Dynamic Optimization: Problem Set #1

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Problem 1

Credit: QuantEcon https://python.quantecon.org/finite_markov.html#exercises

Let y_t denote the employment / earnings state of an individual. Consider the state space $y_t \in Y = \{y^U, y^E\}$, where y^U corresponds to unemployment and y^E corresponds to employment. Let $\boldsymbol{y}$ denote the column vector $(y^U, y^E)'$ representing this state space (this is the grid you would construct on a computer). Suppose the employment dynamics of the individual are characterized by the invariant transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

We interpret a time period as a quarter.

(a) Give economic interpretations of $\alpha = P_{11}$ and $\beta = P_{22}$
$\alpha = P_{11}$: Probability that an unemployed doesn't find a job and remains unemployed at t.
$\beta = P_{22}$: Probability that an employed doesn't lose the job and remains employed at t.

(b) Why do the rows of P sum to 1?
Because each row is a probability distribution. Intuitively: in the model either you are employed or unemployed

(c) Is there an absorbing state in this model?
We say x is an absorbing state if

$$P(X_{t+1} = x | X_t = x) = 1$$

There are no absorbing states in this model if $\beta,\alpha \in (0,1)$.

(d) Compute the probability of being unemployed two quarters after being employed.

$$P(y_{t+2} = y^U, y_{t+1} = y^U | y_t = y^E) + P(y_{t+2} = y^U, y_{t+1} = y^E | y_t = y^E) =$$
$$\alpha(1-\beta) + (1-\beta)\beta = (\alpha+\beta)(1-\beta)$$

(e) Denote the *marginal (probability) distribution* of y_t at time t by ψ_t. $\psi_t(y^L)$ is the probability that process y_t is in state y^L at time t. It is easiest to think of ψ_t as a time-varying row vector. Use the law of total probability to decompose $y_{t+1} = y^L$, accounting for all the possible ways in which state y^L can be reached at time $t+1$.

$$\psi_{t+1}(y^L) = P(y_{t+1} = y^L) = \psi_t(y^L)P(y_t + 1 = y^L | y_t = y^L) + (1 - \psi_t(y^L))P(y_{t+1} = y^L | y_t \neq y^L)$$

(f) Show that the resulting equation can be written as the vector-matrix product

$$\psi_{t+1} = \psi_t P.$$

Therefore: The evolution of the marginal distribution of a Markov chain is obtained by post-multiplying by the transition matrix.

It is clear that the conditional probabilities form question e) are obtained from one column of the transition matrix P, so we can write it as the vector-matrix product.

(g) Show that

$$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t,$$

where $\sim$ reads "is distributed according to".
First $\psi_1 = \psi_0 P$, then also $\psi_2 = \psi_1 P = \psi_0 P^2$. Iterating we get the result

(h) We call ψ^* a *stationary distribution* of the Markov chain if it satisfies

$$\psi^* = \psi^* P.$$

Compute the probability of being unemployed n quarters after being employed. Take $n \to \infty$ and find the stationary distribution of this Markov chain. Find the stationary distribution by alternatively plugging into the above equation for ψ^*.
As a summary (Check the link form quantecon for more detail): A Markov Chain is irreducible if all states communicate, that is there is a positive probability of going from any y to any x (possibly in many steps). Then an irreducible MC converges to a stationary (or ergodic) distribution. So if we take $n \to \infty$ the marginal distribution converges to the stationary: $\psi_{t+n} \to \psi^*$.

To find the stationary distribution we need to solve the equation above. Note $\psi = 0$ is a solution but it's not a probability distribution. Let $\psi^* = (\psi^*(y^U), 1 - \psi^*(y^U))$, then solving the equation we get

$$\psi^*(y^U) = \frac{1-\beta}{2-\alpha-\beta}$$

Economic intuition: Spend more time unemployed (or unemployment rate higher) if the probability of finding a job is lower (the job finding rate) or the probability of losing job is higher (the separation rate).

(i) Suppose $y_0 = y^H$. Solve for $\mathbb{E}_0(y_t)$. Use the law of total / iterated expectation to relate expectation to probabilities. Then use the formulas for marginal (probability) distributions derived above.

$$\mathbb{E}_0(y_t) = P(y_t = y^U | y_0 = y^E) y^U + P(y_t = y^E | y_0 = y^E) y^E$$

And we get the transition probabilities from P^t

Problem 2

Consider the first-order linear *homogeneous* difference equation

$$x_{t+1} = \rho x_t.$$

We parameterize the initial condition by $x_0 = c$.

(a) Prove by induction that the *general solution* (for arbitrary c) is given by

$$x_t = \rho^t c.$$

Guess $x_t = \rho^t c$. Check true at $t = 0 : x_0 = \rho^0 c = c$. If true at t also true at $t+1$:

$$x_{t+1} = \rho x_t = \rho \rho^t c = \rho^{t+1} c$$

(b) Show that the *particular solution* for initial value $x_0 = x$ is given by $x_t = \rho^t x$. If $x_0 = x$ then set $c = x$.

(c) Show that this also implies

$$x_t = \rho^{t-s} x_s$$

for $t > s$.

We have $c = \frac{x_s}{\rho^s}$ for any s, substitute in the general solution from (a) and we get the result.

(d) Prove that this difference equation satisfies the Markov property.

Note we can think of an equation in difference as a stochastic process where the noise term is always zero. So $\mathbb{P}(x_{t+1} = \rho x | x_t = x) = 1$. Because x_{t+1} is a function of x_t but not of $(x_{t-1}, x_{t-2}...)$ this difference equation satisfies the Markov property.

Problem 3

Credit: Klaus Neusser http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf on p. 19 (of the PDF)

We study the dynamics of loan amortization. Denote by D_t the amount of debt owed at time t. The debt contract is serviced by paying an amount Z_t each period. Z_t is given exogenously for this problem (you could imagine some agent optimizing in the background).

(a) Explain why debt dynamics are characterized by the equation

$$D_{t+1} = RD_t - Z_t$$

where R is the constant gross interest rate associated with the loan. What kind of difference equation is this? Is this a forward or a backward equation?

The gross interest rate is $R = 1 + r$, so debt next period is debt at the current period plus the interest rate on debt minus repayment. It is a linear (time-homogeneous ??) first order difference equation. It is a forward equation, start with D_0.

(b) Suppose we start with an initial loan D_0. Solve iteratively (by induction) for D_t. You should get two terms — explain the economics for both terms.

$$D_{t+1} = R^{t+1} D_0 - \sum_{i=0}^{t} R^i Z_{t-i}$$

. First term: Growth in debt if no repayment. Second term: NPV repayments (going back on time), as payment Z at $t = 0$ reduces debt at t by R^t compared to payment at t

(c) Suppose the loan needs to be repaid at time T. Solve for the constant repayment schedule $Z_t = Z$ such that the loan is repaid in period T.

$$D_{t+1} = R^{t+1} D_0 - \left(R^{t+1} - 1\right) \frac{Z}{R-1}$$

. If debt repaid at T, then must have $D_{T+1} = 0$. Therefore

$$Z = \frac{R-1}{1 - R^{-T-1}} D_0$$

(d) What is the condition on constant repayment rate Z relative to D_0 such that the loan is repaid in finite time?
Taking $T \to \infty$ we get $Z = (R-1)D_0$. So the payment is just the interest accruing in each period.

Problem 4

Credit: Klaus Neusser http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf on p. 38 (of the PDF)

We study Cagan (1956)'s model for hyperinflation. The model is summarized by the three equations

$$m_t^d - p_t = \alpha(p_{t+1}^e - p_t)$$

$$m_t^s = m_t^d$$

$$p_{t+1}^e - p_t = \gamma(p_t - p_{t-1})$$

where (all in logs) m_t^d is money demand, m_t^s is money supply, p_t is the price level and p_{t+1}^e is private agents' expectations for the price level in period $t+1$. The first equation of the above system characterizes money demand and the third equation characterizes *adaptive* inflation expectations. Assume that $\alpha < 0$ and $\gamma > 0$.

(a) Characterize a first-order difference equation that solves for p_t as a function of p_{t-1} and m_{t-1}. What kind of difference equation is this?
Substitute the 3rd constraint into the 1st, $m_t^d - p_t = \alpha\gamma(p_t - p_{t-1})$. Let $m_t = m_t^d = m_t^s$. Then rearranging

$$p_t = \frac{\alpha\gamma}{1+\alpha\gamma} p_{t-1} + \frac{1}{1+\alpha\gamma} m_t \equiv \phi p_{t-1} + Z_t$$

This is a linear (time-homogeneous ??) first order difference equation

(b) Using the tools already developed, solve for p_t in terms of some initial conditions on the system.
Let p_0 be the initial price level, then iterating forward (as in the previous exercise)

$$p_t = \phi^t p_0 + \sum_{i=0}^{t-1} \phi^i Z_{t-i}$$

(c) Characterize the stability condition such that if $m_t \to m$ in the long run, p_t converges to a steady state p. Interpret the economics of this stability condition.

To have both terms finite as $t \to \infty$ we need

$$|\phi| = |\frac{\alpha\gamma}{1+\alpha\gamma}| < 1$$

To have stability we need that money demand m_t^d does not respond too much to current current inflation. This is the case if inflation expectations do not respond a lot to current inflation (small γ) and (or) the money demand is inelastic to inflation expectations (small $|\alpha|$). (REVISE)

(d) Explain why this equation should be thought of as a *forward* equation.

Because in this model inflation expectations are adaptive, i.e. they are a function of past inflation. In the NK Phillips curve seen in class is the opposite because there we have rational expectations.

(e) Now assume that agents form expectations rationally instead of adaptively. That is, replace the third equation above by

$$p_{t+1}^e = p_{t+1}.$$

Simplify the model equations again to obtain a difference equation for p_t in terms of m_t. What kind of difference equation is this?

Substitute into the first equation and rearrange

$$p_{t+1} = \frac{\alpha - 1}{\alpha} p_t + \frac{m_t}{\alpha} = \phi p_t + Z_t$$

This is again a linear (time-homogeneous) first order difference equation. However now it is unstable because $\phi > 1$

(f) Argue that we should think of this equation now as a *backward* equation. Solve again for the *general* solution of this difference equation (backwards), i.e., express p_t in terms of p_{t+s} and m_{t+s}.

With rational expectations, expected prices are a function of future prices. Therefore we need a terminal condition for the prices and this is a backward equation. Solving backwards from period $t + h$ up to t

$$p_t = \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{h-1} \phi^{-i} Z_{t+i}$$

(g) Solve for a *particular* solution by imposing some transversality (terminal) condition on $\lim_{T\to\infty} p_T$.

Taking the limit in the equation above

$$p_t = \lim_{h\to\infty} \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} Z_{t+i}$$

We impose the transversality condition $\lim_{T\to\infty} p_T = p$ and as $0 < \phi^-1 < 1$ the limit remains finite.

Problem 5

Take the continuous-time limit of the following equations:

(a) $a_{t+1} = R_t a_t + y_t - c_t$

Add and subtract a_t and let $R_t = (1 + r_t)a_t$. We have

$$a_{t+\Delta} - a_t = \Delta(r_t a_t + y_t - c_t)$$

Divide by Δ and take the limit

$$\dot{a}_t = r_t a_t + y_t - c_t$$

(Note its exactly the same steps as for the capital accumulation equation in the lectures).

(b) $u'(c_t) = \beta R_t u'(c_{t+1})$

Let $R_t = 1 + r_t\Delta$ and $\beta = 1 - \rho\Delta$, the Euler equation for time step Δ is

$$u'(c_t) = (1 + r_t\Delta)(1 - \rho\Delta)u'(c_{t+\Delta})$$

The first order approximation of the last term around c_t is

$$u'(c_{t+\Delta}) = u'(c_t + (c_{t+\Delta} - c_t)) \approx u'(c_t) + u'(c_t)(c_{t+\Delta} - c_t)$$

Substitute back

$$1 = (1 + r_t\Delta)(1 - \rho\Delta)(1 + \frac{u''(c_t)}{u'(c_t)}(c_{t+\Delta} - c_t))$$

Assume also CRRA utility so that $\gamma \equiv -\frac{u''(c_t)c_t}{u'(c_t)}$ is constant, then

$$1 = (1 + r_t\Delta)(1 - \rho\Delta)(1 - \gamma\frac{c_{t+\Delta} - c_t}{c_t})$$

$$(1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2)\gamma\frac{c_{t+\Delta} - c_t}{c_t} = r_t\Delta - \rho\Delta - r_t\rho\Delta^2$$

Divide by Δ each side

$$(1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2)\gamma\frac{(c_{t+\Delta} - c_t)/\Delta}{c_t} = (r_t\Delta - \rho\Delta - r_t\rho\Delta^2)/\Delta$$

Taking $\Delta \to 0$ we get

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\gamma}$$

(c) $\pi_t = \beta\pi_{t+1} + \kappa x_t$

As before, let $\beta = 1 - \rho\Delta$ then the NKPC with time step Δ is

$$\pi_{t+\Delta} - \pi_t = \rho\Delta\Pi_{t+\Delta} - \Delta\kappa x_t$$

Divide by Δ and take $\Delta \to 0$

$$\dot{\pi}_t = \rho\pi_t - \kappa x_t$$

Problem 6

Revert back from continuous to discrete time by plugging in for the definition of first-order derivative

$$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

for small Δ.

(a) $\dot{K}_t = I_t - \delta K_t$

$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$. Take Δ to 1 and rearrange.

(b) $\dot{a}_t = r_t a_t + y_t - c_t$

Same as a)

(c) $\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho}{\gamma}$

Basically we reverse the order of the steps in 5.b). We have

$$\frac{\dot{C}_t}{C_t} = \lim_{\Delta \to 0}(1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2)\frac{(c_{t+\Delta} - c_t)/\Delta}{c_t}$$

$$\frac{r_t - \rho}{\gamma} = \lim_{\Delta \to 0} \frac{(r_t\Delta - \rho\Delta - r_t\rho\Delta^2)/\Delta}{\gamma}$$

Combining the two and substituting for gamma

$$(1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2)(c_{t+\Delta} - c_t)u''(c_t) = -(r_t\Delta - \rho\Delta - r_t\rho\Delta^2)u'(c_t)$$

$$(1 + r_t\Delta)(1 - \rho\Delta)(c_{t+\Delta} - c_t)u''(c_t) = (1 - (1 + r_t\Delta)(1 - \rho\Delta))u'(c_t)$$

Using the first order approximation

$$(1 + r_t\Delta)(1 - \rho\Delta)u'(c_{t+\Delta}) = u'(c_t)$$

Taking $\Delta \to 1$ we get back the discrete time Euler equation

(d) $\dot{\pi}_t = \rho\pi_t - \kappa x_t$

We have

$$\frac{\pi_{t+\Delta} - \pi_t}{\Delta} = \rho\pi_{t+1} - \kappa x_t$$

setting $\Delta = 1$ and rearranging

$$\pi_t = (1 - \rho)\pi_{t+1} + \kappa x_t = \beta\pi_{t+1} + \kappa x_t$$

Problem 7

Consider the equation for wealth dynamics

$$\dot{a}_t = r_t a_t + y_t - c_t.$$

We take $\{r_t\}$ and $\{y_t - c_t\}$ as exogenously given.

(a) Solve for the lifetime budget constraint.

(b) Solve the ODE for its general solution using the integrating factor method introduced in class, i.e., find an expression for a_t in terms of the exogenous processes $\{r_t, y_t, c_t\}$ and some arbitrary initial condition $a_0 = c$.

We do the same steps as in class only with the difference that r (or δ) is not constant:

$$e^{-\int_0^t r_s ds}\dot{a}_t - e^{-\int_0^t r_s ds}a_t = e^{-\int_0^t r_s ds}y_t - e^{-\int_0^t r_s ds}c_t$$

Notice the LHS is equal to $\frac{d}{dt}(a_t e^{-\int_0^t r_s ds})$. Integrate

$$a_t e^{-\int_0^t r_s ds} = C + \int_0^t e^{-\int_0^t r_s ds}(y_s - c_s)ds$$

Setting $t = 0$ we find $a_0 = C$. For the lifetime budget, take $t \to \infty$ to get

$$a_0 = \int_0^\infty e^{-\int_0^\infty r_s ds}(c_s - y_s)ds$$

Problem 8

Credit: Miranda Holmes-Cerfon https://cims.nyu.edu/~holmes/teaching/asa19/handout_Lecture4_2019.pdf

In this problem, we will prove the Chapman-Kolmogorov Equation for a *time-homogeneous* continuous-time Markov chains. Denote the *transition probability* as

$$P_{ij}(t+s) = \mathbb{P}(X_{t+s} = j \mid X_t = i)$$

where i and j should be thought of as indices on the state space, i.e., the ith value of the finite state space $\mathcal{X}$ of the Markov chain.

Denote I the indices associated with the state space $\mathcal{X}$. The Chapman-Kolmogorov equation is:

$$P_{ij}(t+s) = \sum_{k \in I} P_{ik}(t) P_{kj}(s).$$

(a) Use the law of total probability to show that

$$\mathbb{P}(X_{t+s} = j \mid X_0 = i) = \sum_k \mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i)\mathbb{P}(X_t = k \mid X_0 = i)$$

This is a direct application of the law of total probability, i.e. $P(A) = \sum_n P(A \mid B_n) P(B_n)$, applied with the probability mass function $\mathbb{P}(. \mid X_0 = i)$.

(b) Use the Markov property

By the Markov property $\mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i) = \mathbb{P}(X_{t+s} = j \mid X_t = k)$

(c) Invoke time homogeneity to arrive at the result

The time-homogeneity implies that $\mathbb{P}(X_{t+s} = j \mid X_t = k) = P(X_s = j \mid X_0 = k) = P_{kj}(s)$

Problem 9

This problem collects several exercises on Brownian motion and stochastic calculus.

(a) Show that $\mathbb{C}ov(B_s, B_t) = \min\{s, t\}$ for two times $0 \leq s < t$. Use the following tricks: Use the covariance formula $\mathbb{C}ov(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B)$. Use $B_t \sim \mathcal{N}(0, t)$ as well as $B_t - B_s \sim \mathcal{N}(0, t - s)$. And use $B_t = B_s + (B_t - B_s)$.

First we have $\mathbb{C}ov(B_s, B_t) = \mathbb{E}(B_s B_t) - \mathbb{E}(B_s)\mathbb{E}(B_t) = \mathbb{E}(B_s B_t)$. Then

$$\mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + (B_t - B_s))) = \mathbb{E}(B_s^2) + \mathbb{E}(B_s(B_t - B_s))$$

Notice that B_s and $(B_t - B_s)$ are two independent random variables, hence

$$\mathbb{C}ov(B_s, B_t) = s + \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) = s$$

(b) Geometric Brownian motion evolves as: $dX_t = \mu X_t dt + \sigma X_t dB_t$. Show that

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$$

for a given initial value X_0.

To derive the solution it will be useful to apply Ito's lemma to the function $f = log(X_t)$. Recall with a function of stochastic process we cannot use standard calculus, instead we can use Ito's lemma as a stochastic version of the chain rule. Applying Ito's lemma to f

$$df = dlog(X_t) = \partial_t f dt + \partial_X f \mu X_t dt + \frac{1}{2}\partial_{XX} f \sigma^2 X_t^2 dt + \sigma X_t \partial_X f dB_t$$

The partial derivatives are: $\partial_t f = \frac{\partial log(X_t)}{\partial t} = 0$, $\partial_X f = \frac{1}{X_t}$ and $\partial_{XX} f = -\frac{1}{X_t^2}$. Substituting

$$dlog(X_t) = (\mu - \frac{\sigma^2}{2}) + \sigma dB_t$$

Integrate (and use $B_0 = 0$)

$$log(X_t) - log(X_0) = (\mu - \frac{\sigma^2}{2})t + \sigma B_t$$

$$e^{log(\frac{X_t}{X_0})} = e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$$

Rearranging we get the result.

(c) For Geometric Brownian motion as defined above, show that $\mathbb{E} = X_0 e^{\mu t}$.

Taking expectations

$$\mathbb{E}(X_t) = X_0 e^{\mu t - \frac{\sigma^2}{2}t}\mathbb{E}(e^{\sigma B_t})$$

Recall $B_t \sim \mathcal{N}(0,t)$, it is useful to substitute $\sigma B_t = \sigma\sqrt{t}Z$ where $Z \sim \mathcal{N}(0,1)$. Then

$$\mathbb{E}(e^{\sigma B_t}) = \int e^{\sigma\sqrt{t}Z}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$

For the term in the exponential we have

$$\sigma\sqrt{t}Z - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z) = -\frac{1}{2}(z-\sigma\sqrt{t})^2 + \frac{\sigma^2 t}{2}$$

Substitute back

$$\mathbb{E}(e^{\sigma B_t}) = e^{\frac{\sigma^2 t}{2}}\int \frac{e^{-\frac{(z-\sigma\sqrt{t})^2}{2}}}{\sqrt{2\pi}}dz = e^{\frac{\sigma^2 t}{2}}$$

Because the second term is the density of a $\mathcal{N}(\sigma\sqrt{t},1)$ which integrates to one. Substituting into the first equation we get the result.

(d) The Ornstein-Uhlenbeck (OU) process is like a continuous-time variant of the AR(1) process. It evolves as $dX_t = -\mu X_t dt + \sigma dB_t$ for drift parameter μ, diffusion parameter σ, and some X. Show that it solves

$$X_t = X_0 e^{-\mu t} + \sigma\int_0^t e^{-\mu(t-s)}dB_s.$$

As before we derive the solution by applying the Ito's lemma to a properly chosen function. Let $f = e^{\mu t}X_t$, applying Ito's lemma

$$df = de^{\mu t}X_t = \partial_t f dt + \partial_X f(-\mu X_t)dt + \frac{1}{2}\partial_{XX} f\sigma^2 dt + \sigma\partial_X f dB_t$$

Substitute the partial derivatives

$$de^{\mu t}X_t = \mu e^{\mu t}X_t dt + -\mu X_t e^{\mu t}dt + 0 + \sigma e^{\mu t}dB_t$$

Integrate:

$$e^{\mu t}X_t - X_0 = \sigma\int_0^t e^{\mu s}dB_s$$

Multiplying by $e^{-\mu t}$ we get the result.