# Lecture 2

## Dynamics: Continuous Time

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# Outline of today's lecture

1. Ordinary differential equations
2. Prominent examples of differential equations in macro
3. Partial differential equations
4. Solow growth model
5. Continuous-time Markov chains
6. Brownian motion and stochastic differential equations

# 1. Ordinary differential equations

- Consider the "discrete-time" equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- *Continuous-time limit*: consider the limit as $\Delta t \to 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \to 0} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$ is *autonomous* and dropping subscripts: $\dot{X} = g(X)$

- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

# Boundary conditions (I)

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval $t \in [0, 1]$. We call $[0, 1]$ the *state space*. $(0, 1)$ is the *interior of the state space* and $\{0, 1\}$ is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

# Boundary conditions (II)

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions
- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for $X_t$ with some *initial condition* $X_0$
  - *Terminal value problems* instead specify $X_T$
- More broadly: We need sufficient information to characterize the function of interest along the boundary
- Types of boundary conditions: Dirichlet ($X_0 = c$), von-Neumann ($\frac{dX_0}{dt} = c$), reflecting boundaries, ...
- Boundary conditions are very important and can be very subtle (especially for PDEs)

# Linear First-Order ODEs

- Consider the equation:

$$\dot{X}(t) = a(t)X(t) + b(t) \tag{1}$$

- If $b(t) = 0$, (1) is a *homogeneous* equation, if $a(t) = a$ and $b(t) = b$ we say (1) has *constant coefficients*

- Start with $\dot{X}(t) = aX(t)$, divide by $X(t)$ and integrate with respect to $t$

$$\int \frac{\dot{X}(t)}{X(t)} dt = \int a dt$$
$$\log X(t) + c_0 = at + c_1$$
$$X(t) = Ce^{at}$$

  where $C = e^{c_1 - c_0}$

- Pin down constant $C$ by using the boundary condition (we need 1)

- Consider time-varying coefficient with $\dot{X}(t) = a(t)X(t)$ with initial condition $X(0) = \bar{x}$

- Dividing by $X(t)$, integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition: $C = \bar{x}$

- Finally, for $\dot{X}(t) = aX(t) + b$, we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

  after using change of variables $Y(t) = X(t) + \frac{b}{a}$

- Many results for systems of linear differential equations: $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}\boldsymbol{X}(t)$

# 2. Examples of differential equations in macro

**Capital accumulation:**

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps, $K_{t+1} = I_t + (1 - \delta)K_t$
- With arbitrary $\Delta$ time step, $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned} K_{t+\Delta} &= K_t + \Delta(I_t + (1 - \delta)K_t \\ K_{t+\Delta} - K_t &= I_t - \delta K_t \\ \dot{K}_t &= I_t - \delta K_t \end{aligned}$$

- Suppose $\{I_t\}_{t\geq 0}$ exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\dot{K}_t + \delta K_t = I_t$$
$$e^{\int_0^t \delta ds}\dot{K}_t + e^{\int_0^t \delta ds}\delta K_t = e^{\int_0^t \delta ds} I_t$$

- Notice that $\int_0^t \delta ds = \delta \int_0^t ds = \delta[s]_0^t = \delta(t-0) = \delta t$, so

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

- We have $e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = \frac{d}{dt}(K_t e^{\delta t})$, integrating:

$$K_t e^{\delta t} = \tilde{C} + \int_0^t e^{\delta s} I_s ds$$
$$K_t = C + \int_0^t e^{-\delta(t-s)} I_s ds$$

- Integrating constant solves initial condition: $C = K_0$

**Wealth dynamics** (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- $r_t$ is the real rate of return on wealth, $y_t$ is income, and $c_t$ is consumption
- Structure of the equation similar to capital accumulation equation

**Consumption Euler equation**:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., $c_T$ must converge to something
- Suppose there exists time $T$ s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

**New Keynesian Phillips curve**:

$$\dot{\pi}_t = \rho \pi_t + \kappa x_t$$

- This is a backward equation that requires a terminal condition
- As in discrete time, we often consider the 0 inflation steady state with $\pi_T \to 0$
- Then we can solve (work this out yourselves):

$$\pi_t = -\kappa \int_t^\infty x_s ds$$

# 3. A brief intro to partial differential equations

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...
  $\Longrightarrow$ increasingly used in economics
- This class: no self-contained treatment of PDEs *but* we will encounter some simple PDEs

- Consider a function $u(x_1, x_2, \dots, x_n)$ where $x_1, \dots, x_n$ are coordinates in $\mathbb{R}$

- Partial derivatives of $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in $u$ and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \dots, \partial_{x_n} u, \partial_{x_1 x_1} u, \dots)$$

- The *order* of the PDE, is the order of the highest partial derivative

- Examples from physics
  - Heat equation: $\partial_t u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Wave equation: $\partial_{tt} u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Transport equation: $\partial_t u = \partial_x u$ (first-order, linear, homogeneous)

- Income distribution "solves heat equation", wealth dynamics "solve transport equations", dynamic programming often transport + heat

# 4. Solow Growth Model

- As before, $Y_t = C_t + I_t$ and

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

- Representative firms operates neoclassical production function

$$Y_t = F(K_t, L_t, A_t)$$

- Normalize labor to $L_t = 1$ and hold TFP constant $A_t = A$

- We again assume constant savings rate: $Y_t - C_t = I_t = sY_t$

- Assume Cobb-Douglas $Y_t = AK_t^\alpha$ so equilibrium allocation

$$\dot{K}_t = sAK_t^\alpha - \delta K_t$$

- Steady state is given by

$$K^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- Key equilibrium condition in $\dot{K}_t$ is *non-linear* — how to proceed?

- Let $X_t = K_t^{1-\alpha}$, then

$$\begin{aligned}\dot{X}_t &= (1-\alpha)K_t^{-\alpha}\dot{K}_t \\ &= (1-\alpha)K_t^{-\alpha}(sAK_t^{\alpha} - \delta K_t) \\ &= (1-\alpha)sA - (1-\alpha)K_t^{1-\alpha}\delta \\ &= (1-\alpha)sA - (1-\alpha)\delta X_t\end{aligned}$$

- Solution with initial condition $X_0$ (work this out):

$$X_t = X^* + e^{-(1-\alpha)\delta t}\left[X_0 - X^*\right], \quad \text{where } X^* = \frac{sA}{\delta}$$

- Transition dynamics (rate of convergence) governed by $-(1-\alpha)\delta$

# 5. Continuous-time Markov chains

- Definition: Let $X = \{X_t\}_{t\geq 0}$ be a sequence of random variables taking values in a finite or countable state space $\mathcal{X}$. Then $X$ is a *continuous-time Markov chain* if it satisfies the *Markov property*: For any sequence $0 \leq t_1 < t_2 < \ldots < t_n$ of times

$$\mathbb{P}(X_{t_n} = x \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \mathbb{P}(X_{t_n} = x \mid X_{t_{n-1}})$$

- Process $X$ is *time-homogeneous* if the conditional probability does not depend on the current time, i.e., for $x, y \in \mathcal{X}$:

$$\mathbb{P}(X_{t+s} = x \mid X_s = y) = \mathbb{P}(X_t = x \mid X_0 = y)$$

- The *transition density* of process $X$ is denoted $p(t, x \mid s, y)$ and is defined as

$$\mathbb{P}(X_t \in A \mid Y_s = y) = \int_A p(t, x \mid s, y) dx$$

  for any (Borel) set $A \subset \mathcal{X}$. In words: $p(t, x \mid s, y)$ is the probability (density) that process $X_t$ ends up at $X_t = x$ at time $t$ if it started at $X_s = y$ at time $s$

- *Condition expectation* can be written as: $\mathbb{E}[f(X_t) \mid X_0 = y] = \int p(t, x \mid 0, y) f(x) dx$

**Example**:

- Consider the two-state employment process $z_t \in \{z^L, z^H\}$ with transition rates $\lambda^{LH}$ (from L to H) and $\lambda^{HL}$ (from H to L)
- The associated transition matrix (*generator*) is

$$\mathcal{A}^z = \begin{pmatrix} -\lambda^{LH} & \lambda^{LH} \\ \lambda^{HL} & -\lambda^{HL} \end{pmatrix}$$

- Interpretation: households transition *out of* state $i$ at rate $\lambda^{ij}$
- Notice: In discrete time, Markov transition matrix rows sum to 1. Here, rows sum to 0 (*mass preservation*)

# 6. Brownian motion and SDEs

**Definition.** Brownian motion $\{B_t\}_{t\geq 0}$ is a stochastic process with properties:

(i) $B_0 = 0$

(ii) (*Independent increments*) For non-overlapping $0 \leq t_1 < t_2 < t_3 < t_4$, we have $B_{t_2} - B_{t_1}$ independent from $B_{t_4} - B_{t_3}$

(iii) (*Normal, stationary increments*) $B_t - B_s \sim \mathcal{N}(0, t-s)$ for any $0 \leq s < t$

(iv) (*Continuity of paths*) The sample paths of $B_t$ are continuous

- Brownian motion is the only stochastic process with stationary and independent increments that's also continuous
- Einstein (1905) uses Brownian motion to model motion of particles
- Brownian motion is a Markov process
- $B_t \sim \mathcal{N}(0, t)$
- Brownian motion is nowhere differentiable

- Stochastic differential equations (SDEs) add noise / uncertainty to ordinary differential equations (ODEs)

- Start with $\dot{X}_t = \mu X_t$ with solution $X_t = X_0 e^{\mu t}$

- Rewrite as $dX_t = \mu X_t dt$ and "add noise" (using Brownian motion):

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

- Important: $dB_t \sim \mathcal{N}(0, dt)$ because

$$dB_t \approx B_{t+\Delta} - B_t \sim \mathcal{N}(0, t + \Delta - t = \mathcal{N}(0, \Delta)$$

  and now take $\Delta \to dt$ (continuous-time limit)

- Alternatively: $B_{t+\Delta} - B_t \sim \mathcal{N}(0, \Delta) \sim \epsilon_t \sqrt{\Delta}$ where $\epsilon_t \sim \mathcal{N}(0, 1)$. So as $\Delta \to dt$,

$$\mathbb{E}(dB_t) = \mathbb{E}(\epsilon_t \sqrt{dt}) = 0$$
$$\mathbb{E}[(dB_t)^2] = \mathbb{E}[(\epsilon_t \sqrt{dt})^2] = dt$$

- Suppose we have a function of Brownian motion, $X_t = f(t, B_t)$
- We know how Brownian motion $dB_t$ evolves, what about $dX_t$? (That's like $\dot{X}_t$)
- Answer: **Ito's lemma** (core building block of stochastic calculus)

$$dX_t = df(t, B_t) = \partial_t f(t, B_t)dt + \frac{1}{2}\partial_{xx} f(t, B_t)dt + \partial_x f(t, B_t)dB_t$$

- Will not prove this, but heuristically: $(dt)^2 \to 0$ and $(dB_t)^2 \to dt$
- For example from previous slide, $dX_t = \mu X_t dt + \sigma X_t dB_t$:

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$$

- This is called *geometric Brownian motion* (used to model stock prices)
- *Ornstein-Uhlenbeck (OU) process* is a popular model for earnings risk and income fluctuations (think: continuous-time AR(1) process):

$$dz_t = \theta(\bar{z} - z_t)dt + \sigma dB_t$$

- Very important class is the **diffusion process**
- They take the form (not the formal definition)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

where $\mu(\cdot)$ is the *drift* and $\sigma(t, X_t)$ the *diffusion* (volatility) parameter of the process

- This is a shorthand for the (stochastic) integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$