Affine transformations in the Euclidean plane

2

Affine transformations

In Chapter 1 we discussed the two fundamental aspects of geometry: the incidence aspect based on the notion of collinearity, and the metric aspect based on the notion of distance. Isometries are the transformations that respect these features.

In this chapter we want to enlarge our world of transformations to include those that respect incidence but do not necessarily preserve distance. There are two reasons for doing this. First, we want to be able to recognize and classify figures according to their shapes rather than insisting on the stronger condition of congruence. For example, we want to have transformations that relate similar triangles. The second reason is computational convenience. The algebraic conditions that determine an isometry are more difficult to work with than those based merely on incidence.

Definition. *A* collineation *is a bijection T: $\mathbf{E}^2 \to \mathbf{E}^2$ satisfying the condition that for all triples P, Q, and R of distinct points, P, Q, and R are collinear if and only if TP, TQ, and TR are collinear.*

Although this definition, like the original definition of isometry, is appealing because of its geometric flavor, it does not lend itself immediately to computation. We need a more algebraic version.

Definition. *A mapping T: $\mathbf{E}^2 \to \mathbf{E}^2$ is called an* affine transformation *if there is an invertible 2 by 2 matrix A and a vector $b \in \mathbf{R}^2$ such that, for all $x \in \mathbf{R}^2$,*

$$Tx = Ax + b.$$

Remark: By Theorem 1.38 every isometry is an affine transformation.

Remark: The matrix A and the vector b mentioned in the definition are uniquely determined by T. In fact, $b = T(0)$ and the columns of A are the vectors $T\varepsilon_i - b$, $i = 1, 2$. We call A the *linear part* of T, and b the *translation part* of T.

Theorem 1. *Every affine transformation is a collineation.*

Proof: The following identity, which can be easily checked, directly shows that affine transformations are surjective.

$$A(A^{-1}(x - b)) + b = x.$$

On the other hand, if $Ax + b = A\tilde{x} + b$, then $A(x - \tilde{x}) = 0$. Because A is invertible, we must have $x = \tilde{x}$. Thus, affine transformations are injective.

Finally, for any points P and Q and any affine transformation T, it is easy to check that

$$T((1 - t)P + tQ) = (1 - t)TP + tTQ \tag{2.1}$$

for all real t. Thus, if R is a point collinear with P and Q, TR will be collinear with TP and TQ. Conversely, if R' is a point collinear with TP and TQ, there is (because T is surjective) a unique point R with $TR = R'$. But now we know that

$$TR = (1 - t)TP + tTQ \tag{2.2}$$

for some number t. Because T is injective, (2.1) and (2.2) yield

$$R = (1 - t)P + tQ,$$

and R is collinear with P and Q. □

Corollary. *Every isometry is a collineation.*

Theorem 2. *Every collineation is an affine transformation.*

The proof of Theorem 2 is too technical to present here but is included in Appendix E. From now on in this chapter we will treat the word "collineation" as a synonym for affine transformation.

Fixed lines

If T is an affine transformation and ℓ is a line, then $T\ell$ is a line. We now show how to compute this line in terms of the data determining T and ℓ.

Theorem 3. *Let T be an affine transformation, and let $\ell = P + [v]$ be a line. Then $T\ell$ is the line $TP + [Av]$, where A is the linear part of T.*

Proof: Let b be the translation part of T. For real t,

$$T(P + tv) = A(P + tv) + b = TP + tAv.$$

From this equation we can see that every point of $T\ell$ lies on $TP + [Av]$, and conversely. Note that $T\ell$ is in fact a line because $Av \neq 0$. □

Corollary. *Let T be an affine transformation with linear part A and translation part b. A line $P + [v]$ is a fixed line of T if and only if v is an eigenvector of A and $(A - I)P + b \in [v]$. (The notion of eigenvector is discussed in Appendix D.)*

Theorem 4.

i. *If two fixed lines of an affine transformation intersect, they do so in a fixed point.*
ii. *If two fixed lines of an affine transformation are parallel, every line in the pencil containing these lines is fixed.*
iii. *If two lines are parallel, their images under any affine transformation are parallel.*

The reader may prove these facts as an exercise. (See Exercise 2.)

We now have the machinery required to prove Theorem 40 of Chapter 1:

Proof (of Theorem 1.40): Let $\ell = P + [v]$ be a line, and let T be an affine transformation with linear part A and translation part b.

CASE 1: T is a nontrivial translation, so $A = I$ and $b \neq 0$. Then v is automatically an eigenvector of A, and ℓ is a fixed line if and only if $b \in [v]$. Thus, the fixed lines of T are those with direction $[b]$.

CASE 2: If T is a half-turn about a point C, then from Exercise 1.26, $A = -I$ and $b = 2C$. Again, v is automatically an eigenvector, and ℓ is a fixed line if and only if $-2P + 2C \in [v]$; that is, $C \in P + [v]$. Thus the fixed lines of T are those that pass through C.

Now consider the case of a rotation having $A = \text{rot}\,\theta \neq \pm I$. Then A has no nonzero eigenvectors (Exercise 1.31); therefore, T can have no fixed lines.

CASE 3: T is a reflection with axis m. Clearly, m is a fixed line. Furthermore, if $m = Q + [w]$, where $|w| = 1$, then for all real t,

$$\Omega_m(Q + tw^\perp) = Q + tw^\perp - 2\langle tw^\perp, w^\perp\rangle w^\perp = Q - tw^\perp.$$

Thus, $Q + [w^\perp]$ is a fixed line. In other words, Ω_m leaves fixed all lines perpendicular to m. By Theorem 4(i), any fixed line not perpendicular to m must meet the pencil of perpendiculars to m in fixed points. Because Ω_m has no fixed points except on m (Theorem 1.21), Ω_m can have no additional fixed lines.

CASE 4: $T = \Omega_{m}\tau_{kw}$ is a glide reflection consisting of reflection in a line m with unit direction vector w and a translation by a nonzero multiple of w.

We first show that m is a fixed line. Let Q be any point of m. Then for real t,

$$\begin{aligned} T(Q + tw) &= \Omega_{m}(Q + tw + kw) = Q + (t + k)w - 2\langle (t + k)w, w^{\perp}\rangle w^{\perp} \\ &= Q + (t + k)w. \end{aligned}$$

Thus, the line m is a fixed line.

Because T has no fixed points, any other fixed line would have to be parallel to m. Let $\ell = Q + sw^{\perp} + [w]$ be a typical line parallel to m. Then

$$\begin{aligned} T(Q + sw^{\perp} + tw) &= \Omega_{m}(Q + sw^{\perp} + (t + k)w) \\ &= Q + sw^{\perp} + (t + k)w - 2\langle sw^{\perp} + (t + k)w, w^{\perp}\rangle w^{\perp} \\ &= Q - sw^{\perp} + (t + k)w. \end{aligned}$$

Note that ℓ cannot be fixed unless $s = 0$. □

The affine group AF(2)

We now look at the result of successively applying two affine transformations. If

$$Tx = Ax + b \quad \text{and} \quad \bar{T}x = \bar{A}x + \bar{b},$$

then

$$T\bar{T}x = A(\bar{A}x + \bar{b}) + b = (A\bar{A})x + A\bar{b} + b.$$

Thus, the composition of two affine transformations is again an affine transformation. One can arrange that $T\bar{T} = I$ by choosing $\bar{A} = A^{-1}$ and $\bar{b} = -A^{-1}b$, thus showing that the inverse of an affine transformation is also an affine transformation. To summarize, we have proved

Theorem 5. *The set* **AF(2)** *of all affine transformations of* $\mathbf{R}^2$ *is a group, called the* affine group *of* $\mathbf{R}^2$.

Elements of **AF(2)** may be conveniently represented by matrices as follows: Write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

If $y = Ax + b$, we may easily check that the 3 by 3 matrix equation

$$\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \tag{2.3}$$

holds and that the composition operation in **AF(2)** corresponds to matrix multiplication of the associated 3 by 3 matrices.

If **GL(3)** denotes the group of all invertible 3 by 3 matrices, then **AF(2)** is a subgroup of **GL(3)**. This representation may be abbreviated as

$$T = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix},$$

where the sizes of the various matrices are understood from the context. These ideas are formalized in Exercise 6.

Fundamental theorem of affine geometry

Affine geometry consists of those facts about $\mathbf{E}^2$ that depend only on incidence properties and not on perpendicularity or distance. Although affine geometry is interesting in its own right, we will be concentrating here on those aspects that will help us to solve problems of congruence and symmetry of figures.

The fundamental theorem gives a clear and simple criterion for existence and uniqueness of affine transformations, namely, that any two triangles can be related by a unique affine transformation.

At this point it is useful to highlight a fact that arose in the proof of Theorem 1 – affine transformations preserve order along lines.

Theorem 6. *Let P and Q be points, and let T be an affine transformation. Then*

i. *For any real number t,*

$$T((1 - t)P + tQ) = (1 - t)TP + tTQ. \tag{2.1}$$

ii. *A point X lies between P and Q if and only if TX lies between TP and TQ. Furthermore,*

$$\frac{d(P, X)}{d(P, Q)} = \frac{d(TP, TX)}{d(TP, TQ)}.$$

We are now in a position to derive an important uniqueness property of affine transformations.

Theorem 7.

i. *If an affine transformation leaves fixed two distinct points, then it leaves fixed every point on the line joining these points.*
ii. *If an affine transformation leaves fixed three noncollinear points, it must be the identity.*

Proof: The first claim is immediate from formula (2.1). Now, let P, Q, and R be three noncollinear points that are left fixed by an affine transformation T. Let X be any point not lying on any of the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, or $\overleftrightarrow{RP}$. Let A be the midpoint of the segment PQ. Now $\overleftrightarrow{XA}$ cannot be parallel to both $\overleftrightarrow{QR}$ and $\overleftrightarrow{RP}$; hence, it meets one of these lines in a point B (distinct from A). Because X is on a line containing two fixed points A and B, X itself must be a fixed point. We conclude that T leaves every point in the plane fixed and therefore is the identity. □

Our proof of Theorem 7(ii) is a synthetic proof. It uses geometric ideas derived earlier and geometric arguments.

In following the proof it is helpful to make your own diagram. An alternative proof using linear algebra is suggested in Exercise 7.

We now come to the fundamental theorem, which asserts the existence of a unique affine transformation relating any two triangles.

Theorem 8. *Given two noncollinear triples of points, PQR and $P'Q'R'$, there is a unique affine transformation T such that $TP = P'$, $TQ = Q'$, and $TR = R'$.*

Proof: Because $\{Q - P, R - P\}$ and $\{Q' - P', R' - P'\}$ are bases for $\mathbf{E}^2$ (Appendix D), there is an invertible 2 by 2 matrix A such that $A(Q - P) = Q' - P'$ and $A(R - P) = R' - P'$. Let $T = \tau_{P'}A\tau_{-P}$. Then $TP = \tau_{P'}A(P - P) = P'$. Similarly, $TQ = Q'$ and $TR = R'$. Thus, we have constructed an affine transformation with the required property.

We now show that there is only one such transformation. Suppose that $\bar{T}$ agrees with T on P, Q, and R. Then $\bar{T}^{-1}T$ is an affine transformation that leaves P, Q, and R fixed. By Theorem 7, $\bar{T}^{-1}T = I$; that is, $T = \bar{T}$. □

Affine Reflections

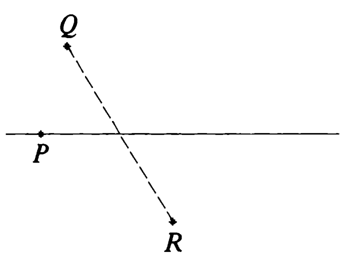

Figure 2.1 The affine reflection $[P; Q \leftrightarrow R]$.

Let P, Q, and R be noncollinear points of $\mathbf{E}^2$. The unique affine transformation (guaranteed to exist by the fundamental theorem) that leaves P fixed while interchanging Q and R is called an *affine reflection* and is denoted by the symbol (used in [8]).

$$[P;\ Q \leftrightarrow R]$$

(see Figure 2.1). Clearly, every ordinary reflection is an affine reflection. In fact, let ℓ be any line, P any point on ℓ, Q any point not on ℓ, and $R = \Omega_\ell Q$. Then it is easy to verify (Exercise 8) that

$$\Omega_\ell = [P;\ Q \leftrightarrow R]. \tag{2.4}$$

We shall soon see that not every affine reflection is an isometry. However, affine reflections share some of the properties of ordinary reflections. To begin with, an affine reflection T must be involutive because T^2 has three noncollinear fixed points. In addition, we have

Theorem 9. *Let M be the midpoint of a segment QR, and let P be any point not collinear with Q and R. Then the affine reflection $[P;\ Q \leftrightarrow R]$ leaves fixed every point of $\overleftrightarrow{PM}$ but no other points.*

Proof: We first check that M is a fixed point. To see this, write $Tx = Ax + b$ as usual. Then

$$Q = TR = AR + b \quad \text{and} \quad R = TQ = AQ + b.$$

Hence,

$$Q + R = A(Q + R) + 2b;$$

that is,

$$M = AM + b = TM,$$

and M is a fixed point. By Theorem 7(i), $\overleftrightarrow{PM}$ consists entirely of fixed points. On the other hand, the affine reflection is not the identity, so it cannot have any additional fixed points, by Theorem 7(ii). □

Theorem 10. *The affine reflection $[P;\ Q \leftrightarrow R]$ leaves fixed the line $\overleftrightarrow{PM}$ and all lines parallel to $\overleftrightarrow{QR}$ and no other lines. (Notation is as for Theorem 9.)*

Proof: Because TQ and TR determine the same line as Q and R, we see that the line $\ell = \overleftrightarrow{QR}$ is fixed. Each line ℓ' parallel to ℓ meets $\overleftrightarrow{PM}$ in a fixed point M'. Thus, $T\ell'$ passes through M' while remaining parallel to $T\ell = \ell$ (Theorems 4(iii) and 1.17). This guarantees that $T\ell' = \ell'$; that is, ℓ' is a fixed line. Finally, suppose that ℓ'' were a fixed line not parallel to ℓ but distinct from $\overleftrightarrow{PM}$. Then ℓ'' meets ℓ' and ℓ in fixed points. This contradicts the fact that all fixed points are on $\overleftrightarrow{PM}$. □

Theorem 11. *The affine reflection $[P;\ Q \leftrightarrow R]$ is an isometry if and only if $\overleftrightarrow{PM} \perp \overleftrightarrow{QR}$.*

Proof: Suppose that the given affine reflection is an isometry. Because it has a line of fixed points, it must be an ordinary reflection with axis $\overleftrightarrow{PM}$, by Theorem 1.39. But $\overleftrightarrow{QR}$ is a fixed line of this reflection, and thus it must be perpendicular to $\overleftrightarrow{PM}$ by Theorem 1.40.

Conversely, suppose that $\overleftrightarrow{PM} \perp \overleftrightarrow{QR}$. Let $m = \overleftrightarrow{PM}$. We show that Ω_m interchanges Q and R and thus, by the fundamental theorem, must coincide with the given affine reflection. To this end, note that

$$\Omega_m Q = Q - 2\langle Q - M, N\rangle N,$$

where N is a unit vector in the direction $[Q - R]$. But

$$Q - M = \tfrac{1}{2}(Q - R).$$

Thus,

$$\Omega_m Q = Q - \langle Q - R, N\rangle N = Q - (Q - R) = R.$$

Also $\Omega_m R = \Omega_m \Omega_m Q = Q$. Hence, Ω_m interchanges Q and R. □

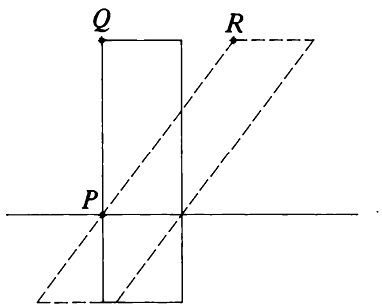

Figure 2.2 The shear $[P; Q \to R]$.

Shears

The fundamental theorem of affine geometry can be used to define other classes of affine transformations. Let P, Q, and R be noncollinear points. The unique affine transformation that leaves fixed every point on the line through P parallel to $\overleftrightarrow{QR}$ and that takes Q to R is denoted by $[P; Q \to R]$ and is called a *shear*. See Figure 2.2.

Theorem 12. *The shear $[P; Q \to R]$ has the line through P parallel to $\overleftrightarrow{QR}$ as its set of fixed points. The fixed lines are those belonging to the pencil of parallels determined by $\overleftrightarrow{QR}$.*

Proof: Let T be the shear in question. T can have no fixed points other than those on the line $m = P + [Q - R]$. Otherwise, it would be the identity.

Let $\ell = Q + [Q - R]$. Because $\ell \parallel m$ and $Tm = m$, $T\ell$ is the unique line through $R = TQ$ parallel to m. In other words, $T\ell = \ell$, and ℓ is a fixed line.

Finally, let X be any point lying neither on ℓ nor on m, and let $X' = TX$. The line $\overleftrightarrow{XX'}$ must be parallel to m; otherwise $\overleftrightarrow{XX'}$ would have to meet m in a fixed point B and $\overleftrightarrow{XB} = \overleftrightarrow{X'B}$ would be a fixed line. But now the fixed lines $\overleftrightarrow{QR}$ and $\overleftrightarrow{XB}$ would have to intersect in a fixed point, which is impossible. This shows that TX lies on $X + [Q - R]$ and, hence, that $X + [Q - R]$ is a fixed line. Our argument also shows that no other lines of $\mathbf{E}^2$ can be fixed. □

Remark: The line of fixed points of a shear is called its *axis*.

Theorem 13. *A shear whose fixed points lie along the x_1-axis has a matrix of the form*

$$s_\lambda = \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5}$$

Proof: Because the origin is a fixed point, the translation part is 0. Also, the vector ε_1 is a fixed point, and this determines the first column of the matrix. Finally, the shear must take ε_2 to a point on the horizontal line $\varepsilon_2 + [\varepsilon_1]$. This determines the form of the second column. □

Remark:

i. Every matrix of the form s_λ, $\lambda \neq 0$, determines a shear.
ii. If T is a shear with axis ℓ, and ρ is any affine transformation, then $\rho T \rho^{-1}$ is a shear with axis $\rho\ell$.

These facts follow easily from the fundamental theorem (Exercise 9). A shear whose axis is a horizontal line through a point P can be written

$$Tx = P + s_\lambda(x - P), \tag{2.6}$$

and a shear whose axis passes through the origin and has direction vector $(\cos\theta, \sin\theta) = (\text{rot}\ \theta)\varepsilon_1$ can be written

$$Tx = (\text{rot}\ \theta)s_\lambda(\text{rot}\ (-\theta))x,$$

so that any shear with axis $P + [(\text{rot}\ \theta)\varepsilon_1]$ can be written in the form

$$Tx = P + (\text{rot}\ \theta)s_\lambda(\text{rot}\ (-\theta))(x - P) \tag{2.7}$$

for some real number $\lambda \neq 0$.

Dilatations

A *dilatation* is an affine transformation with the property that for each line ℓ, either $T\ell = \ell$ or $T\ell \parallel \ell$. The identity is said to be a *trivial dilatation.*

Theorem 14. *A dilatation that leaves two points fixed must be the identity.*

Proof: Suppose that P and Q are distinct fixed points of a dilatation T. Then every point on the line $\overleftrightarrow{PQ}$ is fixed. Let X be any point not on $\overleftrightarrow{PQ}$. Then T takes the line $\overleftrightarrow{PX}$ to a line through P with the same direction. In other words, $\overleftrightarrow{PX}$ is a fixed line. By the same argument $\overleftrightarrow{QX}$ is a fixed line, and so X is a fixed point. Because T has three noncollinear fixed points, it must be the identity.

Thus, a nontrivial dilatation can have at most one fixed point. A dilatation with exactly one fixed point is called a *central dilatation*, and the fixed point is called its *center.* See Figures 2.3 and 2.4.

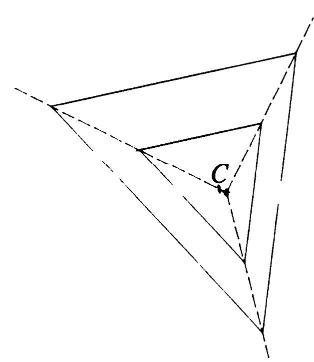

Figure 2.3 Two triangles related by a central dilatation.

Theorem 15.

i. *A central dilatation with center C may be written in the form*

$$Tx = C + \kappa(x - C). \tag{2.8}$$

The number $|\kappa|$ is called the magnification factor of T.

ii. *A dilatation that has no fixed points is a translation.*

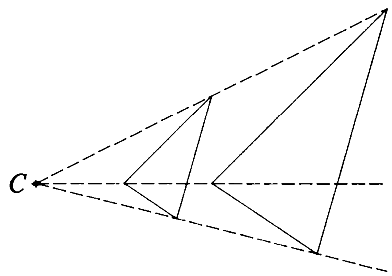

Figure 2.4 Two more triangles related by a central dilatation.

Proof: Let T be a central dilatation with center C. Because T is a dilatation, every vector $v \in \mathbf{R}^2$ must be an eigenvector of A (the linear part of T). Thus, there is a nonzero real number κ such that $A = \kappa I$ (Exercise 10).

Because $TC = C$, the translation part of T is equal to $C - \kappa C$, so that for all $x \in \mathbf{E}^2$,

$$Tx = \kappa x + C - \kappa C = C + \kappa(x - C).$$

This proves (i). Further, if $\kappa \neq 1$, the equation $\kappa x + b = x$ has a solution $x = (-1/(\kappa - 1))b$. Hence, every dilatation is either a translation ($\kappa = 1$) or has a fixed point. □

Theorem 16. *The fixed lines of a central dilatation are precisely those that pass through its center.*

Proof: First, note that

$$T(C + [v]) = TC + [v] = C + [v],$$

so that all lines through C are fixed. On the other hand, if any fixed line ℓ does not pass through C, pick an arbitrary point X on this line. Then $\overleftrightarrow{CX}$ and ℓ are fixed lines intersecting in X. Because X cannot be a fixed point, we have a contradiction. □

Remark: A half-turn is a special central dilatation having $\kappa = -1$.

Similarities

Definition. *A mapping T: $\mathbf{E}^2 \to \mathbf{E}^2$ is called a* similarity *(with magnification factor $\kappa > 0$) if, for all X, $Y \in \mathbf{E}^2$,*

$$d(TX, TY) = \kappa d(X, Y).$$

A similarity can be accomplished in two stages: first, a central dilatation (to make objects the right size); then an isometry (to move objects to the right position).

Theorem 17. *Every similarity is a central dilatation followed by an isometry. In particular, every similarity is an affine transformation.*

Proof: Let T be a similarity with magnification factor κ. Let S be the central dilatation defined by

$$SX = \frac{1}{\kappa}X.$$

Then

$$\begin{aligned} d(TSX, TSY) &= \kappa d(SX, SY) \\ &= \kappa \frac{1}{\kappa} d(X, Y) = d(X, Y). \end{aligned}$$

Thus, TS is an isometry, and, hence, T is this same isometry preceded by the central dilatation S^{-1}. □

Theorem 18.

i. *If* T_1 and T_2 *are similarities with respective magnification factors* κ_1 *and* κ_2, *then* T_1T_2 *is a similarity with magnification factor* $\kappa_1\kappa_2$.

ii. *If* T *is a similarity with magnification factor* κ, *then* T^{-1} *is a similarity with magnification factor* $1/\kappa$.

Proof: Exercise 11. □

Corollary. *The set of similarities of* $\mathbf{E}^2$ *is a group, which we denote by* $\mathbf{Sim}(\mathbf{E}^2)$.

Definition. *Two figures* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *are* similar *if there is a similarity* T *such that* $T\mathscr{F}_1 = \mathscr{F}_2$.

Affine symmetries

Let $\mathscr{F}$ be any figure. An affine transformation leaving $\mathscr{F}$ fixed is called an *affine symmetry* of $\mathscr{F}$, and the set of all affine symmetries is a group called the *affine symmetry group* of $\mathscr{F}$. We use the notation

$$\mathscr{AS}(\mathscr{F}) = \{T \in \mathbf{AF(2)} \mid T\mathscr{F} = \mathscr{F}\}.$$

Because every isometry is an affine transformation, we have

$$\mathscr{S}(\mathscr{F}) \subset \mathscr{AS}(\mathscr{F}) \subset \mathbf{AF(2)}.$$

In the next section we will set up a framework for classifying the affine symmetries of a wide class of figures. In the meantime we will examine the symmetries of some very simple figures.

Theorem 19. *Let $\mathcal{F}$ be the set consisting of a single point. Then $\mathcal{AS}(\mathcal{F}) \cong$* **GL(2)**, *the group of 2 by 2 invertible real matrices.*

Proof: Let P be the point. If $T \in \mathcal{AS}(\mathcal{F})$, then $\tau_{-P}T\tau_P$ leaves 0 fixed, and its translation part is 0. We write $\tau_{-P}T\tau_P = A$, where A is linear, and conclude that $T = \tau_P A \tau_{-P}$; that is, $Tx = P + A(x - P)$ for all $x \in \mathbf{E}^2$. It is now a routine matter to check that the mapping that takes T to A is an isomorphism (Exercise 12). □

Theorem 20. *Let $\mathcal{F}$ be a set consisting of two points. Then $\mathcal{AS}(\mathcal{F})$ is isomorphic to the group of 2 by 2 matrices of the form*

$$\begin{bmatrix} \pm 1 & \lambda \\ 0 & \mu \end{bmatrix} \quad \textit{with} \quad \mu \neq 0. \tag{2.9}$$

Writing

$$T^+_{\lambda,\mu} = \begin{bmatrix} 1 & \lambda \\ 0 & \mu \end{bmatrix} \quad \text{and} \quad T^-_{\lambda,\mu} = \begin{bmatrix} -1 & \lambda \\ 0 & \mu \end{bmatrix},$$

one can verify that the $T^+_{\lambda,\mu}$ form a subgroup of $\mathcal{AS}(\mathcal{F})$. Some other interesting subsets are

i. the subgroup $\mathcal{S}(\mathcal{F})$ determined by $\lambda = 0$, $\mu = \pm 1$,
ii. the subgroup $\{T^+_{\lambda,\mu} | \mu = 1\}$ consisting of all shears leaving the two given points fixed together with the identity,
iii. the set $\{T^+_{\lambda,\mu} | \mu = -1\}$ of affine reflections leaving the two points fixed,
iv. the subgroup $\{T^+_{\lambda,\mu} | \lambda = 0, \mu > 0\}$ consisting of stretches in the direction of the x_2-axis, including the identity as a special case.

Remark: After we have studied projective geometry (Chapter 5), we will be able to see that these transformations are merely the affine versions of the projective collineations leaving a line pointwise fixed. In projective geometry these break down into two types: *homologies* and *elations*.

Rays and angles

Let P be a point of $\mathbf{E}^2$, and let v be a nonzero vector. Then

$$\imath = \{P + tv | t \geqslant 0\} \tag{2.10}$$

is called a *ray* with *origin* P and direction vector v. Clearly, every line through P is the union of two rays with origin P. Their direction vectors are negatives of each other.

The union of two rays $\imath_1$ and $\imath_2$ with common origin P is called an *angle* with vertex P and arms $\mathscr{r}_1$ and $\mathscr{r}_2$. We allow the possibility $\mathscr{r}_1 = \mathscr{r}_2$, in which case we refer to the angle as a *zero angle*. If $\mathscr{r}_1$ and $\mathscr{r}_2$ are two halves of the same line, we say that they are *opposite* rays and that the angle is a *straight angle*. Finally, if $\mathscr{r}_1 \perp \mathscr{r}_2$ we call the angle a *right angle*. (Two rays are perpendicular if their direction vectors are orthogonal.)

Given two distinct points P and Q, there is a unique ray with origin P that passes through Q. We denote this ray by $\overrightarrow{PQ}$. The angle with vertex Q and arms $\overrightarrow{QP}$ and $\overrightarrow{QR}$ is denoted by $\measuredangle PQR$ or, equivalently, $\measuredangle RQP$. Rays and angles may be represented as shown in Figures 2.5–2.8.

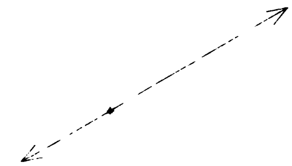

Figure 2.5 A straight angle.

It is now time to define a numerical measure for angles. This must be done in terms of analytic concepts, taking care not to appeal to our pictorial notions of angle measurement.

Definition. *Let $\mathscr{A}$ be an angle whose arms have unit direction vectors u and v. The* radian measure *of $\mathscr{A}$ is defined to be*

$$\cos^{-1}\langle u, v\rangle. \tag{2.11}$$

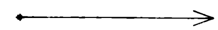

Figure 2.6 A zero angle.

Remark: If we write $u = (\cos\theta, \sin\theta)$ and $v = (\cos\phi, \sin\phi)$, then the radian measure α is the unique number in $[0, \pi]$ such that

$$(\text{rot }\alpha)u = v \quad \text{or} \quad (\text{rot }\alpha)v = u.$$

In other words, there is a rotation by α taking one arm of $\mathscr{A}$ to the other. (See Exercise 14.)

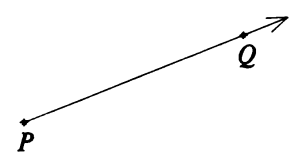

Figure 2.7 The ray $\overrightarrow{PQ}$.

Theorem 21. *Let $\mathscr{A}$ be any angle. Its radian measure α is*
i. 0 *if and only if $\mathscr{A}$ is a zero angle,*
ii. π *if and only if $\mathscr{A}$ is a straight angle,*
iii. *between 0 and π otherwise.*
Furthermore, $\alpha = \pi/2$ if and only if $\mathscr{A}$ is a right angle.

Definition. *An angle $\mathscr{A}$ is* acute *if its radian measure is $<\pi/2$. It is* obtuse *if its radian measure is $>\pi/2$. See Figures 2.9–2.11.*

Theorem 22. *Let $\mathscr{A} = \measuredangle PQR$ be an angle. Then*
i. *$\measuredangle PQR$ is acute if and only if $\langle P - Q, R - Q\rangle$ is positive.*
ii. *$\measuredangle PQR$ is obtuse if and only if $\langle P - Q, R - Q\rangle$ is negative.*

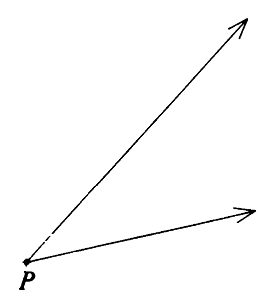

Figure 2.8 An angle with vertex P.

Definition. *Let $\mathscr{A}$ be an angle with vertex P and radian measure α. Let u and v be unit direction vectors of its arms chosen so that* $(\text{rot }\alpha)u = v$. *Then a ray with origin P and direction vector* $\text{rot}(\alpha/2)u$ *is called a* bisector of $\mathscr{A}$.

Remark: A straight angle has two bisectors. Any other angle has a unique bisector.

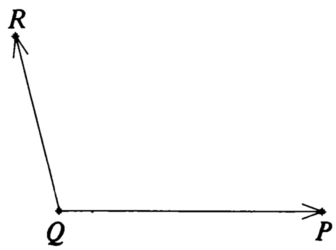

Figure 2.9 An obtuse angle.

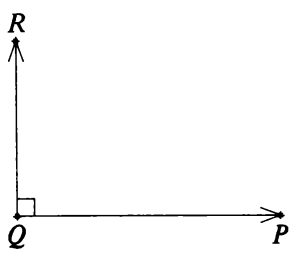

Figure 2.10 A right angle.

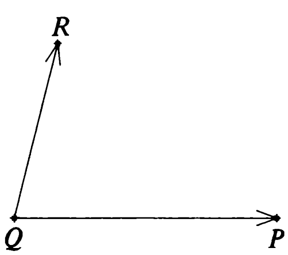

Figure 2.11 An acute angle.

Theorem 23. *For any angle $\mathscr{A}$ there is a unique reflection that interchanges its arms, namely, the reflection in the line containing the bisector(s) of $\mathscr{A}$.*

Proof: Let u, v, and α be as in the previous definition. If P is the vertex of $\mathscr{A}$, set $T = \tau_P(\mathrm{ref}((\theta + \phi)/2))\tau_{-P}$, where $u = (\cos\theta, \sin\theta)$ and $v = (\cos\phi, \sin\phi)$. Then

$$\begin{aligned} T(P + tu) &= P + \left(\mathrm{ref}\left(\frac{\theta+\phi}{2}\right)\right)tu \\ &= P + t\left(\mathrm{ref}\left(\frac{\theta+\phi}{2}\right)\right)(\mathrm{rot}\ \theta)\varepsilon_1 \\ &= P + t(\mathrm{rot}\ \phi)\varepsilon_1 \\ &= P + tv \end{aligned}$$

for all real t. Thus, $Tr_1 = r_2$ and, by symmetry, $Tr_2 = r_1$.

To show uniqueness, let $\bar{T}$ be any other reflection that interchanges r_1 and r_2. Then $\bar{T}T$ leaves the rays r_1 and r_2 fixed. Hence, their point of intersection P is fixed. Because $TP = P$, we have that $\bar{T}P = P$. Thus, the axis of the reflection $\bar{T}$ passes through P, and $\bar{T}T$ is a rotation about P. Now the only rotation that leaves a ray fixed is the identity (Exercise 21), and we conclude that $\bar{T}T$ must be the identity. Thus $T = \bar{T}$. □

Rectilinear figures

A union of finitely many segments, rays, and lines is called a *rectilinear figure*. Familiar examples are triangles, squares, and angles. We will study these in detail later. First, we develop some techniques for computing symmetry groups that are applicable to all rectilinear figures.

Let $\mathscr{F}$ be any rectilinear figure. The figure $\hat{\mathscr{F}}$ consisting of all lines that contain lines, segments, or rays of $\mathscr{F}$ is called the *rectilinear completion* of $\mathscr{F}$. A rectilinear figure $\mathscr{F}$ is said to be *complete* if, whenever a segment is in $\mathscr{F}$, the line containing it is in $\mathscr{F}$. Then $\hat{\mathscr{F}}$ is clearly the smallest complete rectilinear figure containing $\mathscr{F}$. See Figures 2.12 and 2.13.

Theorem 24. *Let T be an affine transformation, and let $\mathscr{F}$ be a rectilinear figure. Then T maps the set of lines of $\hat{\mathscr{F}}$ bijectively to the set of lines of $\widehat{T\mathscr{F}}$.*

Proof: We first show that the map is surjective. Suppose that $T\ell$ is a line of $\widehat{T\mathscr{F}}$, but ℓ is not in $\hat{\mathscr{F}}$. Then for each line m of $\hat{\mathscr{F}}$, Tm meets $T\ell$ in at most one point. Because $T\mathscr{F} \cap T\ell$ is contained in the union of all the Tm, it can contain only finitely many points. But this is impossible because $T\ell$ contains at least a segment of $T\mathscr{F}$ and, hence, an infinite number of points of $T\mathscr{F}$.

It only remains to show that if m is any line of $\hat{\mathscr{F}}$, then Tm is a line of $\widehat{T\mathscr{F}}$. First note that m contains a segment m_0 that is contained in $\mathscr{F}$. Then Tm_0 is a segment in $T\mathscr{F}$. Thus, $\widehat{T\mathscr{F}}$ contains the line determined by Tm_0, namely, Tm. □

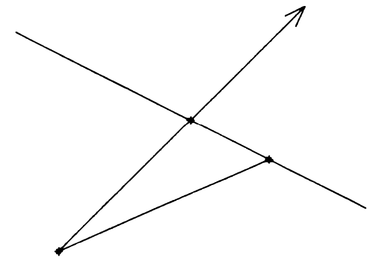

Figure 2.12 A rectilinear figure.

Corollary. *Suppose T is an affine symmetry of a rectilinear figure $\mathscr{F}$. Then T permutes the lines of its rectilinear completion $\hat{\mathscr{F}}$.*

Definition. *Suppose that $\mathscr{F}$ is a rectilinear figure. A point of $\mathscr{F}$ where two lines of $\hat{\mathscr{F}}$ intersect is called a* vertex *of $\mathscr{F}$.*

Theorem 25. *Let $\mathscr{F}$ be a rectilinear figure and T an affine transformation. Then T maps the set of vertices of $\mathscr{F}$ bijectively to the set of vertices of $T\mathscr{F}$. If T is an affine symmetry of $\mathscr{F}$, then T permutes the vertices of $\mathscr{F}$.*

Proof: We need only show that T maps vertices of $\mathscr{F}$ to vertices of $T\mathscr{F}$. The rest is trivial.

Let P be a vertex of $\mathscr{F}$. Then TP is a point of $T\mathscr{F}$. Because P is the intersection of two lines of $\hat{\mathscr{F}}$, TP is the intersection of their images, which, by Theorem 24, are in $\widehat{T\mathscr{F}}$. Thus, TP is a vertex of $T\mathscr{F}$. □

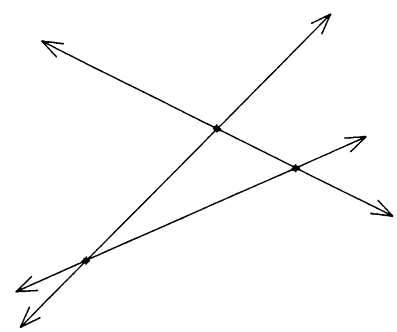

Figure 2.13 Completion of the rectilinear figure in Figure 2.12.

Corollary.

i. *Every affine symmetry T of a rectilinear figure $\mathscr{F}$ is also an affine symmetry of $\hat{\mathscr{F}}$.*
ii. *Every affine symmetry of a rectilinear figure $\mathscr{F}$ permutes the set of vertices of $\hat{\mathscr{F}}$ that are not vertices of $\mathscr{F}$.*

Proof:

i. Because T permutes the lines of $\hat{\mathscr{F}}$ and $\hat{\mathscr{F}}$ is the union of these lines, we must have $T\hat{\mathscr{F}} = \hat{\mathscr{F}}$.
ii. Because T is an affine symmetry of $\hat{\mathscr{F}}$, T must permute the set of vertices of $\hat{\mathscr{F}}$. But T being an affine symmetry of $\mathscr{F}$ must permute the vertices of $\mathscr{F}$ among themselves (i.e., this set is invariant under the permutation). Hence, T must permute the remaining vertices of $\hat{\mathscr{F}}$ among themselves. □

Corollary. *If $\mathscr{F}$ is a rectilinear figure having at least three noncollinear vertices, then $\mathscr{A}\mathscr{S}(\mathscr{F})$ is a finite group.*

Proof: Denote the three noncollinear vertices by P, Q, and R. Then each permutation of the vertices of $\mathscr{F}$ can be realized by *at most* one affine transformation. The only possible candidate is the unique affine transformation (guaranteed by the fundamental theorem) that agrees with the given permutation on P, Q, and R. In general, of course, this candidate may fail to agree with the permutation on some other vertex. Even if it accomplishes the permutation of the vertices, it may fail to be a symmetry of $\mathscr{F}$. In any case, the affine symmetry group has at most $n!$ elements, where n is the number of vertices. $\square$

Remark: If all the vertices of a rectilinear figure $\mathscr{F}$ are collinear, then $\mathscr{A}\mathscr{S}(\mathscr{F})$ is still a finite group, but the preceding proof will not work. Complete figures of this form can be described explicitly, and we shall explore these in Chapter 3, Exercise 7.

The centroid

Let F be a finite set of points of $\mathbf{E}^2$. For $x \in \mathbf{E}^2$ define

$$f(x) = \sum_{P \in F} d(x, P)^2. \tag{2.12}$$

Theorem 26. *There is a unique point of $\mathbf{E}^2$ where the function f achieves its minimum value. This point is called the* centroid of F.

Proof: Let n be the number of points of F. Then

$$\begin{aligned} f(x) &= \sum_P \langle x - P, x - P\rangle \\ &= \sum_P (|x|^2 - 2\langle x, P\rangle + |P|^2) \\ &= n|x|^2 - 2\langle x, \Sigma P\rangle + \Sigma |P|^2. \end{aligned}$$

Write $C = (1/n)\Sigma P$ and $b = (1/n)\Sigma|P|^2$.

Then

$$\begin{aligned} f(x) &= n(|x|^2 - 2\langle x, C\rangle + b) \\ &= n(|x|^2 - 2\langle x, C\rangle + |C|^2 + b - |C|^2) \\ &= n|x - C|^2 + n(b - |C|^2). \end{aligned}$$

Clearly, $f(x)$ is minimum precisely when $x = C$. $\square$

Remark: If $\mathscr{F}$ is a rectilinear figure with a finite number of vertices, the centroid of the set of vertices is often referred to as the centroid of $\mathscr{F}$.

Theorem 27. *Suppose that $\mathscr{F}$ is a rectilinear figure having a finite nonzero number of vertices. Let C be the centroid of $\mathscr{F}$. Then, for any isometry T, TC is the centroid of $T\mathscr{F}$.*

Proof: First, note that P is a vertex of $\mathscr{F}$ if and only if TP is a vertex of $T\mathscr{F}$. Also, the quantity

$$\sum_P |x - TP|^2 = \sum |TT^{-1}x - TP|^2 = \sum |T^{-1}x - P|^2$$

has its minimum value when $T^{-1}x$ is the centroid C of $\mathscr{F}$; that is, $x = TC$. Thus, TC is the centroid of $T\mathscr{F}$. □

Corollary. *If T is a symmetry of a rectilinear figure $\mathscr{F}$ with a finite number of vertices, then T leaves the centroid of $\mathscr{F}$ fixed.*

Symmetries of a segment

Let PQ be a segment. We compute its symmetry group $\mathscr{S}(PQ)$. First of all, we know that any affine transformation T takes the segment PQ to a segment $P'Q'$, where $P' = TP$ and $Q' = TQ$. We first show that T permutes the set $\{P, Q\}$.

Lemma. *A segment determines its end points; that is, if PQ and $\tilde{P}\tilde{Q}$ denote the same segment, then $\{P, Q\} = \{\tilde{P}, \tilde{Q}\}$.*

Proof: Interchanging $\tilde{P}$ and $\tilde{Q}$ if necessary, we may write

$$\tilde{P} = (1 - t)P + tQ \quad \text{and} \quad \tilde{Q} = (1 - s)P + sQ,$$

where $0 \leqslant t < s \leqslant 1$. Now, there exist $\tilde{t}$ and $\tilde{s}$ in $[0, 1]$ such that

$$\begin{aligned} P &= (1 - \tilde{t})\tilde{P} + \tilde{t}\tilde{Q} \\ &= (1 - \tilde{t})((1 - t)P + tQ) + \tilde{t}((1 - s)P + sQ) \\ &= P + (t + \tilde{t}(s - t))(Q - P). \end{aligned}$$

Because $Q \neq P$, we must have $t + \tilde{t}(s - t) = 0$. But the conditions $t \geqslant 0$, $\tilde{t} \geqslant 0$, $s - t > 0$ imply that $t = 0$ and $\tilde{t}(s - t) = 0$; that is, $\tilde{t} = 0$. This proves that $\tilde{P} = P$. The proof that $Q = \tilde{Q}$ is similar. □

There are two isometries that leave $\{P, Q\}$ pointwise fixed: the reflection Ω_ℓ with axis $\ell = \overleftrightarrow{PQ}$, and the identity. On the other hand, there are exactly two isometries that interchange P and Q. Clearly, one is the reflection Ω_m whose axis m is the perpendicular bisector of PQ. But if T is any other isometry interchanging P and Q, the composition $\Omega_m T$ leaves P and Q fixed. This gives

$$\Omega_m T = I \quad \text{and} \quad T = \Omega_m,$$

or

$$\Omega_m T = \Omega_\ell \quad \text{and} \quad T = \Omega_\ell \Omega_m = H_M,$$

where M is the midpoint of PQ. Thus, $\mathscr{S}(PQ)$ consists of four elements: two reflections, a half-turn, and the identity. The multiplication table for this group is

	I	Ω_ℓ	Ω_m	H_M
I	I	Ω_ℓ	Ω_m	H_M
Ω_ℓ	Ω_ℓ	I	H_M	Ω_m
Ω_m	Ω_m	H_M	I	Ω_ℓ
H_M	H_M	Ω_m	Ω_ℓ	I

The abstract group having this multiplication table is called the Klein four-group.

We state the results we have outlined as a theorem. You will be asked to fill in the details of the proof in Exercise 23.

Theorem 28. *The symmetry group of a segment has four elements: two reflections, a half-turn, and the identity. The group multiplication table is as indicated beforehand.*

Symmetries of an angle

Let $\mathscr{A}$ be an angle other than a straight angle. In this section we compute the symmetry group $\mathscr{S}(\mathscr{A})$.

We first prove a uniqueness lemma.

Lemma. *Let $\mathscr{A}$ be an angle with vertex P. Suppose that an affine symmetry T of $\mathscr{A}$ leaves both lines of $\hat{\mathscr{A}}$ fixed. Then T leaves both arms of $\mathscr{A}$ fixed.*

Proof: Let ℓ_1 and ℓ_2 be the lines and $\imath_1$ and $\imath_2$ the associated rays. Let v and w be unit direction vectors of $\imath_1$ and $\imath_2$, respectively. Write $Tx = Ax + b$. Because $TP = P$, we get $AP + b = P$, so that we may write

$$Tx = A(x - P) + P.$$

Because ℓ_1 is a fixed line, $[Av] = [v]$ (Theorem 3). Thus, there is a real number λ such that $Av = \lambda v$. Now $T(P + v) = Av + P = P + \lambda v$. Because T maps $\mathscr{A}$ into $\mathscr{A}$, $P + \lambda v$ must be in $\mathscr{A}$. Hence, λ is positive, and $T\imath_1 = \imath_1$. Similarly, $T\imath_2 = \imath_2$. □

Corollary. *In the lemma if T is an isometry, then T is the identity.*

Proof: The string of equalities

$$1 = |v| = d(P + v, P) = d(T(P + v), TP) = d(P + \lambda v, P) = \lambda|v| = \lambda$$

yields $Av = v$. By symmetry, $Aw = w$, and, hence, A is the identity matrix. Finally, for all x,

$$Tx = A(x - P) + P = x - P + P = x,$$

so that T is the identity. □

Remark: If $\mathscr{A}$ is a straight angle, then (as a set of points) $\mathscr{A}$ is just a line. If $\mathscr{A}$ is a zero angle, then $\mathscr{A}$ is a ray.

Theorem 29. *Suppose that T is an affine symmetry of an angle $\mathscr{A}$. Then T permutes the arms of $\mathscr{A}$.*

Remark: In case $\mathscr{A}$ is a zero angle, we interpret this to mean that T leaves the one arm of $\mathscr{A}$ fixed.

Proof: Let $\imath_1$ and $\imath_2$ be the arms. Let ℓ_1 and ℓ_2 be the lines containing $\imath_1$ and $\imath_2$, respectively. Let m be the line that contains the bisector of $\mathscr{A}$. By the corollary to Theorem 23, T permutes the lines ℓ_1 and ℓ_2. If $T\ell_1 = \ell_1$ and $T\ell_2 = \ell_2$, then the lemma implies that T leaves fixed $\imath_1$ and $\imath_2$. If $T\ell_1 = \ell_2$ and $T\ell_2 = \ell_1$, then $\Omega_m T$ leaves fixed ℓ_1 and ℓ_2 and, hence, $\imath_1$ and $\imath_2$. But then

$$T\imath_1 = \Omega_m\Omega_m T\imath_1 = \Omega_m\imath_1 = \imath_2$$

and

$$T\imath_2 = \Omega_m\Omega_m T\imath_2 = \Omega_m\imath_2 = \imath_1.$$ □

Corollary. *$\mathscr{S}(\mathscr{A})$ consists of two elements Ω_m and I.*

Proof: If $T \in \mathscr{S}(\mathscr{A})$, then either T leaves $\imath_1$ and $\imath_2$ fixed and is therefore the identity, or $\Omega_m T$ leaves $\imath_1$ and $\imath_2$ fixed and is therefore the identity. In the latter case, $T = \Omega_m$. □

Remark: This also shows that if $\imath$ is any ray, then $\mathscr{S}(\imath)$ consists of two elements: reflection in the line of $\imath$, and the identity.

The following will be useful when we discuss triangles.

Theorem 30. *Let $\measuredangle PQR$ be an angle. Suppose $d(P, Q) = d(Q, R)$. Let m be the line containing the bisector of $\measuredangle PQR$. Then Ω_m interchanges P and R while leaving Q fixed.*

Proof: Write $P - Q = |P - Q|u$ and $R - Q = |R - Q|v$, and use the notation of Theorem 23. It is sufficient to check that $\Omega_m P = R$.

$$\Omega_m P = \tau_Q\left(\operatorname{ref}\left(\frac{\theta + \phi}{2}\right)\right)\tau_{-Q}P$$

$$= \tau_P\left(\text{ref}\left(\frac{\theta + \phi}{2}\right)\right)|P - Q|u$$

$$= Q + |P - Q|v = Q + |R - Q|v$$

$$= Q + R - Q = R. \qquad \square$$

Barycentric coordinates

Let P, Q, and R be noncollinear points. For each point $X \in \mathbf{E}^2$ there is a unique triple (λ, μ, ν) of real numbers such that

$$X = \lambda P + \mu Q + \nu R \tag{2.13}$$

and $\lambda + \mu + \nu = 1$. The association

$$X \rightarrow \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix}$$

is called a *barycentric* coordinate system, and PQR is called the triangle of reference. (See Exercise 25.)

Remark:

i. This generalizes the fact that points on a line $\overleftrightarrow{PQ}$ may be uniquely written as $\lambda P + \mu Q$, where $\lambda + \mu = 1$.
ii. We will see that the values of the barycentric coordinates λ, μ, and ν relate in a nice way to the position of X with respect to the triangle of reference.
iii. Barycentric coordinates have a physical interpretation. If weights of λ, μ, and ν are placed at P, Q, and R, respectively, the center of mass of the resulting configuration will be at X. This also applies, of course, to the case of a line, as described in (i).

Theorem 31. *Let $\overleftrightarrow{PQ}$ be a line, and let R be any point not on $\overleftrightarrow{PQ}$. Using PQR as a triangle of reference, we have, for any point X with barycentric coordinates λ, μ, ν,*

i. $\nu = 0$ *if and only if X lies on $\overleftrightarrow{PQ}$.*
ii. $\nu > 0$ *if and only if $XR \cap \overleftrightarrow{PQ} = \varnothing$.*

Proof: If $\nu = 0$, this means that

$$X = \lambda P + \mu Q = (1 - \mu)P + \mu Q, \tag{2.14}$$

and, hence, X lies on $\overleftrightarrow{PQ}$. Conversely, if X lies on $\overleftrightarrow{PQ}$, then (2.14) holds for some value of μ, and by uniqueness of the representation in (2.13), we must have $\nu = 0$.

If $\nu > 0$, then, for $0 \leqslant t \leqslant 1$,

$$(1 - t)X + tR = (1 - t)\lambda P + (1 - t)\mu Q + (1 - t)\nu R + tR.$$

Because $(1 - t)\nu + t > 0$, XR cannot intersect $\overleftrightarrow{PQ}$. On the other hand, if $\nu < 0$, there is a value of t satisfying $(1 - t)\nu + t = 0$; that is,

$$t = \frac{-\nu}{1 - \nu}.$$

Note that $0 < -\nu/(1 - \nu) < 1$. Thus, $\overleftrightarrow{PQ} \cap XR \neq \varnothing$. □

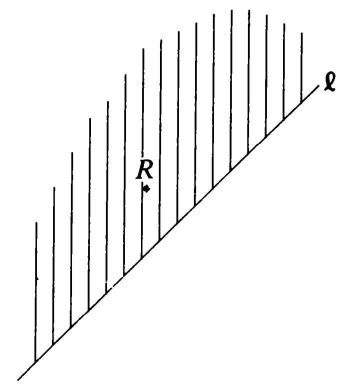

Figure 2.14 A half-plane consisting of all points X such that $XR \cap \ell = \varnothing$.

Definition. *Let ℓ be a line, and let R be a point not on ℓ. The* half-plane *determined by ℓ and R is the set of X such that $XR \cap \ell = \varnothing$. See Figure 2.14.*

Theorem 32.

i. *Every line ℓ determines two half-planes. The reflection Ω_ℓ interchanges the half-planes.*
ii. *Let ℓ be a line, and let P and Q be arbitrary points on ℓ. Let R be any point not on ℓ. Then the half-plane determined by ℓ and R is the set of points having $\nu > 0$. (Again the triangle of reference is PQR.) The set of points having $\nu < 0$ is the half-plane determined by ℓ and $\Omega_\ell R$. The two half-planes are said to be* opposites *of each other.*

Remark: When two points are in the same half-plane, we say that they are *on the same side* of ℓ. Points in opposite half-planes are said to be *on opposite sides* of ℓ.

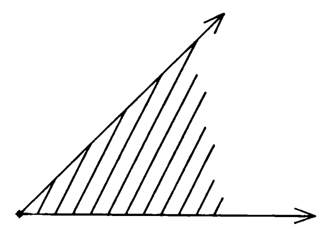

Figure 2.15 The interior of an angle.

Definition. *A point X is said to be in the* interior *of an angle $\measuredangle PQR$ if $\lambda > 0$ and $\nu > 0$. See Figure 2.15.*

Remark: This is the same as saying that X and R are on the same side of $\overleftrightarrow{PQ}$ while X and P are on the same side of $\overleftrightarrow{QR}$.

Theorem 33 (The crossbar theorem). *Let X be a point in the interior of the angle $\measuredangle PQR$. Then the ray $\overrightarrow{QX}$ intersects the segment PR. (See Figure 2.16.)*

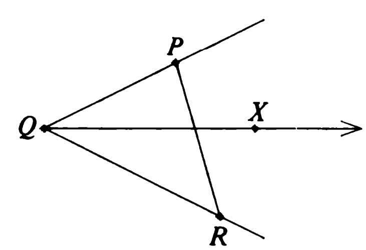

Figure 2.16 The crossbar theorem.

Proof: Using PQR as triangle of reference, we obtain

$$\begin{aligned} Q + t(X - Q) &= (1 - t)Q + t\lambda P + t\mu Q + t\nu R \\ &= t\lambda P + (1 - t + t\mu)Q + t\nu R. \end{aligned}$$

We need to choose a positive value of t so that $1 - t + t\mu = 0$; that is,

$$t = \frac{-1}{\mu - 1} = \frac{1}{1 - \mu}.$$

Because $1 - \mu = \lambda + \nu > 0$, this value of t is indeed positive. Furthermore, $\lambda t = \lambda/(1 - \mu) > 0$ and $\nu t = \nu/(1 - \mu) > 0$, so that $Q + t(X - Q)$ lies on PR. □

Addition of angles

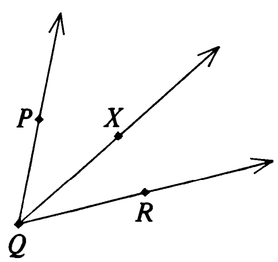

Figure 2.17 Addition of $\measuredangle PQX$ and $\measuredangle XQR$ to form $\measuredangle PQR$.

Theorem 34.

i. *Let $\measuredangle PQR$ be an angle and X a point in its interior. Then the radian measure of $\measuredangle PQR$ is the sum of the radian measures of $\measuredangle PQX$ and $\measuredangle RQX$. (See Figure 2.17.)*
ii. *Let $\measuredangle PQR$ be a straight angle and X any point not on the line $\overleftrightarrow{PQ}$. Then the sum of the radian measures of $\measuredangle PQX$ and $\measuredangle RQX$ is equal to π.*

Remark: In (ii), $\measuredangle PQX$ and $\measuredangle RQX$ are said to be *supplements* of each other. We speak of the pair as a pair of supplementary angles.

Proof: (i) Let θ, θ_1, and θ_2 be the respective radian measures. There is no loss of generality in assuming that $u = P - Q$, $v = R - Q$, and $w = X - Q$ are unit vectors and that $(\text{rot } \theta)u = v$. Because X is in the interior of $\measuredangle PQR$, we may write

$$X - Q = \lambda(P - Q) + \mu(R - Q);$$

that is,

$$w = \lambda u + \mu v,$$

where λ and μ are positive. According to the definition of radian measure, there are four possibilities:

1. $(\text{rot } \theta_1)u = w$ and $(\text{rot } \theta_2)w = v$. Then $(\text{rot}(\theta_1 + \theta_2))u = v$ and $\theta_1 + \theta_2 \equiv \theta \pmod{2\pi}$. But $0 < \theta_1 + \theta_2 < 2\pi$, so that, in fact, $\theta_1 + \theta_2 = \theta$, as required.

The other three possibilities cannot occur. We examine them in turn.

2. $(\text{rot } \theta_1)u = w$ and $(\text{rot } \theta_2)v = w$. Then

$$0 < \sin \theta_1 = \langle u^\perp, w\rangle = \mu\langle u^\perp, v\rangle$$

and

$$0 < \sin \theta_2 = \langle v^\perp, w\rangle = \lambda\langle v^\perp, u\rangle.$$

But $\langle u^\perp, v\rangle = \langle u^{\perp\perp}, v^\perp\rangle = -\langle u, v^\perp\rangle$, so we have a contradiction.

3. $(\text{rot } \theta_1)w = u$ and $(\text{rot } \theta_2)w = v$. This is similar to case (2). We get the same expressions for the negative numbers $\sin(-\theta_1)$ and $\sin(-\theta_2)$ and, thus, a contradiction.

4. (rot θ_1)$w = u$ and (rot θ_2)$v = w$. In this case,

$$0 < \sin\theta_2 = \lambda\langle u, v^\perp\rangle \quad \text{as in (2).}$$

But

$$0 < \sin\theta = \langle u^\perp, v\rangle = -\langle u, v^\perp\rangle,$$

a contradiction.

For part (ii) with $\measuredangle PQR$ a straight angle, we have no expression for w in terms of u and v. But $v = -u$ and $v^\perp = -u^\perp$. This makes the proof easier. For (1), $\theta = \pi$ and $\theta_1 + \theta_2 = \pi$ by the same argument. For (2),

$$0 < \sin\theta_1 = \langle u^\perp, w\rangle = -\langle v^\perp, w\rangle = -\sin\theta_2 < 0,$$

a contradiction. Case (3) is similar. Case (4) can occur and gives

$$\theta_1 + \theta_2 = \pi$$

as in (1). □

Triangles

Let P, Q, and R be noncollinear points of $\mathbf{E}^2$. The *triangle PQR* (sometimes written $\triangle PQR$) is the rectilinear figure consisting of the segments PQ, QR, and PR. These segments are called the *sides* of the triangle.

Theorem 35. *Let PQR be a triangle. Using PQR as the triangle of reference for a barycentric coordinate system, we have that*

i. *A point $X \in \mathbf{E}^2$ is a vertex of $\triangle PQR$ if and only if exactly two barycentric coordinates are zero.*
ii. *A point X is on the figure $\triangle PQR$ if it is a vertex or if one barycentric coordinate is zero and the others are positive.*

Definition. *A point is in the* interior *of $\triangle PQR$ if it is in the interior of all three angles determined by P, Q, and R.*

Remark: Points in the interior of the triangle are characterized by having all three barycentric coordinates positive. Figure 2.18 shows the whole plane divided into seven regions characterized by the signs of the barycentric coordinates λ, μ, ν. For example, the interior of the triangle is characterized by the combination $+++$.

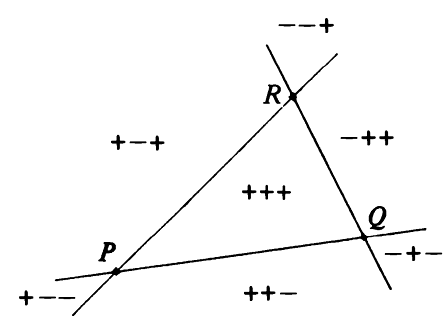

Figure 2.18 Regions of the plane as characterized by the signs of the barycentric coordinates.

Theorem 36. *An affine transformation T takes a triangle $\triangle PQR$ to the triangle $\triangle P'Q'R'$, where $P' = TP$, $Q' = TQ$, and $R' = TR$.*

Proof: $\triangle PQR$ is the union of three segments belonging to three distinct lines. According to Theorem 6, these segments are transformed by T to the respective segments making up $\triangle P'Q'R'$. The fact that P', Q', and R' are noncollinear and thus form a triangle relies on knowing that T^{-1} is an affine transformation and thus preserves collinearity (Theorems 1 and 6). □

Symmetries of a triangle

Let Δ be a triangle. If T is an affine symmetry of Δ, then T permutes the vertices of Δ. Conversely, by the fundamental theorem, every permutation of the vertices is realized by a unique affine transformation. Thus $\mathscr{AS}(\Delta)$ is the group of six elements known as the symmetric group $\mathbf{S}_3$. Algebraically, we may describe the group as $\{I, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$, where $\beta\alpha = \alpha^2\beta$ and $\alpha^3 = I$. In terms of permutations we can set

$$\alpha = (PQR), \quad \beta = (PQ).$$

Clearly, β is the affine reflection $[R; P \leftrightarrow Q]$. Note also that the product of the two affine reflections

$$[R; P \leftrightarrow Q], \quad [P; Q \leftrightarrow R]$$

corresponds to the permutation $(PQR) = \alpha$. Here is the multiplication table for the group $\mathbf{S}_3$:

	I	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
I	I	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	I	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	I	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	I	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	I	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	I

We now investigate $\mathscr{S}(\Delta)$. Clearly, $\mathscr{S}(\Delta)$ is a subgroup of $\mathscr{AS}(\Delta)$.

Definition. *A triangle is*
i. scalene *if all three sides have different lengths;*
ii. isosceles *if exactly two sides have equal lengths;*
iii. equilateral *if all three sides have the same length.*

Theorem 37. $\mathscr{S}(\Delta)$ *consists of*
i. *The identity only if Δ is scalene.*

ii. *$\{I, \Omega\}$ if Δ is isosceles with $d(P, Q) = d(P, R)$. Ω is the affine reflection $[P; Q \leftrightarrow R]$. Of course, Ω is an actual reflection (isometry) in this case.*
iii. *All elements of $\mathscr{A}\mathscr{S}(\Delta)$ if Δ is equilateral. In this case, two elements are nontrivial rotations about the centroid, three are reflections (one in each median), and the sixth is the identity. (A* median *is a line that passes through a vertex and the midpoint of the opposite side.)*

Proof: Because we already know the affine symmetries, it is only necessary to check which of these are isometries. If $T = [P; Q \leftrightarrow R]$ is an isometry, we must have $d(P, Q) = d(TP, TQ) = d(P, R)$, so that at least two sides of Δ must be of equal length. The same holds for the other two affine reflections.

If T is an isometry that permutes the vertices cyclically, then

$$\begin{aligned} d(P, Q) &= d(TP, TQ) = d(Q, R), \quad \text{say}, \\ &= d(TQ, TR) = d(R, P), \end{aligned}$$

so that Δ must be equilateral.

Thus, when Δ is scalene, only the identity can be an isometry. If Δ is isosceles, then $[P; Q \leftrightarrow R]$ is an isometry (Theorem 11 and Exercise 8). Finally, if Δ is equilateral, all three affine reflections are isometries. The cyclic permutations, being products of reflections, are ordinary rotations. □

Corollary. *Let Δ be an equilateral triangle with centroid C. Then $\mathscr{S}(\Delta)$ consists of*

i. *the identity,*
ii. *the three reflections in the medians of Δ,*
iii. *rotations by $\pm 2\pi/3$ about C.*

Proof: First note that

$$C = \tfrac{1}{3}(P + Q + R) = \tfrac{1}{3}P + \tfrac{2}{3}(\tfrac{1}{2}Q + \tfrac{1}{2}R). \tag{2.15}$$

Similarly,

$$C = \tfrac{1}{3}Q + \tfrac{2}{3}(\tfrac{1}{2}P + \tfrac{1}{2}R)$$

and

$$C = \tfrac{1}{3}R + \tfrac{2}{3}(\tfrac{1}{2}P + \tfrac{1}{2}Q).$$

This exhibits C as a point on all three medians. Thus, the product of two reflections in $\mathscr{S}(\Delta)$ is a rotation about C. In fact, if we write

$$T = \tau_M(\text{rot } \theta)\tau_{-M},$$

then

$$T^2 = \tau_M(\text{rot } 2\theta)\tau_{-M} \quad \textit{and} \quad T^3 = \tau_M(\text{rot } 3\theta)\tau_{-M}.$$

Now $T^3 = I$ if and only if rot θ = rot $(\pm 2\pi/3)$. □

We have also deduced the following well-known property of the centroid.

Corollary. *For any triangle the centroid lies on each median and divides it in the ratio of 2:1.*

Congruence of angles

Theorem 38. *Two angles are congruent if and only if they have the same radian measure.*

Proof: Let $\mathscr{A}$ and $\mathscr{B}$ be congruent angles, and let T be an isometry such that $T\mathscr{A} = \mathscr{B}$. By Theorem 24, T maps the two lines of $\hat{\mathscr{A}}$ to the two lines of $\hat{\mathscr{B}}$ and, hence, the vertex of $\mathscr{A}$ to the vertex of $\mathscr{B}$. Let u and v be unit direction vectors for the arms of $\mathscr{A}$. If A is the linear part of T, the arms of $\mathscr{B}$ must have Au and Av as direction vectors. Because A is orthogonal, that is, $\langle Au, Av\rangle = \langle u, v\rangle$, the two angles have the same radian measure.

Conversely, suppose that angles $\mathscr{A}$ and $\mathscr{B}$ have the same radian measure. We may assume that $(\text{rot }\theta)u = v$ and $(\text{rot }\theta)u' = v'$, where the arms of $\mathscr{A}$ (respectively, $\mathscr{B}$) have unit direction vectors u, v (respectively, u', v'). Let ϕ be a number such that

$$u' = (\cos\phi)u + (\sin\phi)u^{\perp} = (\text{rot }\phi)u.$$

Then

$$\begin{aligned} v' &= (\text{rot }\theta)u' = (\text{rot}(\theta+\phi))u \\ &= (\text{rot }\phi)(\text{rot }\theta)u = (\text{rot }\phi)v. \end{aligned}$$

Let P and Q be the respective vertices of $\mathscr{A}$ and $\mathscr{B}$. Then for $t \geqslant 0$,

$$\begin{aligned} \tau_Q(\text{rot }\phi)\tau_{-P}(P + tu) &= \tau_Q(\text{rot }\phi)tu \\ &= \tau_Q(tu') = \text{Q} + (tu') \end{aligned}$$

Similarly,

$$\tau_Q(\text{rot }\phi)\tau_{-P}(P + tv) = Q + tv'.$$

Thus, $\mathscr{A}$ and $\mathscr{B}$ are congruent. □

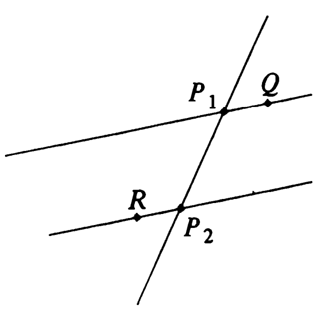

Figure 2.19 A transversal determines two pairs of alternate angles.

A line that intersects two lines in distinct points is called a *transversal* to these lines. Let ℓ_1 and ℓ_2 be parallel lines with direction vector v. Suppose that m is a transversal meeting ℓ_1 and ℓ_2 in P_1 and P_2, respectively. Let $Q = P_1 + v$ and $R = P_2 - v$. Then $\measuredangle P_2P_1Q$ and $\measuredangle RP_2P_1$ are called *alternate* angles. Note that a transversal determines two pairs of alternate angles. (See Figure 2.19.)

Theorem 39. *When a transversal meets two parallel lines, the pairs of alternate angles they determine are congruent.*

Proof: Use the notation introduced in the paragraph preceding Theorem 39. Note that $H_{P_2}\tau_{P_2-P_1}$ takes $\overrightarrow{P_1Q}$ to $\overrightarrow{P_2R}$ and $\overrightarrow{P_1P_2}$ to $\overrightarrow{P_2P_1}$. □

Remark: The isometry $H_{P_2}\tau_{P_2-P_1}$ is in fact a half-turn about the midpoint of P_1P_2.

Congruence theorems for triangles

We now prove the well-known congruence theorems of Euclidean geometry. The first one (sometimes referred to as the SSS theorem) says that two triangles whose vertices can be matched in such a way that corresponding sides have equal length must be congruent.

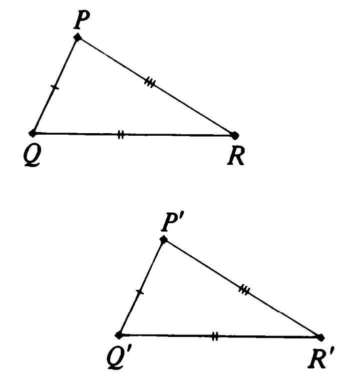

Figure 2.20 The SSS theorem.

Theorem 40. *Let $\triangle PQR$ and $\triangle P'Q'R'$ be such that $d(P, Q) = d(P', Q')$, $d(P, R) = d(P', R')$, and $d(Q, R) = d(Q', R')$. See Figure 2.20. Then there is an isometry T such that $TP = P'$, $TQ = Q'$, and $TR = R'$.*

Proof: Our method of proof is like that of Euclid, making use of isometries to carry out the "superposition" on which the ancient proof relies. The proof is divided into four steps.

1. Given two lines ℓ_1, ℓ_2, there is an isometry T such that $T\ell_1 = \ell_2$ (i.e., $\mathscr{I}(\mathbf{E}^2)$ is transitive on the lines of $\mathbf{E}^2$). To see this, note that if $\ell_1 \| \ell_2$, reflection in the line lying halfway between them will interchange ℓ_1 and ℓ_2. On the other hand, if ℓ_1 intersects ℓ_2, a reflection in any of the bisectors of the angles they form will interchange ℓ_1 and ℓ_2.
2. If PQ and $P'Q'$ are collinear segments of equal length, there is an isometry T such that $TP = P'$ and $TQ = Q'$. This can be done by $\tau_{P'-P}$ or $H_{P'}\tau_{P'-P}$ (Exercise 32).
3. Suppose that $d(P, R) = d(P, R')$ and $d(Q, R) = d(Q, R')$. Then there is an isometry leaving $\ell = \overleftrightarrow{PQ}$ pointwise fixed and taking R to R'. In fact, I or Ω_ℓ will do (Exercise 33).
4. Choose an isometry T_1 taking $\overleftrightarrow{PQ}$ to $\overleftrightarrow{P'Q'}$. Then choose T_2 to take T_1P to P' and T_1Q to Q'. Let T_3 map T_2T_1R to R' while leaving $\overleftrightarrow{P'Q'}$ pointwise fixed. Then $T = T_3T_2T_1$ accomplishes the desired effect. □

Another famous assertion of Euclid proves congruence based on lengths of two sides and the angle they determine (the SAS theorem). In order to prove this, we first derive the so-called Law of Cosines.

Lemma. Let P, Q, and R be three points of $\mathbf{E}^2$. Then

$$d(P, R)^2 = d(P, Q)^2 + d(Q, R)^2 - 2d(P, Q)\,d(Q, R)\cos\theta, \quad (2.16)$$

where θ is the radian measure of $\measuredangle PQR$.

Proof: Apply the polarization identity (Exercise 1.29) with $x = P - Q$ and $y = R - Q$, so that $x - y = P - R$. Also, recall that

$$\cos\theta = \frac{\langle P - Q, R - Q\rangle}{|P - Q||R - Q|}, \quad (2.17)$$

from (2.11). □

Theorem 41. *Let $\triangle PQR$ and $\triangle P'Q'R'$ be such that $d(P, Q) = d(P', Q')$, $d(Q, R) = d(Q', R')$, and $\measuredangle PQR = \measuredangle P'Q'R'$ (in radian measure). Then there is an isometry T such that $TP = P'$, $TQ = Q'$, and $TR = R'$.*

Proof: Apply the Law of Cosines to both triangles. The given conditions say that the right sides of (2.16) are equal. Hence, $d(P, R) = d(P', R')$, and the SSS theorem may be applied. □

Corollary. *The base angles of an isosceles triangle are congruent.*

Proof: Apply the SAS theorem to $\triangle PQR$ and $\triangle RQP$, where $d(P, Q) = d(Q, R)$. □

Angle sums for triangles

The major result of this section is the following:

Theorem 42. *The sum of the radian measures of the three angles in any triangle is equal to π.*

Proof: Let PQR be a triangle. Then the unique line through P parallel to $\overleftrightarrow{QR}$ is the union of the rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$, where $A = P + R - Q$ and $B = P + Q - R$. Note that Q is in the interior of $\measuredangle BPR$ because $Q = B + R - P$. By Theorem 34 the radian measure of $\measuredangle BPR$ is equal to the sum of the radian measures of $\measuredangle BPQ$ and $\measuredangle RPQ$, whereas $\measuredangle BPR$ and $\measuredangle APR$ are supplementary. Finally, we apply Theorems 38 and 39 to the alternate angle pairs $\measuredangle BPQ \simeq \measuredangle PQR$ and $\measuredangle APR \simeq \measuredangle PRQ$ to complete the proof. These constructions are illustrated in Figure 2.21. □

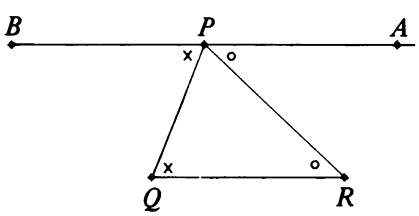

Figure 2.21 Theorem 42. The angle sum of a triangle is π.

Corollary. *If two angles of a triangle are respectively congruent to two angles of another triangle, then the remaining angles are also congruent.*

Remark: When we study non-Euclidean plane geometry, we will discover that Theorem 42 is one of the few results that is false in non-Euclidean planes. In fact, from an axiomatic approach, this criterion can be used to distinguish Euclidean from non-Euclidean planes.

EXERCISES

1. Find the fixed points and fixed lines of the indicated affine transformations.

   i. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   ii. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

   iii. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

2. Prove Theorem 4.
3. Let T be a glide reflection with axis m. Show that a line $\ell \neq m$ satisfies $T\ell \| \ell$ if and only if $\ell \| m$ or $\ell \perp m$.
4. Let ℓ be a line of $\mathbf{E}^2$. Let G be the set of isometries T of $\mathbf{E}^2$ satisfying $T\ell = \ell$. Describe the elements of G explicitly and give the group multiplication table.
5. i. Given two intersecting lines, prove that there is a rotation that takes one to the other. Is it unique?

   ii. Given two parallel lines, prove that there is a translation that takes one to the other. Is it unique?
6. Verify that the mapping described following Theorem 5, which associates to each affine transformation a 3 by 3 matrix, is an injective homomorphism of **AF(2)** into **GL(3)**.
7. Prove Theorem 7(ii) by a direct computation using linear algebra.
8. Verify formula (2.4), which shows that every ordinary reflection is an affine reflection.
9. Verify the statements in the remark following Theorem 13.
10. If every nonzero vector in $\mathbf{R}^2$ is an eigenvector of a 2 by 2 matrix A, show that A is a multiple of the identity matrix.
11. Prove Theorem 18.
12. Complete the proof of Theorem 19.
13. Work out the multiplication table for the group of transformations of Theorem 20 and its subgroups.
14. Verify the remark following the definition of radian measure.

15. Prove Theorem 22.
16. Let $P = (1, 2)$, $Q = (0, 0)$, and $R = (2, 1)$. Show that the radian measure of $\measuredangle PQR$ is $\cos^{-1}(4/5)$.
17. Find the bisectors of a straight angle in terms of its vertex and the unit direction vectors of its arms.
18. What is the bisector of a zero angle?
19. If $\overrightarrow{QX}$ is a bisector of an angle $\measuredangle PQR$, prove that $\measuredangle PQX$ and $\measuredangle RQX$ have the same radian measure, namely, half the measure of $\measuredangle PQR$.
20. Prove that the ray $\overrightarrow{QX}$ of Exercise 19 is the only ray having the property described there, unless $\measuredangle PQR$ is a straight angle.
21. Prove that a rotation about P that leaves a ray with origin P fixed must be the identity.
22. Compute the centroid of the following set of points: $\{(1, 4), (2, 4), (1, 0), (2, 0), (9, 7)\}$.
23. Prove Theorem 28. Fill in the details omitted in the text.
24. Prove that there is a reflection interchanging any two lines. Is it unique?
25. Prove that barycentric coordinates are well-defined.
26. With respect to the triangle of reference $P = (-1, 0)$, $Q = (1, 0)$, $R = (0, 1)$, find the barycentric coordinates of the points: $(0, 0)$, $(1, 1)$, $(\sqrt{2}, \sqrt{2})$, $(0, 5)$, $(2, -1)$, $(-\frac{1}{2}, -\frac{1}{3})$.
27. Let a, b, and c be three numbers, not all zero. Show that the set of all points whose barycentric coordinates satisfy $a\lambda + b\mu + c\nu = 0$ is a line.
28. Let P be a point and N a unit vector. Show that $\{X \mid \langle X - P, N\rangle \geqslant 0\}$ is a half-plane.
29. Prove Theorem 32.
30. Prove that every point in the interior of an angle lies on a segment joining points of the arms of the angle.
31. Prove that every point in the interior of a triangle lies on a segment joining points on two sides of the triangle.
32. If PQ and $P'Q'$ are collinear segments of equal length, prove that either $\tau_{P'-P}$ or $H_{P'}\tau_{P'-P}$ takes PQ to $P'Q'$.
33. Let P, Q, R, and R' be four distinct points such that $d(P, R) = d(P, R')$ and $d(Q, R) = d(Q, R')$. Prove that $R' = \Omega_\ell R$, where $\ell = \overleftrightarrow{PQ}$.
34. Let T be an affine transformation and ℓ a line. Prove that the points $M = \frac{1}{2}(P + TP)$ (as P ranges through ℓ) are all distinct and collinear, or that they all coincide. Express the locus of M (i.e., the line or

point) in terms of the data determining T and ℓ. This result is called Hjelmslev's theorem, although most treatments assume that T is an isometry.

35. If $[C_1;\ P_1 \to Q_1] = [C_2;\ P_2 \to Q_2]$, what relationships must hold among the points in question?

36. Show that Theorem 27 and its corollary hold true for affine transformations and affine symmetries.

37. Suppose that an affine transformation has three concurrent fixed lines. Prove that it is a central dilatation or the identity.

38. Extending the notation of Theorem 13, let

$$s_{\lambda,v} = \tau_v s_\lambda, \quad \text{for} \quad \lambda \in \mathbf{R} \text{ and } \quad v \in \mathbf{R}^2.$$

   i. Verify the identity

$$s_{\lambda,u} s_{\mu,v} = s_{\lambda+\mu,w},$$

   where $w = s_{\lambda,u} v$. Thus show that the set of all $s_{\lambda,v}$ is a group.

   ii. Show that $s_{\lambda,v}$ is a shear with horizontal axis if and only if $[v] = [\varepsilon_1]$.

   iii. Show that every shear with horizontal axis may be written in the form $s_{\lambda,v}$.

   iv. Show that $\{s_{\lambda,v} | \langle v, \varepsilon_2 \rangle = 0\}$ is a group (the case $\lambda = 0$ is included here).

Remark: The group defined in Exercise 38(i) is called the *Galilean group* **GAL(2)**. It arises in classical kinematics when describing uniform motion in a straight line. The subset of affine geometry consisting of those facts of Euclidean geometry that continue to make sense when the figure in question is subjected to transformations by the Galilean group is called Galilean geometry and is the subject of an interesting book by I. M. Yaglom [34].

39. Let $\mathscr{s}_1$ and $\mathscr{s}_2$ be parallel segments. Find a central dilatation taking $\mathscr{s}_1$ to $\mathscr{s}_2$

   i. by a geometric construction,

   ii. in terms of the end points of the given segments.

   Can there be more than one such dilatation?

40. Prove that any affine transformation that preserves perpendicularity must be a similarity.

41. Prove Pasch's theorem: If a line intersects one side of a triangle, it must also intersect one of the other sides.

42. Let PQR be a triangle. Let F be the foot of the perpendicular from P

to $\overleftrightarrow{QR}$. Prove that F is between Q and R if and only if $\measuredangle PQR$ and $\measuredangle PRQ$ are acute angles.

43. Let P and Q be distinct points. Show that $\overrightarrow{PQ} \cap \overrightarrow{QP} = PQ$.

44. Let T be an affine transformation taking $\triangle PQR$ to $\triangle P'Q'R'$ as in Theorem 36. Prove the following facts:
    i. For any $X \in \mathbf{E}^2$ the barycentric coordinates of TX with reference to $\triangle P'Q'R'$ are the same as those of X with reference to $\triangle PQR$.
    ii. If

$$\begin{aligned} P' &= a_{11}P + a_{21}Q + a_{31}R, \\ Q' &= a_{12}P + a_{22}Q + a_{32}R, \\ R' &= a_{13}P + a_{23}Q + a_{33}R, \end{aligned}$$

then the barycentric coordinates of $X' = TX$ (with reference to $\triangle PQR$) are related to those of X by the equation

$$\begin{bmatrix} \lambda' \\ \mu' \\ \nu' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix}.$$

45. Prove Heron's theorem: Let ℓ be a line, and let A and B be points not on ℓ. Among all points X on ℓ, the quantity $d(A, X) + d(X, B)$ is minimum when X is on the segment AB or X is on the segment AB', where $B' = \Omega_\ell B$.