Lecture 2

Dynamics: Continuous Time

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Outline of today's lecture

1. Ordinary differential equations
2. Prominent examples of differential equations in macro
3. Partial differential equations
4. Continuous-time Markov chains
5. Brownian motion and stochastic differential equations
6. Solow growth model

1. Ordinary differential equations

- Consider the "discrete-time" equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- *Continuous-time limit*: consider the limit as $\Delta t \to 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \to 0} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$ is *autonomous* and dropping subscripts: $\dot{X} = g(X)$

- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

Boundary conditions (I)

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval $t \in [0, 1]$. We call $[0, 1]$ the *state space*. $(0, 1)$ is the *interior of the state space* and $\{0, 1\}$ is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

Boundary conditions (II)

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions
- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for X_t with some *initial condition* X_0
  - *Terminal value problems* instead specify X_T
- More broadly: We need sufficient information to characterize the function of interest along the boundary
- Types of boundary conditions: Dirichlet ($X_0 = c$), von-Neumann ($\frac{dX_0}{dt} = c$), reflecting boundaries, ...
- Boundary conditions are very important and can be very subtle (especially for PDEs)

Linear First-Order ODEs

- Consider the equation:

$$\dot{X}(t) = a(t)X(t) + b(t) \tag{1}$$

- If $b(t) = 0$, (1) is a *homogeneous* equation, if $a(t) = a$ and $b(t) = b$ we say (1) has *constant coefficients*

- Start with $\dot{X}(t) = aX(t)$, divide by $X(t)$ and integrate with respect to t

$$\begin{aligned} \int \frac{\dot{X}(t)}{X(t)} dt &= \int a dt \\ \log X(t) + c_0 &= at + c_1 \\ X(t) &= Ce^{at} \end{aligned}$$

  where $C = e^{c_1 - c_0}$

- Pin down constant C by using the boundary condition (we need 1)

- Consider time-varying coefficient with $\dot{X}(t) = a(t)X(t)$ with initial condition $X(0) = \bar{x}$

- Dividing by $X(t)$, integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition: $C = \bar{x}$

- Finally, for $\dot{X}(t) = aX(t) + b$, we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

  after using change of variables $Y(t) = X(t) + \frac{b}{a}$

- Many results for systems of linear differential equations: $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}\boldsymbol{X}(t)$

2. Examples of differential equations in macro

Capital accumulation:

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps, $K_{t+1} = I_t + (1-\delta)K_t$
- With arbitrary Δ time step, $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned} K_{t+\Delta} &= K_t + \Delta(I_t + (1-\delta)K_t \\ K_{t+\Delta} - K_t &= I_t - \delta K_t \\ \dot{K}_t &= I_t - \delta K_t \end{aligned}$$

- Suppose $\{I_t\}_{t\geq 0}$ exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\dot{K}_t + \delta K_t = I_t$$
$$e^{\int_0^t \delta ds}\dot{K}_t + e^{\int_0^t \delta ds}\delta K_t = e^{\int_0^t \delta ds} I_t$$

- Notice that $\int_0^t \delta ds = \delta \int_0^t ds = \delta[s]_0^t = \delta(t-0) = \delta t$, so

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

- We have $e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = \frac{d}{dt}(K_t e^{\delta t})$, integrating:

$$K_t e^{\delta t} = \tilde{C} + \int_0^t e^{\delta s} I_s ds$$
$$K_t = C + \int_0^t e^{-\delta(t-s)} I_s ds$$

- Integrating constant solves initial condition: $C = K_0$

Wealth dynamics (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- r_t is the real rate of return on wealth, y_t is income, and c_t is consumption
- Structure of the equation similar to capital accumulation equation

Consumption Euler equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., c_T must converge to something
- Suppose there exists time T s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

New Keynesian Phillips curve:

$$\dot{\pi}_t = \rho \pi_t + \kappa x_t$$

- This is a backward equation that requires a terminal condition
- As in discrete time, we often consider the 0 inflation steady state with $\pi_T \to 0$
- Then we can solve (work this out yourselves):

$$\pi_t = -\kappa \int_t^\infty x_s ds$$

3. A brief intro to partial differential equations

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...
  $\Longrightarrow$ increasingly used in economics
- This class: no self-contained treatment of PDEs *but* we will encounter some simple PDEs

- Consider a function $u(x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}$
- Partial derivatives of $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in u and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_{x_1 x_1} u, \ldots)$$

- The *order* of the PDE, is the order of the highest partial derivative
- Examples from physics
  - Heat equation: $\partial_t u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Wave equation: $\partial_{tt} u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Transport equation: $\partial_t u = \partial_x u$ (first-order, linear, homogeneous)
- Income distribution "solves heat equation", wealth dynamics "solve transport equations", dynamic programming often transport + heat

5. Solow Growth Model

- Time is discrete and the horizon infinite, $t = 0, 1, 2, \ldots$
- There is a *representative household*: large number of small but identical households
- Assume households have a constant savings rate $s \in (0, 1)$ (out of disposable income)
- A representative firm operates the technology / production function

$$Y_t = F(K_t, L_t, A_t)$$

  where K_t is capital, L_t is labor, A_t is total factor productivity (TFP)

- Capital accumulation: $K_{t+1} = (1 - \delta)K_t + I_t$
- Goods market clearing (*national income accounting identity*): $Y_t = C_t + I_t$

- Feasible allocations in this economy are characterized by

$$K_{t+1} \leq F(K_t, L_t, A_t) + (1-\delta)K_t - C_t$$

- How do we determine the equilibrium allocation among all those allocations that are feasible? $\implies$ assume constant savings rate

$$\begin{aligned} sY_t &= S_t \\ &= I_t = Y_t - C_t \end{aligned}$$

or $C_t = (1-s)Y_t$

- Equilibrium characterized by (non-linear) first-order difference equation:

$$K_{t+1} = sF(K_t, L_t, A_t) + (1-\delta)K_t \tag{2}$$

Definition. (Equilibrium) Given sequences $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial condition for capital K_0, the equilibrium path of the Solow growth model comprises paths for capital, output, consumption and investment $\{K_t, Y_t, C_t, I_t\}_{t=0}^{\infty}$ that satisfy (2), goods market clearing, firm production, and $C_t = sY_t$.

Steady state

- Suppose Cobb-Douglas technology:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}$$

  and no productivity or population growth; also normalize $L_t = 1$

- A steady state is a level of capital K such that

$$K = sAK^{\alpha} + (1-\delta)K$$

- Solving this, we find:

$$K = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Transition dynamics

- The key *degree of freedom* in this economy is the *initial condition* for the (forward) difference equation for capital accumulation: K_0
- Suppose $K_0 < K$ and $K_0 > K$, what happens?
- Read discussion and proofs in Acemoglu, but intuitively:

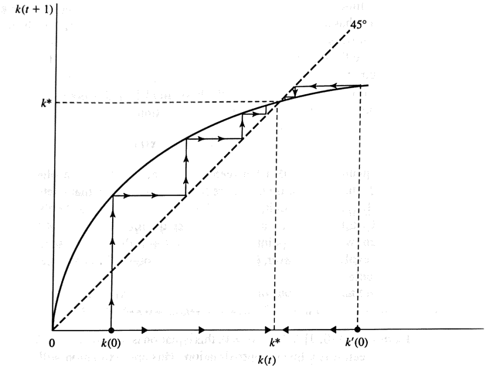

FIGURE 2.7 Transitional dynamics in the basic Solow model.

6. Stochastic Difference Equations

- Consider the process $\{X_t\}$ with

$$X_{t+1} = AX_t + Cw_{t+1} \tag{3}$$

where w_{t+1} is an iid. process with $w_{t+1} \sim \mathcal{N}(0,1)$

- Equation (3) is a *first-order, linear stochastic difference equation*
- Let $\mathbb{E}_t$ the *conditional expectation* operator (conditional on time t information)
- For example:

$$\begin{aligned}\mathbb{E}_t(X_{t+1}) &= \mathbb{E}(X_{t+1} \mid X_t) = \mathbb{E}(AX_t + Cw_{t+1} \mid X_t) \\ &= AX_t + C\mathbb{E}(w_{t+1} \mid X_t) = AX_t + C\mathbb{E}(w_{t+1}) = AX_t\end{aligned}$$

- Rational expectations: agents' beliefs about stochastic processes are consistent with the true distribution of the process

- Consumption Euler equation with uncertainty (e.g., stochastic income):

$$u'(C_t) = \beta R \mathbb{E}_t \left[u'(C_{t+1}) \right]$$

- New Keynesian Phillips curve with uncertainty (e.g., demand shocks):

$$\pi_t = \beta \mathbb{E}_t \left[\pi_{t+1} \right] + \kappa x_t$$