EXERCISES

1. If two maps of the same country on different scales are drawn on tracing paper and superposed, there is just one place that is represented by the same spot on both maps. (It is understood that one of the maps may be turned over before it is superposed on the other.) [Lachlan **1**, pp. 137, 139.]

2. When all the points P on AB are related by a similarity to all the points P' on $A'B'$, the points dividing the segments PP' in the ratio $AB : A'B'$ (internally or externally) are distinct and collinear or else they all coincide.

3. If S is an opposite similarity, S^2 is a dilatation.

4. What is the product (a) of two dilative reflections? (b) of a dilative rotation and a dilative reflection?

5. Let AB and $A'B'$ be two given segments of different lengths. Let A_1 and A_2 divide AA' internally and externally in the ratio $AB{:}A'B'$ (as in Figure 5.6*a*). Let B_1 and B_2 divide BB' in the same manner. Then the lines A_1B_1 and A_2B_2 are at right angles, and are the axes of the dilative reflection that transforms AB into $A'B'$. [Lachlan **1**, p. 134; Johnson **1**, p. 27.] (It has been tacitly assumed that $A_1 \neq B_1$ and $A_2 \neq B_2$. However, if A_2 and B_2 coincide, the axes are A_1B_1 and the perpendicular line through A_2.)

6. Describe the transformation

$$(r, \theta) \to (\mu r, \theta + \alpha)$$

of polar coordinates, and the transformation

$$(x, y) \to (\mu x, -\mu y)$$

of Cartesian coordinates.

6
Circles and spheres

The present chapter shows how Euclidean geometry, in which lines and planes play a fundamental role, can be extended to *inversive* geometry, in which this role is taken over by circles and spheres. We shall see how the obvious statement, that lines and planes are circles and spheres of infinite radius, can be replaced by the sophisticated statement that lines and planes are those circles and spheres which pass through an "ideal" point, called "the point at infinity." In § 6.9 we shall briefly discuss a still more unusual geometry, called *elliptic*, which is one of the celebrated "non-Euclidean" geometries.

6.1 INVERSION IN A CIRCLE

Can it be that all the great scientists of the past were really playing a game, a game in which the rules are written not by man but by God? . . . When we play, we do not ask why we are playing—we just play. Play serves no moral code except that strange code which, for some unknown reason, imposes itself on the play. . . . You will search in vain through scientific literature for hints of motivation. And as for the strange moral code observed by scientists, what could be stranger than an abstract regard for truth in a world which is full of concealment, deception, and taboos? . . . In submitting to your consideration the idea that the human mind is at its best when playing, I am myself playing, and that makes me feel that what I am saying may have in it an element of truth.

J. L. Synge (1897 -)*

All the transformations so far discussed have been similarities, which transform straight lines into straight lines and angles into equal angles. The transformation called *inversion,* which was invented by L. J. Magnus in 1831, is new in one respect but familiar in another: it transforms some

* *Hermathena,* **19** (1958), p. 40; quoted with the editor's permission.

straight lines into circles, but it still transforms angles into equal angles. Like the reflection and the half-turn, it is involutory (that is, of period 2). Like the reflection, it has infinitely many invariant points; these do not lie on a straight line but on a circle, and the center of the circle is "singular:" it has no image!

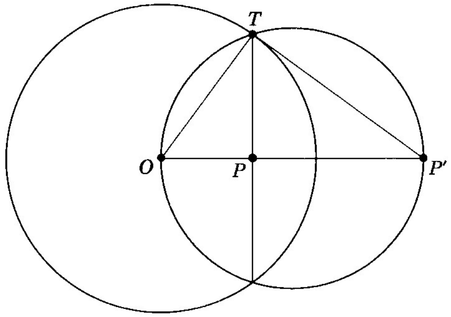

Figure 6.1a

Given a fixed circle with center O and radius k, we define the *inverse* of any point P (distinct from O) to be the point P' on the ray OP whose distance from O satisfies the equation

$$OP \times OP' = k^2.$$

It follows from this definition that the inverse of P' is P itself. Moreover, every point outside the circle of inversion is transformed into a point inside, and every point inside (except the center O) into a point outside. The circle is invariant in the strict sense that every point on it is invariant. Every line through O is invariant as a whole, but not point by point.

To construct the inverse of a given point P (other than O) inside the circle of inversion, let T be one end of the chord through P perpendicular to OP, as in Figure 6.1*a*. Then the tangent at T meets OP (extended) in the desired point P'. For, since the right-angled triangles OPT, OTP' are similar, and $OT = k$,

$$\frac{OP}{k} = \frac{k}{OP'}.$$

To construct the inverse of a given point P' outside the circle of inversion, let T be one of the points of intersection of this circle with the circle on OP' as diameter (Figure 6.1*a*). Then the desired point P is the foot of the perpendicular from T to OP'.

If $OP > \frac{1}{2}k$, the inverse of P can easily be constructed by the use of compasses alone, without a ruler. To do so, let the circle through O with center P cut the circle of inversion in Q and Q'. Then P' is the second intersection of the circles through O with centers Q and Q'. (This is easily seen by considering the similar isosceles triangles POQ, QOP'.)

There is an interesting connection between inversion and dilatation:

6.11 *The product of inversions in two concentric circles with radii k and k' is the dilatation $O(\mu)$ where $\mu = (k'/k)^2$.*

To prove this, we observe that this product transforms P into P'' (on OP) where

$$OP \times OP' = k^2, \qquad OP' \times OP'' = k'^2$$

and therefore

$$\frac{OP''}{OP} = \left(\frac{k'}{k}\right)^2.$$

EXERCISES

1. Using compasses alone, construct the vertices of a regular hexagon.
2. Using compasses alone, locate a point B so that the segment OB is twice as long as a given segment OA.
3. Using compasses alone, construct the inverse of a point distant $\frac{1}{3}k$ from the center O of the circle of inversion. Describe a procedure for inverting points arbitrarily near to O.
4. Using compasses alone, bisect a given segment.
5. Using compasses alone, trisect a given segment. Describe a procedure for dividing a segment into any given number of equal parts.

Note. The above problems belong to the Geometry of Compasses, which was developed independently by G. Mohr in Denmark (1672) and L. Mascheroni in Italy (1797). For a concise version of the whole story, see Pedoe [**1**, pp. 23–25] or Courant and Robbins [**1**, pp. 145–151].

6.2 ORTHOGONAL CIRCLES

A circle is a happy thing to be—
Think how the joyful perpendicular
Erected at the kiss of tangency
Must meet my central point, my avatar.
And lovely as I am, yet only 3
Points are needed to determine me.

Christopher Marley (1890-)

Two circles are said to be *orthogonal* if they cut at right angles, that is, if they intersect in two points at either of which the radius of each is a tangent to the other (Figure 6.2*a*).

By Euclid III.36 (see p. 8) any circle through a pair of inverse points is invariant: the circle of inversion decomposes it into two arcs which invert into each other. Moreover, such a circle is orthogonal to the circle of inversion, and every circle orthogonal to the circle of inversion is invariant in this sense. Through a pair of inverse points we can draw a whole *pencil* of circles (infinitely many), and they are all orthogonal to the circle of inversion. Hence

6.21 *The inverse of a given point P is the second intersection of any two circles through P orthogonal to the circle of inversion.*

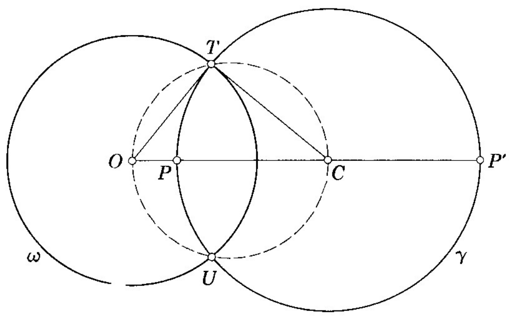

Figure 6.2a

The above remarks provide a simple solution for the problem of drawing, through a given point P, a circle (or line) orthogonal to two given circles. Let P_1, P_2 be the inverses of P in the two circles. Then the circle PP_1P_2 (or the line through these three points, if they happen to be collinear) is orthogonal to the two given circles.

If O and C are the centers of two orthogonal circles ω and γ, as in Figure 6.2*a*, the circle on OC as diameter passes through the points of intersection T, U. Every other point on this circle is inside one of the two orthogonal circles and outside the other. It follows that, if a and b are two perpendicular lines through O and C respectively, either a touches γ and b touches ω, or a cuts γ and b lies outside ω, or a lies outside γ and b cuts ω.

6.3 INVERSION OF LINES AND CIRCLES

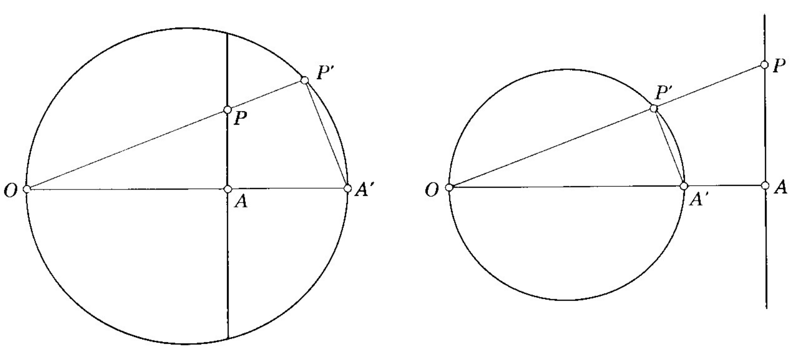

Figure 6.3a

We have seen that lines through O invert into themselves. What about other lines? Let A be the foot of the perpendicular from O to a line not through O. Let A' be the inverse of A, and P' the inverse of any other point P on the line. (See Figure 6.3*a* where, for simplicity, the circle of inversion has not been drawn.) Since

$$OP \times OP' = k^2 = OA \times OA',$$

the triangles OAP, $OP'A'$ are similar, and the line AP inverts into the circle on OA' as diameter, which is the locus of points P' from which OA' subtends a right angle. Thus any line not through O inverts into a circle through O, and vice versa.

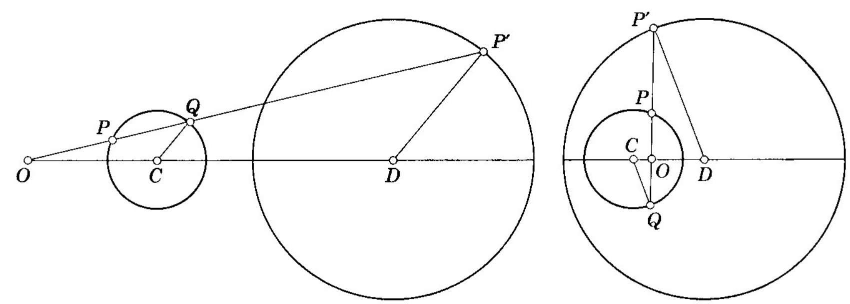

Figure 6.3b

Finally, what about a circle not through O? Let P be any point on such a circle, with center C, and let OP meet the circle again in Q. By Euclid III.35 again, the product

$$p = OP \times OQ$$

is independent of the position of P on the circle. Following Jacob Steiner (1796–1863), we call this product the *power* of O with respect to the circle. It is positive when O is outside the circle, zero when O lies on the circle, and we naturally regard it as being negative when O is inside (so that OP and OQ are measured in opposite directions). Let the dilatation $O(k^2/p)$ transform the given circle and its radius CQ into another circle (or possibly the same) and its parallel radius DP' (Figure 6.3*b*, cf. Figure 5.2*a*), so that

$$\frac{OP'}{OQ} = \frac{OD}{OC} = \frac{k^2}{p}.$$

Since $OP \times OQ = p$, we have, by multiplication,

$$OP \times OP' = k^2.$$

Thus P' is the inverse of P, and the circle with center D is the desired inverse of the given circle with center C. (The point D is usually *not* the inverse of C.)

We have thus proved that the inverse of a circle not through O is another circle of the same kind, or possibly the same circle again. The latter possibility occurs in just two cases: (1) when the given circle is orthogonal to the circle of inversion, so that $p = k^2$ and the dilatation is the identity; (2) when the given circle is the circle of inversion itself, so that $p = -k^2$ and the dilatation is a half-turn.

When p is positive (see the left half of Figure 6.3*b*), so that O is outside the circle with center C, this circle is orthogonal to the circle with center O and radius $\sqrt{p}$; that is, the former circle is invariant under inversion with respect to the latter. In effect, we have expressed the given inversion as the product of this new inversion, which takes P to Q, and the dilatation $O(k^2/p)$, which takes Q to P'. When p is negative (as in the right half of Figure 6.3*b*), P and Q are interchanged by an "anti-inversion:" the product of an inversion with radius $\sqrt{-p}$ and a half-turn [Forder **3**, p. 20].

When discussing isometries and other similarities, we distinguished between *direct* and *opposite* transformations by observing their effect on a triangle. Since we are concerned only with *sense,* the triangle could have been replaced by its circumcircle. Such a distinction can still be made for inversions (and products of inversions), which transform circles into circles. Instead of a triangle we use a circle: not an arbitrary circle but a "small" circle whose inverse is also "small," that is, a circle not surrounding O. Referring again to the left half of Figure 6.3*b*, we observe that P and Q describe the circle with center C in opposite senses, whereas Q and P' describe the two circles in the same sense. Thus the inverse points P and P' proceed oppositely, and

Inversion is an opposite transformation.

It follows that the product of an even number of inversions is direct. One instance is familiar: the product of inversions with respect to two concentric circles is a dilatation.

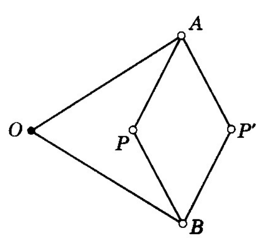

Figure 6.3c

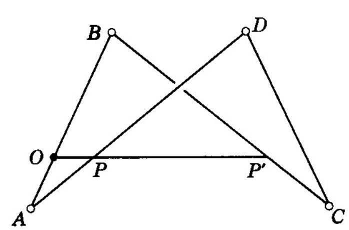

Figure 6.3d

EXERCISES

1. For any two unequal circles that do not intersect, one of the two centers of similitude (§ 5.2) is the center of a circle which inverts either of the given circles into the other. For two unequal intersecting circles, both centers of similitude have this property. What happens in the case of equal intersecting circles?

2. Explain the action of *Peaucellier's cell* (Figure 6.3*c*), an instrument invented by A. Peaucellier in 1864 for the purpose of drawing the inverse of any given locus. It is formed by four equal rods, hinged at the corners of a rhombus $APBP'$, and two equal (longer) rods connecting two opposite corners, A and B, to a fixed pivot O. When a pencil point is inserted at P' and a tracing point at P (or vice versa), and the latter is traced over the curves of a given figure, the pencil point draws the inverse figure. In particular, if a seventh rod and another pivot are introduced so as to keep P on a circle passing through O, the locus of P' will be a straight line. This linkage gives an exact solution of the important mechanical problem of converting circular into rectilinear motion. [Lamb **2**, p. 314.]

3. Explain the action of *Hart's linkage* (Figure 6.3*d*), an instrument invented by H. Hart in 1874 for the same purpose as Peaucellier's cell. It requires only four rods, hinged at the corners of a "crossed parallelogram" $ABCD$ (with $AB = CD$, $BC = DA$). The three collinear points O, P, P', on the respective rods AB, AD, BC, remain collinear (on a line parallel to AC and BD) when the shape of the crossed parallelogram is changed. As before, the instrument is pivoted at O. [Lamb **2**, p. 315.]

4. With respect to a circle γ of radius r, let p be the power of an outside point O. Then the circle with center O and radius k inverts γ into a circle of radius k^2r/p.

6.4 THE INVERSIVE PLANE

Whereupon the Plumber said in tones of disgust:
"I suggest that we proceed at once to infinity."

J. L. Synge [**2**, p. 131]

We have seen that the image of a given point P by reflection in a line (Figure 1.3*b*) is the second intersection of any two circles through P orthogonal to the mirror, and that the inverse of P in a circle is the second intersection of any two circles through P orthogonal to the circle of inversion. Because of this analogy, inversion is sometimes called "reflection in a circle" [Blaschke **1**, p. 47], and we extend the definition of a circle so as to include a straight line as a special (or "limiting") case: a circle of infinite radius. We can then say that *any* three distinct points lie on a unique circle, and that any circle inverts into a circle.

In the same spirit, we extend the Euclidean plane by inventing an "ideal" *point at infinity* O', which is both a common point and the common center of all straight lines, regarded as circles of infinite radius. Two circles with a common point either touch each other or intersect again. This remains obvious when one of the circles reduces to a straight line. When both of them are straight, the lines are either parallel, in which case they touch at O', or intersecting, in which case O' is their second point of intersection [Hilbert and Cohn-Vossen **1**, p. 251].

We can now assert that *every* point has an inverse. All the lines through O, being "circles" orthogonal to the circle of inversion, meet again in O', the inverse of O. When the center O is O' itself, the "circle" of inversion is straight, and the inversion reduces to a reflection.

The Euclidean plane with O' added is called the *inversive* (or "conformal") *plane.** It gives inversion its full status as a "transformation" (§ 2.3): a one-to-one correspondence without exception.

Where two curves cross each other, their angle of intersection is naturally defined to be the angle between their tangents. In this spirit, two intersecting circles, being symmetrical by reflection in their line of centers, make equal angles at the two points of intersection. This will enable us to prove

6.41 *Any angle inverts into an equal angle (or, more strictly, an opposite angle).*

We consider first an angle at a point P which is not on the circle of inversion. Since any direction at such a point P may be described as the direction of a suitable circle through P and its inverse P', two such directions are determined by two such circles. Since these circles are self-inverse, they serve to determine the corresponding directions at P'. To show that an angle at P is still preserved when P is self-inverse, we use 6.11 to express the given inversion as the product of a dilatation and the inversion in a concentric circle that does not pass through P. Since both these transformations preserve angles, their product does likewise.

In particular, right angles invert into right angles, and

6.42 *Orthogonal circles invert into orthogonal circles* (*including lines as special cases*).

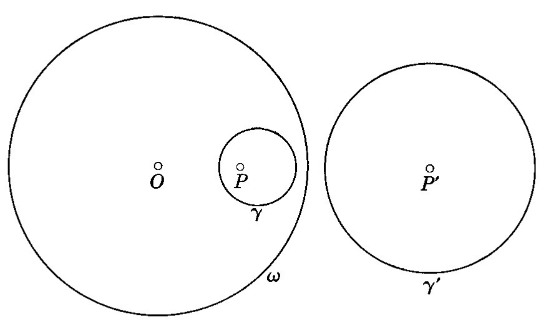

Figure 6.4a

By 6.21, inversion can be defined in terms of orthogonality. Therefore a circle and a pair of inverse points invert (in another circle) into a circle and a pair of inverse points. More precisely, if a circle γ inverts P into Q and

* M. Bôcher, *Bulletin of the American Mathematical Society,* **20** (1914), p. 194.

a circle ω inverts γ, P, Q into γ', P', Q', then the circle γ' inverts P' into Q'. An important special case (Figure 6.4*a*) arises when Q coincides with O, the center of ω, so that Q' is O', the point at infinity. Then P is the inverse of O in γ, and P' is the center of γ'. In other words, if γ inverts O into P, whereas ω inverts γ and P into γ' and P', then P' is the center of γ'.

Two circles either touch, or cut each other twice, or have no common point. In the last case (when each circle lies entirely outside the other, or else one encloses the other), we may conveniently say that the circles *miss* each other.

If two circles, α_1 and α_2, are both orthogonal to two circles β_1 and β_2, we can invert the four circles in a circle whose center is one of the points of intersection of α_1 and β_1, obtaining two orthogonal circles and two perpendicular diameters, as in the remark at the end of § 6.2. Hence, either α_1 touches α_2 and β_1 touches β_2, or α_1 cuts α_2 and β_1 misses β_2, or α_1 misses α_2 and β_1 cuts β_2.

6.5 COAXAL CIRCLES

In this section we leave the inversive plane and return to the Euclidean plane, in order to be able to speak of distances.

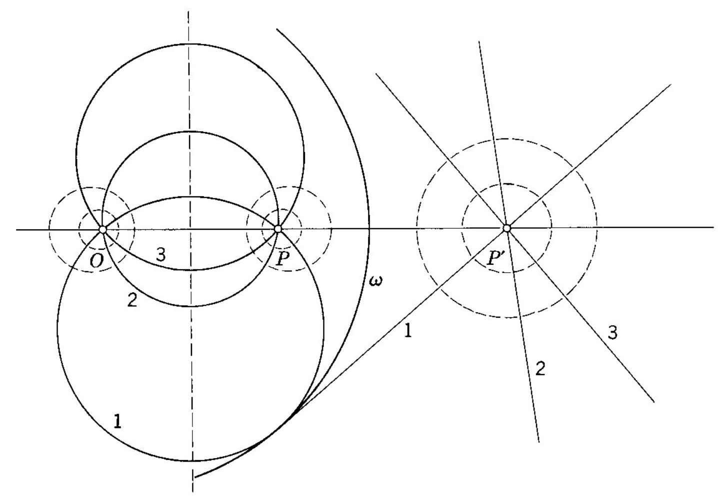

Figure 6.5a

If P and P' are inverse points in the circle ω (with center O), as in Figure 6.5*a*, all the lines through P' invert into all the circles through O and P: an *intersecting* (or "elliptic") *pencil of coaxal circles,* including the straight line OPP' as a degenerate case. The system of concentric circles with center P', consisting of circles orthogonal to these lines, inverts into a *nonintersecting* (or "hyperbolic") *pencil of coaxal circles* (drawn in broken lines). These circles all miss one another and are all orthogonal to the intersecting pencil. One of them degenerates to a (vertical) line, whose inverse is the circle (with center P') passing through O.

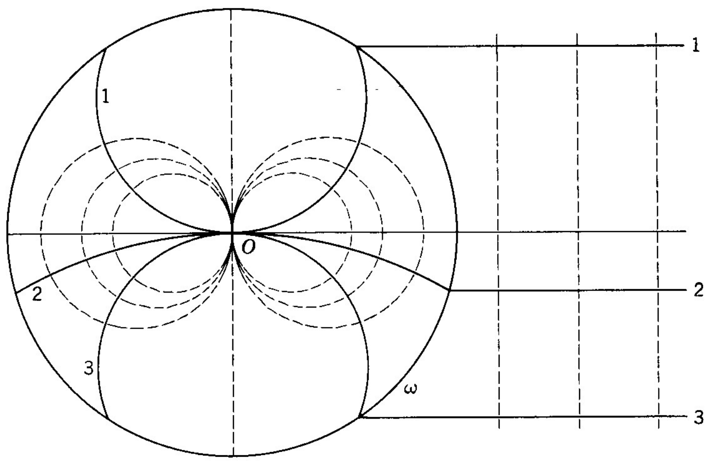

Figure 6.5b

As a kind of limiting case when O and P coincide (Figure 6.5*b*), the circles that touch a fixed line at a fixed point O constitute a *tangent* (or "parabolic") *pencil of coaxal circles.* They invert (in a circle with center O) into all the lines parallel to the fixed line. Orthogonal to these lines we have another system of the same kind, inverting into an orthogonal tangent pencil of coaxal circles. Again each member of either pencil is orthogonal to every member of the other.

Any two given circles belong to a pencil of coaxal circles of one of these three types, consisting of *all the circles orthogonal to both of any two circles orthogonal to both the given circles.* (More concisely, the coaxal circles consist of all the circles orthogonal to all the circles orthogonal to the given circles.) Two circles that cut each other belong to an intersecting pencil (and can be inverted into intersecting lines); two circles that touch each other belong to a tangent pencil (and can be inverted into parallel lines); two circles that miss each other belong to a nonintersecting pencil (by the remark at the end of § 6.4).

Each pencil contains one straight line (a circle of infinite radius) called the *radical axis* (of the pencil, or of any two of its members).* For an intersecting pencil, this is the line joining the two points common to all the circles (OP for the "unbroken" circles in Figure 6.5*a*). For a tangent pencil,

* Louis Gaultier, *Journal de l'École Polytechnique,* **16** (1813), p. 147.

it is the common tangent. For a nonintersecting pencil, it is the line midway between the two *limiting points* (or circles of zero radius) which are the common points of the orthogonal intersecting pencil. For each pencil there is a *line of centers,* which is the radical axis of the orthogonal pencil. Hence

6.51 *If tangents can be drawn to the circles of a coaxal pencil from a point on the radical axis, all these tangents have the same length.*

The radical axis of two given circles may be defined as the locus of points of equal power (§ 6.3) with respect to the two circles. This power can be measured as the square of a tangent except in the case when the given circles intersect in two points O, P, and we are considering a point A on the segment OP; then the power is the negative number $AO \times AP$.

It follows that, for three circles whose centers form a triangle, the three radical axes (of the circles taken in pairs) concur in a point called the *radical center,* which has the same power with respect to all three circles. If this power is positive, its square root is the length of the tangents to any of the circles, and the radical center is the center of a circle (of this radius) which is orthogonal to all the given circles. But if the power is negative, no such orthogonal circle exists.

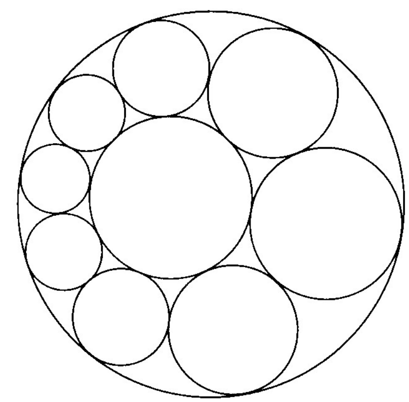

Figure 6.5c

The possibility of inverting any two nonintersecting circles into concentric circles (by taking O at either of the limiting points) provides a remarkably simple proof for Steiner's porism:* If we have two (nonconcentric) circles, one inside the other, and circles are drawn successively touching them and one another, as in Figure 6.5*c*, it may happen that the ring of touching circles closes, that is, that the last touches the first. Steiner's statement is that, if this happens once, it will always happen, whatever be the position of the first circle of the ring. To prove this we need only invert the original two circles into concentric circles, for which the statement is obvious.

* Forder [**3**, p. 23]. See also Coxeter, Interlocked rings of spheres, *Scripta Mathematica,* **18** (1952), pp. 113–121, or Yaglom [**2**, p. 199].

EXERCISES

1. In a pencil of coaxal circles, each member, used as a circle of inversion, interchanges the remaining members in pairs and inverts each member of the orthogonal pencil into itself.

2. The two limiting points of a nonintersecting pencil are inverses of each other in any member of the pencil.

3. If two circles have two or four common tangents, their radical axis joins the midpoints of these common tangents. If two circles have no common tangent (i.e., if one entirely surrounds the other), how can we construct their radical axis?

4. When a nonintersecting pencil of coaxal circles is inverted into a pencil of concentric circles, what happens to the limiting points?

5. In Steiner's porism, the points of contact of successive circles in the ring all lie on a circle, and this will serve to invert the two original circles into each other. Do the *centers* of the circles in the ring lie on a circle?

6. For the triangle considered in Exercise 10 of § 1.5 (page 16), the incircle is coaxal with the "two other circles" (Soddy's circles).

6.6 THE CIRCLE OF APOLLONIUS

The analogy between reflection and inversion is reinforced by the following

PROBLEM. *To find the locus of a point P whose distances from two fixed points A, A′ are in a constant ratio* $1 : \mu$, *so that*

$$A'P = \mu AP.$$

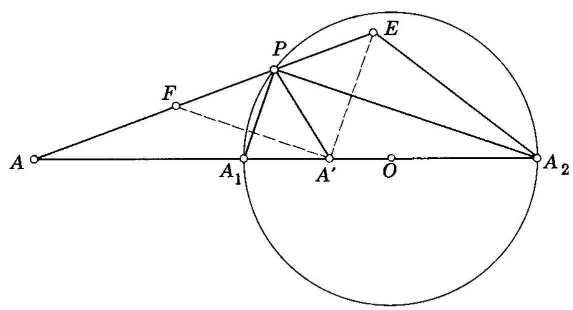

Figure 6.6a

When $\mu = 1$, the locus is evidently the perpendicular bisector of AA', that is, the line that reflects A into A'. We shall see that for other values of μ it is a circle that inverts A into A'. (Apollonius of Perga, *c.* 260–190 B.C.)

Assuming $\mu \neq 1$, let P be any point for which $A'P = \mu AP$. Let the internal and external bisectors of $\angle APA'$ meet AA' in A_1 and A_2 (as in Figure 6.6a, where $\mu = \frac{1}{2}$). Take E and F on AP so that $A'E$ is parallel to A_1P and $A'F$ is parallel to A_2P, that is, perpendicular to A_1P. Since $FP = PA' = PE$, we have

$$\frac{AA_1}{A_1A'} = \frac{AP}{PE} = \frac{AP}{PA'}, \qquad \frac{AA_2}{A'A_2} = \frac{AP}{FP} = \frac{AP}{PA'}.$$

(The former result is Euclid VI.3.) Thus A_1 and A_2 divide the segment AA' internally and externally in the ratio $1 : \mu$, and their location is independent of the position of P. Since $\angle A_1PA_2$ is a right angle, P lies on the circle with diameter A_1A_2.

Conversely, if A_1 and A_2 are defined by their property of dividing AA' in the ratio $1 : \mu$, and P is any point on the circle with diameter A_1A_2, we have

$$\frac{AP}{PE} = \frac{AA_1}{A_1A'} = \frac{1}{\mu} = \frac{AA_2}{A'A_2} = \frac{AP}{FP}.$$

Thus $FP = PE$, and P, being the midpoint of FE, is the circumcenter of the right-angled triangle EFA'. Therefore $PA' = PE$ and

$$\frac{AP}{PA'} = \frac{AP}{PE} = \frac{1}{\mu}$$

[Court **2**, p. 15].

Finally, the *circle of Apollonius* A_1A_2P inverts A into A'. For, if O is its center and k its radius, the distances $a = AO$ and $a' = A'O$ satisfy

$$\frac{a-k}{k-a'} = \frac{AA_1}{A_1A'} = \frac{AA_2}{A'A_2} = \frac{a+k}{a'+k},$$

whence

$$aa' = k^2.$$

EXERCISES

1. When μ varies while A and A' remain fixed, the circles of Apollonius form a nonintersecting pencil with A and A' for limiting points.

2. Given a line l and two points A, A' (not on l), locate points P on l for which the ratio $A'P/AP$ is maximum or minimum. (*Hint*: Consider the circle through A, A' with its center on l. The problem is due to N. S. Mendelsohn, and the hint to Richard Blum.)

3. Express k/AA' in terms of μ.

4. In the notation of Figure 5.6a (which is embodied in Figure 6.6b), the circles on A_1A_2 and B_1B_2 as diameters meet in two points O and $\bar{O}$, such that the triangles OAB and $OA'B'$ are similar, and likewise the triangles $\bar{O}AB$ and $\bar{O}A'B'$. Of the two similarities

$$OAB \to OA'B' \quad \text{and} \quad \bar{O}AB \to \bar{O}A'B',$$

one is opposite and the other direct. In fact, O is where A_1B_1 meets A_2B_2, and $\bar{O}$ lies on the four further circles $AA'P$, $BB'P$, ABT, $A'B'T$ (cf. Ex. 2 of § 5.5). [Casey **1**, p. 185.] If A' coincides with B, O lies on AB'.

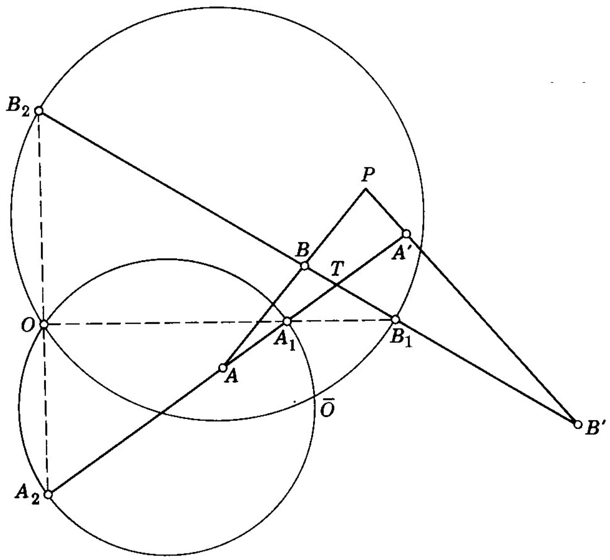

Figure 6.6b

5. Let the *inversive distance* between two nonintersecting circles be defined as the natural logarithm of the ratio of the radii (the larger to the smaller) of two concentric circles into which the given circles can be inverted. Then, if a nonintersecting pencil of coaxal circles includes α_1, α_2, α_3 (in this order), the three inversive distances satisfy

$$(\alpha_1, \alpha_2) + (\alpha_2, \alpha_3) = (\alpha_1, \alpha_3).$$

6. Two given unequal circles are related by infinitely many dilative rotations and by infinitely many dilative reflections. The locus of invariant points (in either case) is the circle having for diameter the segment joining the two centers of similitude of the given circles. (This locus is known as the *circle of similitude* of the given circles.) What is the corresponding result for two given *equal* circles?

7. The inverses, in two given circles, of a point on their circle of similitude, are images of each other by reflection in the radical axis of the two circles [Court **2**, p. 199].

6.7 CIRCLE-PRESERVING TRANSFORMATIONS

Having observed that inversion is a transformation of the whole inversive plane (including the point at infinity) into itself, taking circles into circles, we naturally ask what is the most general transformation of this kind. We distinguish two cases, according as the point at infinity is, or is not, invariant.

In the former case, not only are circles transformed into circles but also lines into lines. With the help of Euclid III.21 (see p. 7) we deduce that equality of angles is preserved, and consequently the measurement of angles is preserved, so that every triangle is transformed into a similar triangle, and the transformation is a similarity (§ 5.4).

If, on the other hand, the given transformation T takes an ordinary point

O into the point at infinity O', we consider the product J_1T, where J_1 is the inversion in the unit circle with center O. This product J_1T, leaving O' invariant, is a similarity. Let k^2 be its ratio of magnification, and J_k the inversion in the circle with center O and radius k. Since, by 6.11, J_1J_k is the dilatation $O(k^2)$, the similarity J_1T can be expressed as J_1J_kS, where S is an isometry. Thus

$$T = J_kS,$$

the product of an inversion and an isometry.

To sum up,

6.71 *Every circle-preserving transformation of the inversive plane is either a similarity or the product of an inversion and an isometry.*

It follows that every circle-preserving transformation is the product of at most four inversions (provided we regard a reflection as a special kind of inversion) [Ford **1**, p. 26]. Such a transformation is called a *homography* or an *antihomography* according as the number of inversions is even or odd. The product of two inversions (either of which could be just a reflection) is called a *rotary* or *parabolic* or *dilative* homography according as the two inverting circles are intersecting, tangent, or nonintersecting (i.e., according as the orthogonal pencil of invariant circles is nonintersecting, tangent, or intersecting). As special cases we have, respectively, a rotation, a translation, and a dilatation. The most important kind of rotary homography is the *Möbius involution,* which, being the inversive counterpart of a half-turn, is the product of inversions in two orthogonal circles (e.g., the product of the inversion in a circle and the reflection in a diameter). Any product of four inversions that cannot be reduced to a product of two is called a *loxodromic* homography [Ford **1**, p. 20].

EXERCISE

When a given circle-preserving transformation is expressed as JS (where J is an inversion and S an isometry), J and S are unique. There is an equally valid expression SJ′, in which the isometry precedes the inversion. Why does this revised product involve the same S? Under what circumstances will we have J′ = J?

6.8 INVERSION IN A SPHERE

By revolving Figures 6.1*a*, 6.2*a*, 6.3*a*, 6.3*b*, and 6.4*a* about the line of centers (OP or OA or OC), we see that the whole theory of inversion extends readily from circles in the plane to spheres in space. Given a sphere with center O and radius k, we define the inverse of any point P (distinct from O) to be the point P' on the ray OP whose distance from O satisfies

$$OP \times OP' = k^2.$$

Alternatively, P' is the second intersection of three spheres through P orthogonal to the sphere of inversion. Every sphere inverts into a sphere, rovided we include, as a sphere of infinite radius, a plane, which is the inverse of a sphere through O. Thus, inversion is a transformation of *inversive* (or "conformal") *space,* which is derived from Euclidean space by postulating a *point at infinity,* which lies on all planes and lines.

Revolving the circle of Apollonius (Figure 6.6*a*) about the line AA', we obtain the *sphere of Apollonius,* which may be described as follows:

6.81 *Given two points A, A' and a positive number μ, let A_1 and A_2 divide AA' internally and externally in the ratio $1 : \mu$. Then the sphere on A_1A_2 as diameter is the locus of a point P whose distances from A and A' are in this ratio.*

EXERCISES

1. If a sphere with center O inverts A into A' and B into B', the triangles OAB and $OB'A'$ are similar.

2. In terms of $a = OA$ and $b = OB$, we have (in the notation of Ex. 1)

$$A'B' = \frac{k^2}{ab} AB.$$

3. The "cross ratio" of any four points is preserved by any inversion:

$$\frac{AB/BD}{AC/CD} = \frac{A'B'/B'D'}{A'C'/C'D'}.$$

[Casey **1**, p. 100.]

4. Two spheres which touch each other at O invert into parallel planes.

5. Let α, β, γ be three spheres all touching one another. Let σ_1, σ_2, . . . be a sequence of spheres touching one another successively and all touching α, β, γ. Then σ_6 touches σ_1, so that we have a ring of six spheres interlocked with the original ring of three.* (*Hint:* Invert in a sphere whose center is the point of contact of α and β.)

6.9 THE ELLIPTIC PLANE

In some unaccountable way, while he [Davidson] moved hither and thither in London, his sight moved hither and thither in a manner that corresponded, about this distant island. . . . When I said that nothing would alter the fact that the place [Antipodes Island] is eight thousand miles away, he answered that two points might be a yard away on a sheet of paper, and yet be brought together by bending the paper round.

H. G. Wells (1866-1946)

(The Remarkable Case of Davidson's Eyes)

Let S be the foot of the perpendicular from a point N to a plane σ, as in Figure 6.9*a*. A sphere (not drawn) with center N and radius NS inverts the plane σ into the sphere σ' on NS as diameter [Johnson **1**, p. 108]. We have

* Frederick Soddy, The Hexlet, *Nature,* **138** (1936), p. 958; **139** (1937), p. 77.

seen that spheres invert into spheres (or planes); therefore circles, being intersections of spheres, invert into circles (or lines). In particular, all the circles in σ invert into circles (great or small) on the sphere σ', and all the lines in σ invert into circles through N. Each point P in σ yields a corresponding point P' on σ', namely, the second intersection of the line NP with σ'. Conversely, each point P' on σ', except N, corresponds to the point P in which NP' meets σ. The exception can be removed by making σ an inversive plane whose point at infinity is the inverse of N.

This inversion, which puts the points of the inversive plane into one-to-one correspondence with the points of a sphere, is known as *stereographic projection.* It serves as one of the simplest ways to map the geographical globe on a plane. Since angles are preserved, small islands are mapped with the correct shape, though on various scales according to their latitude.

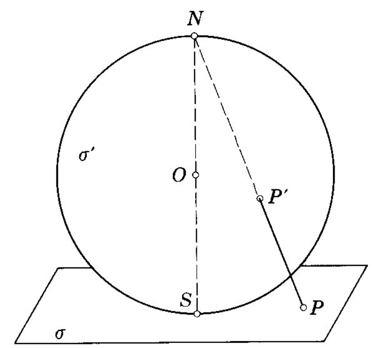

Figure 6.9a

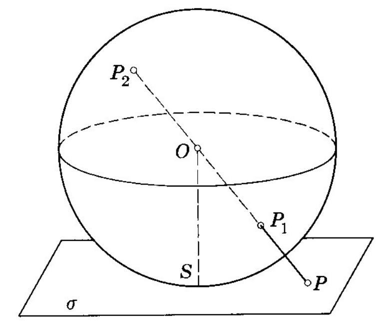

Figure 6.9b

Another way is by *gnomonic* (or central) *projection,* in which the point from which we project is not N but O, the center of the sphere, as in Figure 6.9*b*. Each point P in σ yields a line OP, joining it to O. This diameter meets the sphere in two antipodal points P_1, P_2, which are both mapped on the same point P. Each line m in σ yields a plane Om, joining it to O. This diametral plane meets the sphere in a *great circle.* Conversely, each great circle of the sphere, except the "equator" (whose plane is parallel to σ), corresponds to a line in σ. This time the exception can be removed by adding to the Euclidean plane σ a *line at infinity* (representing the equator) with all its points, called *points at infinity,* which represent pairs of antipodal points on the equator. Thus, all the lines parallel to a given line contain the same point at infinity, but lines in different directions have different points at infinity, all lying on the same line at infinity. (This idea is due to Kepler and Desargues.)

When the line at infinity is treated just like any other line, the plane so extended is called the *projective plane* or, more precisely, the *real* projective plane [Coxeter **2**]. Two parallel lines meet in a point at infinity, and an ordinary line meets the line at infinity in a point at infinity. Hence

6.91 *Any two lines of the projective plane meet in a point.*

Instead of taking a section of all the lines and planes through O, we could more symmetrically (though more abstractly) declare that, by definition, the points and lines of the projective plane *are* the lines and planes through O. The statement 6.91 is no longer surprising; it merely says that any two planes through O meet in a line through O.

Equivalently we could declare that, by definition, the lines of the projective plane are the great circles on a sphere, any two of which meet in a pair of antipodal points. Then the points of the projective plane are the pairs of antipodal points, abstractly identified. This abstract identification was vividly described by H. G. Wells in his short story, *The Remarkable Case of Davidson's Eyes.* (A sudden catastrophe distorted Davidson's field of vision so that he saw everything as it would have appeared from an exactly antipodal position on the earth.)

When the inversive plane is derived from the sphere by stereographic projection, distances are inevitably distorted, but the angle at which two circles intersect is preserved. In this sense, the inversive plane has a partial metric: angles are measured in the usual way, but distances are never mentioned [Graustein **1**, pp. 377, 388, 395].

On the other hand, gnomonic projection enables us, if we wish, to give the *projective* plane a *complete* metric. The distance between two points P and Q in σ (Figure 6.9a) is defined to be the angle POQ (in radian measure), and the angle between two lines m and n in σ is defined to be the angle between the planes Om and On. (This agrees with the customary measurement of distances and angles on a sphere, as used in spherical trigonometry.) We have thus obtained the *elliptic* plane* or, more precisely, the real projective plane with an elliptic metric [Coxeter **3**, Chapter VI; E. T. Bell **2**, pp. 302–311; Bachmann **1**, p. 21].

Since the points of the elliptic plane are in one-to-two correspondence with the points of the unit sphere, whose total area is 4π, it follows that the total area of the elliptic plane (according to the most natural definition of "area") is 2π. Likewise, the total length of a line (represented by a "great semicircle") is π. The simplification that results from using the elliptic plane instead of the sphere is well illustrated by the problem of computing the area of a spherical triangle ABC, whose sides are arcs of three great circles. Figure 6.9c shows these great circles, first in stereographic projection and then in gnomonic projection. The elliptic plane is decomposed, by the three lines BC, CA, AB, into four triangular regions. One of them is the given triangle Δ with angles A, B, C; the other three are marked α, β, γ in Figure 6.9c. (On the sphere, we have, of course, not only four regions but eight.) The two lines CA, AB decompose the plane into two *lunes* whose areas, being proportional to the supplementary angles A and $\pi - A$, are exactly $2A$ and $2(\pi - A)$. The lune with angle A is made up of the two regions Δ and α. Hence

$$\Delta + \alpha = 2A.$$

Similarly $\Delta + \beta = 2B$ and $\Delta + \gamma = 2C$. Adding these three equations and subtracting

$$\Delta + \alpha + \beta + \gamma = 2\pi,$$

we deduce Girard's "spherical excess" formula

6.92 $$\Delta = A + B + C - \pi,$$

which is equally valid for the sphere and the elliptic plane. (A. Girard, *Invention nouvelle en algèbre,* Amsterdam, 1629.)

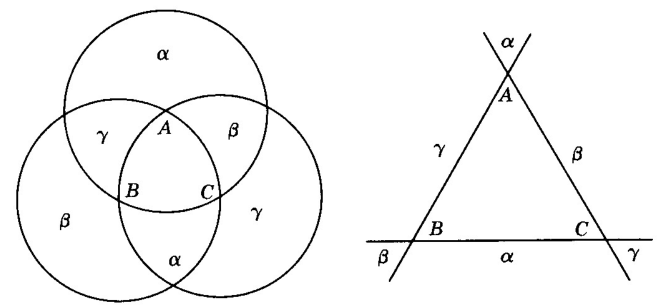

Figure 6.9c

* The name "elliptic" is possibly misleading. It does not imply any direct connection with the curve called an ellipse, but only a rather far-fetched analogy. A central conic is called an ellipse or a hyperbola according as it has no asymptote or two asymptotes. Analogously, a non-Euclidean plane is said to be elliptic or hyperbolic (Chapter 16) according as each of its lines contains no point at infinity or two points at infinity.

EXERCISES

1. Two circles in the elliptic plane may have as many as four points of intersection.
2. The area of a p-gon in the elliptic plane is equal to the excess of its angle sum over the angle sum of a p-gon in the Euclidean plane.