Lecture 1

Dynamics: Discrete Time

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Introduction

- Many (most) economic phenomena are about dynamics
- Dynamics $\sim$ changes in the behavior of economic agents over time
- This is a course about **dynamic optimization**: main approach to study dynamic models in economics
- Goal of this course: hard skills + empower you to do research
  $\implies$ *focus on tools, but with tons of applications*
- The more you can "tool up" during your first two years, the better
- Specifically: **dynamic programming** one of most powerful tools in economic analysis

Dynamic programming will empower you

- Macro: consumption, investment, portfolio allocation, ...
- Labor: search, wage bargaining, ...
- IO: competition and games, pricing, ...
- Finance: asset pricing, dynamic capital structure, portfolio choice, ...
- Growth: technology adoption, poverty traps, firm innovation, ...
- Urban: migration in spatial models, ...
- Inequality: evolution of income-wealth distribution, ...
- Game theory: dynamic games, ...
- ... much, much more

Outline of today's lecture

- Administrative course overview
- Dynamics in discrete time
  1. Stochastic processes
  2. Markov chains
  3. Difference equations
  4. Prominent examples of difference equations in macro
  5. Solow growth model
  6. Stochastic difference equations

GitHub

- One super useful *hard skill* for research / code development: version control
- You should start using git (via GitHub) in your own work from the start
- Course material will be available via a GitHub organization at: `https://github.com/schaab-teaching/DynamicProgramming`
- You should create a GitHub account if you don't already have one
- If this is new for you, I also recommend GitKraken
- You can get Pro versions of GitHub and GitKraken for free after verifying your academic affiliation
- Great resource: `https://git-scm.com/book/en/v2` (Especially chapters 2, 3, 5, and 6)

Admin

- Fall semester macro: Andreas + Gerard, then Fabrice + Oscar
- PSETs every week, solutions will be provided
  $\Longrightarrow$ *You are responsible from now on*
- Final exam will be close to the PSETs
- Recommended books:
  - Ljungqvist and Sargent
  - Stokey and Lucas
  - Romer
  - Acemoglu
  - Oksendal
  - Stachurski / Miao / ...

1. Stochastic processes

- Let X_t be a random variable that is time t adapted
- Discrete time: We index time discretely $t = 0, 1, 2, \ldots, T \leq \infty$
- Stochastic process in discrete time: a sequence of random variables indexed by t, $\{X_t\}_{t=0}^{T}$
- Continuous time: We index time continuously $t \in [0, T]$ with $T \leq \infty$
- Stochastic process in continuous time: a sequence of random variables indexed by t, $\{X_t\}_{t \geq 0}$

2. Markov chains

- A stochastic process $\{X_t\}$ has the *Markov property* if for all $k \geq 1$ and all t:

$$\mathbb{P}(X_{t+1} = x \mid X_t, X_{t-1}, \ldots, X_{t-k}) = \mathbb{P}(X_{t+1} = x \mid X_t)$$

- *State space* of the Markov process = set of events or states that it visits

- A Markov chain is a Markov process (stochastic process with Markov property) that visits a finite number of states (*discrete state space*)

- Simplest example: Individual i is randomly hit by earnings (employment) shocks and switches between $X_t \in \{X^L, X^H\}$

- Markov chains have a *transition matrix* P that describes the probability of transitioning from state i to state j

- Simplest example with state space $\{X^L, X^H\}$

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$$

- This says: P of staying in employment state = 0.8, P of switching = 0.2

- P_{ij} is the probability of switching from state i to state j (one period)

- P^2 characterizes transitions over two periods: $(P^2)_{ij}$ is prob of going from i to j in two periods

- The rows of the transition matrix have to sum to 1 (definition of probability measure)

3. Difference Equations

- We start with deterministic (non-random) dynamics and then conclude with stochastic (random) dynamics

- The *first-order linear difference equation* is defined by

$$x_{t+1} = bx_t + cz_t \tag{1}$$

  where $\{z_t\}$ is an exogenously given, bounded sequence

- For now, all objects are (real) scalars (easy to extend to vectors and matrices)

- Suppose we have an *initial condition* (i.e., given initial value) x_0

- When $c = 0$, (1) is a *time-homogeneous* difference equation

- When cz_t is constant for all t, (1) is an *autonomous* difference equation

Autonomous equations

- Consider the autonomous equation with $z_t = 1$
- A particular solution is the constant solution with $x_t = \frac{c}{1-b}$ when $b \neq 1$
- Such a point is called a *stationary point* or *steady state*
- General solution of the autonomous equation (for some constant x):

$$x_t = (x_0 - x)b^t + x \tag{2}$$

- Important question is long-run behavior (stability / convergence)
- When $|b| < 1$, (2) converges asymptotically to steady state x for any initial value x_0 (steady state x is globally stable)
- If $|b| > 1$, (2) explodes and is not stable (except when $x_0 = x$)

4. Difference Equations: Examples in Macro

Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + I_t$$

- δ is depreciation and I_t is investment
- This is a *forward equation* and requires an initial condition K_0
- If $I_t = 0$ and $0 < \delta < 1$, $K_t \to 0$
- If $I_t = c$ constant, then K_t converges to $\frac{c}{\delta}$: $K_{t+1} = (1-\delta)\frac{c}{\delta} + c = \frac{c}{\delta}$

Wealth dynamics:

$$a_{t+1} = R_t a_t + y_t - c_t$$

- R_t is the gross real interest rate, y_t is income, c_t is consumption
- This is a *forward equation* and requires an initial condition a_0
- We will study this as a *controlled* process because c_t will be chosen optimally
- Work out the following: $R_t = R$ and $y_t = y$ constant, and

$$c_t = \left(1 - \frac{1}{R}\right)\left(a_t + \sum_{s=t}^{\infty} R^{-(t-s)} y\right)$$

  What are the dynamics of a_t?

Consumption Euler equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., c_T must converge to something
- Suppose there exists time T s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

New Keynesian Phillips curve:

$$\pi_t = \beta\pi_{t+1} + \kappa x_t$$

- π_t is inflation, κ is the slope of the PC, x_t is output gap
- This is a *backward equation* and requires a terminal condition
- NK analysis often studies the case $\lim_{T\to\infty} \pi_T = 0$ (0 inflation steady state)
- Suppose output gap $\{x_t\}$ exogenously given and there exists T s.t. for $t \geq T$, $\pi_t = 0$ and $x_t = 0$
- Then we solve backwards: $\pi_{T-1} = \beta\pi_T + \kappa x_{T-1}$
- The *initial value* π_0 is *endogenous*: backward equations solve for initial value π_0, forward equations solve for long run (e.g., K_T)

5. Solow Growth Model

- Time is discrete and the horizon infinite, $t = 0, 1, 2, \ldots$

- There is a *representative household*: large number of small but identical households

- Assume households have a constant savings rate $s \in (0, 1)$ (out of disposable income)

- A representative firm operates the technology / production function

$$Y_t = F(K_t, L_t, A_t)$$

  where K_t is capital, L_t is labor, A_t is total factor productivity (TFP)

- Capital accumulation: $K_{t+1} = (1 - \delta)K_t + I_t$

- Goods market clearing (*national income accounting identity*): $Y_t = C_t + I_t$

- Feasible allocations in this economy are characterized by

$$K_{t+1} \leq F(K_t, L_t, A_t) + (1-\delta)K_t - C_t$$

- How do we determine the equilibrium allocation among all those allocations that are feasible? $\implies$ assume constant savings rate

$$\begin{aligned} sY_t &= S_t \\ &= I_t = Y_t - C_t \end{aligned}$$

or $C_t = (1-s)Y_t$

- Equilibrium characterized by (non-linear) first-order difference equation:

$$K_{t+1} = sF(K_t, L_t, A_t) + (1-\delta)K_t \tag{3}$$

Definition. (Equilibrium) Given sequences $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial condition for capital K_0, the equilibrium path of the Solow growth model comprises paths for capital, output, consumption and investment $\{K_t, Y_t, C_t, I_t\}_{t=0}^{\infty}$ that satisfy (3), goods market clearing, firm production, and $C_t = sY_t$.

Steady state

- Suppose Cobb-Douglas technology:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}$$

and no productivity or population growth; also normalize $L_t = 1$

- A steady state is a level of capital K such that

$$K = sAK^{\alpha} + (1-\delta)K$$

- Solving this, we find:

$$K = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Transition dynamics

- The key *degree of freedom* in this economy is the *initial condition* for the (forward) difference equation for capital accumulation: K_0
- Suppose $K_0 < K$ and $K_0 > K$, what happens?
- Read discussion and proofs in Acemoglu, but intuitively:

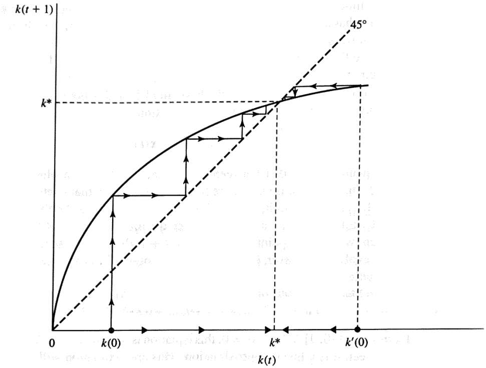

FIGURE 2.7 Transitional dynamics in the basic Solow model.

6. Stochastic Difference Equations

- Consider the process $\{X_t\}$ with

$$X_{t+1} = AX_t + Cw_{t+1} \tag{4}$$

  where w_{t+1} is an iid. process with $w_{t+1} \sim \mathcal{N}(0, 1)$

- Equation (4) is a *first-order, linear stochastic difference equation*

- Let $\mathbb{E}_t$ the *conditional expectation* operator (conditional on time t information)

- For example:

$$\begin{aligned}\mathbb{E}_t(X_{t+1}) &= \mathbb{E}(X_{t+1} \mid X_t) = \mathbb{E}(AX_t + Cw_{t+1} \mid X_t) \\ &= AX_t + C\mathbb{E}(w_{t+1} \mid X_t) = AX_t + C\mathbb{E}(w_{t+1}) = AX_t\end{aligned}$$

- Rational expectations: agents' beliefs about stochastic processes are consistent with the true distribution of the process
- Consumption Euler equation with uncertainty (e.g., stochastic income):

$$u'(C_t) = \beta R \mathbb{E}_t \left[u'(C_{t+1}) \right]$$

- New Keynesian Phillips curve with uncertainty (e.g., demand shocks):

$$\pi_t = \beta \mathbb{E}_t \left[\pi_{t+1} \right] + \kappa x_t$$