M2: Lecture 12

Portfolio Choice and Asset Pricing

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Outline of today's lecture

1. Equity premium
2. Merton model: portfolio choice in continuous time

1. Equity premium

- Consider $i \in \mathcal{I}$ assets with stochastic return processes $\{R_{i,t+1}\}$, with

$$R_{i,t+1} = e^{r_{i,t+1} + \sigma_i \epsilon_{i,t+1} - \frac{1}{2}\sigma_i^2}$$

  where $\epsilon_{i,t} \sim \mathcal{N}(0, 1)$

- For any asset i the household can freely trade, an Euler equation must hold:

$$u'(c_t) = \beta \mathbb{E}_t \Big(R_{i,t+1} u'(c_{t+1}) \Big)$$

- Using CRRA with $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$, we get

$$1 = \mathbb{E}_t \left[e^{-\rho + r_{i,t+1} + \sigma_i \epsilon_{i,t+1} - \frac{1}{2}\sigma_i^2} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$$

- Note that

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \exp\left\{-\gamma \log \frac{c_{t+1}}{c_t}\right\} = e^{-\gamma \Delta \log(c_{t+1})}$$

- Also, for $X \sim \mathcal{N}(\mu, \sigma)$, we have $\mathbb{E}[e^{bx}] = e^{b\mu + \frac{1}{2}b^2\sigma^2}$

- Assuming log-normal consumption growth, we have

$$\mathbb{E}_t\left[e^{\sigma_i \epsilon_{i,t+1} - \gamma \Delta \ln(c_{t+1})}\right] = e^{-\gamma \mathbb{E}_t \Delta \ln(c_{t+1}) + \frac{1}{2}\mathbb{V}ar(\sigma_i \epsilon_{i,t+1} - \gamma \Delta \ln(c_{t+1}))}$$

- Putting it all together:

$$\begin{aligned} 1 &= e^{-\rho + r_{i,t+1} - \frac{1}{2}\sigma_i^2}\mathbb{E}_t\left[e^{\sigma_i \epsilon_{i,t+1} - \gamma \Delta \log(c_{t+1})}\right] \\ &= e^{-\rho + r_{i,t+1} - \frac{1}{2}\sigma_i^2 - \gamma \mathbb{E}_t \Delta \ln(c_{t+1}) + \frac{1}{2}\mathbb{V}ar(\sigma_i \epsilon_{i,t+1} - \gamma \Delta \ln(c_{t+1}))} \end{aligned}$$

- Read about variance and covariance
- We have

$$\mathbb{V}ar(A+B) = \mathbb{V}ar(A) + \mathbb{V}ar(B) + 2\mathbb{C}ov(A,B)$$

- So

$$1 = e^{-\rho + r_{i,t+1} - \frac{1}{2}\sigma_i^2 - \gamma \mathbb{E}_t \Delta \ln(c_{t+1}) + \frac{1}{2}\sigma_i^2 + \frac{\gamma^2}{2}\mathbb{V}ar(\Delta \ln(c_{t+1})) - \gamma \mathbb{C}ov(\sigma_i \epsilon_{i,t+1}, \Delta \ln(c_{t+1}))}$$

  and finally:

$$r_{i,t+1} = \rho + \gamma \mathbb{E}_t \Delta \ln(c_{t+1}) - \frac{\gamma^2}{2}\mathbb{V}ar(\Delta \ln(c_{t+1})) + \gamma \mathbb{C}ov(\sigma_i \epsilon_{i,t+1}, \Delta \ln(c_{t+1}))$$

- Riskfree bond is therefore priced according to:

$$r^f_{t+1} = \rho + \gamma \mathbb{E}_t \Delta \ln(c_{t+1}) - \frac{\gamma^2}{2}\mathbb{V}ar(\Delta \ln(c_{t+1}))$$

- Let asset i denote the stock market
- Equity premium is then given by

$$r_{i,t+1} - r^f_{t+1} = \gamma \mathbb{C}ov(\sigma_i \epsilon_{i,t+1}, \Delta \ln(c_{t+1}))$$

- The equity premium is equal to the coefficient of relative risk aversion (γ) times the covariance of stock returns with consumption growth.
- Key insight: risk is about *covariance*, and in particular covariance with consumption growth demands compensation. (γ is like the price of risk and the covariance is like the quantity of risk.)
- Mehra-Prescott (1986): Rearrange and use post-war data

$$\gamma = \frac{r_{equity} - r^f}{\sigma_{equity,\Delta c}} \approx \frac{0.06}{0.0003} = 200.$$

- Mankiw-Zeldes (1991): What value of X makes you indifferent between the following two gambles:

$$\text{gamble 1} = \begin{cases} \$50,000 & \text{with probability 0.5} \\ \$100,000 & \text{with probability 0.5} \end{cases}$$

$$\text{gamble 2} = \$X \quad \text{with probability1}$$

- Implied risk aversion coefficient:

$$\begin{aligned} \gamma = 1 \quad &\Longrightarrow \quad X = \$70,711 \\ \gamma = 3 \quad &\Longrightarrow \quad X = \$63,246 \\ \gamma = 20 \quad &\Longrightarrow \quad X = \$51,858 \\ \gamma = 200 \quad &\Longrightarrow \quad X = \$50,174 \end{aligned}$$

2. Portfolio choice in continuous time

- Economy has two assets, populated by a representative household
- **Lucas tree**:
  - Tree is in fixed unit supply
  - Pays dividend D_t that evolves as $\frac{dD}{D} = \mu_D dt + \sigma_D dB$
  - Trades at price Q_t, which follows $\frac{dQ}{Q} = \mu_Q dt + \sigma dB$ (conjecture)
- We guess and verify that holding return evolves according to

$$dR = \frac{Ddt + dQ}{Q} = \frac{D}{Q}dt + \mu_Q dt + \sigma dB,$$

  where $\rho_D = D/Q$ is the *constant* dividend-price ratio and $\mu = D/Q + \mu_Q$

- **Riskfree bond**: pays no dividend but price evolves as $\frac{dP}{P} = rdt$

- Household's lifetime value given by

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

- Denote by b_t and k_t household's bond and stock holdings
- Budget constraint:

$$Qdk + Pdb + cdt = Dkdt$$

- Define *liquid net worth* n and the *risky portfolio share* θ implicity via

$$\begin{aligned} \theta n &= Qk \\ (1-\theta)n &= Pb \end{aligned}$$

- Can rearrange budget constraint now to obtain

$$dn = \left[rn + \theta n \Big(\mu - r \Big) - c \right] dt + \theta n \sigma dB$$

Recursive representation yields HJB equation:

$$\rho V(n) = \max_{c,\theta} \left\{ u(c) + \left[rn + \theta n \left(\mu - r \right) - c \right] V'(n) + \frac{1}{2} (\sigma \theta n)^2 V''(n) \right\}$$

with first-order conditions

$$\begin{aligned} u'(c) &= V'(n) \\ \theta &= -\frac{V'(n)}{nV''(n)} \cdot \frac{\mu - r}{\sigma^2} \end{aligned}$$

Lemma. The Euler equation for marginal utility is given by

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu - r}{\sigma} dB.$$

Lemma. With CRRA utility, the Euler equation for consumption is

$$\frac{dc}{c} = \frac{r - \rho}{\gamma} dt + \frac{1 + \gamma}{2} \left(\frac{\mu - r}{\gamma \sigma} \right)^2 dt + \frac{\mu - r}{\gamma \sigma} dB.$$

- The representative household must consume the dividend (fruit) of the tree, so

$$c_t = D_t$$

- Recall:

$$\begin{aligned} \frac{dD}{D} &= \mu_D dt + \sigma_D dB \\ \frac{dc}{c} &= \frac{r-\rho}{\gamma} dt + \frac{1+\gamma}{2}\left(\frac{\mu - r}{\gamma\sigma}\right)^2 dt + \frac{\mu - r}{\gamma\sigma} dB \end{aligned}$$

- So matching coefficients:

$$\begin{aligned} \mu_D &= \frac{r-\rho}{\gamma} + \frac{1+\gamma}{2}\left(\frac{\mu - r}{\gamma\sigma}\right)^2 \\ \sigma_D &= \frac{\mu - r}{\gamma\sigma} \end{aligned}$$

- Riskfree rate: $r = \rho + \gamma\mu_D - \frac{\gamma(1+\gamma)}{2}\sigma_D^2$.

- We can do even better!
- Guess and verify that

$$V(n) = \frac{1}{1-\gamma}\kappa^{-\gamma} n^{1-\gamma}$$

where $\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right]$

- Solution of the model is then given by:

$$\begin{aligned} c &= \kappa n \\ \theta &= \frac{\mu - r}{\gamma\sigma^2} \\ r &= \rho + \gamma\mu_D - \frac{\gamma(1+\gamma)}{2}\sigma_D^2 \\ \mu - r &= \gamma\sigma\sigma_D = \gamma\mathbb{C}ov\left(\frac{dc}{c}, dR\right) \end{aligned}$$

- What does this model tell us? Interesting special case: $\gamma = 1$
- Consumption collapses to $c = \rho n \implies$ consume constant fraction ρ of lifetime net worth
- How much should we invest in the stock market?

$$\theta = \frac{\mu - r}{\gamma \sigma^2} \approx \frac{0.06}{1 \cdot (0.16)^2} = 2.34$$

- In the absence of frictions and with $\gamma = 1$, this model tells us to invest 2.34 times our total net worth (including human capital) in the stock market