M2: Lecture 7

Stochastic Dynamic Programming

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Outline of today's lecture

1. What is the generator of a stochastic process?
2. Stochastic neoclassical growth model
3. Stochastic neoclassical growth with diffusion process
4. Stochastic neoclassical growth with Poisson process
5. Many examples

1. The generator of a stochastic process

- We start with the diffusion process

$$dX = \mu(t, X)dt + \sigma(t, X)dB$$

  where dB is a standard Brownian motion

- The generator $\mathcal{A}$ tells us how the stochastic process is *expected* to evolve

- The generator $\mathcal{A}$ is a functional operator

- Formally, for $f(t, X)$, we have

$$\mathcal{A}f = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(t + \Delta t, X(t + \Delta t)) - f(t, X(t))}{\Delta t}$$

- We will now show that:

$$\mathcal{A}f = \partial_t f(t, X) + \mu(t, X)\partial_X f(t, X) + \frac{1}{2}\sigma(t, X)^2 \partial_{XX} f(t, X)$$

- For the general / multi-dimensional version see Oksendal

- Next, we consider the poisson process $\{Y_t\}$ where $Y_t \in \{Y^1, Y^2\}$. This is a two-state Markov chain in continuous time.

- We assume that the Poisson intensity / arrival rate / hazard rate is λ

- The generator is now given by

$$\mathcal{A}f(Y^j) = \lambda\Big[f(Y^{-j}) - f(Y^j)\Big]$$

- Intuition: at rate λ you transition, so you lose the value of your current state, $f(Y^j)$, and obtain the value of the new state, $f(Y^{-j})$

- Again see Oksendal for general version of this and more details

2. Neoclassical stochastic growth

- Time is continuous and the horizon is infinite, $t \in [0, \infty)$
- Economy populated by representative household that operates production technology $F(k_t, z_t)$ where z_t is exogenous productivity
- Assume $F(\cdot)$ is well behaved: $F(0, z) = 0$, as well as F_k, $F_z > 0$, and $F_{kk} < 0$
- At time $t = 0$, economy's initial state is (k_0, z_0)
- Lifetime value of household is given by

$$V(k_0, z_0) = \max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

  where $u(\cdot)$ is instantaneous utility flow and $\{c_t\}_{t \geq 0}$ is stochastic consumption process. $\mathbb{E}_0$ denotes expectation over future productivity realizations. We assume labor supply is inelastic and normalized to 1

- Capital evolves as before: $\frac{d}{dk} k_t = F(k_t, z_t) - \delta k_t - c_t$

3. Productivity as a diffusion process

- We start with diffusion process:

$$dz_t = -\theta z_t dt + \sigma dB_t,$$

where θ and σ are constants

- This is a continuous-time, mean-reverting AR(1) process called the Ornstein-Uhlenbeck process

- State space is now given by

$$\left\{(k, z) \mid k \in [0, \bar{k}] \text{ and } z \in [\underline{z}, \bar{z}]\right\}$$

- In discrete time, we would have

$$V(k_t, z_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\}$$

- Difference from previous lecture: $\mathbb{E}$ because there is uncertainty

$$\begin{aligned}
(1+\rho\Delta t)V(k_t, z_t) &= \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\} \\
\rho\Delta t V(k_t, z_t) &= \max_c \left\{ u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t) \right\} \\
\rho V(k_t, z_t) &= \max_c \left\{ u(c) + \mathbb{E}_t \frac{V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t)}{\Delta t} \right\}
\end{aligned}$$

- Take limit $\Delta t \to 0$ and drop time subscripts:

$$\rho V(k, z) = \max_c \left\{ u(c) + \mathbb{E} \frac{dV(k, z)}{dt} \right\}$$

- What remains? Characterizing continuation value $\frac{d}{dt}V(k, z)$ (i.e., characterizing how process dV evolves)

- The generator $\mathcal{A}$ is exactly the answer to this question! I.e.,

$$\begin{aligned}\mathbb{E}\frac{dV(k,z)}{dt} &= \mathcal{A}V(k,z) \\ &= \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\end{aligned}$$

- Therefore, we arrive at the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\rho V(k,z) = \max_c \Big\{ u(c) + \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) \\ - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\Big\}\end{aligned}$$

with first-order condition

$$u'(c(k,z)) = \partial_k V(k,z)$$

4. Productivity as a Poisson process

- Next, consider Poisson process for $\{z_t\}$ with $z_t \in \{z^L, z^H\}$
- Generator now given by

$$\mathcal{A}V(k, z^j) = \Big(F(k, z) - \delta z - c\Big)\partial_k V(k, z) + \lambda\Big[V(k, z^{-j}) - V(k, z^j)\Big]$$

- Note: derivation of HJB exactly as before *up to* characterizing $\mathbb{E}[dV]$
- With Poisson process, HJB becomes

$$\rho V(k, z^j) = \max_c \Big\{u(c) + \Big(F(k, z) - \delta z - c\Big)\partial_k V(k, z) \\ + \lambda\Big[V(k, z^{-j}) - V(k, z^j)\Big]\Big\}$$

with first-order condition

$$u'(c(k, z^j)) = \partial_k V(k, z^j)$$

5.1. Example: income fluctuations

- Economy is populated by representative household that faces income risk
- Household accumulates wealth according to

$$\dot{a}_t = ra_t + e^{z_t} - c_t$$

subject to borrowing constraint $a_t \geq 0$

- Preferences again: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$
- Income follows diffusion process: $dy_t = -\theta y_t dt + \sigma dB_t$
- Away from borrowing constraint, HJB given by

$$\rho V = \max_c \left\{ u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz} \right\}$$

with $V_a = \partial_a V(a, z)$ (you'll see this often)

5.2. Example: firm profit maximization

- Firm maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$
- For now, profit given by: $\pi_t = A_t n_t^\alpha - w_t n_t$ where firm chooses labor n_t
  Assume $\alpha < 1$, so this is a decreasing-returns production function
- Firm is small and takes wage $\{w_t\}$ as given (wages determined in general equilibrium)
- Productivity follows two-state high-low process, with $A_t \in \{A^{\text{rec}}, A^{\text{boom}}\}$
- Recursive representation: A is only state variable, $w_t = w(A_t)$

$$rV(A^{\text{boom}}) = \max_n \left\{ A^{\text{boom}} n^\alpha - w(A^{\text{boom}}) n + \lambda \left[V(A^{\text{rec}}) - V(A^{\text{boom}}) \right] \right\}$$

with first-order condition

$$n = \left(\frac{\alpha A^j}{w(A^j)} \right)^{\frac{1}{1-\alpha}}$$

5.3. Example: capital investment with adjustment cost

- Firm again maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$
- Now: let $\psi(\cdot)$ denote an adjustment cost

$$\begin{aligned} \pi_t &= e^{A_t} k_t^\alpha - Q_t \iota_t - \psi(\iota_t, k_t) \\ dk_t &= (\iota_t - \delta k_t) dt \\ dA_t &= -\theta A_t dt + \sigma dB_t \end{aligned}$$

- Firm is small and takes capital price as given
- Recursive representation in terms of (k, A), i.e., $Q_t = Q(k_t, A_t)$

$$\begin{aligned} rV(k, A) = \max_\iota \Big\{ & e^{A_t} k_t^\alpha - Q(A) \iota_t - \psi(\iota_t, k_t) + (\iota - \delta k) \partial_k V(k, A) \\ & - \theta A \partial_A V(k, A) + \frac{\sigma^2}{2} \partial_{AA} V(k, A) \Big\} \end{aligned}$$

with first-order condition: $Q(k, A) + \partial_\iota \psi(\iota(k, A), k) = \partial_k V(k, A)$

5.4. Example: investing in stocks

- Suppose you optimize lifetime utility $V_0 = \mathbb{E}_0 \int_0^\infty u(c_t)dt$
- You can trade two assets: riskfree bond (return rdt), and risky stock

$$dR = (r + \pi)dt + \sigma dB, \text{ where } \pi \text{ is the equity premium}$$

- You have wealth a_t and invest a share θ_t in stocks, thus,

$$da_t = \theta_t a_t dR_t + (1 - \theta_t) a_t r_t dt + y - c_t$$

or, rearranging, and dropping t subscripts

$$da = ra + \theta a \pi dt + y - c + \theta a \sigma dB$$

- HJB becomes:

$$\rho V(a) = \max_{c,\theta} \left\{ u(c) + (ra + \theta a \pi dt + y - c)V'(a) + \frac{1}{2}(\sigma \theta a)^2 V''(a) \right\}$$

with FOCs: (i) $u'(c) = V'(a)$ and (ii) $\theta = -\frac{\pi}{\sigma^2} \frac{V'(a)}{aV''(a)}$

5.5. Example: tax competition

- Two countries, $i \in \{A, B\}$, setting corporate tax rates τ_t^i on firms operating / headquartered in country i

- Mass of multinational firms j, with μ_t denoting % in country A at time t

- Firms relocate activity / headquarters at rate θ towards low-tax country:

$$d\mu_t = \theta\mu_t(\tau_t^B - \tau_t^A)^\gamma dt$$

- Country A maximizes tax revenue: max $\int_0^\infty e^{-\rho t}\tau_t^A \mu_t dt$. Countries compete over taxes $\{\tau_{it}\}$

- Dynamic Nash: country A sets τ_t^A as best response taking τ_t^B as given

- Recursive representation: the only state variable is μ_t

$$\rho V^A(\mu) = \max_{\tau^A} \left\{ \tau^A \mu + \theta\mu \Big(\tau^B(\mu) - \tau^A\Big)^\gamma \partial_\mu V^A(\mu) \right\}$$

  Best response strategies: $0 = \mu + \gamma\theta\mu(\tau^B(\mu) - \tau^A)^{\gamma-1} V_\mu^A(\mu)$