Lesson 32
The Laplace Transform

Specific Objectives:

At the end of the lesson, the students are expected to:

- define Laplace Transforms and its properties
- find the Laplace Transforms of different types of functions

Let f(t) be a function of t defined for each positive values of t. Then the Laplace transform of f(t), donated by $L\{f(t)\}$, is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \text{ ———— } (1)$$

provided that the integral exists. s is a parameter, which may be a real or complex number. $L\{f(t)\}$ being clearly a function of s is briefly written as F(s). That is,

$$L\{f(t)\} = \int_0^\infty e^{-st} dt = F(s)$$

where $L\{f(t)\}$ or $F(s)$ – Laplace Transform of f(t).

Transforms of Elementary Functions

The Laplace transforms of some exponential, trigonometric, polynomials and hyperbolic functions are:

1. $L\{e^{at}\} = \dfrac{1}{(s-a)},\ s > a$

   $L\{1\} = \dfrac{1}{s},\ s > 0$

2. $L\{\sin kt\} = \dfrac{k}{s^2 + k^2},\ s > 0$

   $L\{\cos kt\} = \dfrac{s}{s^2 + k^2},\ s > 0$

3. $L\{t^n\} = \dfrac{n!}{s^{n+1}}$ when $n = 0,1,2,3,...$

4. $L\{\sinh kt\} = \dfrac{k}{s^2 - k^2},\ s > |k|$

$$L\{\cosh kt\} = \frac{s}{s^2 - k^2},\ s > |k|$$

Properties of Laplace Transforms

1. Linearity Property

   If a, b, c be any constants and f, g, h be any functions of t, then

$$L\{af(t) + bg(t) - ch(t)\} = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

2. First Shifting Property

   If $L\{f(t)\} = F(s)$, then

$$L\{e^{at}f(t)\} = F(s-a)$$

Application of this property leads us to the following useful results:

1. $L\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

2. $L\{e^{at}\sin kt\} = \dfrac{k}{(s-a)^2 + k^2}$

3. $L\{e^{at}\cos kt\} = \dfrac{(s-a)}{(s-a)^2 + k^2}$

4. $L\{e^{at}\sinh kt\} = \dfrac{k}{(s-a)^2 - k^2}$

5. $L\{e^{at}\cosh kt\} = \dfrac{(s-a)}{(s-a)^2 - k^2}$

   where in each case, s > a.

Example: Find the Laplace transform of the following:

$$1. L\{f(t)\} = L\{2e^{2t} + 5t^3 - 3\sin 3t + 7\cos 3t\}$$

$$= 2L\{e^{2t}\} + 5L\{t^3\} - 3L\{\sin 3t\} + 7L\{\cos 3t\}$$

$$= 2\left[\frac{1}{s-2}\right] + 5\left[\frac{3!}{s^{3+1}}\right] - 3\left[\frac{3}{s^2+3^2}\right] + 7\left[\frac{s}{s^2+3^2}\right]$$

$$= \frac{2}{s-2} + \frac{5(1)(2)(3)}{s^4} - \frac{9}{s^2+9} + \frac{7s}{s^2+9}$$

$$= \underline{\underline{\frac{2}{s-2} + \frac{30}{s^4} - \frac{9}{s^2+9} + \frac{7s}{s^2+9}}}$$

$$2. L\{f(t)\} = L\{t^3 e^{-3t}\}$$

$$= L\{e^{-3t} t^3\}$$

$$= \frac{3!}{(\mathrm{s}+3)^{3+1}}$$

$$= \frac{(1)(2)(3)}{(s+3)^4}$$

$$= \underline{\underline{\frac{6}{(s+3)^4}}}$$

$$3.L\{f(t)\} = L\{e^{-2t}\sin 4t\}$$

$$= \frac{4}{(s+2)^2 + 4^2}$$

$$= \frac{4}{s^2 + 4s + 4 + 16}$$

$$= \underline{\underline{\frac{4}{s^2 + 4s + 20}}}$$

Seatwork: Find the Laplace transform of the following:

1. $f(t) = e^{-3t}(2\cos 5t - 3\sin 5t)$ *ans:* $\frac{(2s-9)}{(s^2+6s+34)}$

2. $f(t) = \sin 2t \sin 3t$ *ans:* $\frac{12s}{(s^2+25)(s^2+1)}$

Homework: Find the Laplace transform of the following:

1. $f(t) = te^{-4t}\sin 3t$
2. $f(t) = 2t^2 + \sin t + 2e^{3t}$