Historical introduction

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In the beginning, geometry was a collection of rules for computing lengths, areas, and volumes. Many were crude approximations arrived at by trial and error. This body of knowledge, developed and used in construction, navigation, and surveying by the Babylonians and Egyptians, was passed on to the Greeks. Blessed with an inclination toward speculative thinking and the leisure to pursue this inclination, the Greeks transformed geometry into a deductive science. About 300 B.C., Euclid of Alexandria organized some of the knowledge of his day in such an effective fashion that all geometers for the next 2000 years used his book, *The Elements*, as their starting point.

First he defined the terms he would use – points, lines, planes, and so on. Then he wrote down five postulates that seemed so clear that one could accept them as true without proof. From this basis he proceeded to derive almost 500 geometrical statements or theorems. The truth of these was in many cases not at all self-evident, but it was guaranteed by the fact that all the theorems had been derived strictly according to the accepted laws of logic from the original (self-evident) assertions.

Although a great breakthrough in their time, the methods of Euclid are imperfect by modern standards. To begin with, he attempted to define everything in terms of a more familiar notion, sometimes creating more confusion than he removed. The following examples provide an illustration:

A *point* is that which has no part. A *line* is breadthless length. A straight line is a line which lies evenly with the points on itself. A *plane angle* is the inclination to one another of two lines which meet. When a straight line set upon a straight line makes adjacent angles equal to one another, each of the equal angles is a *right angle*.

Euclid did not define length, distance, inclination, or "set upon." Once having made his definitions, Euclid never used them. He used instead the "rules of interaction" between the defined objects as set forth in his five postulates and other postulates that he implicitly assumed but did not state. Euclid's five postulates were the following:

I. To draw a straight line from any point to any other point.
II. To produce a finite straight line continuously in a straight line.
III. To describe a circle with any center and distance.
IV. That all right angles are equal to each other.
V. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Euclid did not feel it necessary to enunciate the following postulate, even though he used it in his very first theorem.

Two circles, the sum of whose radii is greater than the distance between their centers, and the difference of whose radii is less than that distance, must have a point of intersection.

It is natural to ask why Euclid singled out his five postulates for explicit mention. After Euclid, mathematicians attempted to make explicit the assumptions that Euclid had neglected to mention. The fifth postulate attracted much attention. It was cumbersome but intuitively appealing, and people felt that it might be deduced from the other assumptions of Euclid. Many "proofs" of the fifth postulate were proposed, but they usually contained a hidden assumption equivalent to what was to be proved. Three such equivalent conditions were:

i. Two intersecting straight lines cannot be parallel to the same straight line. (Playfair)
ii. Parallel lines remain at a constant distance from each other. (Proclus)
iii. The interior angles of a triangle add up to two right angles. (Legendre)

In 1763 a man named Klügel wrote a dissertation at Göttingen in which he evaluated all significant attempts to prove the parallel postulate in the 2000 years since Euclid had stated it. Of the 28 proofs he examined, not one was found to be satisfactory. Of particular interest was the work of the Jesuit Saccheri (1667–1733). Saccheri assumed the negation of the fifth postulate and deduced the logical consequences, hoping to arrive at a contradiction. He derived many strange-looking results, some of which he claimed were inconsistent with Euclid's other postulates. Actually, he had discovered some fundamental facts about what we now call hyperbolic geometry.

Gauss (1777–1855) was apparently the first mathematician to whom it occurred that this negation might never lead to a contradiction and that geometries differing from that of Euclid might be possible. The thought struck him as being so revolutionary that he would not make it public. In 1829 he wrote that he feared the "screams of the dullards," so entrenched were the ideas of Euclid. Lobachevsky (1793–1856) and Bolyai (1802–1860) independently worked out geometries that seemed consistent and yet negated Euclid's fifth postulate. These works were published in 1829 and 1832, respectively. Experience proved that Gauss had overestimated the dullards. They paid no attention to the new theories.

Almost 40 years later Beltrami (1835–1900) and Klein (1849–1925) produced models within Euclidean geometry of the geometry of Bolyai and Lobachevsky (now called *hyperbolic geometry*). It was thus established that if Euclid's geometry was free of contradiction, then so was hyperbolic geometry. Because hyperbolic geometry satisfied all the assumptions of Euclid except the parallel postulate, it was finally determined that a proof of the postulate was impossible.

With this branching of geometry into Euclidean and non-Euclidean, it became useful to categorize results according to their dependence on the fifth postulate. Any theorem of Euclid that made no use of the parallel postulate was called a theorem of *absolute geometry*. It was equally valid in Euclidean and hyperbolic geometry. By contrast, certain Euclidean theorems that depended only on postulates I, II, and V became known as *affine geometry*. Theorems common to absolute and affine geometry are called theorems of *ordered geometry*.

The study of central projection was forced upon mathematicians by the problems of perspective faced by artists such as Leonardo da Vinci (1452–1519). The image made by a painter on canvas can be regarded as a projection of the original onto the canvas with the center of projection at the eye of the painter. In this process, lengths are necessarily distorted in a way that depends on the relative positions of the various objects depicted. How is it possible that the geometric structure of the original can still usually be recognized on the canvas? It must be because there are geometric properties invariant under central projection. *Projective geometry* is the body of knowledge that developed from these considerations. Many of the basic facts of projective geometry were discovered by the French engineer Poncelet (1788–1867) in 1813 while a prisoner of war, deprived of books, in Russia. Affine and projective geometry are also closely related, because the study of those properties of figures that remain invariant under parallel projection also leads to affine geometry. This aspect of affine geometry was recognized by Euler (1707–1783).

Because progress in geometry had been frequently hampered by lack of computational facility, the invention of *analytic geometry* by Descartes (1596–1650) made simple approaches to more problems possible. For instance, it allowed an easy treatment of the theory of conics, a subject which had previously been very complicated. Since the time of Descartes, analytic methods have continued to be fruitful because they have allowed geometers to make use of new developments in algebra and calculus.

The scope of geometry was greatly enlarged by Riemann (1826–1866). He realized that the geometry of surfaces provided numerous examples of new geometries. Suppose that a curve lying on the surface is called a line if each small segment of it is the shortest curve joining its end points. Then, for instance, if the surface is a sphere, the lines are the great circles. In this geometry, called *double elliptic geometry*, the following theorems are valid:

i. Every pair of lines has two points of intersection. These points are antipodal; that is, they lie at the opposite ends of the same diameter.
ii. Every pair of nonantipodal points determines exactly one line. An antipodal pair has many lines through them.
iii. The sum of the angles of a triangle is greater than π. It is possible for a triangle to have three right angles.

Riemann and Schläfli (1814–1895) considered higher-dimensional Euclidean and spherical spaces, and in his celebrated inaugural lecture at Göttingen in 1854, Riemann laid the foundations of geometry as a study of general spaces of any dimension, which are now called Riemannian manifolds. These spaces are the principal objects of study in modern *differential geometry*. As the name suggests, the methods used depend on calculus. The geometry of Riemann was used by Einstein (1879–1955) as a basis for his general theory of relativity (1916).

Although Gauss observed the relationship between the angle sum of a triangle and the curvature of the surface on which it occurs, Riemann and those who followed him carried these ideas over to Riemannian manifolds. Thus, curvature is still an important phenomenon in differential geometry, and it indicates how much the geometry of the space being studied differs from being Euclidean.

Although Euclid believed that his geometry contained true facts about the physical world, he realized that he was dealing with an idealization of reality. He did not mean that there was such a thing physically as breadthless length. But he was relying on many of the intuitive properties of real objects. In order to free geometry from reliance on physical concepts for its proofs, Hilbert (1862–1943) rewrote the foundations of geometry in 1899. Hilbert started with undefined objects (e.g., points, lines, planes), undefined relations (e.g., collinearity, congruence, betweenness), and certain axioms expressed in terms of the undefined objects and relations. Anything that could be deduced from this by the usual rules of logic was a geometrical theorem valid in that particular geometry. The choice of axioms was a matter of taste. Of course, some geometries would be interesting and some not, but that is a subjective judgment. The theorems do not depend on the nature of the undefined objects but only on the axioms they satisfy.

Seeing all these geometries around him, Klein, in 1872, proposed to classify them according to the groups of transformations under which their propositions remain true. Since then, group theory has been of increasing importance to geometers. The new geometries of Riemann gave rise to complicated groups of transformations. Soon techniques were developed to study these groups in their own right. Much work on the subject was done by Sophus Lie (1842–1899), and these groups became known as *Lie groups* in his honor.

Lie groups and differential geometry are active areas of current mathematical research.

Three approaches to the study of geometry

1. THE AXIOMATIC APPROACH

Following Hilbert, we start with some undefined objects, relations, and an axiom system. Then we deduce the logical consequences. We shall make some use of this approach. However, we need some motivation in order to know which axioms to choose and how to interpret our results. Without this, the study will not be very interesting.

2. THE ANALYTIC APPROACH

A point is represented by an ordered pair, triple, and so forth, of real numbers (or, more generally, elements of some other algebraic structure). Points are defined to be collinear if they satisfy an equation of a certain type. Then every algebraic equation that one can derive will have some geometrical interpretation. In this approach, linear algebra and matrices are used to facilitate computation.

3. THE EMPIRICAL APPROACH

Our goal is to discover geometrical facts about the world we live in. We use only those facts that we can observe and their logical consequences. Thus, one can conceive of trying to discover whether the parallel postulate is true or false in the world of physical space. Gauss, in fact, tried to do this by locating mirrors on three distant mountain peaks and measuring the sum of the angles of the large triangle formed by light rays sent from one peak to another. His results were inconclusive because the limits of experimental error were larger than the deviation of his measurement from π.

An example from empirical geometry

Our experience of the external world comes to us through our senses, especially vision and touch. As we move around and view objects from various places, the objects usually appear to change shape. An important exception is the straight line whose shape appears unchanged by a change in viewpoint. Three aspects of collinearity present themselves. Three points are collinear if they appear to be "in line;" that is, viewing one from another obscures the third. Secondly, if one begins at point A and moves "straight ahead" towards B, one will traverse all points on the line segment AB. Finally, if we stretch an elastic band from A to a suitable close point B, the elastic will fall along the segment AB.

If our main goal were to describe the properties of physical space, it would be valuable to construct and study many axiomatic geometries to see which ones best fit our observation. In physics one can never be completely sure that a certain theory is right. One can only say that it fits the observations better than any other known theory.

A basic question is this: What can we rightfully deduce about the nature of our space by observation? We can shed light on this problem by proposing a hypothetical universe and studying the system from the outside. For a long time it was believed that the earth was flat. However, if we go to a point P in the ocean, sail straight ahead 500 miles to Q, then turn right and go 500 miles to R, then return straight to P, we will find that the distance from R to P is about 667 miles. Checking our results with Pythagoras' theorem, we see that it does not hold for the right-angled triangle PQR. Thus, we see that it is possible to conduct an experiment to show that the geometry of our earth is not Euclidean.

Suppose now that our earth had been a circular cylinder rather than a sphere. If we had performed the same experiment, we would have found that the distance from R to P was about 707 miles, as predicted by Pythagoras. Our experiment would not prove, of course, that our earth was a plane. However, it would not contradict that hypothesis. A more ambitious experiment would be to try to answer the following question. If you start at a point and go straight ahead, is it possible that after a while you will begin to get closer to your starting point? Can you actually reach your starting point in this way?

Nature of the book

Although we will be dealing with many of the aspects of geometry mentioned in the historical sketch, we will not discuss them in chronological order. We will rely heavily on analytic techniques that, of course, were not available to Euclid. The group concept will frequently be used to make our discussions more transparent. Linear algebra, an indispensible tool for any modern treatment of geometry, will be used on almost every page.

The book begins with a thorough investigation of the Euclidean plane. Here we set the pattern for our study of the non-Euclidean geometries. Points, lines, reflections, and distance are defined. Questions of parallelism, perpendicularity, and symmetry are studied. Isometries (distance-preserving transformations) are classified, and the structure of the isometry group is determined.

Many of the facts derived about the Euclidean plane are already familiar to those who have studied geometry from another approach. However, the same format can be used to investigate the projective and hyperbolic planes. The results are beautiful and, in some cases, surprising.

When we have completed our construction of the three consistent models of plane geometry, we will have some appreciation for the kind of experiments in empirical geometry with which two competing models of the universe could be tested. Although we have limited ourselves to the two-dimensional case by studying planes, it is not too hard to see how higher-dimensional Euclidean, elliptic, and hyperbolic spaces could be studied. Modern cosmology attempts to describe the universe in terms of the geometrical properties of a four-dimensional "spacetime." Although discussion of such models is beyond the scope of this book, we hope that the techniques and thought patterns developed by studying this book will be useful to those who might later wish to work in this area. An interesting nontechnical reference is Rucker [28].

For further reading on the ideas discussed in this introduction, readers are referred to Coxeter [8], Faber [13], Greenberg [16], Meschkowski [23], Tietze [30], and Euclid's *Elements* as presented by Heath [18].