5 The projective plane $\mathbf{P}^2$

Introduction

We now come to a geometric structure that is more abstract than the previous two we have dealt with. The geometry of the projective plane will resemble that of the sphere in many respects. However, we regain the Euclidean phenomenon that two lines can intersect only once. The projective plane will also be a foundation for our study of hyperbolic geometry in Chapter 7.

Although many of the properties of the projective plane are familiar, one that will appear strange is that of nonorientability. In $\mathbf{P}^2$ every reflection may be regarded as a rotation. This has the intuitive consequence that an outline of a left hand can be moved continuously to coincide with its mirror image, the outline of a right hand.

The abstraction is involved in the fact that every point of $\mathbf{P}^2$ is a *pair of points* of $\mathbf{S}^2$. Two antipodal points of $\mathbf{S}^2$ are considered to be the same point of $\mathbf{P}^2$.

Definition. *The projective plane* $\mathbf{P}^2$ *is the set of all pairs* $\{x, -x\}$ *of antipodal points of* $\mathbf{S}^2$.

Remark: Two alternative definitions of $\mathbf{P}^2$, equivalent to the preceding one are

i. The set of all lines through the origin in $\mathbf{E}^3$.
ii. The set of all equivalence classes of ordered triples (x_1, x_2, x_3) of numbers (i.e., vectors in $\mathbf{E}^3$) not all zero, where two vectors are equivalent if they are proportional.

Let $\pi: \mathbf{S}^2 \to \mathbf{P}^2$ be the mapping that sends each x to $\{x, -x\}$. Then π is a two-to-one map of $\mathbf{S}^2$ onto $\mathbf{P}^2$.

A *line* of $\mathbf{P}^2$ is a set of the form $\pi\ell$, where ℓ is a line of $\mathbf{S}^2$. If ξ is a pole of ℓ, then $\pi\xi$ is called the *pole* of $\pi\ell$. Clearly, πx lies on $\pi\ell$ if and only if $\langle \xi, x \rangle = 0$. Two points are perpendicular if their representatives on $\mathbf{S}^2$ are perpendicular. Two lines are perpendicular if their poles are perpendicular.

Incidence properties of $\mathbf{P}^2$

Theorem 1.

i. *Two lines of* $\mathbf{P}^2$ *have exactly one point of intersection.*
ii. *Two points of* $\mathbf{P}^2$ *lie on exactly one line.*

Proof:

i. Let $\pi\xi$ and $\pi\eta$ be poles of lines of $\mathbf{P}^2$. Because $\pi\xi \neq \pi\eta$, ξ and η are not antipodal. Thus, $\xi \times \eta$ and $-\xi \times \eta$ determine the two points of intersection of the corresponding lines of $\mathbf{S}^2$ (Theorem 4.7). But $\pi(\xi \times \eta)$ and $\pi(-\xi \times \eta)$ are the same point of $\mathbf{P}^2$.
ii. Again, let πX and πY be points of $\mathbf{P}^2$. Then X and Y are not antipodal, so they lie on a unique line ℓ of $\mathbf{S}^2$ (Theorem 4.6). Thus, πX and πY lie on $\pi\ell$. □

Homogeneous coordinates

Let $\{e_1, e_2, e_3\}$ be a basis of $\mathbf{R}^3$. Then every vector $x \in \mathbf{R}^3$ determines a unique triple (x_1, x_2, x_3) of real numbers according to the equation

$$x = x_1e_1 + x_2e_2 + x_3e_3.$$

If πx is a point of $\mathbf{P}^2$, λ is any nonzero real number, and

$$\lambda x = u_1e_1 + u_2e_2 + u_3e_3,$$

then (u_1, u_2, u_3) is called a *homogeneous coordinate vector* of πx. We say that u_1, u_2, u_3 are *homogeneous coordinates* of πx.

Let $\xi = (\xi_1, \xi_2, \xi_3)$ and $x = (x_1, x_2, x_3)$. Then $\langle \xi, x\rangle = 0$ becomes the equation of the line with pole $\pi\xi$. Homogeneous coordinates are often a useful computational device. Their usefulness is primarily due to the following result.

Theorem 2. *Let P, Q, R, and S be four points of* $\mathbf{P}^2$, *no three of which are collinear. Then there is a basis of* $\mathbf{R}^3$ *with respect to which the four points have coordinates* $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, *and* $(1, 1, 1)$.

Proof: Let v_1, v_2, and v_3 be any vectors in $\mathbf{R}^3$ that are representatives of P, Q, and R, respectively. Because P, Q, and R are not collinear, these three vectors are linearly independent. Let v_4 be any representative of S. Now there must exist real numbers k_1, k_2, k_3, none of which is zero, such that

$$v_4 = k_1 v_1 + k_2 v_2 + k_3 v_3.$$

Put $e_1 = k_1 v_1$, $e_2 = k_2 v_2$, and $e_3 = k_3 v_3$. Then $\{e_1, e_2, e_3\}$ is the required basis. $\square$

Theorem 3. *Let x and y be homogeneous coordinate vectors of two points of $\mathbf{P}^2$. Then $\lambda x + \mu y$ (λ, μ real) is a typical point on the line they determine.*

Two famous theorems

Having introduced the incidence structure of $\mathbf{P}^2$ and having defined the notion of homogeneous coordinates, we turn to two fundamental classical theorems in projective geometry: Desargues' theorem and Pappus' theorem. The elegance of the statements testifies to the unifying power of projective geometry. Analogous results in $\mathbf{E}^2$ would have to make allowances for many special cases. The elegance of the proofs (which follow Coxeter [7]) testifies to the power of the method of homogeneous coordinates. In this section the word "triangle" denotes a set of three noncollinear points. We have not yet defined segments in $\mathbf{P}^2$, so our old notion of triangle does not apply.

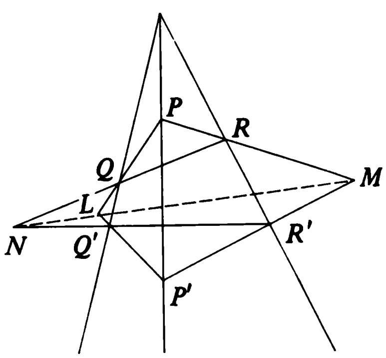

Figure 5.1 Desargues' theorem.

Theorem 4 (Desargues' theorem). *Let PQR and $P'Q'R'$ be triangles in $\mathbf{P}^2$. Suppose $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$ are concurrent. Then $\overleftrightarrow{PQ} \cap \overleftrightarrow{P'Q'}$, $\overleftrightarrow{QR} \cap \overleftrightarrow{Q'R'}$, and $\overleftrightarrow{PR} \cap \overleftrightarrow{P'R'}$ are collinear. (See Figure 5.1.)*

Proof: We may choose a basis for $\mathbf{R}^3$ such that in the associated homogeneous coordinate system $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$, and $X = (1, 1, 1)$, where X is the given point of concurrence. If X were collinear with any two of these points, then two sides (such as $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$) would coincide, leaving the conclusion meaningless. Thus, we may assume that no three of P, Q, R, and X are collinear. Now P' may be given coordinates $(p, 1, 1)$ because

$$\lambda(1, 0, 0) + \mu(1, 1, 1) = (\lambda + \mu, \mu, \mu),$$

which is equivalent to

$$\left(1 + \frac{\lambda}{\mu}, 1, 1\right).$$

Similarly, $Q' = (1, q, 1)$ and $R' = (1, 1, r)$. Now the equation of $\overleftrightarrow{PQ}$ is $x_3 = 0$, and that of $\overleftrightarrow{P'Q'}$ is

$$(1 - q)x_1 + (1 - p)x_2 + (pq - 1)x_3 = 0.$$

These lines intersect in $L = (p - 1, 1 - q, 0)$. Similarly, the other two points of intersection are $M = (1 - p, 0, r - 1)$ and $N = (0, q - 1, 1 - r)$. The three points L, M, and N are collinear because the sum of the three coordinate vectors is zero. □

Theorem 5 (Pappus' theorem). *Let $A_1B_1C_1$ and $A_2B_2C_2$ be collinear triples of points. Then the points $\overleftrightarrow{A_1B_2} \cap \overleftrightarrow{A_2B_1} = C_3$, $\overleftrightarrow{B_2C_1} \cap \overleftrightarrow{B_1C_2} = A_3$, and $\overleftrightarrow{A_1C_2} \cap \overleftrightarrow{A_2C_1} = B_3$ are collinear. (See Figure 5.2.)*

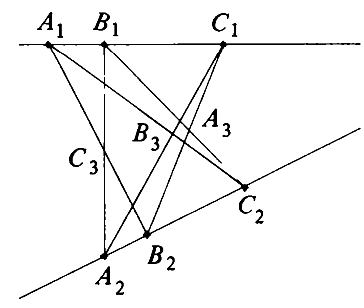

Figure 5.2 Pappus' theorem.

Proof: Assign homogeneous coordinates as follows:

$$A_1 = (1, 0, 0), \quad A_2 = (0, 1, 0), \quad A_3 = (0, 0, 1),$$
$$C_1 = (1, 1, 1), \quad B_1 = (p, 1, 1), \quad B_3 = (1, q, 1),$$
$$B_2 = (1, 1, r).$$

Then

$$C_2 = \overleftrightarrow{B_1A_3} \cap \overleftrightarrow{B_3A_1} = (pq, q, 1),$$
$$C_3 = \overleftrightarrow{A_1B_2} \cap \overleftrightarrow{A_2B_1} = (pr, 1, r).$$

Because A_2, B_2, and C_2 are collinear, we must have

$$C_2 = (0, \lambda, 0) + (1, 1, r) = (1, \lambda + 1, r).$$

On the other hand,

$$C_2 = (pq, q, 1) = (pqr, qr, r).$$

Thus, we must have $pqr = 1$.

Now $\overleftrightarrow{A_3B_3}$ consists of points of the form $(0, 0, \lambda) + (1, q, 1) = (1, q, 1 + \lambda)$. Because $C_3 = (pqr, q, rq) = (1, q, rq)$, it must be on this line. □

Applications to $\mathbf{E}^2$

One of the reasons for the invention of $\mathbf{P}^2$ was to simplify the incidence geometry of $\mathbf{E}^2$. To illustrate this, consider the following picture in $\mathbf{E}^3$. We regard the plane $x_3 = 1$ consisting of all points in $\mathbf{E}^3$ of the form $(x_1, x_2, 1)$ as a model of $\mathbf{E}^2$. Every line through the origin of $\mathbf{E}^3$ that is not parallel to $\mathbf{E}^2$ meets $\mathbf{E}^2$ in a unique point. If (x_1, x_2, x_3) are homogeneous coordinates for such a point of $\mathbf{P}^2$, then

$$\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right) \tag{5.1}$$

is the corresponding point of $\mathbf{E}^2$. Conversely, each point of $\mathbf{E}^2$ determines a unique point of $\mathbf{P}^2$.

Every line of $\mathbf{E}^2$ determines a unique plane through the origin in $\mathbf{E}^3$ and, hence, a unique line of $\mathbf{P}^2$. Every line of $\mathbf{P}^2$ determines a unique plane through the origin in $\mathbf{E}^3$ and, hence (with one exception), a unique line of $\mathbf{E}^2$. The exception is the plane through the origin parallel to $\mathbf{E}^2$.

Let T: $\mathbf{E}^2 \to \mathbf{P}^2$ be the map we have been discussing.

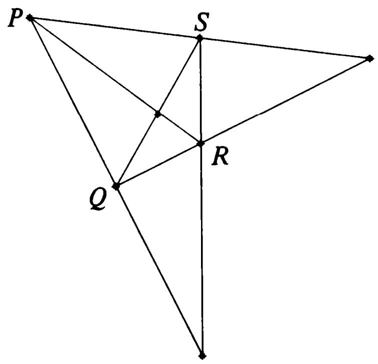

Figure 5.3 Quadrangle: Case 1.

Theorem 6.

i. *Denote by ℓ_∞ the exceptional line of $\mathbf{P}^2$. Then T maps $\mathbf{E}^2$ bijectively to $\mathbf{P}^2 - \ell_\infty$.*
ii. *Let P and Q be points of $\mathbf{E}^2$. Then TP and TQ determine a line ℓ' of $\mathbf{P}^2$, and T maps $\ell = \overleftrightarrow{PQ}$ bijectively to $\ell' - \ell_\infty$.*
iii. *Let ℓ and m be lines of $\mathbf{E}^2$. If $\ell \cap m = P$, then $\ell' \cap m' = TP$. If $\ell \parallel m$, then $\ell' \cap m'$ lies on ℓ_∞.*

Remark: What this theorem says is that $\mathbf{P}^2$ contains a subset that has the same incidence structure as $\mathbf{E}^2$. Two lines will be parallel on $\mathbf{E}^2$ if and only if they correspond to lines meeting on ℓ_∞.

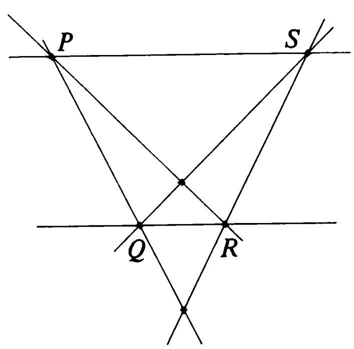

Figure 5.4 Quadrangle: Case 2.

Example: A quadrangle $PQRS$ in $\mathbf{P}^2$ consists of four points, no three collinear, and the six lines drawn through pairs of vertices. The three points $\overleftrightarrow{PQ} \cap \overleftrightarrow{RS}$, $\overleftrightarrow{PR} \cap \overleftrightarrow{QS}$, and $\overleftrightarrow{PS} \cap \overleftrightarrow{QR}$ are called *diagonal* points of the quadrangle. Now the corresponding figure in $\mathbf{E}^2$ can take on many forms, depending on where ℓ_∞ intersects the figure. We list the possibilities. They are illustrated in Figures 5.3–5.7.

1. ℓ_∞ contains no vertex (P, Q, R, or S) and no diagonal point. In this case we have an ordinary Euclidean quadrangle.
2. ℓ_∞ contains no vertex but one diagonal point. In this case two sides of the quadrangle are parallel; the other two are not.
3. ℓ_∞ contains no vertex but two diagonal points. In this case we have a parallelogram.
4. ℓ_∞ contains one vertex and no diagonal points. Here we have three ordinary points Q, R, and S, the lines $\overleftrightarrow{QR}$ and $\overleftrightarrow{RS}$, together with parallel lines through Q and S, respectively.
5. ℓ_∞ contains two vertices P and Q. In this case one diagonal point is forced to be on ℓ_∞.

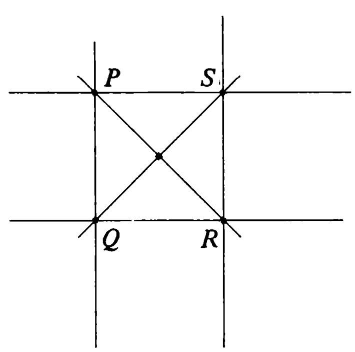

Figure 5.5 Quadrangle: Case 3.

If we start with the general case (1) and gradually turn one of the lines, say $\overleftrightarrow{PS}$, while leaving the others fixed, the point of intersection $\overleftrightarrow{PS} \cap \overleftrightarrow{QR}$ gets farther and farther away in $\mathbf{E}^2$. On $\mathbf{P}^2$ the corresponding point is getting closer to ℓ_∞. This is why ℓ_∞ is sometimes called "the line at ∞." This also accounts for the statement "parallel lines meet at ∞."

Desargues' theorem in $\mathbf{E}^2$

The projective version of Desargues' theorem has many interpretations in $\mathbf{E}^2$, depending on where the various lines cut ℓ_∞. For instance, if X is on ℓ_∞, the theorem would read as follows.

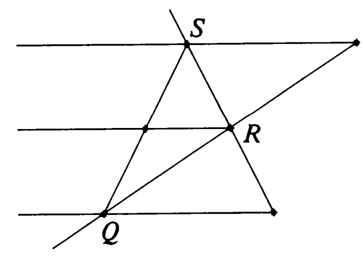

Figure 5.6 Quadrangle: Case 4.

Theorem 7. *Let PQR and $P'Q'R'$ be triangles in $\mathbf{E}^2$. Suppose that $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$ are parallel. Then*

i. *If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$ and $\overleftrightarrow{QR} \parallel \overleftrightarrow{Q'R'}$, then $\overleftrightarrow{PR} \parallel \overleftrightarrow{P'R'}$ (Figure 5.8).*
ii. *If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$ but $\overleftrightarrow{QR} \cap \overleftrightarrow{Q'R'} = N$, then $\overleftrightarrow{PR}$ and $\overleftrightarrow{P'R'}$ meet (say in M), and $\overleftrightarrow{MN}$ is parallel to $\overleftrightarrow{PQ}$ (Figure 5.9).*
iii. *If $\overleftrightarrow{PQ} \cap \overleftrightarrow{P'Q'} = L$, $\overleftrightarrow{QR} \cap \overleftrightarrow{Q'R'} = M$ and $\overleftrightarrow{PR} \cap \overleftrightarrow{P'R'} = N$, then L, M, and N are collinear (Figure 5.10).*

Observe that the three cases correspond to the following in $\mathbf{P}^2$.

i. ℓ_∞ contains all three of L, M, and N.
ii. ℓ_∞ contains one of L, M, or N.
iii. ℓ_∞ contains none of L, M, or N.

If X is not on ℓ_∞ in Desargues' theorem, it would read as follows.

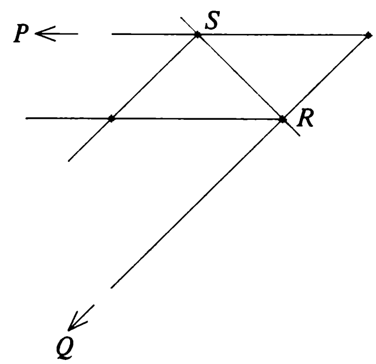

Figure 5.7 Quadrangle: Case 5.

Theorem 8. *Let PQR and $P'Q'R'$ be triangles in $\mathbf{E}^2$. Suppose $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$ meet in X. Then the conclusions of Theorem 7 hold.*

If we take ℓ_∞ to be the line $PP'X$ in Desargues' theorem, we get the following:

Theorem 9. *Let $QRR'Q'$ be a trapezoid ($\overleftrightarrow{QQ'} \| \overleftrightarrow{RR'}$). Let ℓ and m be parallel lines through Q and R. Let ℓ' and m' be parallel lines through Q' and R'. Let $X = \ell \cap \ell'$, $Y = m \cap m'$, and $Z = \overleftrightarrow{QR} \cap \overleftrightarrow{Q'R'}$. Then X, Y, and Z are collinear.*

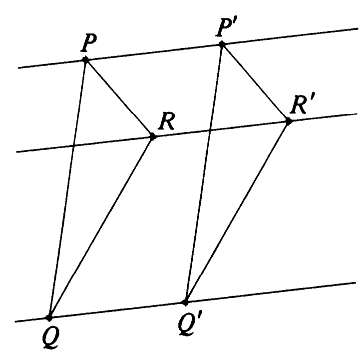

Figure 5.8 Affine consequences of Desargues' theorem: Case 1.

Theorem 10. *Let $QRR'Q'$ be a parallelogram in $\mathbf{E}^2$ ($\overleftrightarrow{QQ'} \| \overleftrightarrow{RR'}$ and $\overleftrightarrow{Q'R'} \| \overleftrightarrow{QR}$). Let ℓ and m be parallel lines through Q and R. Let ℓ' and m' be parallel lines through Q' and R'. If $X = \ell \cap \ell'$ and $Y = m \cap m'$, then $\overleftrightarrow{XY}$ is parallel to $\overleftrightarrow{QR}$.*

The projective group

Let **PGL(2)** be the group of collineations of $\mathbf{P}^2$. (Use the same definition as for $\mathbf{E}^2$.) Each invertible linear map A: $\mathbf{R}^3 \to \mathbf{R}^3$ determines a unique collineation $\tilde{A}$ in **PGL(2)** according to the definition

$$\tilde{A}\pi x = \pi A x. \tag{5.2}$$

The mapping $A \to \tilde{A}$ is a homomorphism of **GL(3)** $\to$ **PGL(2)** whose kernel is

$$K = \{kI | k \neq 0 \in \mathbf{R}\}.$$

It is a fact that this map is surjective (Exercise 17), so that

$$\mathbf{GL(3)}/K \cong \mathbf{PGL(2)}.$$

This fact is equivalent to the characterization of affine transformations in Theorem 2.2, whose proof is given in Appendix E.

Now if $A = kI$, then $\det A = k^3$. Because $k^3 = 1$ if and only if $k = 1$, we see that

1. Every member of **GL(3)** is equivalent to (is a multiple of) some member of **SL(3)**. In particular, if $k = (\det A)^{-1/3}$, we see that $\det(kA) = 1$, so that $kA \in \mathbf{SL(3)}$.
2. $\mathbf{SL(3)} \cap K = \{I\}$.

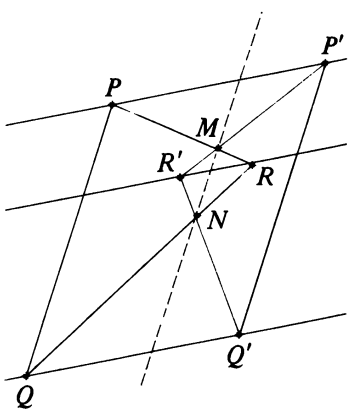

Figure 5.9 Affine consequences of Desargues' theorem: Case 2.

Thus, the homomorphism restricted to **SL(3)** is an isomorphism, and

$$\mathbf{SL(3)} \cong \mathbf{PGL(2)}.$$

The subgroup of **PGL(2)** that fixes ℓ_∞ may be identified with **AF(2)**. In fact, it is the image of **AF(2)** under the composition of the usual mappings:

$$\mathbf{AF(2)} \to \mathbf{GL(3)} \to \mathbf{PGL(2)}. \tag{5.3}$$

It is easy to check that this composite mapping is injective.

An element of **PGL(2)** is called a *projective collineation* or projective transformation.

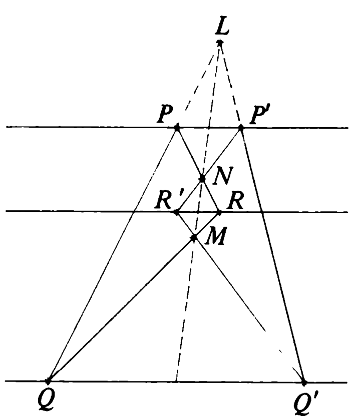

Figure 5.10 Affine consequences of Desargues' theorem: Case 3.

The fundamental theorem of projective geometry

Theorem 11. *Let PQRS and P'Q'R'S' be quadrangles. Then there is a unique* $T \in \mathbf{PGL(2)}$ *such that* $TP = P'$, $TQ = Q'$, $TR = R'$, *and* $TS = S'$.

Proof: Choose homogeneous coordinates of (1, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 1) for P, Q, R, and S, respectively. Then let A be a matrix whose columns are coordinate vectors for P', Q', and R', respectively. Call them x, y, and z. Now

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}.$$

However,

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ x_3 + y_3 + z_3 \end{bmatrix},$$

and, for any λ, μ, ν,

$$\begin{bmatrix} \lambda x_1 & \mu y_1 & \nu z_1 \\ \lambda x_2 & \mu y_2 & \nu z_2 \\ \lambda x_3 & \mu y_3 & \nu z_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 + \mu y_1 + \nu z_1 \\ \lambda x_2 + \mu y_2 + \nu z_2 \\ \lambda x_3 + \mu y_3 + \nu z_3 \end{bmatrix}.$$

Choose λ, μ, and ν so that

$$w = \lambda x + \mu y + \nu z$$

is a coordinate vector for S'. The projective collineation whose matrix with respect to P, Q, R, and S is

$$\begin{bmatrix} \lambda x_1 & \mu y_1 & \nu z_1 \\ \lambda x_2 & \mu y_2 & \nu z_2 \\ \lambda x_3 & \mu y_3 & \nu z_3 \end{bmatrix}$$

is the required transformation T. Uniqueness will be proved in Exercise 16. □

Corollary. *Let $\{P, Q, R\}$ and $\{P', Q', R'\}$ be two noncollinear triples of points. Let ℓ be a line not containing any of these points. Then there is a unique projective collineation T such that $TP = P'$, $TQ = Q'$, $TR = R'$, and $T\ell = \ell$.*

Proof: Let $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ meet ℓ in A and A', respectively. Let $\overleftrightarrow{PR}$ and $\overleftrightarrow{P'R'}$ meet ℓ in B and B', respectively. Then $RQAB$ and $R'Q'A'B'$ are quadrangles to which the fundamental theorem may be applied. The unique T so determined leaves ℓ fixed. Furthermore, because $P = \overleftrightarrow{RB} \cap \overleftrightarrow{QA}$, TP must be $\overleftrightarrow{R'B'} \cap \overleftrightarrow{Q'A'} = P'$.

Conversely, any projective collineation satisfying the stated conditions must take A to A' and B to B' and so must coincide with T. □

Remark: When a choice of ℓ_∞ has been made, this corollary with $\ell = \ell_\infty$ is just the fundamental theorem of affine geometry (Theorem 2.8).

A survey of projective collineations

In this section we will outline some of the facts about projective collineations. This material would occupy a whole chapter if done in detail.

Because our main emphasis in this book is on metric geometry, we will present the results with a minimum of discussion. All the necessary background for proving the theorems as exercises has already been developed.

Theorem 12. *Every projective collineation has at least one fixed point and one fixed line.*

In view of Theorem 12 it is useful to choose a point P and a line ℓ and examine the group of all projective collineations, leaving them fixed. We choose a homogeneous coordinate system in which P and ℓ have simple representations. If P lies on ℓ, let $P = (1, 0, 0)$. If P does not lie on ℓ, take $P = (0, 0, 1)$. In either case we can arrange that ℓ has the equation $x_3 = 0$. The next few theorems assume a homogeneous coordinate system satisfying these conditions.

Theorem 13. *If P does not lie on ℓ, the group of projective collineations leaving P and ℓ fixed is isomorphic to* **GL(2)**. *Each such collineation can be uniquely written in the form*

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad ad - bc \neq 0.$$

Theorem 14. *Suppose that P lies on ℓ. Then every projective collineation leaving P and ℓ fixed is uniquely represented by a matrix of the form*

$$\begin{bmatrix} a & b & p \\ 0 & c & q \\ 0 & 0 & 1 \end{bmatrix}, \quad ac \neq 0.$$

Conversely, each such matrix determines a projective collineation leaving P and ℓ fixed.

Taking $\ell = \ell_\infty$ allows us to regard this group as the group of affine transformations leaving fixed one particular pencil of parallels, namely, the lines parallel to the x_1-axis. In fact, if two points (λ, μ) and $(\tilde{\lambda}, \mu)$ in $\mathbf{E}^2$ have the same x_2-coordinate, then

$$\begin{bmatrix} a & b & p \\ 0 & c & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & \tilde{\lambda} \\ \mu & \mu \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a\lambda + b\mu + 1 & a\tilde{\lambda} + b\mu + 1 \\ c\mu + q & c\mu + q \\ 1 & 1 \end{bmatrix},$$

so that their images also have the same x_2-coordinate. In affine terms this transformation is a central dilatation (centered at the origin), followed by a shear, and then a translation.

Of course, the transformation of Theorem 14 may have fixed points other than P and/or fixed lines other than ℓ. In fact,

Theorem 15. *The transformation of Theorem 14 has exactly one fixed point and one fixed line if and only if* $a = c = 1$ *and* $bq \neq 0$.

Theorem 16. *A projective collineation with two fixed points may be written in the form*

$$\begin{bmatrix} a & 0 & p \\ 0 & c & q \\ 0 & 0 & 1 \end{bmatrix}.$$

Such a collineation has at least two fixed lines.

Corollary. *A projective collineation with two fixed points may be written in the form*

$$\begin{bmatrix} a & 0 & 0 \\ 0 & c & q \\ 0 & 0 & 1 \end{bmatrix}.$$

In this representation $(1, 0, 0)$ *and* $(0, 1, 0)$ *are fixed points. The lines* $x_1 = 0$ *and* $x_3 = 0$ *are fixed lines.*

Theorem 17. *If a projective collineation has three collinear fixed points, it may be written*

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & q \\ 0 & 0 & 1 \end{bmatrix}, \quad a \neq 0.$$

Every point on the line $x_1 = 0$ *is fixed. In addition, the line* $x_3 = 0$ *is a fixed line.*

The transformations of Theorem 17 are called perspective collineations.

A *perspective collineation* with axis ℓ and center P is a projective collineation that leaves fixed every point on ℓ and every line through P. We may regard the identity as the trivial perspective collineation. All other perspective collineations have a unique axis and a unique center. A nontrivial perspective collineation is called an *elation* if its axis and center are incident; otherwise, it is called a *homology*.

Theorem 18. *When a perspective collineation is represented as in Theorem 17, it is*

i. *an elation if* $a = 1$ *and* $q \neq 0$;

ii. *the identity if* $a = 1$ *and* $q = 0$;
iii. *a homology if* $a \neq 1$.

Remark. If ℓ_∞ is taken to be the axis of a perspective collineation, elations become translations and homologies become central dilatations. On the other hand, if ℓ_∞ is one of the other fixed lines, elations become shears and homologies become stretches along one direction (see Theorem 2.20, case (iv)) possibly composed with an affine reflection. The special homology giving rise to an affine reflection is called a *harmonic homology*.

The term "perspective collineation" is explained by the following theorem.

Theorem 19. *Suppose that a nontrivial perspective collineation with center* P *takes* X *to* X'. *Then* P, X, *and* X' *are collinear.*

Theorem 20. *Let* P *be a point and* ℓ *a line. Let* X *and* X' *be points collinear with* P. *Assume that* X *and* X' *are not on* ℓ *and not equal to* P. *Then there is a unique perspective collineation with center* P *and axis* ℓ *that takes* X *to* X'.

Polarities

Let b be a real-valued, symmetric, nondegenerate, bilinear function on $\mathbf{E}^3$. If $\{e_1, e_2, e_3\}$ is a basis for $\mathbf{R}^3$. we have

$$\begin{aligned} b(x, y) &= \sum_{i,j=1}^{3} x_i y_j b(e_i, e_j) \\ &= \sum_{i,j=1}^{3} b_{ij} x_i y_j = x^t B y = \langle x, By \rangle, \end{aligned} \tag{5.4}$$

where $B = [b_{ij}] = [b(e_i, e_j)]$.

Each such b determines a relation

$$\bar{b} \subset \mathbf{P}^2 \times \mathbf{P}^2$$

consisting of those pairs $(\pi x, \pi y)$ such that $b(x, y) = 0$.

The relation $\bar{b}$ is called a *polarity*. If $b(x, y) = 0$, we say that πx and πy are *conjugate*. For a given y the set

$$\{\pi x | b(x, y) = 0\}$$

is a line called the *polar line* of πy. We call πy the *pole* of the line with respect to b.

Some polarities have self-conjugate points. The set of self-conjugate points is called a *conic* determined by the polarity. For example, if

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

then the conic determined by $\tilde{b}$ is

$$\{\pi x | x^t B x = 0\} = \{\pi x | x_1^2 + x_2^2 - x_3^2 = 0\}.$$

This conic in $\mathbf{P}^2$ corresponds by formula (5.1) to the unit circle $x_1^2 + x_2^2 = 1$ in $\mathbf{E}^2$.

Similarly,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \text{gives the ellipse} \quad 2x_1^2 + 3x_2^2 = 4,$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \text{gives the hyperbola} \quad -2x_1^2 + 3x_2^2 = 4,$$

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \quad \text{gives the parabola} \quad x_2^2 = 4x_1.$$

Some polarities do not have self-conjugate points. For example, if

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$b(x, y) = x_1y_1 + x_2y_2 + x_3y_3 = \langle x, y \rangle,$$

and $b(x, y) = 0$ if and only if $x_1^2 + x_2^2 + x_3^2 = 0$. The fact the $\tilde{b}$ has no self-conjugate points translates to the fact that no line can be perpendicular to itself in $\mathbf{E}^3$.

A polarity also induces a relation among the lines of $\mathbf{P}^2$. Two lines are said to be *conjugate* if the pole of one lies on the other. A line that passes through its own pole is said to be *self-conjugate.*

Theorem 21. *Let P and Q be points of $\mathbf{P}^2$ with respective polar lines $\not{p}$ and $\not{q}$. Then P lies on $\not{q}$ if and only if Q lies on $\not{p}$.*

Proof: Let $P = \pi x$ and $Q = \pi y$. Then P lies on $\not{q}$ if and only if $b(x, y) = 0$. By symmetry this is also the condition for Q to lie on $\not{p}$. □

Theorem 22. *Let P and Q be self-conjugate points of $\mathbf{P}^2$. Then $\overleftrightarrow{PQ}$ cannot be a self-conjugate line.*

Proof: Let $\mathscr{p}$ and $\mathscr{q}$ be the respective polar lines of P and Q. The lines $\mathscr{p}$ and $\mathscr{q}$ are distinct because P and Q are distinct. Let R be the point where $\mathscr{p}$ and $\mathscr{q}$ intersect. This point is not on $\overleftrightarrow{PQ}$. Because R is conjugate to both P and Q, its polar line $\mathscr{r}$ must pass through both P and Q; that is, $\mathscr{r} = \overleftrightarrow{PQ}$. The line $\mathscr{r}$ is not self-conjugate because it does not pass through R. □

Theorem 23. *A line contains exactly one self-conjugate point if and only if it is a self-conjugate line.*

Proof: Let ℓ be a line with exactly one self-conjugate point $P = \pi x$. Let $Q = \pi y$ be any other point of ℓ. Then for any real number λ,

$$\begin{aligned} b(x + \lambda y, x + \lambda y) &= b(x, x) + 2\lambda b(x, y) + \lambda^2 b(y, y) \\ &= \lambda(2b(x, y) + \lambda b(y, y)). \end{aligned}$$

Because there is only one self-conjugate point on ℓ, we must have $b(x, y) = 0$. Otherwise, one could solve the equation for a nonzero value of λ. Because the equation $b(x, y) = 0$ holds for all x with πx on ℓ, the pole of ℓ is πy. Hence, ℓ is a self-conjugate line.

Conversely, if a line is self-conjugate, its pole is self-conjugate. By Theorem 22 the line can have no other self-conjugate points. □

Definition. *Let $\tilde{b}$ be a polarity defining a conic $\mathscr{C}$. A line that is self-conjugate with respect to $\tilde{b}$ is called a* tangent *to the conic $\mathscr{C}$. The pole of this line is called the* point of contact. *(See Figure 5.11 in which ℓ and $\mathscr{m}$ are tangents having respective points of contact L and M.)*

Corollary. *A line meets a conic in at most two points.*

Proof: This follows from considering a quadratic function of the type occurring in Theorem 23. □

Definition. *A line that meets a conic twice is called a* secant.

Cross products

Conjugacy with respect to a polarity is a generalization of the theory of perpendicularity with respect to an inner product. We recall that in order to find a vector in $\mathbf{E}^3$ that is perpendicular to two given vectors, we construct the cross product.

If u and v are vectors in $\mathbf{R}^3$, there is a unique vector w in $\mathbf{R}^3$ such that, for all $z \in \mathbf{R}^3$,

$$b(w, z) = \sqrt{|\det B|}\,\det(z, u, v).$$

Here we may compute the right side by writing z, u, and v as column vectors and taking the determinant of the resulting 3×3 matrix.

We call w the cross product (of u and v) with respect to b and write $w = u \times_b v$ or simply $w = u \times v$ if b is clear from the context.

Clearly, the formulas

$$b(u \times v, w) = b(u, v \times w)$$

and

$$b(u, u \times v) = b(v, u \times v) = 0$$

are true. Thus the cross product is a device for computing poles of lines. The following proposition is obvious.

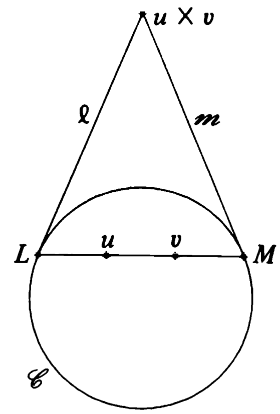

Figure 5.11 Conic, tangents, pole, and polar.

Theorem 24. *Let πu and πv be points in $\mathbf{P}^2$. Then the line joining πu and πv has pole $\pi(u \times v)$. (Again see Figure 5.11.)*

Definition *A triangle $\triangle PQR$ of $\mathbf{P}^2$ is said to be* self-polar *if each vertex is the pole of the side opposite it. Any self-polar triangle gives rise to a basis $\{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ such that $b(e_i, e_j) = 0$ for $i \neq j$ and $b(e_i, e_i) = \pm 1$. Such a basis is said to be* orthonormal *with respect to b.*

Theorem 25.

i. *Let $\{e_1, e_2, e_3\}$ be orthonormal with respect to b. Then, after replacing e_3 by its negative if necessary, we have*

$$e_1 \times e_2 = b(e_3, e_3)e_3,$$

$$e_2 \times e_3 = b(e_1, e_1)e_1,$$

$$e_3 \times e_1 = b(e_2, e_2)e_2.$$

ii. *For a given b the number of occurrences of -1 among the $b(e_i, e_i)$ is independent of the choice of orthonormal basis.*

Definition. *Let b be a nondegenerate, bilinear, symmetric function. Suppose that $\{e_i\}$ is a basis orthonormal with respect to b. Suppose that $+1$ occurs r times and -1 occurs s times among the $b(e_i, e_i)$. Then the ordered pair (r, s) is called the* signature *of b.*

The following (vector triple product) formula is indispensable for computation.

Theorem 26. *$(u \times v) \times w = (-1)^s(b(u, w)v - b(v, w)u)$, where the signature of b is (r, s).*

Proof: Choose a basis of the type used in Theorem 25. Then

$$(e_1 \times e_2) \times e_2 = b(e_3, e_3)e_3 \times e_2 = -b(e_3, e_3)b(e_1, e_1)e_1,$$
$$(-1)^s(b(e_1, e_2)e_2 - b(e_2, e_2)e_1) = (-1)^s(-1)b(e_2, e_2)e_1.$$

These are equal if and only if

$$b(e_1, e_1)b(e_3, e_3) = (-1)^s b(e_2, e_2);$$

that is,

$$b(e_1, e_1)b(e_2, e_2)b(e_3, e_3) = (-1)^s.$$

The other combinations can be checked similarly. □

EXERCISES

1. Prove Theorem 3.
2. Let $x = (1, 0, 0)$, $y = (1, 1, 0)$, $z = (1, 0, 1)$, $w = (1, 1, 1)$. Let ℓ be the line joining πx and πy, and let m be the line joining πz and πw. Find $\ell \cap m$.
3. Draw diagrams illustrating the various possibilities in Theorem 8.
4. Draw a diagram illustrating Theorem 9.
5. Draw a diagram illustrating Theorem 10.
6. Pappus' theorem yields many distinct results in $\mathbf{E}^2$ depending on the position of ℓ_∞. State as many of these results as you can.
7. Let ℓ and ℓ' be distinct lines, and let C be a point not on either line. The *perspectivity* $[C; \ell \to \ell']$ is the mapping α that sends each point $P \in \ell$ to the intersection of $\overleftrightarrow{PC}$ with ℓ'.
   i. Verify that the mapping α is well-defined.
   ii. Verify that α is a bijection with exactly one fixed point.
   iii. Verify that α^{-1} is a perspectivity.
   iv. Show that the composition of two perspectivities need not be a perspectivity.
   v. Given four distinct points P, Q, P', and Q', prove that there is a unique perspectivity taking P to P' and Q to Q'.
8. A *projectivity* is a composition of finitely many perspectivities. Each projectivity relates a pair of (not necessarily distinct) lines. For each line ℓ prove that the set of all projectivities that take ℓ to itself is a group. With respect to an appropriate choice of homogeneous coordinates, find a matrix representation for this group.

9. Let P, Q, and R be distinct points on a line ℓ, and let P', Q', and R' be distinct points on a line ℓ'. Prove that there is a unique projectivity sending P to P', Q to Q', and R to R'.
10. If ℓ and ℓ' are distinct in Exercise 9, show that the required projectivity may be expressed as the product of two perspectivities.
11. Show that any projectivity is the product of three or fewer perspectivities.
12. Let A, B, C, and D be four collinear points. Show that there is a unique projectivity that interchanges A and B and also interchanges C and D.
13. Show that a projectivity relating distinct lines is a perspectivity if and only if it has a fixed point.
14. Classify the projectivities of a given line ℓ in terms of their fixed point behavior.
15. Prove that a projective collineation that leaves fixed four points, no three of which are collinear, must be the identity. (*Hint*: Choose ℓ_∞ to be one of the fixed lines, and apply Theorem 2.2 to $\mathbf{P}^2 - \ell_\infty$.)
16. Prove the uniqueness part of Theorem 11 – there is only one projective collineation relating two specified quadrangles.
17. Prove that every projective collineation is of the form $\tilde{A}$ for some $A \in \mathbf{GL(3)}$.
18. The fixed lines of a projective collineation $\tilde{A}$ can be found by computing the eigenvectors of A^t. Justify this statement and use it to prove Theorem 12.
19. Prove Theorem 13.
20. If T is a projective collineation, prove that the restriction of T to one line ℓ is a projectivity. Prove also that every projectivity arises in this way.
21. Prove Theorem 14.
22. Show that the transformation of Theorem 14 preserves the relationships $(a - 1)x_1 + px_3 = 0$ and $(c - 1)x_2 + qx_3 = 0$, in addition to preserving the line $x_3 = 0$. Thus, unless $a = c = 1$, there is an additional fixed line. Use this to prove Theorem 15.
23. Prove Theorem 16 and its corollary.
24. Prove Theorem 17.
25. Prove Theorem 18.
26. A perspective collineation induces a projectivity on any fixed line. Discuss the fixed point behavior of such a projectivity.
27. Show that the set of perspective collineations with a given axis and

center (with the identity thrown in) is a group. Do these groups have any finite subgroups?

28. i. Verify the remarks following Theorem 18.
    ii. Show that the harmonic homologies are those having $a = -1$ and $q = 0$ in Theorem 18.
    iii. Prove that the only projective collineations that are involutions are the harmonic homologies.
29. When ℓ_∞ is taken to be the axis of a harmonic homology, what affine transformation of $\mathbf{P}^2 - \ell_\infty$ results?
30. i. Prove that there is a unique harmonic homology with a given center and axis.
    ii. If α is a harmonic homology with center P and axis ℓ and β is a harmonic homology with center Q and axis m, prove that $\alpha\beta = \beta\alpha$ if and only if Q lies on ℓ and P lies on m.
31. Prove Theorem 19.
32. Prove Theorem 20.
33. Let $\tilde{b}$ be a polarity, and let ℓ be a non-self-conjugate line. For each $X \in \ell$, let $\alpha(X)$ be the point where the polar line of X intersects ℓ. Prove that α is a projectivity. Show further that $\alpha^2 = I$; that is, α is an involution.
34. Prove Theorem 23.
35. Prove Theorem 25.