Dynamic Optimization: Problem Set #2

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Problem 1: proof of contraction mapping theorem

Credit: David Laibson

In class, we defined the Bellman operator B, which operates on functions w, and is defined by

$$(Bw)(x) \equiv \max_{x' \in \Gamma(x)} \Big\{ F(x, x') + \beta w(x') \Big\}$$

for all $x \in \mathcal{X}$ in the state space, where $\Gamma(x)$ is some constraint set—in our case, this was the budget constraint. The definition is expressed pointwise, but it applies to all possible values in the state space. We call B an operator because it maps a function w to a new function Bw. So both w and Bw map $\mathcal{X}$ into $\mathbb{R}$. Operator B maps *functions* and is therefore called a functional operator. In class, we showed that the solution of the Bellman equation—the value function—is a fixed point of the Bellman operator.

What does it mean to *iterate* $B^n w$?

$$\begin{aligned}
(Bw)(x) &= \max_{x' \in \Gamma(x)} \Big\{ F(x, x') + \beta w(x') \Big\} \\
(B(Bw))(x) &= \max_{x' \in \Gamma(x)} \Big\{ F(x, x') + \beta (Bw)(x') \Big\} \\
(B(B^2 w))(x) &= \max_{x' \in \Gamma(x)} \Big\{ F(x, x') + \beta (B^2 w)(x') \Big\} \\
&\vdots \\
(B(B^n w))(x) &= \max_{x' \in \Gamma(x)} \Big\{ F(x, x') + \beta (B^n w)(x') \Big\}.
\end{aligned}$$

What does it mean for functions to converge to a limiting function? Let v_0 be some guess for the value function, then convergence would mean

$$\lim_{n\to\infty} B^n v_0 = v.$$

And why might $B^n w$ converge as $n \to \infty$? The answer is that B is a *contraction mapping*.

Definition 1. Let (S, d) be a metric space and $B : S \to S$ be a function that maps S into intself. B is a contraction mapping if for some $\beta \in (0, 1)$, $d(Bf, Bg) \leq \beta d(f, g)$, for any two functions f and g.[1]

Intuitively, B is a contraction mapping if applying the operator B to any two functions f and g (that are not the same) moves them strictly closer together. Bf and Bg are strictly closer together than f and g. We can now state the contraction mapping theorem.

Theorem 2. *If (S, d) is a complete metric space and $B : S \to S$ is a contraction mapping, then:*

(i) B has exactly one fixed point $v \in S$

(ii) For any $v_0 \in S$, $\lim_{n\to\infty} B^n v_0 = v$

(iii) $B^n v_0$ has an exponential convergence rate at least as great as $-\ln(\beta)$

In this problem, we will illustrate and prove the contraction mapping theorem.

(a) Consider the contraction mapping $(Bw)(x) \equiv h(x) + \alpha w(x)$ with $\alpha \in (0, 1)$. Iteratively apply the operator B and show that

$$\lim_{n\to\infty} (B^n f)(x) = \frac{h(x)}{1-\alpha}$$

Argue that this shows that the fixed point of this operator B is consequently the function $v(x) = \frac{1}{1-\alpha} h(x)$. Show that $(Bv)(x) = v(x)$.

(b) Now we will prove the contraction mapping theorem in 3 steps (we will not prove the convergence rate). Show that $\{B^n f_0\}_{n=0}^{\infty}$ is a Cauchy sequence. (Cauchy sequence definition: Fix any $\epsilon > 0$. Then there exists N such that $d(B^m f_0, B^n f_0) \leq \epsilon$ for all $m, n \leq N$.)

Solution. Choose some $f_0 \in S$. Let $f_n = B^n f_0$. Since B is a contraction

$$d(f_2, f_1) = d(Bf_1, Bf_0) \leq \delta d(f_1, f_0).$$

[1] A metric d is a way of representing the distance between two functions, or two members of (metric) space S. One example: the supremum pointwise gap.

Continuing by induction,

$$d(f_{n+1}, f_n) \leq \delta^n d(f_1, f_0) \quad \forall n$$

We can now bound the distance between f_n and f_m when $m > n$.

$$\begin{aligned} d(f_m, f_n) &\leq d(f_m, f_{m-1}) + ... + d(f_{n+2}, f_{n+1}) + d(f_{n+1}, f_n) \\ &\leq \left[\delta^{m-1} + ... + \delta^{n+1} + \delta^n\right] d(f_1, f_0) \\ &= \delta^n \left[\delta^{m-n-1} + ... + \delta^1 + 1\right] d(f_1, f_0) \\ &< \frac{\delta^n}{1-\delta} d(f_1, f_0) \end{aligned}$$

So $\{f_n\}_{n=0}^{\infty}$ is Cauchy. ✓

(c) Show that the limit point v is a fixed point of B.

Solution. Since S is complete $f_n \rightarrow v \in S$.

We now have a candidate fixed point $v \in S$.

To show that $Bv = v$, note

$$\begin{aligned} d(Bv, v) &\leq d(Bv, B^n f_0) + d(B^n f_0, v) \\ &\leq \delta d(v, B^{n-1} f_0) + d(B^n f_0, v) \end{aligned} .$$

These inequalities must hold for all n.

And both terms on the RHS converge to zero as $n \rightarrow \infty$.

So $d(Bv, v) = 0$, implying that $Bv = v$. ✓

(d) Show that only one fixed point exists.

Solution. Now we show that our fixed point is unique.

Suppose there were two fixed points: $v \neq v^*$.

Then $Bv = v$ and $Bv^* = v^*$ (since fixed points).

Also have $d(Bv, Bv^*) < d(v, v^*)$ (since B is contraction)

So, $d(v, v^*) = d(Bv, Bv^*) < d(v, v^*)$.

Contradiction.

So the fixed point is unique. ✓

Problem 2: Blackwell's sufficiency conditions

Credit: David Laibson

We now show that there are in fact sufficient conditions for an operator to be contraction mapping.

Theorem 3. *(Blackwell's sufficient conditions) Let $X \subset \mathbb{R}^l$ and let $C(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup-metric. Let $B : C(X) \to C(X)$ be an operator satisfying two conditions:*

1. *monotonicity: if $f, g \in C(X)$ and $f(x) \leq g(x)$ $\forall x \in X$,*
   then $(Bf)(x) \leq (Bg)(x)$, $\forall x \in X$

2. *discounting: there exists some $\delta \in (0, 1)$ such that*

$$[B(f + a)](x) \leq (Bf)(x) + \delta a \ \ \forall\, f \in C(X),\ a \geq 0,\ x \in X.$$

Then, B is a contraction with modulus δ.

Note that a is a constant and $(f + a)$ is the function generated by adding a to the function f. Blackwell's conditions are sufficient but not necessary for B to be a contraction.

In this problem, we will prove these sufficient conditions.

(a) Let d be the sup-metric and show that, for any $f, g \in C(X)$, we have $f(x) \leq g(x) + d(f, g)$ for all x

Solution. Follows directly from definition of sup-metric.

(b) Use monotonicity and discounting to show that, for any $f, g \in C(X)$, we have $(Bf)(x) \leq (Bg)(x) + \delta d(f, g)$ and $(Bg)(x) \leq (Bf)(x) + \delta d(f, g)$.

Solution. Using monotonicity and discounting, we have for all x

$$(Bf)\,(x) \leq [B(g + d(f, g))](x) \leq (Bg)\,(x) + \delta d(f, g)$$
$$(Bg)\,(x) \leq [B(f + d(f, g))](x) \leq (Bf)\,(x) + \delta d(f, g)$$

(c) Combine these to show that $d(Bf, Bg) \leq \delta d(f, g)$.

Solution.

$$(Bf)\,(x) - (Bg)\,(x) \leq \delta d(f, g)$$
$$(Bg)\,(x) - (Bf)\,(x) \leq \delta d(f, g)$$
$$|(Bf)(x) - (Bg)(x)| \leq \delta d(f, g)$$
$$\sup_x |(Bf)(x) - (Bg)(x)| \leq \delta d(f, g)$$
$$d(Bf, Bg) \leq \delta d(f, g) \quad \checkmark$$

Problem 3: example of Blackwell's conditions

Credit: David Laibson

We will now work out a simple example to illustrate these sufficient conditions. In particular, consider the Bellman operator in a consumption problem (with stochastic asset returns, stochastic labor income, and a liquidity constraint).

$$(Bf)(x) = \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \quad \forall x$$

1. Check the first of Blackwell's conditions: monotonicity

   Solution. Assume $f(x) \leq g(x)\ \forall x$. Suppose c_f^* is the optimal policy when the continuation value function is f.

$$\begin{aligned}
(Bf)(x) &= \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \\
&= u(c_f^*) + \delta E f(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}) \\
&\leq u(c_f^*) + \delta E g(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}) \\
&\leq \sup_{c\in[0,x]} \left\{u(c) + \delta E g(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \\
&= (Bg)(x)
\end{aligned}$$

2. Check the second of Blackwell's conditions: discounting

   Solution. Adding a constant (Δ) to an optimization problem does not affect optimal choice, so

$$\begin{aligned}
[B(f+\Delta)](x) &= \sup_{c\in[0,x]} \left\{u(c) + \delta E\left[f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) + \Delta\right]\right\} \\
&= \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} + \delta\Delta \\
&= (Bf)(x) + \delta\Delta
\end{aligned}$$

Problems 4 – 8

For all other solutions, you can check here (**credit** David Laibson): https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson