M2: Lecture 5

Dynamic Optimization in Continuous Time

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Outline of today's lecture

1. Neoclassical growth model in continuous time
2. Calculus of variations
3. Optimal control theory
4. Example:

1. Neoclassical growth model in continuous time

- The lifetime value of the representative household is

$$v(k_0) = \max_{\{c_t\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\begin{aligned} \dot{k}_t &= F(k_t) - \delta k_t - c_t \\ k_0 &\text{ given ,} \end{aligned}$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now

- This is the **sequence problem** in continuous time

2. Calculus of variations

- Resources:
  - LeVeque: Finite Difference Methods for Ordinary and Partial Differential Equations
  - Kamien and Schwartz: Dynamic Optimization
  - Gelfand and Fomin: Calculus of Variations
- This dynamic optimization problem is associated with the Lagrangian

$$L = \int_0^\infty e^{-\rho t} \left[u(c_t) + \mu_t \Big(F(k_t) - \delta k_t - c_t - \dot{k}_t \Big) \right] dt$$

- μ_t is the Lagrange multiplier on the capital accumulation ODE
- What do we do with $\dot{k}_t$??

- Integrate by parts:

$$\int_0^\infty e^{-\rho t}\mu_t \dot{k}_t dt = e^{-\rho t}\mu_t k_t\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(e^{-\rho t}\mu_t\right)k_t dt$$
$$= -\mu_0 k_0 + \int_0^\infty e^{-\rho t}\rho\mu_t k_t dt - \int_0^\infty e^{-\rho t}\dot{\mu}_t k_t dt$$

- Plugging into Lagrangian:

$$L = \int_0^\infty e^{-\rho t}\left[u(c_t) + \mu_t\Big(F(k_t) - \delta k_t - c_t\Big) - \rho\mu_t k_t + \dot{\mu}_t k_t\right]dt + \mu_0 k_0$$

- What have we accomplished?

- Notice $\mu_0 k_0$, this is crucial. What's intuition?

$$L = \int_0^\infty e^{-\rho t}\Big[u(c_t) + \mu_t\Big(F(k_t) - \delta k_t - c_t\Big) - \rho\mu_t k_t + \dot{\mu}_t k_t\Big]dt + \mu_0 k_0$$

- The planner optimizes over paths $\{c_t\}$ and $\{k_t\}$
- At an optimum, there cannot be *any* small perturbation in these paths that the planner finds preferable
- Let $\{c_t\}$ and $\{k_t\}$ be *candidate* optimal paths. Consider $\hat{c}_t = c_t + \alpha h_t^c$ and $\hat{k}_t = k_t + \alpha h_t^k$ for arbitrary functions h_t^c and h_t^k

$$\begin{aligned} L(\alpha) = \int_0^\infty e^{-\rho t}\Big[&u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta\alpha h_t^k - c_t - \alpha h_t^c\Big) \\ &- \rho\mu_t(k_t + \alpha h_t^k) + \dot{\mu}_t(k_t + \alpha h_t^k)\Big]dt + \mu_0(k_0 + \alpha h_0^k) \end{aligned}$$

- What about *boundary conditions*? At $t = 0$, capital stock is fixed (k_0 given) while consumption is free. So must have: $h_0^k = 0$ while h_0^c is free

Necessary condition for optimality: $\frac{d}{d\alpha}L(0) = 0$

$$L(\alpha) = \int_0^\infty e^{-\rho t} \Big[u(c_t + \alpha h_t^c) + \mu_t \Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta \alpha h_t^k - c_t - \alpha h_t^c \Big) - \rho \mu_t (k_t + \alpha h_t^k) + \dot{\mu}_t (k_t + \alpha h_t^k) \Big] dt + \mu_0 (k_0 + \alpha h_0^k)$$

Work this out yourselves (many times, in many applications!)

$$\frac{d}{d\alpha} L(0) = \int_0^\infty e^{-\rho t} \Big[u'(c_t) h_t^c + \mu_t \Big(F'(k_t) h_t^k - \delta h_t^k - h_t^c \Big) - \rho \mu_t h_t^k + \dot{\mu}_t h_t^k \Big] dt + \mu_0 h_0^k$$

where $h_0^k = 0$ because k_0 is fixed

Group terms:

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\left[\Big(u'(c_t) - \mu_t\Big)h_t^c + \Big(\mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t\Big)h_t^k\right]dt$$

Fundamental Theorem of the Calculus of Variations: Since h_t^c and h_t^k were arbitrary, we must have *pointwise*

$$0 = u'(c_t) - \mu_t$$
$$0 = \mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t$$

Proposition. (Euler equation for marginal utility)

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dot{u}_{c,t}}{u_{c,t}} = \rho - F'(k_t) + \delta = \rho - r_t$$

- We have now solved the neoclassical growth model in continuous time. Its solution is given by a system of two ODEs.

- Suppose $u(c) = \log(c)$ and $F(k) = k^{\alpha}$, then:

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \delta - \rho$$
$$\dot{k}_t = k_t^{\alpha} - \delta k_t - c_t$$

  with k_0 given

- Derive the consumption Euler equation yourselves!

- What are the boundary conditions? (Always ask about BCs!)
  - Initial condition on capital: k_0 given
  - Terminal condition on consumption : $\lim_{T\to\infty} c_T = c_{ss}$

3. Optimal control theory

- Optimal control theory emerged from the calculus of variations
- Applies to dynamic optimization problems in continuous time that feature (ordinary) differential equations as constraints
- Again the neoclassical growth model:

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

  subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t, \quad k_0 \text{ given}$$

- Three new terms:
  - **State variable**: k_t
  - **Control variable**: c_t
  - **Hamiltonian**: $H(c_t, k_t, \mu_t) = u(c_t) + \mu_t \left[F(k_t) - \delta k_t - c_t\right]$

- With Hamiltonian in hand, *copy-paste* formula that we can always use:

  Optimality condition: $\frac{\partial}{\partial c}H = 0$

  Multiplier condition: $\rho\mu_t - \dot{\mu}_t = \frac{\partial}{\partial k}H$

  State condition: $\dot{k}_t = \frac{\partial}{\partial \mu}H$

- This gives us the same equations that we derived using calc of variations:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \rho\mu_t - \dot{\mu}_t &= \mu_t(F'(k_t) - \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

- We again get system of Euler equation and capital accumulation:

$$\begin{aligned} \dot{c}_t &= \frac{u'(c_t)}{u''(c_t)}\Big(\rho - F'(k_t) + \delta\Big) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

4. Simple example

- Credit: Kamien-Schwartz p. 129
- Simple problem: not much intuition, but illustrates mechanics

$$\max \int_0^1 (x+u)dt$$

  subject to $\dot{x} = 1 - u^2$ and initial condition $x_0 = 1$

- Step 1: form Hamiltonian $H(t,x,u,\lambda) = x + u + \lambda(1-u^2)$
- Step 2: necessary conditions (note: no discounting here)

$$\begin{aligned} 0 &= H_u = 1 - 2\lambda u \\ -\dot{\lambda} &= H_x = 1 \end{aligned}$$

  and terminal condition $\lambda_1 = 0$ (because u_1 is *free*)

- Step 3: manipulate necessary conditions:

$$\lambda = 1 - t$$
$$u = \frac{1}{2\lambda}$$

and therefore: $u = \frac{1}{2}(1 - t)$

- Finally: solve for all paths (control, state, multiplier)

$$x_t = t - \frac{1}{4}(1 - t) + \frac{5}{4}$$
$$\lambda_t = 1 - t$$
$$u_t = \frac{1}{2}(1 - t)$$