Lesson 33
Inverse Laplace Transforms

Specific Objectives:

At the end of the lesson, the students are expected to:

- define Inverse Laplace Transform
- find the inverse Laplace transforms of a function

Note: The Table of Laplace transform can be used to find the inverse Laplace transforms

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $L\{f(t)\} = F(s)$ then $f(t)$ is the inverse Laplace transform of $F(s)$, that is, $L^{-1}\{F(s)\}$.

$$L^{-1}\{F(s)\} = f(t)$$

Table of Inverse Laplace Transform

1. $L^{-1}\left\{\dfrac{1}{s}\right\} = 1$
2. $L^{-1}\left\{\dfrac{1}{(s-a)}\right\} = e^{at}$
3. $L^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!},\ n = 1, 2, 3, \ldots$
4. $L^{-1}\left\{\dfrac{1}{(s-a)^n}\right\} = e^{at}\dfrac{t^{n-1}}{(n-1)!}$
5. $L^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\} = t^n$
6. $L^{-1}\left\{\dfrac{k}{s^2+k^2}\right\} = \sin kt$
7. $L^{-1}\left\{\dfrac{s}{s^2+k^2}\right\} = \cos kt$
8. $L^{-1}\left\{\dfrac{k}{s^2-k^2}\right\} = \sinh kt$
9. $L^{-1}\left\{\dfrac{s}{s^2-k^2}\right\} = \cosh kt$
10. $L^{-1}\left\{\dfrac{1}{(s-a)^2+k^2}\right\} = \dfrac{1}{k}e^{at}\sin kt$

11. $L^{-1}\left\{\dfrac{(s-a)}{(s-a)^2+k^2}\right\}=e^{at}\cos kt$

12. $L^{-1}\left\{\dfrac{s}{\left(s^2+k^2\right)^2}\right\}=\dfrac{1}{2k}t\sin kt$

When evaluating inverse transforms, it often happens that a function of s under consideration does not match exactly the form of a Laplace transform $F(s)$ given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by an appropriate constant.

Linearity Property: L^{-1} is a Linear Transform, that is, for constants α and β:

$$L^{-1}\{\alpha f(s)+\beta g(s)\}=\alpha L^{-1}\{f(s)\}+\beta L^{-1}\{g(s)\}$$

Partial fractions play an important role in finding Laplace transforms when the denominator of f(s) is factorable into distinct linear factors.

Example: Evaluate the following

$$1. L^{-1}\left\{\frac{1}{s^5}\right\}=\frac{t^{5-1}}{(5-1)!}$$

$$=\frac{t^4}{4!}$$

$$=\frac{t^4}{(1)(2)(3)(4)}$$

$$=\frac{1}{24}t^4$$

$$2. L^{-1}\left\{\frac{1}{s^2+7}\right\}=L\left\{\frac{1}{s^2+\left(\sqrt{7}\right)^2}\right\}$$

$$=\frac{1}{\sqrt{7}}L^{-1}\left\{\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right\}$$

$$=\frac{1}{\sqrt{7}}\sin\sqrt{7}t$$

$$3. L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = L^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}$$

$$= -2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \underline{\underline{-2\cos 2t + 3\sin 2t}}$$

$$4. L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$consider\ \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

multiply both sides by $(s-1)(s-2)(s+4)$

$$s^2+6s+9 = A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

$$\text{s}^2+6s+9 = A(s^2+2s-8)+B(s^2+3s-4)+C(s^2-3s+2)$$

compare coeff :

$$s^2 : 1 = A+B+C \qquad \text{- a}$$

$$s:\ 6 = 2A+3B-3C \qquad \text{-} b$$

$$\text{k}:\ 9 = -8A-4B+2C \qquad \text{- c}$$

a multiply by 3 and add to b

$3A+3B+3C=3$

$\underline{2A+3B-3C=6}$

$5A+6B \quad =9 \quad \text{-} d$

$a-2, add\ \ c$

$-2A-2B-2C=-2$

$\underline{-8A-4B+2C=9}$

$-10A-6B \quad =7 \quad \text{-} e$

add d & e

$$5A + 6B = 9$$
$$\underline{-10A - 6B = 7}$$
$$-5A \qquad = 16$$
$$A = -\frac{16}{5}$$

substitute in d

$$5\left(-\frac{16}{5}\right) + 6B = 9$$
$$6B = 16 + 9 = 25$$
$$B = \frac{25}{6}$$

substitute in a

$$\left(\frac{-16}{5} + \frac{25}{6} + C = 1\right)30$$
$$-96 + 125 + 30C = 30$$
$$30C = 30 + 96 - 125$$
$$C = \frac{1}{30}$$

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s=4)} = \frac{\frac{-16}{5}}{s-1} + \frac{\frac{25}{6}}{s-2} + \frac{\frac{1}{30}}{s+4}$$

$$L^{-1}\left\{\frac{s^2 + 6s + 9}{(s-1)(s-2)(s=4)}\right\}$$
$$= \frac{-16}{5} L^{-1}\left(\frac{1}{s-1}\right) + \frac{25}{6} L^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{30} L^{-1}\left(\frac{1}{s+4}\right)$$
$$= -\frac{16}{5} e^{t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

Seatwork: Evaluate the following:

1. $L^{-1}\left\{\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right\}$

Homework: Evaluate the following

1. $L^{-1}\left\{\dfrac{s^2 - 3s + 4}{s^3}\right\}$
2. $L^{-1}\left\{\dfrac{s - 2}{s^2 - 4s + 13}\right\}$