Dynamic Optimization: Problem Set #4

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Problem 1 (hard)

In this problem, we work through a simple variant of the seminal Brunnermeier and Sannikov (2014, AER) paper.

We study the consumption, savings, and investment problem of an agent. We denote by $\{D_t\}_{t\geq 0}$ the *dividend stream* and by $\{Q_t\}_{t\geq 0}$ the asset price. Assume that the asset price evolves according to

$$\frac{dQ}{Q} = \mu_Q dt + \sigma_Q dB,$$

where you can interpret μ_Q and σ_Q as simple constants (alternatively, think of them as more complicated objects that would be determined in general equilibrium, which we abstract from here).

This is a model of two assets, capital and bonds. Bonds pay the riskfree rate of return r_t. Capital is accumulated and owned by the agent. Capital is traded at price Q_t and yields dividends at rate D_t.

The key interesting feature of this problem is that the agent faces both (idiosyncratic) earnings risk and (aggregate) asset price risk.

Gerard: clean up the following Households take as given all aggregate prices and behave according to preferences given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

Households consume and save, investing their wealth into bonds and capital. Letting k denote a household's units of capital owned and b units of bonds, the budget constraint is characterized by

$$\begin{aligned} dk_t &= \Phi(\iota_t) k_t - \delta k_t \\ db_t &= r_t b_t + D_t k_t + w_t z_t - c_t - \iota_t k_t. \end{aligned}$$

The rate of investment is given by ι_t. Investment adjustment costs are captured by the concave technology Φ. Dividends are paid to households in units of the numeraire, thus entering the equation for db. The law of motion for household earnings are given by

$$dz = \mu_z dt + \sigma_z dW.$$

I will rewrite the household problem in terms of liquid net worth, defined by the equations

$$\theta n = Qk$$
$$(1-\theta)n = b,$$

so that total liquid net worth is $n = Qk + b$. Thus, $dn = kdQ + Qdk + (dk)(dQ) + db$. For now, I assume that households' capital accumulation is non-stochastic. That is, there is no capital quality risk, and therefore $(dk)(dQ) = 0$. Plugging in and simplifying,

$$dn = rn + \theta n \left[\frac{D-\iota}{Q} + \frac{dQ}{Q} + \Phi(\iota) - \delta - r \right] + wz - c.$$

Two observations are in order. First, the choice of ι is entirely static in this setting, yielding the optimality condition

$$\Phi'(\iota) = \frac{1}{Q},$$

so that optimal household investment is only a function of the price of capital, $\iota = \iota(Q)$. Second, households take as given aggregate "prices" (r, w, D, Q).

These two features of the household problem allow me to work with the following, simplified representation. Define

$$dR = \underbrace{\frac{D-\iota(Q)}{Q}}_{\text{Dividend yield}} dt + \underbrace{\Big[\Phi(\iota(Q)) - \delta\Big] dt + \frac{dQ}{Q}}_{\text{Capital gains}} \equiv \mu_R dt + \sigma_R dB$$

to be the effective rate of return on households' capital investments. I use

$$\mu_R = \frac{D-\iota(Q)}{Q} + \Phi(\iota(Q)) - \delta + \mu_Q$$
$$\sigma_R = \sigma_Q.$$

After internalizing the optimal internal rate of capital investment, $\iota = \iota(Q)$, this return is exogenous from the perspective of the household: it depends on macro conditions and prices, but not on the particular portfolio composition of the household. I can therefore rewrite the law of motion of the household's liquid net worth as $dn = rn + \theta n(dR - r) + wz - c$, or

$$dn = rn + \theta n(\mu_R - r) + wz - c + \theta n \sigma_R dB.$$

State space. We denote the agent's individual states by (n, z). There is a variable Γ_t that evolves according to the diffusion process

$$d\Gamma = \mu_\Gamma dt + \sigma_\Gamma dB.$$

We call Γ the *aggregate state of the economy*. And we assume that all capital prices are functions of it, that is,

$$r_t = r(\Gamma_t), \quad D_t = D(\Gamma_t), \quad Q_t = Q(\Gamma_t).$$

This now allows us to write the household problem recursively.

Recursive representation. Assume on top that dividends and prices The household problem can therefore be written in terms of the household state variables (n, z) as well as the aggregate state space Γ. We have

$$\rho V(n, z, \Gamma) = \max_{c,\theta} \Big\{ u(c) + V_n \Big[rn + \theta n(\mu_R - r) + wz - c \Big] + \frac{1}{2} V_{nn} (\theta n \sigma_R)^2 + V_z \mu_z + \frac{1}{2} V_{zz} \sigma_z^2$$
$$+ V_{n\Gamma} \theta n \sigma_R \sigma_\Gamma + V_\Gamma \mu_\Gamma + \frac{1}{2} \sigma_\Gamma^T V_{\Gamma\Gamma} \sigma_\Gamma \Big\},$$

where I assume that $\mathbb{E}(dWdB) = 0$ so that households' earnings risk is, for now, uncorrelated with the aggregate productivity risk factor. The first-order conditions for consumption and portfolio choice are given by

$$u_c = V_n$$
$$\theta = -\left(\frac{V_n}{nV_{nn}} \frac{\mu_R - r}{\sigma_R^2} + \frac{V_{n\Gamma}}{nV_{nn}} \frac{\sigma_\Gamma}{\sigma_R} \right).$$

The household HJB and the corresponding optimality condition hold everywhere in the interior of the household state space.

Lemma 1. *The household Euler equation for marginal utility is given by*

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu_R - r}{\sigma_R} dB - \gamma \frac{c_z}{c} \sigma_z dW.$$

Proof. The HJB envelope condition is given by

$$(\rho - r)V_n = V_{nn}s + V_n \theta(\mu_R - r) + \frac{1}{2} V_{nnn} (\theta n \sigma_R)^2 + V_{nn} n (\theta \sigma_R)^2 + V_{ny} \mu_y + \frac{\sigma_y^2}{2} V_{yny}$$
$$+ V_{nn\Gamma} \theta n \sigma_R \boldsymbol{\sigma}_\Gamma + V_{n\Gamma} \theta \sigma_R \boldsymbol{\sigma}_\Gamma + V_{n\Gamma} \boldsymbol{\mu}_\Gamma + \frac{1}{2} \boldsymbol{\sigma}_\Gamma^T V_{n\Gamma\Gamma} \boldsymbol{\sigma}_\Gamma.$$

Applying Ito's lemma to $V_n(n, y, \Gamma)$, we have

$$\begin{aligned}
dV_n &= V_{nn}dn + \frac{1}{2}V_{nnn}(dn)^2 + V_{ny}dy + \frac{1}{2}V_{nyy}(dy)^2 + V_{n\Gamma}d\Gamma + \frac{1}{2}V_{n\Gamma\Gamma}(d\Gamma)^2 + V_{nn\Gamma}(dn)(d\Gamma) \\
&= V_{nn}(s + \theta n \sigma_R dB) + \frac{1}{2}V_{nnn}(\theta n \sigma_R)^2 + V_{ny}(\mu_y + \sigma_y dW) + \frac{1}{2}V_{nyy}\sigma_y^2 \\
&\quad + V_{n\Gamma}(\boldsymbol{\mu}_\Gamma + \boldsymbol{\sigma}_\Gamma dB) + \frac{1}{2}\boldsymbol{\sigma}_\Gamma^T V_{n\Gamma\Gamma}\boldsymbol{\sigma}_\Gamma + V_{nn\Gamma}(\theta n \sigma_R)(\boldsymbol{\sigma}_\Gamma).
\end{aligned}$$

Putting this together with

$$\begin{aligned}
(\rho - r)V_n - V_n\theta(\mu_R - r) - V_{nn}n(\theta\sigma_R)^2 - V_{n\Gamma}\theta\sigma_R\boldsymbol{\sigma}_\Gamma =& V_{nn}s + \frac{1}{2}V_{nnn}(\theta n\sigma_R)^2 + V_{ny}\mu_y + \frac{\sigma_y^2}{2}V_{yny} \\
&+ V_{nn\Gamma}\theta n\sigma_R\boldsymbol{\sigma}_\Gamma + V_{n\Gamma}\boldsymbol{\mu}_\Gamma + \frac{1}{2}\boldsymbol{\sigma}_\Gamma^T V_{n\Gamma\Gamma}\boldsymbol{\sigma}_\Gamma,
\end{aligned}$$

we have

$$\begin{aligned}
dV_n =& V_{nn}\theta n\sigma_R dB + V_{ny}\sigma_y dW + V_{n\Gamma}\boldsymbol{\sigma}_\Gamma dB \\
&+ (\rho - r)V_n - V_n\theta(\mu_R - r) - V_{nn}n(\theta\sigma_R)^2 - V_{n\Gamma}\theta\sigma_R\boldsymbol{\sigma}_\Gamma.
\end{aligned}$$

Using

$$\theta n V_{nn} = -\left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{n\Gamma}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right),$$

we have

$$\begin{aligned}
dV_n =& -\left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{n\Gamma}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right)\sigma_R dB + V_{ny}\sigma_y dW + V_{n\Gamma}\boldsymbol{\sigma}_\Gamma dB \\
&+ (\rho - r)V_n - V_n\theta(\mu_R - r) + \left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{n\Gamma}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right)\theta\sigma_R^2 - V_{n\Gamma}\theta\sigma_R\boldsymbol{\sigma}_\Gamma.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
dV_n =& -V_n\frac{\mu_R - r}{\sigma_R}dB - V_{n\Gamma}\boldsymbol{\sigma}_\Gamma dB + V_{ny}\sigma_y dW + V_{n\Gamma}\boldsymbol{\sigma}_\Gamma dB \\
&+ (\rho - r)V_n - V_n\theta(\mu_R - r) + V_n(\mu_R - r)\theta + V_{n\Gamma}\boldsymbol{\sigma}_\Gamma\theta\sigma_R - V_{n\Gamma}\theta\sigma_R\boldsymbol{\sigma}_\Gamma.
\end{aligned}$$

Combining terms yield the result. ■

Lemma 2. *The household Euler equation for consumption is given by*

$$\frac{dc}{c} = \frac{r - \rho}{\gamma}dt + \frac{1}{2}(1 + \gamma)\left[\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 + \left(\frac{c_z}{c}\sigma_z\right)^2\right]dt + \frac{\mu_R - r}{\gamma\sigma_R}dB + \frac{c_z}{c}\sigma_z dW.$$

Proof. The consumption policy function is given by $c = c(n, y, \Gamma)$. Thus,

$$dc = c_n dn + c_y dy + c_\Gamma d\Gamma + \frac{1}{2}c_{nn}(dn)^2 + \frac{1}{2}c_{yy}\sigma_y^2 + \frac{1}{2}\boldsymbol{\sigma}_\Gamma^T c_{\Gamma\Gamma}\boldsymbol{\sigma}_\Gamma + c_{n\Gamma}\theta n\sigma_R\boldsymbol{\sigma}_\Gamma,$$

and so

$$\begin{aligned}
(dc)^2 &= \Big(c_n\theta n\sigma_R dB + c_y\sigma_y dW + c_\Gamma\boldsymbol{\sigma}_\Gamma dB\Big)^2 \\
&= \Big(c_n\theta n\sigma_R dB + c_\Gamma\boldsymbol{\sigma}_\Gamma dB\Big)^2\Big(c_y\sigma_y dW\Big)^2 \\
&= \Big[(c_n\theta n\sigma_R)^2 + 2c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma + (c_\Gamma\boldsymbol{\sigma}_\Gamma)^2 + (c_y\sigma_y)^2\Big]dt.
\end{aligned}$$

Let's simplify this expression a bit. Notice that we have

$$\begin{aligned}
(c_n n\sigma_R)^2\theta\theta + 2c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma &= -(c_n n\sigma_R)^2\theta\left(\frac{u_c}{nu_{cc}c_n}\frac{\mu_R - r}{\sigma_R^2} + \frac{u_{cc}c_\Gamma}{nu_{cc}c_n}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right) + 2c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma \\
&= -(c_n n\sigma_R)^2\theta\frac{1}{nc_n}\left(-c\frac{\mu_R - r}{\gamma\sigma_R^2} + c_\Gamma\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right) + 2c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma \\
&= (c_n n\sigma_R)^2\theta\frac{1}{nc_n}c\frac{\mu_R - r}{\gamma\sigma_R^2} - (c_n n\sigma_R)\theta c_\Gamma\boldsymbol{\sigma}_\Gamma + 2c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma \\
&= \left(\frac{\mu_R - r}{\gamma\sigma_R^2} - \frac{c_\Gamma}{c}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right)c^2\frac{\mu_R - r}{\gamma} + c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma \\
&= c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - c_\Gamma\boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} + c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma.
\end{aligned}$$

Therefore, we have

$$(dc)^2 = \left[c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - c_\Gamma\boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} + c_n c_\Gamma\theta n\sigma_R\boldsymbol{\sigma}_\Gamma + (c_\Gamma\boldsymbol{\sigma}_\Gamma)^2 + (c_y\sigma_y)^2\right]dt$$

We can take one last step here, noting that

$$\begin{aligned}
(dc)^2 &= \left[c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - c_\Gamma\boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} + c_\Gamma\boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} - (c_\Gamma\boldsymbol{\sigma}_\Gamma)^2 + (c_\Gamma\boldsymbol{\sigma}_\Gamma)^2 + (c_y\sigma_y)^2\right]dt \\
&= \left[c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 + (c_y\sigma_y)^2\right]dt.
\end{aligned}$$

I haven't used this yet in the rest of the proof, but it's the same thing...

Using Ito's lemma for $u_c(c)$, we have

$$du_c = u_{cc}dc + \frac{1}{2}u_{ccc}(dc)^2.$$

Plugging in and using CRRA, we have

$$(\rho - r)dt - \frac{\mu_R - r}{\sigma_R}dB - \gamma\frac{c_y}{c}\sigma_y dW$$
$$= \frac{u_{cc}}{u_c}dc + \frac{1}{2}\frac{u_{ccc}}{u_c}\left[c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - c_\Gamma \boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} + c_n c_\Gamma \theta n \sigma_R \boldsymbol{\sigma}_\Gamma + (c_\Gamma \boldsymbol{\sigma}_\Gamma)^2 + (c_y\sigma_y)^2\right]dt.$$

Plugging in for CRRA,

$$\frac{r-\rho}{\gamma}dt + \frac{\mu_R - r}{\gamma\sigma_R}dB + \frac{c_y}{c}\sigma_y dW$$
$$= \frac{dc}{c} - \frac{1}{2}\frac{1+\gamma}{c^2}\left[c^2\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - c_\Gamma \boldsymbol{\sigma}_\Gamma c\frac{\mu_R - r}{\gamma\sigma_R} + c_n c_\Gamma \theta n \sigma_R \boldsymbol{\sigma}_\Gamma + (c_\Gamma \boldsymbol{\sigma}_\Gamma)^2 + (c_y\sigma_y)^2\right]dt.$$

Therefore, we have

$$\begin{aligned}\frac{dc}{c} =& \frac{r-\rho}{\gamma}dt + \frac{\mu_R - r}{\gamma\sigma_R}dB + \frac{c_y}{c}\sigma_y dW \\ &+ \frac{1}{2}(1+\gamma)\left[\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - \frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\frac{\mu_R - r}{\gamma\sigma_R} + \frac{1}{c^2}c_n c_\Gamma \theta n \sigma_R \boldsymbol{\sigma}_\Gamma + \left(\frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\right)^2 + \left(\frac{c_y}{c}\sigma_y\right)^2\right]dt.\end{aligned}$$

Lastly, I can substitute in for θ again, so that this term becomes

$$\begin{aligned}\frac{1}{c^2}c_n c_\Gamma \theta n \sigma_R \boldsymbol{\sigma}_\Gamma &= \frac{1}{c^2}c_n c_\Gamma n \sigma_R \boldsymbol{\sigma}_\Gamma \frac{c}{nc_n}\left(\frac{\mu_R - r}{\gamma\sigma_R^2} - \frac{c_\Gamma}{c}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right) \\ &= \frac{1}{c}c_\Gamma \sigma_R \boldsymbol{\sigma}_\Gamma\left(\frac{\mu_R - r}{\gamma\sigma_R^2} - \frac{c_\Gamma}{c}\frac{\boldsymbol{\sigma}_\Gamma}{\sigma_R}\right) \\ &= \frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\frac{\mu_R - r}{\gamma\sigma_R} - \left(\frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dc}{c} =& \frac{r-\rho}{\gamma}dt + \frac{\mu_R - r}{\gamma\sigma_R}dB + \frac{c_y}{c}\sigma_y dW \\ &+ \frac{1}{2}(1+\gamma)\left[\left(\frac{\mu_R - r}{\gamma\sigma_R}\right)^2 - \frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\frac{\mu_R - r}{\gamma\sigma_R} + \frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\frac{\mu_R - r}{\gamma\sigma_R} - \left(\frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\right)^2 + \left(\frac{c_\Gamma}{c}\boldsymbol{\sigma}_\Gamma\right)^2 + \left(\frac{c_y}{c}\sigma_y\right)^2\right]dt.\end{aligned}$$

Simplifying yields the result.

■

Comparison to Brunnermeier and Sannikov. It turns out that I already derived the analog to Bru-San's Proposition II.2 a long time ago. I have

$$\frac{dV_n}{V_n} = (\rho - r) - \frac{\mu_R - r}{\sigma_R}dB + \frac{V_{nz}}{V_n}\sigma_z dW,$$

where they call $V_n = \theta$. For the sake of comparison, let $V_n = \theta^{\text{BS}}$. Then,

$$\frac{d\theta^{\text{BS}}}{\theta^{\text{BS}}} = \mu_\theta dt + \sigma_\theta^B dB + \sigma_\theta^W dW,$$

where

$$\begin{aligned} \mu_\theta &= \rho - r \\ \underbrace{-\sigma_Q \sigma_\theta^B}_{\text{Risk premium}} &= \underbrace{\frac{D - \iota(Q)}{Q} + \Phi(\iota(Q)) - \delta + \mu_Q - r}_{\text{Expected excess return on capital}} \\ \sigma_\theta^W &= \frac{V_{nz}}{V_n} \sigma_z. \end{aligned}$$

Of course, they don't have earnings risk.

Problem 2

Credit: Gabriel Chodorow-Reich

Gerard: add "Homework4" here (whatever you find valuable)

Problem 3

We now solve a version of the income fluctuations problem in continuous time. In discrete time, the problem is as follows:

Discrete time. The canonical buffer stock model in discrete-time is a variant of the life-cycle model of consumption featuring idiosyncratic income risk. A household's preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

and the critical assumption is that $\beta R = 1$, where β is the household's discount parameter and R is the gross interest rate. In a model without uncertainty, this assumption would imply a constant consumption profile over time.

The household's budget constraint is encoded in its evolution of wealth, given by

$$a_{t+1} = R(y_t + a_t - c_t), \tag{2}$$

where $y_t \sim^{iid} F$ is an income shock that is independent and identically distributed over time. We may furthermore assume that agents face an exogenously determined, uniform borrowing constraint, $a_t \geq \underline{a}$, that is tighter than the natural borrowing constraint.

The recursive problem of the household can then be written using the Bellman equation

$$v(a_t) = \max_{c_t} u(c_t) + \beta E_t[v(a_{t+1})], \tag{3}$$

subject to

$$a_{t+1} = R(a_t + y_t - c_t)$$
$$a_t \geq 0.$$

Continuous time without borrowing constraint. In continuous time, the evolution of wealth is given by

$$da_t = (ra_t - c_t)dt + \sigma dB_t, \tag{4}$$

where the Brownian term is the appropriate analog to independent and identically distributed income shocks in discrete time.

We now assume that households are not subject to a borrowing constraint.

Hence, the stationary Hamilton-Jacobi-Bellman (HJB) equation can be written as

$$\rho v(a_t) = \max_{c_t} u(c_t) + v'(a_t)[ra_t - c_t] + \frac{1}{2}v''(a_t)\sigma^2. \tag{5}$$

The HJB in our simple setting is given by an ordinary instead of a partial differential equation because we only have one state variable. The implied first-order condition for consumption is given by

$$u'(c_t) = v'(a_t), \tag{6}$$

where c_t now denotes the optimal consumption policy function. Differentiating the FOC again with respect to wealth, we have $u''(c_t)c'(a_t) = v''(a_t)$. Similarly, differentiating the HJB with respect to wealth yields

$$\rho v'(a_t) = u'(c_t)c'(a_t) + v''(a_t)[ra_t - c(a_t)] + [r - c'(a_t)]v'(a_t) + \frac{1}{2}v'''(a_t)\sigma^2,$$

where we could drop the max operator since $c(a_t)$ denotes the policy function. Simplifying and using the first-order conditions, we have the following HJB envelope condition

$$(\rho - r)v'(a_t) = v''(a_t)[ra_t - c(a_t)] + \frac{1}{2}v'''(a_t)\sigma^2.$$

To obtain a closed-form solution to the buffer-stock model, let utility be log so that $u(c_t) = \ln(c_t)$. We will write the consumption policy function as $c(a_t)$. The first-order condition for consumption then implies that

$$v'(a_t) = \frac{1}{c(a_t)}. \tag{7}$$

Integrating both sides with respect to a_t yields

$$v(a_t) = \frac{1}{c'(a_t)} \ln[c(a_t)] + \kappa. \tag{8}$$

The HJB must therefore satisfy

$$\rho\kappa + \frac{\rho}{c'(a_t)} \ln[c(a_t)] = \ln[c(a_t)] + \frac{ra_t}{c(a_t)} - 1 - \frac{\sigma^2}{2} \frac{c'(a_t)}{c(a_t)^2}.$$

We can see immediately that the $c(a_t)^2$ term in the denominator on the RHS is going to make solving for a policy function c very difficult. It also implies a simple fix, however: Consider an alternative wealth evolution equation given by $da_t = (ra_t - c_t)dt + \sigma a_t dB_t$. Then the HJB becomes $\rho v(a_t) = u(c_t) + v'(a_t)[ra_t - c_t] + \frac{\sigma^2}{2} a_t^2 v''(a_t)$. Using the Ansatz that the policy function for consumption is linear in wealth, in particular $c(a_t) = \rho a_t$, we have

$$\rho\kappa + \ln(\rho a_t) = \ln(\rho a_t) + \frac{r-\rho}{\rho} - \frac{\sigma^2 a_t^2}{2\rho a_t^2}.$$

The postulated value function (8) therefore satisfies the HJB for $\kappa = \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}$. The closed-form solution of the buffer stock model with log utility is then given by

$$c(a_t) = \rho a_t \tag{9}$$

$$v(a_t) = \frac{1}{\rho} \ln(\rho a_t) + \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}. \tag{10}$$

Problem 4

Credit: David Laibson

Solve the *"eat the pie problem"* on problem set (PSET) #2 here: https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson.

Problem 5

To illustrate this approach, I will consider a simple problem where household preferences are

$$\max_{\{c_t\}} \int_0^\infty e^{-\rho t} u(c_t) dt.$$

There is no uncertainty. The household budget constraint is given by

$$da_t = r_t a_t + w_t - c_t.$$

Aggregate prices follow a deterministic process, $\boldsymbol{r} = \{r_t\}$ and $\boldsymbol{w} = \{w_t\}$. Finally, the household starts with an initial wealth position a_0, and wealth is the only state variable.

Step #1: Lifetime budget constraint

Lemma 3. *(Lifetime Budget Constraint) For any linear ODE*

$$\frac{dy}{dt} = r(t)y(t) + x(t)$$

we have the integration result

$$y(T) = y(0)e^{\int_0^T r(s)ds} + \int_0^T e^{\int_t^T r(s)ds} x(t)dt.$$

Using this formula for the household budget constraint, I arrive at the lifetime budget constraint

$$\lim_{T\to\infty} a(T) = a(0) \lim_{T\to\infty} e^{\int_0^T r(s)ds} - \lim_{T\to\infty} \int_0^T e^{\int_t^T r(s)ds} \Big[c(t) - w_t\Big] dt.$$

Under the no-Ponzi assumption of $\lim_{T\to\infty} a(T) = 0$*, this further simplifies to*

$$\int_0^\infty e^{-\int_0^t r(s)ds} c(t)dt = a(0) + \int_0^\infty e^{-\int_0^t r(s)ds} w(t)dt \equiv W.$$

Proof. Consider any ODE

$$\frac{dy}{dt} = r(t)y(t) + x(t).$$

Using an integrating factor approach, we have

$$e^{\int -r(s)ds}\frac{dy}{dt} - e^{\int -r(s)ds} r(t)y(t) = e^{\int -r(s)ds} x(t).$$

The LHS can then be written as a product rule, so that

$$\frac{d}{dt}\left(y(t)e^{\int -r(s)ds}\right) = \frac{dy}{dt}e^{\int -r(s)ds} + y(t)e^{\int -r(s)ds}\frac{d}{dt}\left(\int -r(s)ds\right) = e^{\int -r(s)ds}x(t).$$

The last derivative follows from the fundamental theorem of calculus for indefinite integrals.

Alternatively, since I know that I will work on the definite time horizon $t \in [0, T]$, I can choose a slightly different integrating factor: I can write $u(t) = e^{-\int_0^t r(s)ds}$, so

$$e^{\int_0^t -r(s)ds}\frac{dy}{dt} - e^{\int_0^t -r(s)ds} r(t)y(t) = e^{\int_0^t -r(s)ds} x(t).$$

Using Leibniz rule, I have

$$\frac{d}{dt}\left(y(t)e^{\int_0^t -r(s)ds}\right) = \frac{dy}{dt}e^{\int_0^t -r(s)ds} + y(t)e^{\int_0^t -r(s)ds}\frac{d}{dt}\left(\int_0^t -r(s)ds\right) = \frac{dy}{dt}e^{\int_0^t -r(s)ds} - y(t)e^{\int_0^t -r(s)ds}r(t).$$

Now, I have

$$\frac{d}{dt}\left(y(t)u(t)\right) = u(t)x(t).$$

Finally, this implies

$$y(T)u(T) - y(0)u(0) = \int_0^T u(t)x(t),$$

or, noting $u(0) = 1$,

$$y(T)e^{-\int_0^T r(s)ds} = y(0) + \int_0^T e^{-\int_0^t r(s)ds}x(t)dt.$$

Rearranging,

$$y(T) = y(0)e^{\int_0^T r(s)ds} + \int_0^T e^{\int_t^T r(s)ds}x(t)dt.$$

■

Exercise: Use this result to characterize the initial lifetime wealth W of the household. We have $W = W(a_0, \boldsymbol{r}, \boldsymbol{w})$.

It will be a useful exercise to characterize the response of initial lifetime wealth dW to a general perturbation of this economy, $\{da_0, d\boldsymbol{r}, d\boldsymbol{w}\}$. We have:

$$dW = W_{a_0}da_0 + W_{\boldsymbol{r}}d\boldsymbol{r} + W_{\boldsymbol{w}}d\boldsymbol{w}.$$

Exercise: work out each of these derivatives and interpret

Example: Let $R_{0,t} = e^{\int_0^t r_s ds}$ for ease of exposition. The only difficult term is $W_{\boldsymbol{r}}$. We have

$$\begin{aligned}
W_{r_\tau} &= \frac{\partial}{\partial r_\tau}\left[\int_0^\infty e^{-\int_0^t r_s ds}w_t dt\right] \\
&= \int_0^\infty \frac{\partial}{\partial r_\tau}\left[e^{-\int_0^t r_s ds}\right]w_t dt \\
&= \int_\tau^\infty \frac{\partial}{\partial r_\tau}\left[e^{-\int_0^t r_s ds}\right]w_t dt \\
&= -\int_\tau^\infty e^{-\int_0^t r_s ds}\left[\frac{\partial}{\partial r_\tau}\int_0^t r_s ds\right]w_t dt \\
&= -\int_\tau^\infty e^{-\int_0^t r_s ds}w_t dt \\
&= -\int_\tau^\infty e^{-(\int_0^t r_s ds + \int_0^\tau r_s ds - \int_0^\tau r_s ds)}w_t dt \\
&= -e^{-\int_0^\tau r_s ds}\int_\tau^\infty e^{-\int_\tau^t r_s ds}w_t dt.
\end{aligned}$$

Rewriting, this yields

$$W_{r_\tau} = -\frac{1}{R_{0,\tau}} \int_\tau^\infty \frac{w_t}{R_{\tau,t}} dt.$$

Interpret: This is the effect on lifetime wealth of a change in interest rates at time τ.

Step #2: Euler equation

To derive the continuous-time Euler equation for the household problem, I start by writing down the HJB. It is

$$\rho V(t,a) = V_t + u(c) + V_a \Big[r_t a + w_t - c \Big].$$

The envelope condition is

$$(\rho - r_t) V_a = V_{at} + V_{aa} \Big[r_t a + w_t - c \Big].$$

Next, note that since $c = c(t,a)$ we also have $u_c = u_c(t,a)$. Using Ito's lemma,

$$dV_a = V_{at} dt + V_{aa} \Big[r_t a + w_t - c \Big] dt.$$

And so directly I obtain

$$\frac{dV_a}{V_a} = \frac{du_c}{u_c} = (\rho - r_t) dt.$$

In this setting, where dr_t and dw_t are entirely deterministic, the Euler equation is of course also a deterministic equation. Using separation of variables, I can integrate and obtain

$$\int_0^t \frac{1}{u_c} du_c = \int_0^t (\rho - r_s) ds$$

$$\ln(u_c(t)) - \ln(u_c(0)) = \int_0^t (\rho - r_s) ds,$$

or simply

$$u_c(t) = u_c(0) e^{\int_0^t (\rho - r_s) ds} = u_c(0) e^{\int_0^t \rho ds} e^{-\int_0^t r_s ds} = u_c(0) \frac{1}{e^{-\rho t} R_{0,t}}.$$

Thus, we have integrated back up to the discrete-time Euler equation.

Lemma 4. *The continuous-time Euler equation in this simple setting without uncertainty between two dates $t > s$ is given by*

$$u_c(c_0) = e^{-\rho(t-s)} R_{s,t} u_c(c_t).$$

In particular, as long as households remain unconstrained I can express consumption at time t in terms of the consumption choice at any other date. In particular,

$$c_t = (u_c)^{-1} \Big[u_c(c_0) \frac{1}{e^{-\rho t} R_{0,t}} \Big].$$

Now the only thing I need to be really careful about is whether I want preference parameters (other than ρ) to be time-varying. That is, do I want $u_t = u(c_t)$, or $u_t = u(t, c_t)$? If I decide that preferences are only a function of consumption but otherwise constant, then so is u_c and

$$c_t = c_0 (u_c)^{-1} \left[\frac{1}{e^{-\rho t} R_{0,t}} \right].$$

In particular, under CRRA where $u_c = c^{-\gamma}$, I get

$$c_t = c_0 \left[e^{-\rho t} R_{0,t} \right]^{\frac{1}{\gamma}}.$$

Step #3: MPC

All that is left for me to do is put together the lifetime budget constraint with the Euler equation, and then take a derivative. To that end, define the household's MPC as

$$\text{MPC}_{0,t} = \frac{\partial c_t}{\partial a_0} = \frac{\partial c_t}{\partial W}.$$

This definition of course captures the intuition that the household experiences a marginal change in assets (or wealth or unearned income) in period 0, and then changes his path of consumption expenditures $\{c_t\}$ accordingly.

I have

$$\int_0^\infty e^{-\int_0^t r(s)ds} c(t) dt = \int_0^\infty \frac{c_t}{R_{0,t}} dt = W.$$

Substituting in for c_t, I obtain

$$\begin{aligned} W &= \int_0^\infty \frac{c_0 \left[e^{-\rho t} R_{0,t} \right]^{\frac{1}{\gamma}}}{R_{0,t}} dt \\ &= c_0 \int_0^\infty e^{-\frac{\rho}{\gamma} t} R_{0,t}^{\frac{1-\gamma}{\gamma}} dt \end{aligned}$$

To get started with a simple case, assume that $r_t = r$ is constant. Then

$$R_{0,t} = e^{\int_0^t r ds} = e^{rt}.$$

Differentiating with respect to W,

$$1 = \frac{\partial c_0}{\partial W} \int_0^\infty e^{-\frac{1}{\gamma} \left[\rho - (1-\gamma) r \right] t} dt.$$

Solving this out,[1] assuming for now that $\kappa > 0$,

$$\text{MPC} = \left[\int_0^\infty e^{-\kappa t} dt\right]^{-1} = \left[-\frac{1}{\kappa} e^{-\kappa x}\Big|_0^\infty\right]^{-1} = -\kappa\left[e^{-\kappa\infty} - e^{-\kappa 0}\right]^{-1} = -\kappa\left[0 - 1\right]^{-1} = \kappa.$$

Lemma 5. *The MPC in this setting is constant and given by*

$$MPC = \kappa,$$

confirming that I get the same result as I did before.

Problem 6

For the discrete-time, representative agent economy, the Euler equation is given by

$$U'(C_t) = \beta E[R^j_{t+1} U'(C_{t+1})]$$

for the potentially stochastic return R^j_{t+1} of any asset j, and where $\beta = e^{-\rho}$. We can write

$$1 = e^{-\rho} E\left[R^j_{t+1} \frac{U'(C_{t+1})}{U'(C_t)}\right].$$

Starting with a CRRA utility function, we can write

$$1 = e^{-\rho} E\left[R^j_{t+1} \frac{C^{-\gamma}_{t+1}}{C^{-\gamma}_t}\right] = E\left[e^{-\rho} R^j_{t+1} e^{-\gamma \ln(C_{t+1}/C_t)}\right] = E\left[R^j_{t+1} e^{-\rho-\gamma\Delta \ln C_{t+1}}\right].$$

Finally, denote by $r^j_{t+1} = \ln R^j_{t+1}$ the log return of the asset, then we arrive at

$$1 = E\left[e^{r^j_{t+1} - \rho - \gamma \Delta \ln C_{t+1}}\right].$$

Euler equation under log-normality. A log-normal RV is characterized via the representation

$$X = e^{\mu + \sigma Z},$$

where Z is a standard normal random variable, and (μ, σ) are the parameters of the log-normal. The mean of the log-normal is given by

$$E(X) = e^{\mu + \frac{1}{2}\sigma^2}$$

[1] Mathematically, for any $c \neq 0$, we have

$$\int e^{cx} dx = \frac{1}{c} e^{cx} + C.$$

and its variance by

$$\mathrm{Var}(X) = [e^{\sigma^2} - 1]e^{2\mu+\sigma^2}.$$

The Euler equation can be further simplified when we assume

$$R^j_{t+1} = e^{r^j_{t+1} + \sigma^j \epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2},$$

where $\epsilon^j_{t+1} \sim \mathcal{N}(0,1)$, so that

$$R^j_{t+1} \sim \log \mathcal{N}\left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2, \sigma^j\right).$$

Assume also that $\Delta \ln C_{t+1}$ is conditionally normal, with mean $\mu_{C,t}$ and variance $\sigma^2_{C,t}$. Furthermore assume that the two normals are also jointly, conditionally normal. Then we have the asset pricing equation

$$1 = E_t[\exp(X_t)],$$

where

$$X_t = -\rho + r^j_{t+1} + \sigma^j \epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma \Delta \ln C_{t+1} \sim -\rho + \mathcal{N}\left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma \mu_{C,t} \;\;,\;\; (\sigma^j)^2 + \gamma^2 \sigma^2_{C,t} - 2\rho_{j,C}\gamma\sigma^j\sigma_{C,t}\right).$$

Since $\exp(X_t)$ is log-normal, we have

$$1 = \exp\left\{ -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma\mu_{C,t} + \frac{1}{2}(\sigma^j)^2 + \frac{1}{2}\gamma^2\sigma^2_{C,t} - \gamma\rho_{j,C}\sigma^j\sigma_{C,t}\right\}.$$

Taking the log from both sides, we finally arrive at

$$0 = -\rho + r^j_{t+1} - \gamma\mu_{C,t} + \frac{\gamma^2}{2}\sigma^2_{C,t} - \gamma\rho_{r,C}\sigma^j\sigma_{C,t}. \tag{11}$$

More generall, we can leave this as

$$0 = -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma E_t(\Delta \ln C_{t+1}) + \frac{1}{2}\mathrm{Var}_t(\sigma^j \epsilon^j_{t+1} - \gamma \Delta \ln C_{t+1}) \tag{12}$$

Risk-free rate. For the risk-free rate, simply plug in $\sigma^f = 0$. Then we have

$$0 = -\rho + r^f_{t+1} - \gamma E_t(\Delta \ln C_{t+1}) + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1}). \tag{13}$$

Equity premium. For the asset class of equities, which we denote by the return R^E_{t+1}, we have

$$\pi^E_{t+1} \equiv r^E_{t+1} - r^f_{t+1} = \frac{1}{2}(\sigma^E)^2 - \frac{1}{2}\mathrm{Var}_t(\sigma^E \epsilon_{t+1} - \gamma \Delta \ln C_{t+1}) + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1}).$$

Now use the formula

$$\mathrm{Var}(aX - bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) - 2ab\mathrm{Cov}(X,Y).$$

Thus,

$$\pi^E_{t+1} = \frac{1}{2}(\sigma^E)^2 - \frac{1}{2}\left(\mathrm{Var}_t(\sigma^E\epsilon_{t+1}) + \mathrm{Var}_t(\gamma\Delta\ln C_{t+1}) - 2\sigma^E\gamma\mathrm{Cov}_t(\epsilon_{t+1}, \Delta\ln C_{t+1})\right) + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta\ln C_{t+1}).$$

Rewriting, we arrive at

$$\pi^E_{t+1} = \gamma\sigma_{C,E} \tag{14}$$

where $\sigma_{C,E}$ is the covariance between equity returns and log consumption growth.