Dynamic Optimization: Problem Set #2

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Problem 1: Going on the job market

Credit: David Laibson

Consider the following "job market" optimization problem. A graduate student works on a project with current market value x, where x is an Ito process with drift α and standard deviation σ :

$$dx = \alpha dt + \sigma dz$$

The student works on the project until she chooses to either (i) take the paper to the job market, generating termination payoff x, or (ii) abandon the project and start a new project that has current market value $x = 0$. This replacement of projects is made at zero cost. (We assume that the student can only work on one project at a time.)

The optimal policy is a two-sided threshold rule. The student stops when (i) $x \geq M$, or (ii) $x \leq 0$. Here M is an endogenous boundary representing the job M arket. The zero bound follows from the economics of the problem (free substitution).

Assume that time has opportunity cost w (annual wage) and the student has an annual discount rate ρ.

(a) Show that in the continuation region the HJB satisfies

$$\rho V = -w + \alpha V' + \left(\sigma^2/2\right) V''$$

(b) This is the so-called complete equation. The associated reduced equation is

$$\left(\sigma^2/2\right) V'' + \alpha V' - \rho V = 0$$

Guess the solution of the reduced form is $V = \exp(rx)$ and verify, show there are two solutions.

A particular solution of the complete equation is generated by the policy of never stopping:

$$V(x) = -w/\rho.$$

The general solution of the complete equation is:

$$V(x) = C^{+} \exp\left(r^{+}x\right) + C^{-} \exp\left(r^{-}x\right) - w/\rho$$

Remember that the superscripts $+$ and $-$ are not arithmetic operators.

(c) We have three boundary conditions (one value matching condition and two smooth pasting conditions; why aren't we using the other value matching condition at 0?).

$$V(M) = M$$

$$V'(M) = 1$$

$$V'(0) = 0$$

Derive the two smooth pasting conditions (with the V'), interpret each of the boundary conditions.

(d) We can now solve for our three free variables: C^{+}, C^{-}, M. Show the solution must satisfy

$$C^{-} = -r^{+}C^{+}/r^{-}$$

$$C^{+} = \frac{1}{r^{+}\left[\exp\left(r^{+}M\right) - \exp\left(r^{-}M\right)\right]}$$

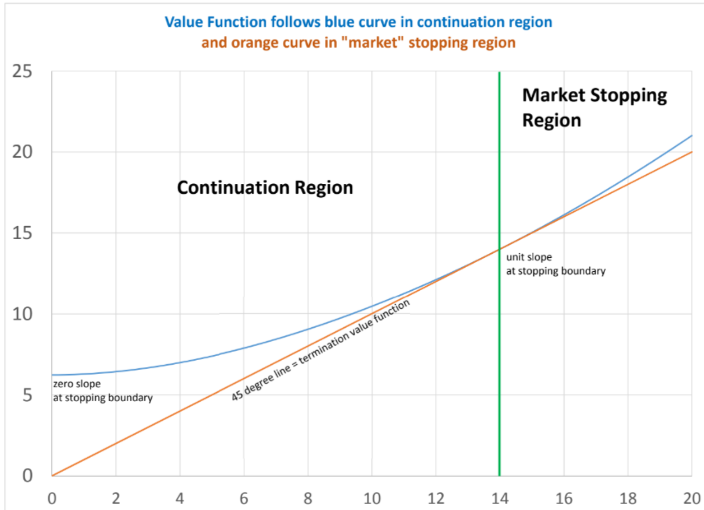
$$C^{-} = \frac{1}{r^{-}\left[\exp\left(r^{-}M\right) - \exp\left(r^{+}M\right)\right]}$$

$$M = \frac{r^{+}\exp\left(r^{+}M\right) - r^{-}\exp\left(r^{-}M\right)}{\exp\left(r^{+}M\right) - \exp\left(r^{-}M\right)} - w/\rho$$

$$C^{+} + C^{-} = \frac{1/r^{+} - 1/r^{-}}{\exp\left(r^{+}M\right) - \exp\left(r^{-}M\right)}$$

The solution of the value function looks as follows (for parameters $\alpha = 1, \sigma = 5, w = 1$, and $\rho = 0.05$)

(e) If you could pick projects with less positive drift (smaller α) and more Brownian noise (larger σ) would you? Why or why not?



Problem 2: The Equity Premium

Credit: Pablo Kurlat

Suppose that consumption growth is:

$$\frac{c_{t+1}}{c_t} = \begin{cases} 1+g & \text{with probability } 1-\mu \\ 1-D & \text{with probability } \mu \end{cases}$$

and the (gross) return on the stock market is:

$$R_{t+1} = \begin{cases} 1+y & \text{with probability } 1-\mu \\ 1-F & \text{with probability } \mu \end{cases}$$

Furthermore, suppose the event $\frac{c_{t+1}}{c_t} = 1-D$ and the event $R_{t+1} = 1-F$ always coincide, i.e. $\frac{c_{t+1}}{c_t}$ and R_t are perfectly correlated. The risk-free (gross) real interest rate is $1+r$. There is a representative household, with preferences given by:

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right)$$

with

$$u\left(c_t\right) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

(a) Write down the Euler equations for risk-free assets and for the stock market.

(b) Find explicit expressions in terms of parameters for the following quantities:

a) $\mathbb{E}\left(\frac{u'(c_{t+1})}{u'(ct)}\right)$

b)$\mathbb{E}\left(R_{t+1}\right)$

c) $Cov\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right)$

(c) Find an explicit expression in terms of parameters for the value of β that is consistent with the household's Euler equation for risk-free assets, given that the risk-free rate is $1+r$.

(d) Derive a joint restriction on the values of $\beta, \gamma, \mu, g, D, y$ and F that needs to be satisfied for the household's Euler equation for the stock market to hold.

(e) Set the following parameter values:

r	γ	μ	g	D	F
0.01	3	0.02	0.015	0.3	0.45

What must be the values of β and y for the household's Euler equations to hold?

(f) What is the value of $\mathbb{E}\left(R_{t+1}\right)$? What is the equity premium?

(g) Suppose a researcher is trying to measure the equity premium empirically in a sample the event "$\frac{c_{t+1}}{c_t} = 1 - D$ and $R_{t+1} = 1 - F$" has never been realized. What would the researcher's estimates be for:

(a) $\mathbb{E}\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)$

(b) $\mathbb{E}\left(R_{t+1}\right)$

(c) $\text{Cov}\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right)$

Would the researcher conclude that there is an equity premium "puzzle"? Explain.

Problem 3

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Please solve Problem set 4 of David.

Problem 4

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Please solve Problem 1 of David's problem set (PSET) #5.