Lesson 25
Higher–Order, First Degree Differential Equations
(Linear Differential Equations with Constant Coefficients)

Specific Objectives:

At the end of the lesson, the students are expected to:

- define differential operator D
- differentiate homogeneous and non-homogeneous higher order first degree DE
- perform the operations on differential operators

Standard Form:

$$a_0\left(\frac{d^n y}{dx^n}\right)+a_1\left(\frac{d^{n-1} y}{dx^{n-1}}\right)+a_2\left(\frac{d^{n-2} y}{dx^{n-2}}\right)+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=R(x)$$

where a_0, a_1, a_2. . . a_{n-1} are constants and R(x) is a function of x.

Preliminary Theory

1. Initial–Value and Boundary Value Problems

For a linear differential equation, an nth–order initial–value problem is

Solve: $a_0\left(\frac{d^n y}{dx^n}\right)+a_1\left(\frac{d^{n-1} y}{dx^{n-1}}\right)+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=R(x)$

Subject to: $y=y_0, y'=y_1,\ldots, y_{(n-1)}=y_{n-1}$

Another type of problem in which the dependent variable y or its derivatives are specified at different points is called a boundary–value problem.

Solve: $a_2\frac{d^2 y}{dx^2}+a_1\frac{dy}{dx}+a_0 y=R(x)$

Subject to: $y(a)=y_0, y(b)=y_1$

The prescribed values $y(a)=y_0$ and $y(b)=y_1$ are called boundary conditions.

2. Homogeneous Equations

Standard Form:

$$a_0\left(\frac{d^n y}{dx^n}\right)+a_1\left(\frac{d^{n-1}y}{dx^{n-1}}\right)+a_2\left(\frac{d^{n-2}y}{dx^{n-2}}\right)+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=0 \text{ —— } (1)$$

where a_0, a_1, a_2, . . ., a_{n-1} are constants and R(x) = 0

(1) is a homogeneous linear differential equation since the degree of the derivatives of each term is one.

3. Non–Homogeneous Equations

Standard Form:

$$a_0\left(\frac{d^n y}{dx^n}\right)+a_1\left(\frac{d^{n-1}y}{dx^{n-1}}\right)+a_2\left(\frac{d^{n-2}y}{dx^{n-2}}\right)+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=R(x)$$

where a_0, a_1, a_2. . . a_{n-1} are constants and R(x) is a function of x.

4. Differential Operators, D–Operators

D–Operator

An operator is a symbol indicating an operation to be performed. For D–operator, it means taking the derivative of a function with respect to x. Thus,

$$y' = \frac{dy}{dx} = Dy$$
$$y'' = \frac{d^2y}{dx^2} = D^2y$$
$$y''' = \frac{d^3y}{dx^3} = D^3y$$
$$y^n = \frac{d^ny}{dx^n} = D^ny$$

Algebraic Properties of D–Operator

1. Commutative with respect to Addition/Multiplication

$$\left(D^m + D^n\right)y = \left(D^n + D^m\right)y$$

$$\left(D^m D^n\right)y = \left(D^n D^m\right)y$$

for differential operatos with constant coefficients

2. Associative with respect to Addition/ Multiplication

$$\left[\left(D^m + D^n\right) + D^p\right]y = \left[D^m + \left(D^n + D^p\right)\right]y$$

$$\left[\left(D^m D^n\right)D^p\right]y = \left[D^m\left(D^n D^p\right)\right]y$$

3. Distributive with respect to Addition

$$\left[D^m\left(D^n + D^p\right)\right]y = \left(D^m D^n + D^m D^p\right)y$$

4. Linearity

$$Q(D)\left[y_1 + y_2 + \ldots + y_n\right] = Q(D)y_1 + Q(D)y_2 + \ldots + Q(D)y_n$$

where Q(D) is a polynomial in D

Example: Perform the indicated operations:

1. $(D-2)x^3$

SOLUTION:

$$(D-2)x^3 = D\left(x^3\right) - 2x^3$$

$$(D-2)x^3 = 3x^2 - 2x^3$$

$$\underline{\underline{(D-2)x^3 = x^2(3-2x)}}$$

2. $(D+1)^2 xe^{-x}$

SOLUTION:

$$(D+1)^2 xe^{-x} = \left(D^2 + 2D + 1\right)xe^{-x}$$

$$(D+1)^2 xe^{-x} = D^2\left(xe^{-x}\right) + 2D\left(xe^{-x}\right) + xe^{-x}$$

But $D\left(xe^{-x}\right) = x\left(-e^{-x}\right) + e^{x}(1)\ orD\left(xe^{-x}\right) = -xe^{-x} + e^{-x}$

$$D^2\left(xe^{-x}\right) = -\left[x\left(-e^{-x}\right) + e^{-x}\right] + \left(-e^{-x}\right)orD^2\left(xe^{-x}\right) = xe^{-x} - 2e^{-x}$$

Substituting, $(D+1)^2 xe^{-x} = xe^{-x} - 2e^{-x} + 2\left(-xe + e^{-x}\right) + xe^{-x}$

$$(D+1)^2 xe^{-x} = xe^{-x} - 2e^{-x} - 2xe^{-x} + 2e^{-x} + xe^{-x}$$

$$\underline{\underline{(D+1)^2 xe^{-x} = 0}}$$

3. $(D^2 - 36)\sin 2x$

SOLUTION:

$$(D^2 - 36)\sin 2x = D^2(\sin 2x) - 36\sin 2x$$
$$D(\sin 2x) = 2\cos 2x$$
$$D^2(\sin 2x) = 2(-2\sin 2x) = -4\sin 2x$$
Substituting, $(D^2 - 36)\sin 2x = -4\sin 2x - 36\sin 2x$
$$\underline{\underline{(D^2 - 36) = -40\sin 2x}}$$

4. $D^8(D-m)e^{mx}$

SOLUTION:

$$D^8(D-m)e^{mx} = D^9(e^{mx}) - mD^8(e^{mx})$$
$$D^8(D-m)e^{mx} = m^9 e^{mx} - m(m^8 e^{mx})$$
$$D^8(D-m)e^{mx} = m^9 e^{mx} - m^9 e^{mx}$$
$$\underline{\underline{D^8(D-m)e^{mx} = 0}}$$

since $D(e^{mx}) = me^{mx}$
$$D^2(e^{mx}) = m(me^{mx}) = m^2 e^{mx}$$
$$D^3(e^{mx}) = m^2(me^{mx}) = m^3 e^{mx}$$

In general, $D^n(e^{mx}) = m^n e^{mx}$

Seatwork: Perform the indicated operations:

1. $D^2(2x^3)$ ans: $12x$
2. $(D^2 - 4D + 3)(3x^2 + 2\cos x)$ ans: $9x^2 - 24x + 6 + 8\sin x + 4\cos x$

Homework: Perform the indicated operations:

1. $D(x \ln x)$ ans: $1 + \ln x$
2. $D(x^2 + \sin 4x - \ln x)$ ans: $2x + 4\cos 4x - 1/x$

3. $(2D^2 - 3D + 5)(x \cos x - 3)$ ans: $3x\sin x + 3x\cos x - 4\sin x - 3\cos x - 15$

4. $(D^3 + 2D - 4)(e^{-x}\sin x + e^{2x})$ ans: $4e^{-x}(\cos x - \sin x) + 8e^{2x}$