Dynamic-Stochastic Weights: A Numerical User Guide

Eduardo Dávila*

Andreas Schaab†

August, 2022

Abstract

These notes provide a pedagogical user guide for the numerical implementation of welfare assessments with DS-weights, building on Dávila and Schaab (2021). We develop a series of applications. This user guide also introduces the `DS-weights` code repository that contains all codes and replication materials, which can be accessed at https://github.com/schaab-lab/DS-weights.

*Yale University and NBER. Email: eduardo.davila@yale.edu. Website: https://eduardodavila.com.

†Columbia Business School. Email: ajs2428@columbia.edu. Website: https://andreasschaab.com.

1 Social Transfers in an Endowment Economy

In this introductory section, we study social transfer policy in a simple endowment economy and demonstrate how to numerically implement welfare assessments using DS-weights. The environment we introduce below is arguably the simplest example of a dynamic stochastic economy.[1]

1.1 Endowment Economy

Time is continuous and indexed by $t \in [0, T]$, with $T \leq \infty$. The economy is populated by a continuum (of measure 1) of households that face idiosyncratic endowment risk. We denote a household's endowment process by z_t. There are no financial markets and, consequently, no prices or macroeconomic aggregates that would be determined by general equilibrium forces. Finally, there is no ex ante (or type) heterogeneity, and so households can be indexed by their sole state variable z_t.

Endowment process and consumption. We assume that the household's endowment process z_t follows a two-state Markov chain (or Poisson process), with $z_t \in \{z^L, z^H\}$. The transition rate in both states is simply given by λ. We normalize $\bar{z} = \frac{z^L + z^H}{2} = 1$.

Households cannot trade in asset markets and consume hand-to-mouth, that is,

$$c_t = z_t + \theta T_t,$$

where T_t is a social transfer policy. We assume the specific form $T_t = \bar{z} - z_t = 1 - z_t$. Consumption can therefore be rewritten as $c_t = \theta \bar{z} + (1 - \theta) z_t$. When $\theta = 0$, financial autarky implies that households simply consume their endowment. As $\theta \to 1$, social transfer policy fully insures households against income fluctuations.

Preferences. For a given θ, household preferences are encoded in the definition of private lifetime value

$$V_0(z; \theta) = \mathbb{E}^{z,\theta} \int_0^\infty e^{-\rho t} u(c_t) dt,$$

where z denotes the initial endowment state of the household and ρ is the discount rate. We index the lifetime value function $V_0(\cdot)$ by θ to make explicit the dependence of the allocation on the social transfer policy. Finally, $\mathbb{E}^{z,\theta}$ denotes the conditional expectation operator (over future realizations of z_t) for a given θ, assuming that the household is initialized at $z_0 = z$. That is, we denote conditional expectations as $\mathbb{E}^{z,\theta} f(z_t) = \mathbb{E}[f(z_t; \theta) \,|\, z_0 = z]$ following Oksendal (2013).

Recursive representation and equilibrium. This endowment economy is a particularly simple environment because the law of motion of the household's state variable, z_t in this case, does not de-

[1] This section develops a continuous-time variant of Application 1 in Dávila and Schaab (2021).

pend on the stance of policy θ. In other words, the transition probabilities of z_t are fully exogenous, as we explained above. From an implementation perspective, this is the key difference between this and subsequent applications below: Once we allow households to save and accumulate wealth, the transition dynamics of the household state variables will become endogenous to θ. In this endowment economy, therefore, we simply have $\mathbb{E}^{z,\theta} = \mathbb{E}^z$, which just says that the probability measure (for z_t) itself does not depend on θ.

We can now write the stationary consumption policy function as

$$c(z;\theta) = z + \theta(1-z). \tag{1}$$

Notationally, the consumption policy function $c(\cdot)$ is a function of the endowment state z. We think of θ as a fixed parameter here. The consumption policy function is therefore stationary (or time-homogeneous) because, given a state z, it depends on no time-varying object. Using (1), we can rewrite the individual's private lifetime value as

$$V(z;\theta) = \mathbb{E}^z \left[\int_0^\infty e^{-\rho t} u(c(z_t;\theta)) dt \right].$$

Given a fixed θ, the definition of competitive equilibrium is trivial: Households consume their exogenous endowment at all times. The goods market clears as a result.

Finally, we denote the initial cross-sectional distribution of households (over their endowment states z) by $g_0(z)$.[2]

Transition densities. Let Y_t be a (strong) Markov process (z_t above is such a process). With Y_t initialized at time 0 at the value $Y_0 = y$, we adopt the notation for conditional expectation (as above) $\mathbb{E}^y f(Y_t) = \mathbb{E}[f(Y_t) \mid Y_0 = y]$. We now introduce the *transition density* of the process Y_t, which will be crucial in what follows.

Definition 1. (**Transition Density**) *We denote the transition density of a (strong) Markov process Y_t by $p(t, x \mid s, y)$ and it is defined implicitly by*

$$\mathbb{P}(Y_t \in A \mid Y_s = y) = \int_A p(t, x \mid s, y) dx$$

for any (Borel) set A.

In words, $p(t, x \mid s, y)$ characterizes the probability (density for continuous state spaces) that process Y_t, initialized at $Y_s = y$ at time s, ends up at $Y_t = x$ at time t. The following Lemma is crucial for our purposes.

[2] The time evolution of $g_t(z)$ is characterized by a simple Kolmogorov forward equation that encodes the endowment transitions of households. There exists a stationary distribution, which we here denote simply by $g(z)$.

Lemma 1. *Conditional expectation can be written in terms of transition densities as*

$$\mathbb{E}^y f(Y_t) = \int p(t, x \,|\, 0, y)\, f(x)\, dx.$$

The transition densities of the economic processes we are interested in are typically general equilibrium objects and may depend on the policy θ. To make this potential dependence explicit, we will oftentimes write $p(t, x \mid 0, y; \theta)$. For the endowment economy of this section, however, the transition density associated with the process z_t is completely exogenous and therefore independent of θ. In other words, we have

$$p(t, x \,|\, 0, z; \theta) = p(t, x \,|\, 0, z).$$

Concluding our discussion thus far, we note that a household's private lifetime value can be written in terms of the transition density $p(t, x \,|\, 0, z)$ as

$$V(z; \theta) = \int_0^\infty e^{-\rho t} \int p(t, x \,|\, 0, z)\, u(c(x; \theta))\, dx\, dt,$$

This follows directly from Lemma 1 because the private lifetime value is simply a conditional expectation.[3] Crucially, note that θ only enters the expression for the private lifetime value via the consumption policy function $c(\cdot)$ but not via the transition density $p(\cdot)$.

1.2 Welfare Assessments Using DS-Weights

We are now ready to define DS-weights in continuous time for our endowment economy. For completeness, we formally state the definition as follows.

Definition 2. (**DS-weights**) *A DS-planner finds a transfer policy $d\theta$ desirable if and only if $\frac{dW^{DS}}{d\theta} > 0$, where*

$$\frac{dW^{DS}}{d\theta} = \int \left[\int_0^\infty \int \omega(t, x \,|\, 0, z; \theta)\, \frac{dc(x; \theta)}{d\theta}\, dx\, dt \right] g_0(z)\, dz,$$

where $\omega(t, x \,|\, 0, z)$ denotes the DS-weight associated with an individual that starts in state $z_0 = z$ at time 0 and transitions to state $z_t = x$ at time t.

Since households consume a single good and do not work in this economy, the *instantaneous consumption-equivalent effect* defined in the main text of Dávila and Schaab (2021) is simply given by $\frac{dc(x;\theta)}{d\theta}$ here. For example, the DS-weights for an unnormalized utilitarian planner (denoted with a

[3] The integral $\int dx$ is taken over the set of values that z_t can take on. Since the household state variables z_t follow a discrete-state Markov chain in this section, the integral is really a sum over the finite values that z_t can take on. Since we will work with continuous-state processes in later sections, we use the integral notation throughout.

superscript UU) are given by

$$\omega^{UU}(t, x \mid 0, z; \theta) = \alpha(z)\, e^{-\rho t} p(t, x \mid 0, z)\, \frac{\partial u(c(x;\theta))}{\partial c}$$

where $\alpha(z)$ is an individual-specific Pareto weight distribution across households—indexed by their initial state z—and $\frac{\partial u(c(x;\theta))}{\partial c}$ denotes marginal utility in state x.

As in discrete-time, DS-weights generically admit an individual multiplicative decomposition of the form

$$\omega(t, x \mid 0, z; \theta) = \underbrace{\tilde{\omega}(0, z)}_{\text{Individual Component}} \cdot \underbrace{\tilde{\omega}(t \mid 0, z)}_{\text{Dynamic Component}} \cdot \underbrace{\tilde{\omega}(t, x \mid 0, z)}_{\text{Stochastic Component}}$$

For an unnormalized utilitarian planner, for example, one individual multiplicative decomposition of DS-weights is simply given by

$$\begin{aligned}
\tilde{\omega}^{UU}(0, z) &= \alpha(z) \\
\tilde{\omega}^{UU}(t \mid 0, z) &= e^{-\rho t} \\
\tilde{\omega}^{UU}(t, x \mid 0, z; \theta) &= p(t, x \mid 0, z) \frac{\partial u(c(x;\theta))}{\partial c}.
\end{aligned}$$

In the following, we leave the dependence of equilibrium objects on θ implicit. We now introduce the normalized utilitarian planner.

Proposition 2. (**Normalized Utilitarian Planners**) *The unique normalized individual multiplicative decomposition of a utilitarian DS-planner (denoted NU) is given by*

$$\begin{aligned}
\tilde{\omega}^{NU}(t, x \mid 0, z) &= \frac{p(t, x \mid 0, z) \frac{\partial u(c(x))}{\partial c}}{\int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c}\, dx} = \frac{p(t, x \mid 0, z) \frac{\partial u(c(x))}{\partial c}}{\mathbb{E}^{z}\left[\frac{\partial u(c(z_t))}{\partial c}\right]} \\
\tilde{\omega}^{NU}(t \mid 0, z) &= \frac{e^{-\rho t} \int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c}\, dx}{\int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c}\, dx\, dt} \\
\tilde{\omega}^{NU}(0, z) &= \frac{\alpha(z) \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c}\, dx\, dt}{\int \left(\alpha(z) \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c}\, dx\, dt\right) g_0(z)\, dz}
\end{aligned}$$

Notice that under the assumption of Markov / recursive equilibrium, marginal utility is a function of (t, x) and, in particular in this time-homogeneous setting, of x. That is,

$$\frac{\partial u(\cdot)}{\partial c} \equiv u_c(x) : \mathbb{R} \to \mathbb{R}.$$

1.3 Aggregate Additive Decomposition

Finally, we present our aggregate additive decomposition in continuous time. Using the individual multiplicative decomposition, we can rewrite the welfare assessment of the DS-planner as

$$\frac{dW^{DS}}{d\theta} = \int \tilde{\omega}(0,z)\left[\int_0^\infty \tilde{\omega}(t\,|\,0,z)\left(\int \tilde{\omega}(t,x\,|\,0,z)\,\frac{dc(x)}{d\theta}\,dx\right)dt\right] g_0(z)\,dz.$$

In the main text of Dávila and Schaab (2021), we derive the aggregate additive decomposition (Proposition 1) by applying a cross-sectional covariance decomposition *on individuals i*. In these notes, we have dropped this individual i notation and instead work with a standard *state space* notation. That is, we associate individuals with their states. To derive the aggregate additive decomposition of welfare assessments, we now have to map this state space representation back to a representation in terms of individuals.

The state of an individual in this endowment economy at time t is z_t. We denote by $\boldsymbol{z} = \{z_t\}_{t\geq 0}$ a *path* (or stochastic process) of endowment realizations. And we denote by $\boldsymbol{z} \in \boldsymbol{Z}$ the space of such paths. $\boldsymbol{z}$ is the continuous-time analog of a history z^t in discrete time. Now, in the absence of ex ante type heterogeneity, individual i in our economy can be (uniquely almost surely) identified with the realized path $\boldsymbol{z}$ that this individual experiences. In the main text, we introduce the cross-sectional operator $\mathbb{E}^i f(\cdot) = \int f(\cdot) di$ to aggregate over individuals *at a specific date / state / history*. The analog in this setting, therefore, is an expectation operator

$$\mathbb{E}^{\boldsymbol{z}} f(\cdot) = \int f(\cdot)\mu(\boldsymbol{z})d\boldsymbol{z}$$

where by $\mu(\boldsymbol{z})$ we denote the measure of households that have experienced path $\boldsymbol{z}$ up to the date / aggregate state that is under consideration.

Why do we need to introduce such complicated notation, especially considering that the equilibrium of our economy is Markov and admits a recursive representation? The key insight here is that, while allocations are recursive, welfare assessments are not. And, consequently, DS-weights must be non-recursive / non-Markov. So we need this added complexity to handle aggregation over DS-weights, not aggregation over equilibrium objects.

To make this very concrete, consider date t and suppose we are interested in the effect of policy $d\theta$ on aggregate consumption at date t. Starting from the more complex notation, there is a measure $\mu(\boldsymbol{z})$ of households (over paths $\boldsymbol{z}$) in this economy. Our object of interest is therefore

$$\frac{dC_t}{d\theta} = \int \frac{dc_t(\boldsymbol{z})}{d\theta}\mu(\boldsymbol{z})d\boldsymbol{z},$$

where $\frac{dc_t(\boldsymbol{z})}{d\theta}$ is the change in consumption at date t by the individual identified by lifetime history $\boldsymbol{z}$. And $\mu(\boldsymbol{z})$ is the measure of such individuals in the economy. As far as equilibrium objects are concerned, we can leverage the Markov property of general equilibrium. That is, we have

$$c_t(\boldsymbol{z}) = c(z_t),$$

where z_t is the value of path $\boldsymbol{z}$ at date t. In other words, to determine consumption at date t of an individual that experiences lifetime path $\boldsymbol{z}$, all we need to know is the value of that process at date t, i.e., z_t. Similarly, the measure of households over paths, $\mu(\boldsymbol{z})$, at date t simply coincides with the cross-sectional distribution at date t, $g_t(z)$.[4] Therefore, we also have

$$\frac{dC_t}{d\theta} = \int \frac{dc(z)}{d\theta} g_t(z) dz,$$

which is the standard recursive (state space) notation: The point of recursivity is precisely that we do not need to carry around histories.[5]

Welfare assessments on the other hand—and consequently DS-weights—are not Markov. This is because any welfare assessment must specify the date or initial state at which the welfare assessment was made. Welfare assessments can generally depend on the entire history $\boldsymbol{z}$. An important observation, however, is that the DS-weights associated with welfarist planners (in particular the normalized utilitarian planner) admit a special structure, in that they only depend on two states: the state *at which* the initial assessment is made, and the state *for which* the assessment is

[4] This statement formally requires a law of large numbers.

[5] Another way to make this point is as follows. Suppose we adopt discrete-time notation. Then we have

$$\begin{aligned}\frac{dC_t}{d\theta} &= \int \frac{dc(z^t)}{d\theta} \mu(z^t) dz^t \\ &= \int \dots \int \frac{dc(z^t)}{d\theta} \mu(z_0) \dots \mu(z_t)\, dz_0 \dots dz_t\end{aligned}$$

Now if consumption was non-recursive, there would be nothing else we could do. But because of the Markov property of general equilibrium in this economy, we can further simplify this using $\frac{dc(z^t)}{d\theta} = \frac{dc(z_t)}{d\theta}$, so

$$\begin{aligned}\frac{dC_t}{d\theta} &= \int \frac{dc(z_t)}{d\theta} \dots \left(\int \left(\int \mu(z_0) dz_0 \right) \mu(z_1) dz_1 \right) \dots \mu(z_t) dz_t \\ &= \int \frac{dc(z_t)}{d\theta} \mu(z_t) dz_t\end{aligned}$$

The first line follows by bringing $\frac{dc(z_t)}{d\theta}$ out of all the integrals except the outermost one, precisely because consumption at date t does not depend on past earnings realizations given z_t. The second line follows by using the fact that $\mu(\cdot)$ is always a probability measure (or a cross-sectional distribution) that integrates to 1.

made. The aggregate welfare assessment made at time 0 by such a DS-planner is therefore given by

$$
\begin{aligned}
\frac{dW^{DS}}{d\theta} &= \iiint \omega(\boldsymbol{z}^t \,|\, z_0) \frac{dc(\boldsymbol{z}^t \,|\, z_0)}{d\theta} d\mu_{\boldsymbol{z}^t|z_0}\, dt\, d\mu_{z_0} \\
&= \int \tilde{\omega}(0,z) \left[\int_0^\infty \tilde{\omega}(t \,|\, 0,z) \left(\int \tilde{\omega}(t,x \,|\, 0,z)\, \frac{dc(x)}{d\theta}\, dx \right) dt \right] g_0(z)\, dz \\
&= \int \tilde{\omega}(0,z) \int_0^\infty \tilde{\omega}(t \,|\, 0,z) \int \frac{\tilde{\omega}(t,x \,|\, 0,z)}{p(t,x \,|\, 0,z)}\, \frac{dc(x)}{d\theta}\, p(t,x \,|\, 0,z)\, g_0(z)\, dx\, dt\, dz \\
&= \iint \tilde{\omega}(0,z) \int_0^\infty \tilde{\omega}(t \,|\, 0,z) \frac{\tilde{\omega}(t,x \,|\, 0,z)}{p(t,x \,|\, 0,z)}\, \frac{dc(x)}{d\theta}\, g(t,x \,|\, 0,z)\, dt\, dx\, dz.
\end{aligned}
$$

In the first line, $\int d\mu_{\boldsymbol{z}^t|z_0}$ integrates over all possible paths that proceed from z_0 and reach $\boldsymbol{z}^t$ at date t according to the probability measure of such paths. In subsequent lines, $g(t,x \,|\, 0,z)$ is the joint density of households that started in state z at time 0 and reached state x at time t. Crucially, the second line exploits the property that DS-weights only depend on the initial state $(0,z)$ and the state for which the assessment is made (t,x). (All other terms come out of the integrals. And since we're integrating probability measures, these other terms are all equal to 1.)

So building on our discussion above, the notion of *cross-sectional covariance decomposition* we want to use here is a different one. We now define the inner product

$$
\mathbb{E}^i[f(t,x \,|\, 0,z)] = \iint f(t,x \,|\, 0,z)\, g(t,x \,|\, 0,z) dx\, dz
$$

and the cross-sectional covariance decomposition

$$
\begin{aligned}
\mathbb{C}ov^i(X,Y) &= \mathbb{E}^i(XY) - \mathbb{E}^i(X)\mathbb{E}^i(Y) \\
&= \iint X(t,x \,|\, 0,z) Y(t,x \,|\, 0,z) g(t,x \,|\, 0,z)\, dx\, dz \\
&\quad - \left(\iint X(t,x \,|\, 0,z) g(t,x \,|\, 0,z)\, dx\, dz \right) \left(\iint Y(t,x \,|\, 0,z) g(t,x \,|\, 0,z)\, dx\, dz \right).
\end{aligned}
$$

Suppose $Y(t,x\,|\,0,z) = Y(x)$. Then we have

$$
\begin{aligned}
\mathbb{C}ov^i(X,Y) &= \iint X(t,x\,|\,0,z)Y(x)p(t,x\,|\,0,z)\, g(0,z)\, dx\, dz \\
&\quad - \left(\iint X(t,x\,|\,0,z)g(t,x\,|\,0,z)\, dx\, dz \right) \left(\iint Y(x)p(t,x\,|\,0,z)\, g(0,z)\, dx\, dz \right) \\
&= \int Y(x) \int X(t,x\,|\,0,z)p(t,x\,|\,0,z)\, g(0,z)\, dz\, dx \\
&\quad - \left(\iint X(t,x\,|\,0,z)g(t,x\,|\,0,z)\, dx\, dz \right) \left(\int Y(x) \int p(t,x\,|\,0,z)\, g(0,z)\, dz\, dx \right) \\
&= \int Y(x) \int X(t,x\,|\,0,z)p(t,x\,|\,0,z)\, g(0,z)\, dz\, dx \\
&\quad - \left(\iint X(t,x\,|\,0,z)g(t,x\,|\,0,z)\, dx\, dz \right) \left(\int Y(x)\, g(t,x)\, dx \right),
\end{aligned}
$$

and where $\int Y(x)\, g(t,x)\, dx = \mathbb{E}^i(Y)$ in this case.

The aggregate additive decomposition then follows:

$$
\begin{aligned}
\frac{dW^{DS}}{d\theta} &= \iint \tilde{\omega}(0,z) \int_0^\infty \tilde{\omega}(t\,|\,0,z) \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dt\, dx\, dz \\
&\quad - \iint \tilde{\omega}(0,z)\, g(t,x\,|\,0,z)\, dx\, dz \int_0^\infty \iint \tilde{\omega}(t\,|\,0,z) \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dt\, dx\, dz \\
&\quad + \iint \tilde{\omega}(0,z)\, g(t,x\,|\,0,z)\, dx\, dz \int_0^\infty \iint \tilde{\omega}(t\,|\,0,z) \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dt\, dx\, dz
\end{aligned}
$$

where the first two lines define the $\mathbb{C}ov^i$ of the *redistribution* term. Also note that

$$
\iint \tilde{\omega}(0,z)\, g(t,x\,|\,0,z)\, dx\, dz = \int \tilde{\omega}(0,z)g(0,z) \int p(t,x\,|\,0,z)\, dx\, dz = \int \tilde{\omega}(0,z)g(0,z)dz.
$$

Next up, we decompose the third line into

$$
\begin{aligned}
&\int \tilde{\omega}(0,z)\, g(0,z)\, dz \int_0^\infty \left(\iint \tilde{\omega}(t\,|\,0,z) \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dx\, dz \right) dt \\
&- \int \tilde{\omega}(0,z)\, g(0,z)\, dz \int_0^\infty \left(\iint \tilde{\omega}(t\,|\,0,z)\, g(t,x\,|\,0,z)\, dx\, dz \right) \left(\iint \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dx\, dz \right) dt \\
&+ \int \tilde{\omega}(0,z)\, g(0,z)\, dz \int_0^\infty \left(\iint \tilde{\omega}(t\,|\,0,z)\, g(t,x\,|\,0,z)\, dx\, dz \right) \left(\iint \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\, g(t,x\,|\,0,z)\, dx\, dz \right) dt
\end{aligned}
$$

where again the first two lines define the $\mathbb{C}ov^i$ decomposition associated with the *intertemporal-*

sharing term. Notice that

$$\iint \tilde{\omega}(t\,|\,0,z)\, g(t,x\,|\,0,z)\, dx\, dz = \int \tilde{\omega}(t\,|\,0,z) g(0,z) \int p(t,x\,|\,0,z)\, dx\, dz = \int \tilde{\omega}(t\,|\,0,z)\, g(0,z)\, dz.$$

Finally, we again decompose the third line into

$$\begin{aligned}
&A \int_0^\infty B \left(\iint \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta} g(t,x\,|\,0,z)\, dx\, dz \right) dt \\
-&A \int_0^\infty B \left(\iint \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} g(t,x\,|\,0,z)\, dx\, dz \right) \left(\iint \frac{dc(x)}{d\theta} g(t,x\,|\,0,z)\, dx\, dz \right) dt \\
+&A \int_0^\infty B \left(\iint \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} g(t,x\,|\,0,z)\, dx\, dz \right) \left(\iint \frac{dc(x)}{d\theta} g(t,x\,|\,0,z)\, dx\, dz \right) dt
\end{aligned}$$

where for compactness $A = \int \tilde{\omega}(0,z)\, g(0,z)\, dz$ and $B = \int \tilde{\omega}(t\,|\,0,z)\, g(0,z)\, dz$. The first two lines correspond to *risk-sharing* and the third line to *aggregate efficiency*. Notice that

$$\iint \frac{dc(x)}{d\theta} g(t,x\,|\,0,z)\, dx\, dz = \int \frac{dc(x)}{d\theta} \int g(t,x\,|\,0,z)\, dx\, dz = \int \frac{dc(x)}{d\theta} g(t,x)\, dx.$$

We summarize this derivation in the following Proposition.

Proposition 3. **(Aggregate Additive Decomposition)** *In continuous time, the welfare assessment of the DS-planner satisfies the following aggregate additive decomposition*

$$\frac{dW^{DS}}{d\theta} = \Xi_{AE} + \Xi_{RS} + \Xi_{IS} + \Xi_{RE}$$

where

$$\begin{aligned}
\Xi_{AE} &= \mathbb{E}^i\Big[\tilde{\omega}(0,z)\Big] \int_0^\infty \mathbb{E}^i\Big[\tilde{\omega}(t\,|\,0,z)\Big]\; \mathbb{E}^i\left[\frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)}\right]\; \mathbb{E}^i\left[\frac{dc(x)}{d\theta}\right] dt \\
\Xi_{RS} &= \mathbb{E}^i\Big[\tilde{\omega}(0,z)\Big] \int_0^\infty \mathbb{E}^i\Big[\tilde{\omega}(t\,|\,0,z)\Big]\; \mathbb{C}ov^i\left(\frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)}, \frac{dc(x)}{d\theta}\right) dt \\
\Xi_{IS} &= \mathbb{E}^i\Big[\tilde{\omega}(0,z)\Big] \int_0^\infty \mathbb{C}ov^i\left(\tilde{\omega}(t\,|\,0,z), \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta}\right) dt \\
\Xi_{RE} &= \mathbb{C}ov^i\left(\tilde{\omega}(0,z), \int_0^\infty \tilde{\omega}(t\,|\,0,z) \frac{\tilde{\omega}(t,x\,|\,0,z)}{p(t,x\,|\,0,z)} \frac{dc(x)}{d\theta} dt\right)
\end{aligned}$$

1.4 Numerical Implementation

Once DS-weights are computed, leveraging them to study welfare assessments is straightforward—for example, using Propositions 2 and 3. Indeed, computing the aggregate additive decomposition of welfare assessment $\frac{dW^{DS}}{d\theta}$ using Proposition 3 requires only two objects, DS-weights and the instantaneous consumption-equivalent effect. In this subsection, we discuss how to compute both of these objects numerically. We start by considering a marginal policy change $d\theta$ (locally) around some given policy $\bar{\theta}$, and afterwards show how to make (global) welfare assessments for discrete policy changes.

High-level roadmap. Generically, DS-weights are functions of allocations, prices, and other equilibrium objects. While the exact form that DS-weights take depends on the welfare criterion that the researcher wants to consider, we illustrate several important benchmarks in these notes.

Our first step, therefore, is to compute the competitive equilibrium of our economy (i.e., allocations, prices, and other equilibrium objects) for the given policy $\bar{\theta}$, around which we want to consider a marginal policy change. Following a finite-difference approach, when then re-compute the competitive equilibrium for the policy $\bar{\theta} + h$, where h is small. For example, this yields the two consumption policy functions $c(x; \bar{\theta})$ and $c(x; \bar{\theta} + h)$, and we can then compute the instantaneous consumption-equivalent effect, evaluated at $\bar{\theta}$, as

$$\frac{dc(x; \bar{\theta})}{d\theta} = \frac{c(x; \bar{\theta} + h) - c(x; \bar{\theta})}{h}.$$

We follow this approach for all those equilibrium objects that are required to compute DS-weights.

Summary of competitive equilibrium. To compute the competitive equilibria of the applications we study in these notes, we follow the approach of Achdou et al. (2021) and Schaab and Zhang (2021). These papers develop finite-difference methods to solve dynamic programming problems and, in particular, heterogeneous-agent economies. Continuous-time dynamic programming problems give rise to partial differential equations, and finite-difference methods are the most widely used approach to solve such equations.

Concretely, for the endowment economy we study in this section, computing competitive equilibrium at $\bar{\theta}$ requires computing the stationary household value function $V(z; \bar{\theta})$, the stationary consumption policy function $c(z; \bar{\theta})$, and the stationary cross-sectional distribution $g(z; \bar{\theta})$. The consumption policy function is trivially given by $c(z; \bar{\theta}) = z + \bar{\theta}(1 - z)$. The value function solves the equation

$$\rho V(z; \bar{\theta}) = u(c(z; \bar{\theta})) + \lambda \Big[V(z^-; \bar{\theta}) - V(z; \bar{\theta}) \Big],$$

where we (slightly informally) denote by z^- the endowment value that is not z. In fact, in the simplest calibration where z_t follows a two-state Markov chain, the value function is characterized by the system

$$\rho V(z^L; \bar{\theta}) = u\Big(z^L + \bar{\theta}(1 - z^L)\Big) + \lambda \Big[V(z^H; \bar{\theta}) - V(z^L; \bar{\theta}) \Big]$$

$$\rho V(z^H; \bar{\theta}) = u\Big(z^H + \bar{\theta}(1 - z^H)\Big) + \lambda \Big[V(z^L; \bar{\theta}) - V(z^H; \bar{\theta}) \Big].$$

Both Achdou et al. (2021) and Schaab and Zhang (2021) discuss how the *generator* of the stochastic process for the state vector, which we denote $\mathcal{A}$, can be leveraged to solve the differential equations that characterize heterogeneous-agent economies. Since the state vector in this application follows a stationary Markov chain, the generator is simply given by

$$\mathcal{A}f(z) = \lambda\Big[f(z^-) - f(z)\Big].$$

For additional details, please refer to Achdou et al. (2021) and Schaab and Zhang (2021).[6] The *adjoint* of the generator, which we denote $\mathcal{A}^*$, can then be used to compute the cross-sectional distribution.[7] In particular, $g(z;\bar{\theta})$ is characterized by the Kolmogorov forward equation

$$0 = \mathcal{A}^* g(z;\bar{\theta}).$$

Discretization on a grid. We implement these equations numerically on a grid, i.e., a collection of points or nodes that discretize the state space of the economy. The state space of our endowment economy is given by $\mathcal{X} = \{z^L, z^H\}$ since z_t is the only state variable and it follows a two-state Markov chain. We always use $\mathcal{X}$ to denote the state space. In this case, we use the grid $\boldsymbol{G} = \{z^L, z^H\}$ to represent the state space. We think of grids as column vectors of coordinates, i.e., collections of grid points that are represented by their d-dimensional coordinates, where $d = 1$ in this case. We use bold-faced notation throughout to denote vectors and matrices that correspond to discretized representations of our equilibrium objects on a grid. In the case of our endowment economy, the state space $\mathcal{X}$ is already discrete, but we adopt this notational distinction to remain consistent with the notation we use in later sections. That is, $\boldsymbol{z} = (z^L, z^H)'$ is the vector of endowment states on the grid, and $\boldsymbol{V}$, $\boldsymbol{c}$, and $\boldsymbol{g}$ respectively denote the vectors that represent the value function, the consumption policy function, and the cross-sectional distribution on the grid. Denoting by $\boldsymbol{A}$ the matrix corresponding to the generator $\mathcal{A}$, the linear system of equations that characterizes the value function can be written as

$$\rho \boldsymbol{V} = \boldsymbol{u} + \boldsymbol{A}\boldsymbol{V}, \tag{2}$$

where $\boldsymbol{u} = u(\boldsymbol{c})$. We can solve equation (2) using standard linear equation solvers.

With these equations, we can numerically solve the competitive equilibrium of our endowment economy for a given $\bar{\theta}$. We then also solve the the equilibrium objects for $\bar{\theta} + h$, and compute finite-differences as necessary.

Transition densities. DS-weights often depend on the transition density of the (controlled) stochastic process that characterizes the economy's state variables. In fact, this will always be the

[6] For an accessible textbook treatment, see Oksendal (2013).

[7] More generally, the law of motion of the state vector, and therefore the generator and adjoint, will depend on $\bar{\theta}$. In subsequent sections, we will therefore often use the more general notation $\mathcal{A}_t(\bar{\theta})$ and $\mathcal{A}_t^*(\bar{\theta})$ to signify that the stochastic processes described by these operators may be time-inhomogeneous, and therefore depend on t, and vary with the policy stance $\bar{\theta}$. In this simple endowment economy, the process z_t is fully exogenous and does not depend on $\bar{\theta}$.

case for any non-paternalistic welfare criterion when individuals have rational expectations. Recall, for example, that the stochastic component of DS-weights for an unnormalized utilitarian planner takes the form $\tilde{\omega}^{UU}(t, x \mid 0, z; \bar{\theta}) = p(t, x \mid 0, z) \frac{\partial u(c(x;\bar{\theta}))}{\partial c}$, where $p(\cdot)$ is the transition density of the state variable z_t. With the consumption policy function $c(z; \bar{\theta})$ in hand, we can already compute the marginal utility term. We now discuss how to compute transition densities, which are a core ingredient for computing DS-weights.

Proposition 4. *The transition density of the Markov process z_t, denoted $p(t, x \mid 0, z)$, satisfies the Kolmogorov forward equation*

$$\partial_t p = \mathcal{A}^* p \tag{3}$$

subject to the initial condition $p(0, x \mid 0, z) = \delta(x - z)$.

Proposition 4 tells us that all we need to compute the transition density $p(\cdot)$ is the adjoint $\mathcal{A}^*$. We have already computed $\mathcal{A}^*$ as part of the competitive equilibrium, leveraging the methods developed by Achdou et al. (2021) and Schaab and Zhang (2021). Therefore, computing transition densities requires no additional work.

It is important to note that equation (3) is time-inhomogeneous, i.e., it depends explicitly on calendar time t, even though competitive equilibrium is stationary. Even when the economy is at its stationary equilibrium, the transition density of the stochastic process z_t depends explicitly on time t. The process z_t is persistent. Consider a household starting in state $z_0 = z$. Intuitively, it is more likely that $z_t = z$ for t close to 0. As $t \to \infty$, we have $\lim_{t\to\infty} \mathbb{P}(z_t = z^L \mid z_0 = z) = 0.5$ because the process is symmetric.

Therefore, the grid representation of $p(\cdot)$ does not simply take the form of a vector, as the stationary cross-sectional distribution $g(\cdot)$ does. Equation (3) holds for a given starting time s, here assumed to be time $s = 0$, and a given starting point, denoted $z_0 = z$. The partial derivatives in the time and state dimensions are taken with respect to the first two arguments of $p(\cdot)$, i.e., t and x, rather than s and z. The (discretized) grid representation of $p(\cdot)$ is therefore that of a time-varying vector for each distinct starting value z, which we may denote $\boldsymbol{p}_t(z)$. That is, element j of $\boldsymbol{p}_t(z)$ characterizes the probability (density) of process z_t reaching state z^j when starting at $z_0 = z$. We can stack these column vectors into a matrix, which we denote $\boldsymbol{P}_t$, where the ith column represents the transition density vector $\boldsymbol{p}_t(z^i)$. Denoting by $\boldsymbol{D}_t$ the finite-difference matrix that corresponds to the partial derivative ∂_t and by $\boldsymbol{A}^T$ the transpose of $\boldsymbol{A}$, which discretizes the adjoint $\mathcal{A}^*$, we can write the discretized version of equation (3) as

$$\boldsymbol{D}_t \boldsymbol{P}_t = \boldsymbol{A}^T \boldsymbol{P}_t. \tag{4}$$

DS-weights. We now have all the ingredients we need to compute the DS-weights for a normalized utilitarian planner. In practice, we simply plug into the formulas of Proposition 2. We discuss each of the components in turn.

The individual component of DS-weights, $\tilde{\omega}(0, z)$, is a function of the initial state of the household under consideration, i.e., $z_0 = z$. We always adopt the convention that assessments are made at time 0, and we make no explicit reference to DS-weights also being a function of the calendar time at which the initial assessment is made. In other words, $\tilde{\omega}(0, z) = \tilde{\omega}(z) : \mathcal{X} \to \mathbb{R}$. The grid representation of the individual component is therefore that of a vector, which we denote

$$\tilde{\boldsymbol{\omega}}^{\text{ind}}$$

The dynamic component of DS-weights, $\tilde{\omega}(t \mid 0, z)$, is a function of both the intial state of the household, z, and the calendar time t for which (not at which!) the assessment is made. We store the data for the discretized grid representation in the form of a time-varying vector, which we denote

$$\tilde{\boldsymbol{\omega}}_t^{\text{dyn}}$$

Finally, the stochastic component of DS-weights, $\tilde{\omega}(t, x \mid 0, z)$, is a function of the initial state of the household, z, the calendar time for which the assessment is made, t, and finally the state for which (not at which!) the assessment is made, x. Intuitively, therefore, the information structure of the stochastic component of DS-weights is the same as that of the transition density $p(t, x \mid 0, z)$. We can therefore denote the discretized grid representation of the stochastic component as

$$\tilde{\boldsymbol{\omega}}_t^{\text{sto}}(z),$$

i.e., as a time-varying vector for each initial state z. Alternatively, we can again stack these vectors—as we did for $\boldsymbol{p}_t(z)$—and collect them in a matrix

$$\tilde{\boldsymbol{\omega}}_t^{\text{sto}},$$

where we abuse notation slightly.[8]

We are now ready to present a variant of Proposition 2 that summarizes the (discretized) grid representation of DS-weights for a normalized utilitarian planner. These are the formulas that we actually use in practice to compute DS-weights on a grid.

Proposition 5. (**Grid Representation of Normalized DS-Weights**) *The discretized grid representation*

[8] Whenever $\tilde{\boldsymbol{\omega}}_t^{\text{sto}}(z)$ is explicitly written as a function of z, it represents a vector. Otherwise, it represents the stacked matrix.

of the unique normalized individual multiplicative decomposition for a utilitarian DS-planner is given by

$$\tilde{\boldsymbol{\omega}}_n^{\text{sto}}(z) = \frac{\boldsymbol{p}_n(z) \cdot \boldsymbol{u}_c}{\boldsymbol{p}_n(z)' \, \boldsymbol{u}_c}$$

$$\tilde{\boldsymbol{\omega}}_n^{\text{dyn}} = \frac{\tilde{\beta}_n \, \boldsymbol{P}_n' \, \boldsymbol{u}_c}{\sum_{n=0}^{N} \tilde{\beta}_n \, \boldsymbol{P}_n' \, \boldsymbol{u}_c}$$

$$\tilde{\boldsymbol{\omega}}^{\text{ind}} = \frac{\boldsymbol{\alpha} \cdot \sum_{n=0}^{N} \tilde{\beta}_n \, \boldsymbol{P}_n' \, \boldsymbol{u}_c}{\left(\boldsymbol{\alpha} \cdot \sum_{n=0}^{N} \tilde{\beta}_n \, \boldsymbol{P}_n' \, \boldsymbol{u}_c\right)' \tilde{\boldsymbol{g}}}$$

where $\cdot$ denotes elementwise multiplication, $'$ denotes the transpose of a vector or matrix, and $\boldsymbol{u}_c = \frac{\partial}{\partial c} u(\boldsymbol{c})$ denotes the column vector of marginal utilities on the grid. Finally, $\tilde{\beta}_n = d\boldsymbol{t}_n \, e^{-\rho \boldsymbol{t}_n}$ denotes a scaled discount factor for the discretized time grid, and $\tilde{\boldsymbol{g}}$ denotes the discretized, initial cross-sectional distribution scaled by dz.

2 Optimal Savings Taxes in a Huggett Economy

This section studies optimal savings taxation in a continuous-time variant of the Huggett (1993) model. The key departure from the endowment economy of Section 1 is that we allow households to save and accumulate wealth, subject to a borrowing constraint. As a result, the state of a household is now summarized by earnings potential z_t and wealth a_t. While z_t is still exogenous and invariant to policy changes, as in Section 1, the evolution of household wealth a_t is endogenous. Therefore, the transition dynamics of the state vector as a whole now depend on policy changes.

We start with a brief description of the model and then discuss how to compute DS-weights and make welfare assessments when the state vector transition density $p(t, x \mid 0, z; \theta)$ is a function of our policy variable of interesting θ.

2.1 Model

We consider a continuous-time variant of the Huggett (1993) model with inelastic labor supply. We abstract from aggregate uncertainty and focus on the stationary equilibrium of the model where macroeconomic aggregates, and in particular prices, are constant.

Households. There is a continuum of households with preferences over consumption. The household's private lifetime value is defined as

$$V_0(\cdot) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

We assume that households supply one unit of labor inelastically. The budget constraint is given by

$$\dot{a}_t = (1-\theta)\Big(r_t a_t + w_t z_t + \tau_t - c_t\Big),$$

where θ is a constant tax on savings, r_t and w_t denote the real interest and wage rates, τ_t denotes a lump-sum rebate from the government, and c_t is consumption. Households face the borrowing constraint

$$a_t \geq \underline{a}.$$

Earnings risk. Households face idiosyncratic earnings risk that is encoded in the stochastic process z_t, which we model as a two-state Markov chain as in Section 1, with $z_t \in \{z^L, z^H\}$. The transition rate in both states is again given by λ.

Cross-sectional household distribution. We denote the joint density of households over income and wealth by $g_t(a, z)$. This object varies over time according to a Kolmogorov forward equation, which we present below. In the absence of ex ante (or type) heterogeneity, households can be associated with their state variables, (a, z).

Production. There is a representative firm that produces the final consumption good with labor as its only factor of production. And since aggregate labor is normalized to 1, output in the stationary equilibrium is simply given by $Y_t = 1$ for all t. With perfect competition and constant returns to scale, the real wage must be equal to the marginal product of labor, which implies $w_t = 1$.

Government. The fiscal authority balances its budget at all times. It simply rebates its revenues from savings taxes back to households uniformly. This implies

$$\tau_t = \theta \int \Big(r_t a_t + w_t z_t + \tau_t - c_t \Big) g_t(a,z)\, da\, dz.$$

Markets. The goods market clears according to

$$Y_t = C_t = \iint c_t(a,z)\, g_t(a,z)\, da\, dz.$$

The bond market clearing condition is

$$0 = B_t = \iint a\, g_t(a,z)\, da\, dz,$$

where we assume that bonds are in zero net supply. Notice that this assumption also implies $\tau_t = 0$ because savings taxes yield no aggregate revenue.

Competitive equilibrium. We first present the definition of competitive equilibrium and afterwards discuss the stationary equilibrium.

Definition 3. (**Competitive Equilibrium**) Given a savings tax rate θ and an initial distribution $g_0(a,z)$, the competitive equilibrium in sequence space representation is given by time paths for (i) prices $\{r_t, w_t\}$, (ii) aggregate allocations $\{Y_t, C_t, B_t, \tau_t\}$, (iii) functions $c_t(a,z)$ and $V_t(a,z)$ representing household behavior, and (iv) the cross-sectional distribution $g_t(a,z)$ such that households and firms optimize facing prices, markets clear, the government budget constraint holds, and the evolution of the cross-sectional distribution is consistent with household behavior.

In the following, we focus on the stationary equilibrium of this economy. Achdou et al. (2021) prove the existence of a unique stationary equilibrium in a version of this model. In the absence of aggregate uncertainty and other sources of exogenous time variation in aggregates, the stationary equilibrium obtains when we initialize the economy at the stationary cross-sectional distribution, i.e., $g_0(a,z) = g(a,z)$. We denote objects associated with the stationary equilibrium by simply dropping the time subscripts.

Definition 4. (**Stationary Competitive Equilibrium**) Given a savings tax rate θ, the stationary competitive equilibrium comprises stationary (i) prices $\{r, w\}$, (ii) aggregate allocations $\{Y, C, B, \tau\}$,

(iii) functions $c(a,z)$ and $V(a,z)$ representing household behavior, and (iv) cross-sectional distribution $g(a,z)$ such that households and firms optimize facing prices, markets clear, the government budget constraint holds, and the stationary cross-sectional is consistent with household behavior.

Before discussing DS-weights and welfare, we summarize some of the key conditions that define stationary equilibrium. In particular, household behavior is summarized by the stationary Hamilton-Jacobi-Bellman equation for the private lifetime value,

$$\rho V(a,z) = u(c) + \Big[(1-\theta)(ra + z - c(a,z))\Big]\partial_a V(a,z) + \lambda\Big[V(a,z^-) - V(a,z)\Big], \tag{5}$$

where we already make use of the equilibrium conditions $w = 1$ and $\tau = 0$. Equation (5) holds everywhere in the interior of the state space. At the borrowing constraint, $a = \underline{a}$, a state-constraint boundary condition ensures that households cannot move further into debt. See Achdou et al. (2021) for details. The consumption policy function $c(a,z)$ solves the first-order condition

$$u'(c(a,z)) = \partial_a V(a,z). \tag{6}$$

As before, we leverage the analytical properties of the generator and its adjoint. In this application, the vector of state variables is given by (a_t, z_t). The generator of this two-dimensional stochastic process takes the form

$$\mathcal{A}(\theta)f(a,z;\theta) = \Big[(1-\theta)(r(\theta)a + z - c(a,z;\theta))\Big]\partial_a f(a,z;\theta) + \lambda\Big[f(a,z^-;\theta) - f(a,z;\theta)\Big],$$

where we now explicitly account for the endogeneity of household savings decisions and for the possibility that the transition dynamics of the state variables depend on the policy θ—directly through the savings tax rate, and indirectly through the equilibrium real interest rate and consumption decisions. We again denote the adjoint of the generator by $\mathcal{A}^*(\theta)$.

The stationary cross-sectional distribution is characterized by the Kolmogorov forward equation

$$0 = \mathcal{A}^*(\theta)g(a,z;\theta), \tag{7}$$

where we again explicitly acknowledge that different policy stances may be associated with different cross-sectional distributions in stationary equilibrium.

Finally, we denote the transition density associated with the state process (a_t, z_t) by $p(t, x \,|\, 0, z; \theta)$, where $p(\cdot)$ characterizes the probability (density) that a household in state $(a_0, z_0) = z$ at time 0 reaches state x at time t. In this notation, both z and x are now two-dimensional. As before, the transition density solves a time-inhomogeneous Kolmogorov forward equation, given by

$$\partial_t p(t, x \,|\, 0, z; \theta) = \mathcal{A}^*(\theta)p(t, x \,|\, 0, z; \theta), \tag{8}$$

where the partial derivative on the LHS is with respect to t, and the partial derivatives encoded in $\mathcal{A}^*$ are with respect to x (not z).

2.2 DS-Weights with Endogenous Transition Densities

In most economic applications, the transition density associated with the state variables, $p(t, x \mid 0, z; \theta)$, will depend on θ. When considering the effect of a policy change on the private lifetime value of an individual in Section 1, we had

$$\frac{dV(z;\theta)}{d\theta} = \frac{d}{d\theta} \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z)\, u(c(x;\theta))\, dx\, dt = \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z)\, \frac{\partial u(c(x))}{\partial c} \underbrace{\frac{dc(x;\theta)}{d\theta}}_{\substack{\text{Instantaneous Consumption-} \\ \text{Equivalent Effect}}} dx\, dt$$

because the endowment process z_t was fully exogenous. Now, we have instead

$$\begin{aligned}
\frac{dV(z;\theta)}{d\theta} &= \frac{d}{d\theta} \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z; \theta)\, u(c(x;\theta))\, dx\, dt \\
&= \int_0^\infty e^{-\rho t} \int \left[\frac{dp(t, x \mid 0, z; \theta)}{d\theta} p(t, x \mid 0, z; \theta)\, u(c(x;\theta)) + p(t, x \mid 0, z; \theta) \frac{\partial u(c(x;\theta))}{\partial c} \frac{dc(x;\theta)}{d\theta} \right] dx\, dt \\
&= \int_0^\infty e^{-\rho t} \int p(t, x \mid 0, z; \theta)\, \frac{\partial u(c(x;\theta))}{\partial c} \left[\frac{dp(t, x \mid 0, z; \theta)}{d\theta} \, \frac{u(c(x;\theta))}{\frac{\partial u(c(x;\theta))}{\partial c}} + \frac{dc(x;\theta)}{d\theta} \right] dx\, dt.
\end{aligned}$$

We can therefore generalized our definition of DS-weights as follows.

Definition 2. (**DS-Weights: Endogenous Transition Densities**) *A DS-planner finds a policy $d\theta$ desirable if and only if $\frac{dW^{DS}}{d\theta} > 0$, where*

$$\frac{dW^{DS}}{d\theta} = \int \int_0^\infty \int \omega(t, x \mid 0, z; \theta) \left[\underbrace{\frac{dp(t, x \mid 0, y; \theta)}{d\theta} \, \frac{u(c(x;\theta))}{\frac{\partial u(\cdot)}{\partial c}}}_{\text{Transition Density Effect}} + \underbrace{\frac{dc(x;\theta)}{d\theta}}_{\substack{\text{Instantaneous Consumption-} \\ \text{Equivalent Effect}}} \right] g_0(z)\, dx\, dt\, dz,$$

where $\omega(t, x \mid 0, z)$ denotes the DS-weight associated with an individual that starts in state $(a_0, z_0) = z$ at time 0 and transitions to state $(a_t, z_t) = x$ at time t.

The key novel object that appears in this formulation is $\frac{dp}{d\theta}$. Our key result in this subsection is the following Proposition.

Proposition 6. (**Evolution of Transition Density Effect**) *The transition density effect $\frac{d}{d\theta} p(t, x \mid 0, z; \theta)$*

is a weak solution to the Kolmogorov forward equation with forcing term

$$\partial_t p_\theta = \mathcal{A}^* p_\theta + \frac{d}{d\theta}\mathcal{A}^*(\theta)p(\theta)$$

with initial condition $p_\theta(0, x \mid 0, y; \theta) = 0$*. Rearranging, we have*

$$p_\theta = \left[\partial_t - \mathcal{A}^*\right]^{-1} \frac{d}{d\theta}\mathcal{A}^*(\theta)p,$$

where $\frac{d}{d\theta}\mathcal{A}^*(\theta)p$ *is characterized in the proof. For the Huggett economy presented in Section 2.1, we have*

$$\frac{d}{d\theta}\mathcal{A}_t^*(\theta)f_t(a,z) = -\partial_a\left[\left(\frac{dr_t(\theta)}{d\theta}a + \frac{dw_t(\theta)}{d\theta}e^z - \frac{dc_t(a,z;\theta)}{d\theta}\right)f_t(a,z)\right]$$

The proof of Proposition 6 is in Appendix A.1.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2021. Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach.

Dávila, Eduardo and Andreas Schaab. 2021. Welfare Assessments with Heterogeneous Individuals.

Huggett, Mark. 1993. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of economic Dynamics and Control*, 17(5):953–969.

Oksendal, Bernt. 2013. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media.

Schaab, Andreas and Allen Tianlun Zhang. 2021. Dynamic Programming in Continuous Time with Adaptive Sparse Grids.

A Proofs

A.1 Proof of Proposition 6

Proof. For a given θ, the transition density is still characterized by the Kolmogorov forward equation

$$\partial_t p(\theta) = \mathcal{A}^*(\theta)p(\theta).$$

Differentiating fully with respect to θ and denoting $p_\theta = \frac{dp}{d\theta}$, we obtain

$$\partial_t p_\theta = \frac{d}{d\theta}\mathcal{A}^*(\theta)p(\theta) + \mathcal{A}^* p_\theta$$

Consider the Huggett model with

$$\mathcal{A}_t = \Big(r_t a + w_t e^z - c_t(a,z)\Big)\partial_a - \mu z \partial_z + \frac{\sigma^2}{2}\partial_{zz}$$

and

$$\mathcal{A}_t^* f_t(a,z) = -\partial_a\Big[\Big(r_t(\theta)a + w_t(\theta)e^z - c_t(a,z;\theta)\Big)f_t(a,z)\Big] + \partial_z\Big[\mu z f_t(a,z)\Big] + \frac{\sigma^2}{2}\partial_{zz}f_t(a,z).$$

Differentiating, we obtain

$$\frac{d}{d\theta}\mathcal{A}_t^*(\theta)f_t(a,z) = -\partial_a\left[\left(\frac{dr_t(\theta)}{d\theta}a + \frac{dw_t(\theta)}{d\theta}e^z - \frac{dc_t(a,z;\theta)}{d\theta}\right)f_t(a,z)\right]$$

■