Dynamic Optimization: Problem Set #2

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Problem 1: proof of contraction mapping theorem

Credit: David Laibson

In class, we defined the Bellman operator B, which operates on functions w, and is defined by

$$(Bw)(x) \equiv \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta w(x') \right\}$$

for all $x \in \mathcal{X}$ in the state space, where $\Gamma(x)$ is some constraint set—in our case, this was the budget constraint. The definition is expressed pointwise, but it applies to all possible values in the state space. We call B an operator because it maps a function w to a new function Bw. So both w and Bw map $\mathcal{X}$ into $\mathbb{R}$. Operator B maps *functions* and is therefore called a functional operator. In class, we showed that the solution of the Bellman equation—the value function—is a fixed point of the Bellman operator.

What does it mean to *iterate* $B^n w$?

$$(Bw)(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta w(x') \right\}$$

$$(B(Bw))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (Bw)(x') \right\}$$

$$(B(B^2 w))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (B^2 w)(x') \right\}$$

$$\vdots$$

$$(B(B^n w))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (B^n w)(x') \right\}.$$

What does it mean for functions to converge to a limiting function? Let v_0 be some guess for the value function, then convergence would mean

$$\lim_{n\to\infty} B^n v_0 = v.$$

And why might $B^n w$ converge as $n \to \infty$? The answer is that B is a *contraction mapping*.

Definition 1. Let (S, d) be a metric space and $B : S \to S$ be a function that maps S into intself. B is a contraction mapping if for some $\beta \in (0, 1)$, $d(Bf, Bg) \leq \beta d(f, g)$, for any two functions f and g.[1]

Intuitively, B is a contraction mapping if applying the operator B to any two functions f and g (that are not the same) moves them strictly closer together. Bf and Bg are strictly closer together than f and g. We can now state the contraction mapping theorem.

Theorem 2. *If (S, d) is a complete metric space and $B : S \to S$ is a contraction mapping, then:*

(i) B has exactly one fixed point $v \in S$

(ii) For any $v_0 \in S$, $\lim_{n\to\infty} B^n v_0 = v$

(iii) $B^n v_0$ has an exponential convergence rate at least as great as $-\ln(\beta)$

In this problem, we will illustrate and prove the contraction mapping theorem.

(a) Consider the contraction mapping $(Bw)(x) \equiv h(x) + \alpha w(x)$ with $\alpha \in (0, 1)$. Iteratively apply the operator B and show that

$$\lim_{n\to\infty} (B^n f)(x) = \frac{h(x)}{1-\alpha}$$

Argue that this shows that the fixed point of this operator B is consequently the function $v(x) = \frac{1}{1-\alpha} h(x)$. Show that $(Bv)(x) = v(x)$.

(b) Now we will prove the contraction mapping theorem in 3 steps (we will not prove the convergence rate). Show that $\{B^n f_0\}_{n=0}^{\infty}$ is a Cauchy sequence. (Cauchy sequence definition: Fix any $\epsilon > 0$. Then there exists N such that $d(B^m f_0, B^n f_0) \leq \epsilon$ for all $m, n \leq N$.)

Solution. Choose some $f_0 \in S$. Let $f_n = B^n f_0$. Since B is a contraction

$$d(f_2, f_1) = d(Bf_1, Bf_0) \leq \delta d(f_1, f_0).$$

[1] A metric d is a way of representing the distance between two functions, or two members of (metric) space S. One example: the supremum pointwise gap.

Continuing by induction,

$$d(f_{n+1}, f_n) \leq \delta^n d(f_1, f_0) \quad \forall n$$

We can now bound the distance between f_n and f_m when $m > n$.

$$\begin{aligned} d(f_m, f_n) &\leq d(f_m, f_{m-1}) + ... + d(f_{n+2}, f_{n+1}) + d(f_{n+1}, f_n) \\ &\leq \left[\delta^{m-1} + ... + \delta^{n+1} + \delta^n\right] d(f_1, f_0) \\ &= \delta^n \left[\delta^{m-n-1} + ... + \delta^1 + 1\right] d(f_1, f_0) \\ &< \frac{\delta^n}{1-\delta} d(f_1, f_0) \end{aligned}$$

So $\{f_n\}_{n=0}^{\infty}$ is Cauchy. ✓

(c) Show that the limit point v is a fixed point of B.

Solution. Since S is complete $f_n \to v \in S$.

We now have a candidate fixed point $v \in S$.

To show that $Bv = v$, note

$$\begin{aligned} d(Bv, v) &\leq d(Bv, B^n f_0) + d(B^n f_0, v) \\ &\leq \delta d(v, B^{n-1} f_0) + d(B^n f_0, v) \end{aligned}.$$

These inequalities must hold for all n.

And both terms on the RHS converge to zero as $n \to \infty$.

So $d(Bv, v) = 0$, implying that $Bv = v$. ✓

(d) Show that only one fixed point exists.

Solution. Now we show that our fixed point is unique.

Suppose there were two fixed points: $v \neq v^*$.

Then $Bv = v$ and $Bv^* = v^*$ (since fixed points).

Also have $d(Bv, Bv^*) < d(v, v^*)$ (since B is contraction)

So, $d(v, v^*) = d(Bv, Bv^*) < d(v, v^*)$.

Contradiction.

So the fixed point is unique. ✓

Problem 2: Blackwell's sufficiency conditions

Credit: David Laibson

We now show that there are in fact sufficient conditions for an operator to be contraction mapping.

Theorem 3. *(Blackwell's sufficient conditions) Let $X \subset \mathbb{R}^l$ and let $C(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup-metric. Let $B : C(X) \to C(X)$ be an operator satisfying two conditions:*

1. *monotonicity: if $f, g \in C(X)$ and $f(x) \leq g(x)\ \forall x \in X$,*
   then $(Bf)(x) \leq (Bg)(x), \forall x \in X$

2. *discounting: there exists some $\delta \in (0, 1)$ such that*

$$[B(f + a)](x) \leq (Bf)(x) + \delta a \ \ \forall\, f \in C(X),\ a \geq 0,\ x \in X.$$

Then, B is a contraction with modulus δ.

Note that a is a constant and $(f + a)$ is the function generated by adding a to the function f. Blackwell's conditions are sufficient but not necessary for B to be a contraction.

In this problem, we will prove these sufficient conditions.

(a) Let d be the sup-metric and show that, for any $f, g \in C(X)$, we have $f(x) \leq g(x) + d(f, g)$ for all x

Solution. Follows directly from definition of sup-metric.

(b) Use monotonicity and discounting to show that, for any $f, g \in C(X)$, we have $(Bf)(x) \leq (Bg)(x) + \delta d(f, g)$ and $(Bg)(x) \leq (Bf)(x) + \delta d(f, g)$.

Solution. Using monotonicity and discounting, we have for all x

$$(Bf)\,(x) \leq [B(g + d(f, g))](x) \leq (Bg)\,(x) + \delta d(f, g)$$

$$(Bg)\,(x) \leq [B(f + d(f, g))](x) \leq (Bf)\,(x) + \delta d(f, g)$$

(c) Combine these to show that $d(Bf, Bg) \leq \delta d(f, g)$.

Solution.

$$(Bf)\,(x) - (Bg)\,(x) \leq \delta d(f, g)$$

$$(Bg)\,(x) - (Bf)\,(x) \leq \delta d(f, g)$$

$$|(Bf)(x) - (Bg)(x)| \leq \delta d(f, g)$$

$$\sup_x |(Bf)(x) - (Bg)(x)| \leq \delta d(f, g)$$

$$d(Bf, Bg) \leq \delta d(f, g) \quad \checkmark$$

Problem 3: example of Blackwell's conditions

Credit: David Laibson

We will now work out a simple example to illustrate these sufficient conditions. In particular, consider the Bellman operator in a consumption problem (with stochastic asset returns, stochastic labor income, and a liquidity constraint).

$$(Bf)(x) = \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \quad \forall x$$

1. Check the first of Blackwell's conditions: monotonicity

   Solution. Assume $f(x) \leq g(x)\ \forall x$. Suppose c_f^* is the optimal policy when the continuation value function is f.

$$\begin{aligned}
(Bf)(x) &= \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \\
&= u(c_f^*) + \delta E f(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}) \\
&\leq u(c_f^*) + \delta E g(\tilde{R}_{+1}(x-c_f^*) + \tilde{y}_{+1}) \\
&\leq \sup_{c\in[0,x]} \left\{u(c) + \delta E g(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} \\
&= (Bg)(x)
\end{aligned}$$

2. Check the second of Blackwell's conditions: discounting

   Solution. Adding a constant (Δ) to an optimization problem does not affect optimal choice, so

$$\begin{aligned}
[B(f+\Delta)](x) &= \sup_{c\in[0,x]} \left\{u(c) + \delta E\left[f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) + \Delta\right]\right\} \\
&= \sup_{c\in[0,x]} \left\{u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1})\right\} + \delta\Delta \\
&= (Bf)(x) + \delta\Delta
\end{aligned}$$

Problem 4: growth model

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

In class, we studied the growth model with deterministic dynamics. Consider the sequence of the problem with ln utility and full depreciation

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(k_t^{\alpha} - k_{t+1})$$

where $0 < \alpha < 1$, subject to the constraint

$$k_{t+1} \in [0, k_t^{\alpha}] \equiv \Gamma(k_t).$$

Think of k_t^{α} as the resources you have available, and so the most you would be allowed to save is k_t^{α}. We represent this constraint by the *feasibility set* $\Gamma(k_t)$. (This is the more general notation you will find in Stokey-Lucas, for example.)

Also consider the associated Bellman equation

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ \ln(k^{\alpha} - k') + \beta V(k') \right\}.$$

Problem 5: equity model

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Assume that a consumer with only equity wealth must choose period by period consumption in a discrete-time dynamic optimization problem. Specifically, consider the sequence problem:

$$V(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho t} u(c_t)$$

subject to the constraints

$$x_{t+1} = e^{r + \sigma u_t - \frac{\sigma^2}{2}} (x_t - c_t)$$

where u_t is iid and $u_t \sim \mathcal{N}(0,1)$. There is a feasibility constraint $c_t \in [0, x_t]$. And we assume an endowment $x_0 > 0$. Here, x_t represents equity wealth at period t and c_t is consumption in period t. The consumer has discount rate ρ. The consumer can only invest in a risky asset with expected return $e^r = \mathbb{E}e^{r+\sigma u_t - \frac{\sigma^2}{2}}$. And we assume CRRA preferences with $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$, with $\gamma \in [0, \infty]$. We call this *constant* relative risk aversion because the relative risk aversion coefficient

$$-\frac{cu''(c)}{u'(c)} = \gamma$$

is constant.

The associated Bellman equation is

$$V(x) = \max_{x' \in [0,x]} \left\{ u(x - x') + \mathbb{E} e^{-\rho} V\left(e^{r + \sigma u - \frac{\sigma^2}{2}} x' \right) \right\}.$$

(a) Explain all terms in this Bellman equation. Why is u not a state variable, i.e., why don't we have $V(x, u)$?

Problem 6: some true / false questions

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Discuss whether the following are true / false / uncertain:

1. All supremium / max sequence problems have a unique value function solution. (True: Why?)
2. If the flow payoff / instantaneous utility function is bounded, then there exists a unique bounded solution to the Bellman equation.
3. In the growth problem above, for any $\epsilon > 0$, there exists a value T such taht $k_t < 1 + \epsilon$ for all $t > T$. (True: Why?)
4. In the growth problem above, we have $\lim_{n\to\infty} \beta^n V(k_n) \leq 0$.

Problem 7: optimal stopping

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Consider the optimal stopping application from class: Each period $t = 0, 1, \ldots$ the consumer draws a job offer from a uniform distribution with support in the unit interval: $x \sim \text{unif}[0, 1]$. The consumer can either accept the offer and realize net present value x, or the consumer can wait another period and draw again. Once you accept an offer the game ends. Waiting to accept an offer is costly because the value of the remaining offers declines at rate $\rho = -\ln(\beta)$ between periods. The Bellman equation for this problem is:

$$V(x) = \max\left\{x,\ \beta\mathbb{E}V(x')\right\}$$

where x' is your next draw, which is a random variable.

1. Explain the intuition behind this Bellman equation. Explain every term.

2. Consider the associated functional operator:

$$(Bw)(x) = \max\left\{x,\ \beta\mathbb{E}w(x')\right\}$$

for all x. Using Blackwell's conditions, show that this Bellman operator is a contraction mapping.

3. What does the contraction mapping property imply about $\lim_{n\to\infty} B^n w$, where w is *any* arbitrary function?

4. Suppose we make a (bad?) guess $w(x) = 1$ for all x. Analytically iterate on $B^n w$ and show that

$$\lim_{n\to\infty}(B^n w)(x) = V(x) = \begin{cases} x^* & \text{if } x \le x^* \\ x & \text{if } x > x^* \end{cases}$$

where

$$x^* = e^{\rho}\left(1 - \left[1 - e^{-2\rho}\right]^{\frac{1}{2}}\right).$$

Problem 8: optimal investment

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Every period you draw a cost c distributed uniformly between 0 and 1 for completing a project. If you undertake the project, you pay c, and complete the project with probability $1-p$. Each period in which the project remains uncompleted, you pay a late fee of l. The game continues until you complete the project.

(a) Write down the Bellman Equation assuming no discounting. Why is it ok to assume no discounting in this problem?

(b) Derive the optimal threshold: $c^* = \sqrt{2l}$. Explain intuitively, why this threshold does not depend on the probability of failing to complete the project, p.

(c) How would these results change if we added discounting to the framework? Redo steps a and b, assuming that the agent discounts the future with discount factor $0 < \beta < 1$ and assuming that $p = 0$. Show that the optimal threshold is given by

$$c^* = \frac{1}{\beta}\left(\beta - 1 + \sqrt{(1-\beta)^2 + 2\beta^2 l}\right)$$

(d) When $0 < \beta < 1$, is the optimal value of c^* still independent of the value of p? If not, how does c^* qualitatively vary with p? Provide an intuitive argument.

(e) Take the perspective of an agent who has not yet observed the current period's draw of c. Prove that the expected delay until completion is given by:

$$\frac{1}{c^*(1-p)} - 1$$