta = \pm\xi$.

Suppose now that α and β are lines with respective unit normals ξ

and η. Then $\rho = \Omega_\alpha\Omega_\beta$ has a fixed line with unit normal ξ if and only if $\Omega_\alpha\zeta = \pm\Omega_\beta\zeta$. The condition $\Omega_\alpha\zeta = -\Omega_\beta\zeta$ is satisfied only if $\zeta = b(\zeta, \xi)\xi + b(\zeta, \eta)\eta$. But this implies that

$$b(\zeta, \xi) = b(\zeta, \xi)b(\xi, \xi) + b(\zeta, \eta)b(\eta, \xi)$$

and

$$b(\zeta, \eta) = b(\zeta, \xi)b(\zeta, \eta) + b(\zeta, \eta)b(\eta, \eta).$$

These equations in turn give

$$b(\zeta, \eta)b(\eta, \xi) = b(\zeta, \xi)b(\eta, \xi) = 0.$$

Thus, either $\alpha \perp \beta$ or $b(\zeta, \eta) = b(\zeta, \xi) = 0$, which would imply $\zeta = 0$, an impossibility.

We conclude that $\Omega_\alpha\zeta = -\Omega_\beta\zeta$ if and only if $\alpha \perp \beta$ and ζ is in the span of ξ and η. This means that ρ is a half-turn, and the fixed line passes through its center.

We now search for ζ with $b(\zeta, \zeta) = 1$ and $\Omega_\alpha\zeta = \Omega_\beta\zeta$. Then $b(\zeta, \xi)\xi = b(\zeta, \eta)\eta$. If $\alpha \neq \beta$, this implies that $b(\zeta, \xi) = b(\zeta, \eta) = 0$. Thus, the line of $\mathbf{H}^2$ with unit normal ζ is a common perpendicular of α and β.

Summarizing our results concerning isometries that are the product of two reflections, we have

Theorem 37. *Let α and β be distinct lines of $\mathbf{H}^2$. The isometry $\Omega_\alpha\Omega_\beta$ has the following fixed line behavior.*

i. *If α and β intersect at P and $\alpha \perp \beta$, every line through P is fixed. In this case, $\Omega_\alpha\Omega_\beta$ is the half-turn about P.*
ii. *If α and β intersect at P and α is not perpendicular to β, $\Omega_\alpha\Omega_\beta$ has no fixed lines.*
iii. *If α and β are parallel, $\Omega_\alpha\Omega_\beta$ has no fixed lines. Thus, parallel displacements have no fixed lines.*
iv. *If α and β have a common perpendicular ℓ, then $\Omega_\alpha\Omega_\beta$ leaves ℓ fixed but has no other fixed lines.*

Theorem 38. *Let γ be a line perpendicular to two distinct lines α and β. Then the nontrivial glide reflection $\Omega_\alpha\Omega_\beta\Omega_\gamma$ has γ as its only fixed line.*

Proof: We must determine the unit spacelike vectors ζ satisfying

$$\Omega_\alpha\zeta = \pm\Omega_\beta\Omega_\gamma\zeta.$$

A calculation similar to that used in Theorem 37 shows that the positive sign cannot occur. With the negative sign ζ must be a unit normal to the line γ. The details are left to Exercise 38. □

Segments, rays, angles, and triangles

Let P be a point. As we know, a line through P can be parametrized by

$$\alpha(t) = (\cosh t)P + (\sinh t)\xi$$

for a suitable unit spacelike vector ξ. A set of the form $\alpha([0, L])$, $L > 0$, is called a *segment* of length L. The points $\alpha(0)$ and $\alpha(L)$ are called *end points*. The point $M = \alpha(L/2)$ is the *midpoint*, and the usual definition of perpendicular bisector holds. The set $\alpha([0, \infty))$ is called a *ray*. The point $\alpha(0)$ is called the *origin* of the ray. It is a not-quite-obvious fact that these definitions have all the properties we should expect.

Theorem 39.

i. *Two distinct points A and B are the end points of exactly one segment, which we denote by AB or, equivalently, BA. The length of AB is $d(A, B)$.*
ii. *Each ray has exactly one origin. For each pair of points A and B, there is exactly one ray with origin A that passes through B. We denote this ray by $\overrightarrow{AB}$.*

Definition. *The unit spacelike vector ξ occurring in the definition of α is called the* direction vector *of the ray $\alpha([0, \infty))$. Note that $b(P, \xi) = 0$.*

Remark: Each ray has a unique direction vector. Taking our inspiration from the projective model of $\mathbf{H}^2$, we may think of ξ as a point "past infinity" toward which the ray is heading.

Angles and triangles are defined as in $\mathbf{E}^2$ along with the associated terms (straight angles, opposite rays, etc.). The radian measure of an angle is $\cos^{-1} b(\xi, \eta)$, where ξ and η are the direction vectors of the rays making up the angle. In Exercise 41 you will be asked to check that this is equivalent to

$$\cos^{-1} b\left(\frac{Q \times P}{|Q \times P|}, \frac{Q \times R}{|Q \times R|}\right)$$

for the angle $\measuredangle PQR$. Note that $Q \times P$ and $Q \times R$ are spacelike vectors.

Definition. *A* half-plane *bounded by a line ℓ is a set of the form*

$$\{x \in \mathbf{H}^2 | b(\xi, x) > 0\},$$

where ξ is a unit normal of ℓ.

Theorem 40. *Each half-plane is bounded by a unique line. Each line bounds two half-planes. The union of these two half-planes is $\mathbf{H}^2 - \ell$. Two* *points of* $\mathbf{H}^2 - \ell$ *are in the same half-plane if and only if the segment joining them does not meet* ℓ.

Definition. *The* interior *of an angle* $\measuredangle PQR$ *is the intersection of the half-plane bounded by* $\overleftrightarrow{PQ}$ *containing* R *with the half-plane bounded by* $\overleftrightarrow{RQ}$ *containing* P. *(This definition does not make sense for straight angles or zero angles. The interior of such an angle is undefined.)*

Theorem 41. *Let* $\mathscr{A} = \measuredangle PQR$ *be an angle whose interior is defined. Let* ξ *and* η *be direction vectors of its arms. Then the interior of* $\mathscr{A}$ *consists of those points* X *such that the direction vector of* $\overrightarrow{QX}$ *is a positive linear combination of* ξ *and* η.

Proof: Let X be any point other than Q, and let ζ be the direction vector of $\overrightarrow{QX}$. Because the subspace $\{v \in \mathbf{R}^3 | b(v, Q) = 0\}$ is two dimensional, $\{\xi, \eta\}$ is a basis, and there are unique numbers λ and μ such that

$$\zeta = \lambda\xi + \mu\eta.$$

We claim that X is in the interior of $\mathscr{A}$ if and only if λ and μ are positive. To see this, write

$$X = (\cosh t)Q + (\sinh t)\zeta$$

and note that $\xi \times Q$ is a unit normal to one arm, say $\overrightarrow{QP}$. Then

$$\begin{aligned} b(X, \xi \times Q) &= (\sinh t)b(\zeta, \xi \times Q) = \mu(\sinh t)b(\eta, \xi \times Q) \\ &= \mu\frac{\sinh t}{\sinh s}b(R, \xi \times Q), \end{aligned}$$

where s is the number satisfying $R = (\cosh s)Q + (\sinh s)\eta$. Thus, X and R lie on the same side of $\overleftrightarrow{PQ}$ if and only if $\mu > 0$. Similarly, X and P lie on the same side of $\overleftrightarrow{RQ}$ if and only if $\lambda > 0$. □

Addition of angles

Theorem 42.

i. *Let* $\mathscr{A} = \measuredangle PQR$ *be an angle with a point* X *in its interior. Then the radian measure of* $\mathscr{A}$ *is the sum of the radian measures of* $\measuredangle PQX$ *and* $\measuredangle RQX$.
ii. *Let* $\mathscr{A} = \measuredangle PQR$ *be a straight angle, and let* X *be any point not on the line* $\overleftrightarrow{PQ}$. *Then the sum of the radian measures of* $\measuredangle PQX$ *and* $\measuredangle RQX$ *is equal to* π.

Proof:

i. Let ξ, η, and ζ be the respective direction vectors as in Theorem 41. We need to prove the identity

$$\cos^{-1} b(\xi, \zeta) + \cos^{-1} b(\zeta, \eta) = \cos^{-1} b(\xi, \eta), \qquad (7.11)$$

using the fact that $\zeta = \lambda\xi + \mu\eta$, where λ and μ are positive numbers satisfying

$$b(\zeta, \zeta) = \lambda^2 + \mu^2 + 2\lambda\mu b(\xi, \eta) = 1.$$

For convenience write $a = b(\xi, \eta)$. Then $b(\xi, \zeta) = \lambda + \mu a$ and $b(\zeta, \eta) = \lambda a + \mu$, so that our identity reduces to

$$\cos^{-1}(\lambda + \mu a) + \cos^{-1}(\lambda a + \mu) = \cos^{-1} a, \qquad (7.12)$$

which can be verified by calculus (Exercise 44).

ii. When $\measuredangle PQR$ is a straight angle, we have no expression for ζ in terms of ξ and η. We do not need one, however. The required identity reduces to

$$\cos^{-1} b(\xi, \zeta) + \cos^{-1} b(-\xi, \zeta) = \pi,$$

which is just one of the standard properties of the $\cos^{-1}$ function. (See Theorem 2F.) □

Remark: The Euclidean version of this theorem (Theorem 2.14) is essentially the same thing. We could have used the preceding proof in Chapter 2. On the other hand, if we fixed an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathbf{R}^3$ with $e_3 = Q$, the proof given in 2.34 can be easily modified to prove Theorem 42. Both proofs have advantages and disadvantages. The first one is more geometric. The second one is more direct but relies explicitly on a computation involving differentiation, and therefore it is in some sense less elementary.

Remark: All our Euclidean definitions of rectilinear figures and their associated properties hold true in $\mathbf{H}^2$. Because of the incidence structure of $\mathbf{H}^2$, however, some new types of figures are possible.

Triangles and hyperbolic trigonometry

In hyperbolic geometry triangles are easier to deal with than in spherical or elliptic geometry because segments are simple. Each pair of points determines a unique segment. Thus, we can define, as in $\mathbf{E}^2$, the triangle $\triangle PQR$ to be the union of the segments PQ, QR, and PR. Each triangle has three angles. The interior of the triangle is the intersection of the interiors of its three angles.

Theorem 43. *Let ABC be a triangle. Let a, b, and c be the lengths of its sides. Then, using the same notational conventions as in spherical trigonometry (see Figure 7.14), we have*

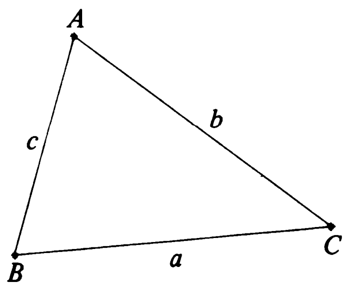

Figure 7.14 Theorem 43. A hyperbolic triangle.

i. $$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}. \tag{7.13}$$

ii. $$\frac{\sin A}{\sinh a} = \frac{2(\sinh s \sinh(s-a) \sinh(s-b) \sinh(s-c))^{1/2}}{\sinh a \sinh b \sinh c}. \tag{7.14}$$

iii. $$\cosh a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \tag{7.15}$$

Remark: Birman and Nomizu [5] have worked out trigonometric formulas for Lorentzian plane geometry. Their formulas bear a relationship to (7.13) and (7.14) analogous to that between plane Euclidean and spherical (Theorem 4.38) formulas, This is related to the fact that $\mathbf{H}^2$ may be regarded as a "sphere" in Lorentzian three-space. (See Exercise 72.)

Asymptotic triangles

Each line belongs to two parallel pencils. However, each ray determines a unique parallel pencil. In fact, if ζ is the direction vector of a ray $\overrightarrow{PX}$, then $P + \zeta$ is a lightlike vector with the property that $\{\xi | b(\xi, P + \zeta) = 0\}$ is the set of unit normal vectors of a unique pencil. (The other pencil to which the line $\overleftrightarrow{PX}$ belongs is determined by $P - \zeta$.)

Let PQ be a segment, and let $\overrightarrow{PX}$ and $\overrightarrow{QY}$ be parallel rays determining the same pencil. Then the union of PQ and the two rays is called *a (singly) asymptotic triangle.* Two views of an asymptotic triangle are shown in Figure 7.15. You may think of an asymptotic triangle as an ordinary triangle with one vertex "at ∞."

A pair of rays $\overrightarrow{PX}$ and $\overrightarrow{PY}$ together with the line common to the parallel pencils they determine is a *doubly asymptotic triangle.* See Figure 7.16. A *triply asymptotic triangle* consists of three lines mutually parallel in pairs. See Figure 7.17.

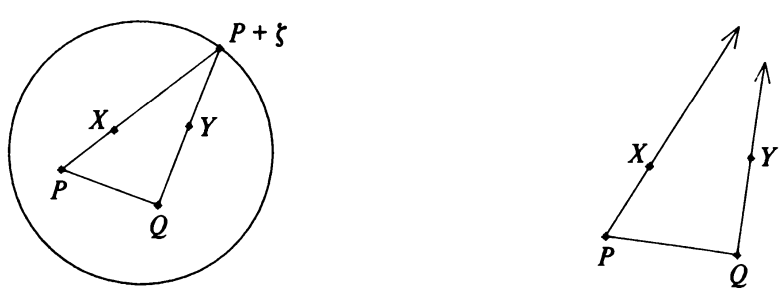

Figure 7.15 A singly asymptotic triangle, two views.

Figure 7.16 A doubly asymptotic triangle, two views.

Figure 7.17 A triply asymptotic triangle, two views.

Quadrilaterals

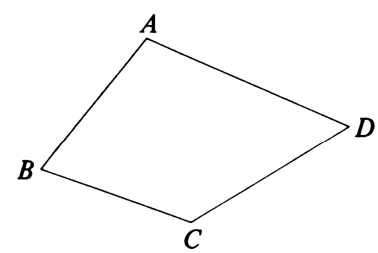

Figure 7.18 A convex quadrilateral.

A *convex quadrilateral* $ABCD$ is the union of four segments (sides) AB, BC, CD, and DA that are placed in such a way that each side determines a half-plane that contains the opposite side (See Figure 7.18.) Note that $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{BD}$ at an interior point of the figure. The other diagonal points (in the sense of projective geometry) can be distributed in six distinct configurations as far as incidence is concerned. This together with the possibilities for equality of various lengths and angles gives us a rich variety of generalizations of the notions of parallelogram, rectangle, rhombus, and square. We will only scratch the surface of this wealth of symmetric figures.

First, consider a convex quadrilateral $ABCD$ in which opposite sides have the same length. This is the best generalization of the Euclidean notion of parallelogram. The special case in which all four sides have equal length is called a *rhombus*. (See Figure 7.19.)

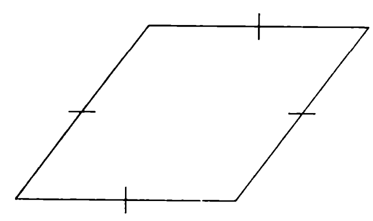

Figure 7.19 A rhombus.

A convex quadrilateral in which all four angles have equal radian measure is called an *equiangular quadrilateral*. The equiangular rhombus is the hyperbolic analogue of the square. (See Figures 7.20 and 7.21.)

Although convex quadrilaterals cannot have four right angles in hyperbolic geometry, there are several different figures that could be considered analogues of the rectangle. The *Saccheri quadrilateral* $ABCD$ has $d(A, B) = d(C, D)$, $AB \perp BC$, and $CD \perp BC$. (See Figure 7.22.) The *Lambert quadrilateral*, on the other hand, has three right angles. It is shown in Figure 7.23.

Regular polygons

A regular polygon with any number of sides can be constructed by taking as vertices the orbit of a point Q under a cyclic subgroup of the group of rotations leaving another point P fixed. The resulting figure has the same symmetry group as in the Euclidean case. However, as the trigonometric formulas show, the angles get smaller as $d(P, Q)$ increases. In general, there are regular m-gons whose angles all have radian measure equal to any number between 0 and $(1 - 2/m)\pi$ that you wish to prescribe. For example, there is a regular 7-gon all of whose angles are right angles. See Figure 7.24.

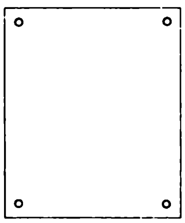

Figure 7.20 An equiangular quadrilateral.

Congruence theorems

Theorem 44. *There is a unique reflection that interchanges a given pair of points of* $\mathbf{H}^2$.

Proof: Let P and Q be the given points. Let $\mathscr{m}$ be the perpendicular bisector of PQ. Note that the midpoint M of PQ is a unit timelike vector in the direction $[P + Q]$ and that the unit normal to $\mathscr{m}$ has direction $[P - Q]$. It is a straightforward exercise (Exercise 51) to verify that $\Omega_{\mathscr{m}}$ interchanges P and Q. On the other hand, if $\Omega_{\mathscr{m}'}$ is any reflection interchanging P and Q, then $\Omega_{\mathscr{m}}\Omega_{\mathscr{m}'}$ must leave P and Q fixed and, hence, by Theorem 36, must be the identity. □

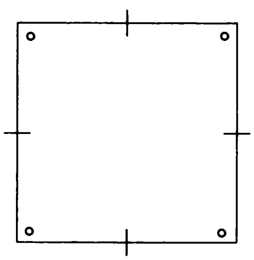

Figure 7.21 An equiangular rhombus.

Theorem 45. *There are precisely two reflections that interchange a given pair of intersecting lines of* $\mathbf{H}^2$.

Proof: Suppose that the two lines have unit normals ξ and η. Let α be the line with unit normal in the direction $[\xi + \eta]$. It is easy (Exercise 52) to verify that Ω_α interchanges the two given lines. On the other hand, if $\Omega_{\alpha'}$ is any other reflection interchanging the given lines, the rotation $\Omega_\alpha\Omega_{\alpha'}$ leaves both lines fixed. By Theorem 37, α and α' must be perpendicular. Note that α' is just the line whose unit normals have direction $[\xi - \eta]$. □

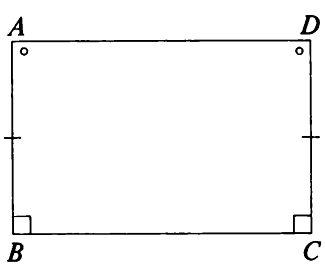

Figure 7.22 A Saccheri quadrilateral.

Classification of isometries of $\mathbf{H}^2$

Our main result is that every isometry of $\mathbf{H}^2$ is a motion. First, we have the following uniqueness theorems.

Theorem 46. *Let T be an isometry that leaves fixed a point P and a line ℓ through P. Let $\mathscr{m}$ be the line through P perpendicular to ℓ. Then either T or $\Omega_{\mathscr{m}}T$ has ℓ as a line of fixed points.*

Proof: Let X be an arbitrary point of ℓ other than P. Let v be the unit direction vector of $\overrightarrow{PX}$. Then for some positive number s,

$$X = (\cosh s)P + (\sinh s)v.$$

Similarly, if Y is a third point on ℓ,

$$Y = (\cosh t)P + (\sinh t)v$$

for some t. Because $T\ell = \ell$, the points TX and TY have similar representations. Using the fact that $b(TX, P) = b(X, P)$ and $b(TY, P) = b(Y, P)$, we see that these representations take the form

$$TX = (\cosh s)P \pm (\sinh s)v, \quad TY = (\cosh t)P \pm (\sinh t)v. \tag{7.16}$$

But now, $b(TX, TY) = b(X, Y)$, and, hence, the signs occurring in (7.16) are either both positive or both negative. In the first case T leaves ℓ pointwise fixed. It is easy to check that $\Omega_m T$ has the same property in the second case. □

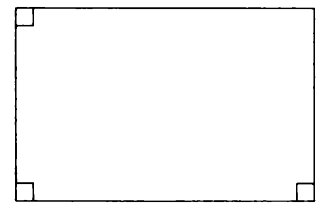

Figure 7.23 A Lambert quadrilateral.

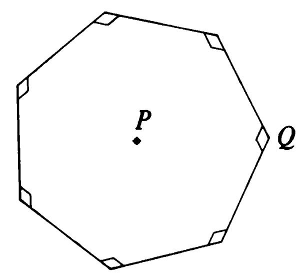

Figure 7.24 A regular 7-gon with seven right angles.

Theorem 47. *Let T be an isometry of $\mathbf{H}^2$. Suppose that T has a line ℓ of fixed points. Then either $T = \Omega_\ell$ or T is the identity.*

Proof: Assume that T is not the identity. Choose any point X not fixed by T. Let A be the foot of the perpendicular from X to ℓ, and let v be the unit direction vector of $\overrightarrow{AX}$. We may then construct an orthonormal basis $\{e_1, e_2, e_3\}$ with $e_3 = A$, $e_2 = v$, and $e_1 = e_2 \times e_3$. Write

$$X = (\cosh t)e_3 + (\sinh t)e_2, \quad t > 0.$$

Choose $s > 0$ and consider on ℓ the points

$$Y = (\cosh s)e_3 + (\sinh s)e_1, \quad Y' = (\cosh s)e_3 - (\sinh s)e_1.$$

Using the fact that $d(X, Y) = d(X, Y')$ and, hence, $d(TX, Y) = d(TX, Y')$, we conclude that $b(TX, e_1) = 0$. Thus, TX must lie on the line $\overleftrightarrow{AX}$. Writing

$$TX = (\cosh u)e_3 + (\sinh u)e_2,$$

and using the fact that $b(TX, A) = b(X, A)$, we get $\cosh s = \cosh u$; that is, $s = \pm u$. From this it is clear that $TX = \Omega_\ell X$ and that T agrees with Ω_ℓ at every nonfixed point of T. It remains only to show that all fixed points of T lie on ℓ. To see this, suppose that Y is a fixed point of T. Then

$$\begin{aligned} b(X, Y) &= b(TX, TY) = b(\Omega_\ell X, Y) \\ &= b(X - 2b(X, e_2)e_2, Y) \\ &= b(X, Y) - 2b(X, e_2)b(Y, e_2). \end{aligned}$$

Because $b(X, e_2) \neq 0$, we must have $b(Y, e_2) = 0$; that is, Y lies on ℓ. □

Theorem 48. *Every isometry of the hyperbolic plane is a motion.*

Proof: Let T be an isometry. We shall construct a sequence of reflections whose product coincides with T. First choose an arbitrary point P, and let T_1 be the reflection that interchanges P and TP. Then T_1T has P as a fixed point (Theorem 44). (In the special case where T already has a fixed point, we may shorten the construction by letting P be the fixed point and by letting $T_1 = I$.)

The second step is to construct T_2, so that T_2T_1T fixes P and a line ℓ through P. This may be arranged by letting T_2 be a reflection interchanging an arbitrary line ℓ through P with $T_1T\ell$ (Theorem 45). (If T_1T already has a fixed line ℓ through P, we may choose $T_2 = I$.)

Now, directly applying Theorem 46, we can choose a suitable reflection T_3 (or possibly $T_3 = I$), so that $T_3T_2T_1T$ leaves ℓ pointwise fixed. Finally, by Theorem 47, we can choose $T_4 = \Omega_\ell$ or $T_4 = I$, so that $T_4T_3T_2T_1T$ is the identity. Because each T_i is its own inverse, this means that $T = T_1T_2T_3T_4$, as required. □

Remark: As we saw in Theorem 35, this product may be written as the product of three or fewer reflections. In this case, however, we can observe this fact more directly as follows. Using Theorem 17, we can see that $T_2T_3T_4$ is either a rotation about P or a reflection in a line through P. Then, depending on T_1, we can conclude that T is a rotation, a reflection, or a glide reflection. Furthermore, we have all the information necessary to explicitly find the transformations T_i.

Corollary. *Every isometry of* $\mathbf{H}^2$ *is one of the following: reflection, rotation, parallel displacement, translation, or glide reflection.*

Circles, horocycles, and equidistant curves

Definition. *Let C be a point and $r \geqslant 0$ a number. Then*

$$\mathscr{C} = \{X \mid d(X, C) = r\} \tag{7.17}$$

is called a circle *with center C and radius r.*

Theorem 49. *Let $\mathscr{P}$ be the pencil of lines through a point C, and let P be any point. Then the orbit of P by* $\mathrm{REF}(\mathscr{P})$ *is the circle with center C and radius $r = d(P, C)$. Conversely, every circle arises in this way.*

Definition. *Let m be a line and r a positive number. The portion of*

$$\{X \mid d(X, m) = r\} \tag{7.18}$$

lying in a half-plane determined by m is called an equidistant curve. *The line m is also (by definition) an equidistant curve corresponding to $r = 0$.*

Theorem 50. *Let $\mathscr{P}$ be the pencil of lines perpendicular to a line m, and let P be any point. Then the orbit of P by* REF($\mathscr{P}$) *is an equidistant curve. Conversely, every equidistant curve arises in this way.*

Definition. *Let $\mathscr{P}$ be a pencil of parallels, and let P be any point. Then the orbit of P by* REF($\mathscr{P}$) *is called a* horocycle.

Remark: The horocycle may be thought of as a limiting case of a circle having its center "at infinity."

Theorem 51. *Let v be a nonzero vector in $\mathbf{R}^3$, and let a be a number. If*

$$\{x \in \mathbf{H}^2 | b(v, x) = a\} \tag{7.19}$$

is nonempty, it is a circle, an equidistant curve, or a horocycle. Conversely, each circle, equidistant curve, and horocycle has an equation of this form.

A higher-dimensional version of the results of this section is found in [6].

EXERCISES

1. Prove Theorem 2.
2. Find spacelike vectors ξ and η such that $\xi \times \eta$ is a nonzero lightlike vector, but $b(\xi, \eta)^2 = b(\xi, \xi)b(\eta, \eta)$.
3. Verify that neither parallel nor ultraparallel lines intersect. (See the remark following Theorem 6.)
4. Verify the remarks following the definition of pencils.
5. i. Let $\xi = (1/\sqrt{2})(1, 1, 0)$ and $\eta = (1/2\sqrt{3})(3, 2, 1)$. Find an orthonormal basis with ξ as one element and a multiple of $\xi \times \eta$ as another.
   ii. Let ξ and η be the respective unit normals of lines of $\mathbf{H}^2$. If the lines intersect, find the point of intersection. If they are ultraparallel, find the common perpendicular.
6. i. Prove Theorem 9.
   ii. Prove that $d(\ell, m) = \cosh^{-1}|\langle \xi, \eta \rangle|$, where ℓ and m are lines with unit normals ξ and η.
7. Complete the proof of Theorem 11 by showing that $d(Q, R) = d(P, R) + d(P, Q)$ implies that P lies between Q and R.
8. Prove Theorem 12.
9. Verify that Theorems 17 and 18 hold.
10. Prove Theorem 19.
11. Prove Theorem 20 and the remark following it.

12. Prove Theorem 21.
13. Prove Theorem 22.
14. Prove Theorem 23 and its corollary.
15. Prove Theorem 24.
16. Check that $\pm\zeta$ must have the form $(1, r, -r)$ as indicated in Theorem 25.
17. Verify formula (7.7).
18. Prove Theorem 27.
19. i. Check that $\mathrm{DIS}(\mathscr{P})$ is a subgroup of $\mathrm{REF}(\mathscr{P})$.
    ii. Show that if α is any line of the pencil $\mathscr{P}$, $\mathrm{REF}(\mathscr{P}) = \mathrm{DIS}(\mathscr{P}) \cup \{\Omega_\alpha D | D \in \mathrm{DIS}(\mathscr{P})\}$.
    iii. Check that the mapping $h \to D_h$ is an isomorphism of $\mathbf{R}$ (the additive group of real numbers) onto $\mathrm{DIS}(\mathscr{P})$.
20. Verify formula (7.9).
21. Prove Theorems 28 and 29.
22. Let ℓ be the common perpendicular of an ultraparallel pencil $\mathscr{P}$.
    i. Check that $\mathrm{TRANS}(\ell)$ is a subgroup of $\mathrm{REF}(\mathscr{P})$.
    ii. Show that if α is any line of the pencil $\mathscr{P}$,
    $$\mathrm{REF}(\mathscr{P}) = \mathrm{TRANS}(\ell) \cup \{\Omega_\alpha \circ T | T \in \mathrm{TRANS}(\ell)\}.$$
    This means that $\mathrm{TRANS}(\ell)$ is a subgroup of index 2. One coset is $\mathrm{TRANS}(\ell)$, and the other is the set of reflections.
    iii. Check that the mapping $h \to T_h$ is an isomorphism of $\mathbf{R}$ onto $\mathrm{TRANS}(\ell)$.
23. If ℓ, m, and n are lines of a pencil, prove that $\Omega_\ell\Omega_m\Omega_n = \Omega_n\Omega_m\Omega_\ell$.
24. If H_1, H_2, and H_3 are distinct half-turns, prove that
    $$H_1H_2H_3 \neq H_3H_2H_1.$$
25. If $T \in \mathrm{TRANS}(m)$ and $\ell \perp m$, show that $\Omega_\ell T = T\Omega_\ell$. Verify formula (7.10).
26. Using the matrix representation (7.10), show that a nontrivial glide reflection
    i. has no fixed points,
    ii. leaves fixed its axis and no other lines.
27. Prove that there is a unique reflection Ω_{pq} interchanging any two lines p and q of a pencil of parallels (respectively, ultraparallels).
28. What is the square of a glide reflection in $\mathbf{H}^2$?
29. Describe the product of two glide reflections in $\mathbf{H}^2$ with perpendicular axes.
30. Let P and Q be points. Show that there is a unique translation taking P to Q.

31. Show that two nontrivial rotations of $\mathbf{H}^2$ commute if and only if they have the same center.
32. Let P, Q, and R be three noncollinear points of $\mathbf{H}^2$. Discuss the product of the half-turns H_P, H_Q, and H_R. Given a rotation, show that it can be expressed as the product of three half-turns.
33. Let γ be a line. Explain why no line can be both parallel to γ and perpendicular to γ.
34. Let v and w be nonproportional lightlike vectors. Prove that $b(v, w) \neq 0$.
35. Prove Theorem 34.
36. Let P, Q, and R be three points lying, respectively, on three members $\mathscr{p}$, $\mathscr{q}$, and $\mathscr{r}$ of a pencil of parallels. If P and Q are interchanged by $\Omega_{\mathscr{p}\mathscr{q}}$, and Q and R are interchanged by $\Omega_{\mathscr{q}\mathscr{r}}$, prove that
    i. P, Q, and R cannot be collinear.
    ii. $\Omega_{\mathscr{p}\mathscr{r}}$ interchanges P and R.
    (Notation is as in Exercise 27.)
37. Prove Theorem 36.
38. Fill in the missing details in the proof of Theorem 38.
39. Prove Theorem 39.
40. Prove that a segment AB consists of A, B and all points between A and B.
41. Verify the statements made in the text about the definition of radian measure of an angle.
42. Prove Theorem 40.
43. Prove the crossbar theorem in $\mathbf{H}^2$.
44. Prove the identity (7.12) for $a \in [-1, 1]$ and $\lambda, \mu \in (0, \infty)$.
45. Prove the formulas of hyperbolic trigonometry (Theorem 43).
46. Let ABC be a triangle in $\mathbf{H}^2$ with sides of lengths $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$. Prove that if AC is perpendicular to AB, then

    $$\cosh a = \cosh b \cosh c.$$

    Find a direct proof that does not make use of Exercise 45.
47. The angle sum for a triangle in $\mathbf{H}^2$ is less than π. Prove this for the special cases of an equilateral triangle and a right-angled triangle.
48. The *defect* of a triangle in $\mathbf{H}^2$ is the amount by which its angle sum differs from π. Let $\triangle ABC$ be a triangle, and let F be a point between A and C that is the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Prove that

$$\text{defect}(\triangle ABF) + \text{defect}(\triangle CBF) = \text{defect}(\triangle ABC).$$

Hence, prove that the defect of any triangle in $\mathbf{H}^2$ is positive.

Remark: The analogous conclusion can be drawn in spherical geometry or elliptic geometry. The angle sum is greater than π, and the amount of the difference is called the *excess*. Defect and excess can be used as measures of area in non-Euclidean geometry. The excess cannot be greater than 2π, and in fact there are spherical triangles whose areas are as close to 2π as we please. On the other hand, the defect of a triangle in $\mathbf{H}^2$ is less than π, and there are triangles whose areas are as close to π as we please.

49. Draw some pictures indicating how four points $ABCD$ might not determine a convex quadrilateral.
50. Show that a Saccheri quadrilateral can be decomposed into two Lambert quadrilaterals.
51. Fill in the missing details in the proof of Theorem 44.
52. Verify that the reflection Ω_α in Theorem 45 interchanges the two given lines.
53. i. Prove that there is a unique reflection interchanging any two distinct rays with common origin.
    ii. Prove that there are exactly two reflections interchanging two intersecting lines.
54. Let AB, BC, and CD be three line segments with $AB \perp BC$ and $BC \perp CD$. Given that AB and CD have equal length, prove that $d(A, C) = d(B, D)$. Work in $\mathbf{H}^2$, although your results should be equally valid in $\mathbf{E}^2$.
55. Find the symmetry group of
    i. the rhombus,
    ii. the equiangular quadrilateral,
    iii. the equiangular rhombus,
    iv. the Saccheri quadrilateral.
56. Formulate Hjelmslev's theorem so that it makes sense in $\mathbf{H}^2$. Is it true?
57. Verify that the SSS, SAS, and AAA congruence theorems hold in $\mathbf{H}^2$.
58. What congruence theorems hold for asymptotic triangles?
59. Verify that the concurrence theorems (4.53 and 4.54) are valid in the hyperbolic plane.
60. Let PQ and PX be perpendicular segments. Show that there is a unique ray $\overrightarrow{QY}$ such that PQ, $\overrightarrow{PX}$, and $\overrightarrow{QY}$ form an asymptotic triangle. If the radian measure of $\measuredangle Q$ is θ and the length of PQ is d,

show that $\sin\theta \cosh d = 1$. The number θ is called the "angle of parallelism" determined by d. See Figure 7.25.

61. Prove Theorem 49.
62. Prove that a circle has only one center and one radius.
63. Discuss the various ways in which a circle can intersect a line or another circle.
64. Prove Theorem 50.
65. Prove that an equidistant curve uniquely determines the line m and the number r in (7.18).
66. Prove that a line meets an equidistant curve in at most two points.
67. Prove that a line meets a horocycle in at most two points.
68. Prove Theorem 51.
69. Identify the following curves in $\mathbf{H}^2$.
    i. $x_1 + x_2 = \sqrt{2}\sinh(2)$.
    ii. $x_3 = 2$.
    iii. $x_1 + x_3 = 2$.
70. Prove that the groups ROT($\mathcal{P}$), TRANS(m), and DIS($\mathcal{P}$) would have worked equally well in characterizing circles, equidistant curves, and horocycles, respectively. What are the stabilizers in these cases?
71. Investigate the status of Theorems 49–50 and Exercises 61–66 in the Euclidean, spherical, and projective settings.
72. Investigate the relationships suggested in the remark following Theorem 43.

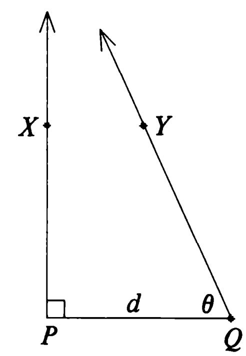

Figure 7.25 The angle of parallelism.