

Dynamic Optimization: Problem Set #1

Andreas Schaab

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DIFFICULTY: * easy, ** medium, *** hard

Problem 1 * (Canonical)

Credit: QuantEcon https://python.quantecon.org/finite_markov.html#exercises

Let y_t denote the employment / earnings state of an individual. Consider the state space $y_t \in Y = \{y^U, y^E\}$, where y^U corresponds to unemployment and y^E corresponds to employment. Let y denote the column vector $(y^U, y^E)'$ representing this state space (this is the grid you would construct on a computer). Suppose the employment dynamics of the individual are characterized by the invariant transition matrix

$$P = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \beta & \beta \end{pmatrix}.$$

We interpret a time period as a quarter.

- (a) Give economic interpretations of $\alpha = P_{11}$ and $\beta = P_{22}$
 $\alpha = P_{11}$: Probability that an unemployed doesn't find a job and remains unemployed at t.
 $\beta = P_{22}$: Probability that an employed doesn't lose the job and remains employed at t.
- (b) Why do the rows of P sum to 1?
Because each row is a probability distribution. Intuitively: in the model either you are employed or unemployed
- (c) Is there an absorbing state in this model?
We say x is an absorbing state if

$$P(X_{t+1} = x | X_t = x) = 1$$

There are no absorbing states in this model if $\beta, \alpha \in (0, 1)$.

- (d) Compute the probability of being unemployed two quarters after being employed.

$$P(y_{t+2} = y^U, y_{t+1} = y^U | y_t = y^E) + P(y_{t+2} = y^U, y_{t+1} = y^E | y_t = y^E) = \\ \alpha(1 - \beta) + (1 - \beta)\beta = (\alpha + \beta)(1 - \beta)$$

- (e) Denote the *marginal (probability) distribution* of y_t at time t by ψ_t . $\psi_t(y^L)$ is the probability that process y_t is in state y^L at time t . It is easiest to think of ψ_t as a time-varying row vector. Use the law of total probability to decompose $y_{t+1} = y^L$, accounting for all the possible ways in which state y^L can be reached at time $t + 1$.

$$\psi_{t+1}(y^L) = P(y_{t+1} = y^L) = \psi_t(y^L)P(y_{t+1} = y^L | y_t = y^L) + (1 - \psi_t(y^L))P(y_{t+1} = y^L | y_t \neq y^L)$$

- (f) Show that the resulting equation can be written as the vector-matrix product

$$\psi_{t+1} = \psi_t P.$$

Therefore: The evolution of the marginal distribution of a Markov chain is obtained by post-multiplying by the transition matrix.

It is clear that the conditional probabilities from question e) are obtained from one column of the transition matrix P , so we can write it as the vector-matrix product.

- (g) Show that

$$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t,$$

where \sim reads “is distributed according to”.

First $\psi_1 = \psi_0 P$, then also $\psi_2 = \psi_1 P = \psi_0 P^2$. Iterating we get the result

- (h) We call ψ^* a *stationary distribution* of the Markov chain if it satisfies

$$\psi^* = \psi^* P.$$

Compute the probability of being unemployed n quarters after being employed. Take $n \rightarrow \infty$ and find the stationary distribution of this Markov chain. Find the stationary distribution by alternatively plugging into the above equation for ψ^* .

As a summary (Check the link from quantecon for more detail): A Markov Chain is irreducible if all states communicate, that is there is a positive probability of going from any y to any x (possibly in many steps). Then an irreducible MC converges to a stationary (or ergodic) distribution. So if we take $n \rightarrow \infty$ the marginal distribution converges to the stationary: $\psi_{t+n} \rightarrow \psi^*$.

To find the stationary distribution we need to solve the equation above. Note $\psi = 0$ is a solution but it's not a probability distribution. Let $\psi^* = (\psi^*(y^U), 1 - \psi^*(y^U))$, then solving the equation we get

$$\psi^*(y^U) = \frac{1 - \beta}{2 - \alpha - \beta}$$

Economic intuition: Spend more time unemployed (or unemployment rate higher) if the probability of finding a job is lower (the job finding rate) or the probability of losing job is higher (the separation rate).

- (i) Suppose $y_0 = y^H$. Solve for $\mathbb{E}_0(y_t)$. Use the law of total / iterated expectation to relate expectation to probabilities. Then use the formulas for marginal (probability) distributions derived above.

$$\mathbb{E}_0(y_t) = P(y_t = y^U | y_0 = y^E) y^U + P(y_t = y^E | y_0 = y^E) y^E$$

And we get the transition probabilities from P^t

Problem 2 *

Consider the first-order linear *homogeneous* difference equation

$$x_{t+1} = \rho x_t.$$

We parameterize the initial condition by $x_0 = c$.

- (a) Prove by induction that the *general solution* (for arbitrary c) is given by

$$x_t = \rho^t c.$$

Guess $x_t = \rho^t c$. Check true at $t = 0$: $x_0 = \rho^0 c = c$. If true at t also true at $t + 1$:

$$x_{t+1} = \rho x_t = \rho \rho^t c = \rho^{t+1} c$$

- (b) Show that the *particular solution* for initial value $x_0 = x$ is given by $x_t = \rho^t x$. If $x_0 = x$ then set $c = x$.

- (c) Show that this also implies

$$x_t = \rho^{t-s} x_s$$

for $t > s$.

We have $c = \frac{x_s}{\rho^s}$ for any s , substitute in the general solution from (a) and we get the result.

- (d) Prove that this difference equation satisfies the Markov property.

Note we can think of an equation in difference as a stochastic process where the noise term is always zero. So $\mathbb{P}(x_{t+1} = \rho x_t | x_t = x) = 1$. Because x_{t+1} is a function of x_t but not of $(x_{t-1}, x_{t-2}, \dots)$ this difference equation satisfies the Markov property.

Problem 3 *

Credit: Klaus Neusser <http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf> on p. 19 (of the PDF)

We study the dynamics of loan amortization. Denote by D_t the amount of debt owed at time t . The debt contract is serviced by paying an amount Z_t each period. Z_t is given exogenously for this problem (you could imagine some agent optimizing in the background).

- (a) Explain why debt dynamics are characterized by the equation

$$D_{t+1} = RD_t - Z_t$$

where R is the constant gross interest rate associated with the loan. What kind of difference equation is this? Is this a forward or a backward equation?

The gross interest rate is $R = 1 + r$, so debt next period is debt at the current period plus the interest rate on debt minus repayment. It is a linear (time-homogeneous ??) first order difference equation. It is a forward equation, start with D_0 .

- (b) Suppose we start with an initial loan D_0 . Solve iteratively (by induction) for D_t . You should get two terms — explain the economics for both terms.

$$D_{t+1} = R^{t+1}D_0 - \sum_{i=0}^t R^i Z_{t-i}$$

. First term: Growth in debt if no repayment. Second term: NPV repayments (going back on time), as payment Z at $t = 0$ reduces debt at t by R^t compared to payment at t

- (c) Suppose the loan needs to be repaid at time T . Solve for the constant repayment schedule $Z_t = Z$ such that the loan is repaid in period T .

$$D_{t+1} = R^{t+1}D_0 - \left(R^{t+1} - 1\right) \frac{Z}{R - 1}$$

. If debt repaid at T , then must have $D_{T+1} = 0$. Therefore

$$Z = \frac{R - 1}{1 - R^{-T-1}} D_0$$

- (d) What is the condition on constant repayment rate Z relative to D_0 such that the loan is repaid in finite time?

Taking $T \rightarrow \infty$ we get $Z = (R - 1)D_0$. So the payment is just the interest accruing in each period.

Problem 4 *

Credit: Klaus Neusser <http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf> on p. 38 (of the PDF)

We study Cagan (1956)'s model for hyperinflation. The model is summarized by the three equations

$$m_t^d - p_t = \alpha(p_{t+1}^e - p_t)$$

$$m_t^s = m_t^d$$

$$p_{t+1}^e - p_t = \gamma(p_t - p_{t-1})$$

where (all in logs) m_t^d is money demand, m_t^s is money supply, p_t is the price level and p_{t+1}^e is private agents' expectations for the price level in period $t + 1$. The first equation of the above system characterizes money demand and the third equation characterizes *adaptive* inflation expectations. Assume that $\alpha < 0$ and $\gamma > 0$.

- (a) Characterize a first-order difference equation that solves for p_t as a function of p_{t-1} and m_{t-1} . What kind of difference equation is this?

Substitute the 3rd constraint into the 1st, $m_t^d - p_t = \alpha\gamma(p_t - p_{t-1})$. Let $m_t = m_t^d = m_t^s$. Then rearranging

$$p_t = \frac{\alpha\gamma}{1 + \alpha\gamma} p_{t-1} + \frac{1}{1 + \alpha\gamma} m_t \equiv \phi p_{t-1} + Z_t$$

This is a linear (time-homogeneous ??) first order difference equation

- (b) Using the tools already developed, solve for p_t in terms of some initial conditions on the system.

Let p_0 be the initial price level, then iterating forward (as in the previous exercise)

$$p_t = \phi^t p_0 + \sum_{i=0}^{t-1} \phi^i Z_{t-i}$$

- (c) Characterize the stability condition such that if $m_t \rightarrow m$ in the long run, p_t converges to a steady state p . Interpret the economics of this stability condition.

To have both terms finite as $t \rightarrow \infty$ we need

$$|\phi| = \left| \frac{\alpha\gamma}{1 + \alpha\gamma} \right| < 1$$

To have stability we need that money demand m_t^d does not respond too much to current current inflation. This is the case if inflation expectations do not respond a lot to current inflation (small γ) and (or) the money demand is inelastic to inflation expectations (small $|\alpha|$). (REVISE)

- (d) Explain why this equation should be thought of as a *forward* equation.

Because in this model inflation expectations are adaptive, i.e. they are a function of past inflation. In the NK Phillips curve seen in class is the opposite because there we have rational expectations.

- (e) Now assume that agents form expectations rationally instead of adaptively. That is, replace the third equation above by

$$p_{t+1}^e = p_{t+1}.$$

Simplify the model equations again to obtain a difference equation for p_t in terms of m_t . What kind of difference equation is this?

Substitute into the first equation and rearrange

$$p_{t+1} = \frac{\alpha - 1}{\alpha} p_t + \frac{m_t}{\alpha} = \phi p_t + Z_t$$

This is again a linear (time-homogeneous) first order difference equation. However now it is unstable because $\phi > 1$

- (f) Argue that we should think of this equation now as a *backward* equation. Solve again for the *general* solution of this difference equation (backwards), i.e., express p_t in terms of p_{t+s} and m_{t+s} .

With rational expectations, expected prices are a function of future prices. Therefore we need a terminal condition for the prices and this is a backward equation. Solving backwards from period $t + h$ up to t

$$p_t = \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{h-1} \phi^{-i} Z_{t+i}$$

- (g) Solve for a *particular* solution by imposing some transversality (terminal) condition on $\lim_{T \rightarrow \infty} p_T$.

Taking the limit in the equation above

$$p_t = \lim_{h \rightarrow \infty} \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} Z_{t+i}$$

We impose the transversality condition $\lim_{T \rightarrow \infty} p_T = p$ and as $0 < \phi^{-1} < 1$ the limit remains finite.

Problem 5 * (** the Euler) (Canonical)

Take the continuous-time limit of the following equations:

(a) $a_{t+1} = R_t a_t + y_t - c_t$

Add and subtract a_t and let $R_t = (1 + r_t)a_t$. We have

$$a_{t+\Delta} - a_t = \Delta(r_t a_t + y_t - c_t)$$

Divide by Δ and take the limit

$$\dot{a}_t = r_t a_t + y_t - c_t$$

(Note its exactly the same steps as for the capital accumulation equation in the lectures).

(b) $u'(c_t) = \beta R_t u'(c_{t+1})$

Let $R_t = 1 + r_t \Delta$ and $\beta = 1 - \rho \Delta$, the Euler equation for time step Δ is

$$u'(c_t) = (1 + r_t \Delta)(1 - \rho \Delta) u'(c_{t+\Delta})$$

The first order approximation of the last term around c_t is

$$u'(c_{t+\Delta}) = u'(c_t + (c_{t+\Delta} - c_t)) \approx u'(c_t) + u''(c_t)(c_{t+\Delta} - c_t)$$

Substitute back

$$1 = (1 + r_t \Delta)(1 - \rho \Delta) \left(1 + \frac{u''(c_t)}{u'(c_t)} (c_{t+\Delta} - c_t)\right)$$

Assume also CRRA utility so that $\gamma \equiv -\frac{u''(c_t)c_t}{u'(c_t)}$ is constant, then

$$1 = (1 + r_t \Delta)(1 - \rho \Delta) \left(1 - \gamma \frac{c_{t+\Delta} - c_t}{c_t}\right)$$

$$(1 + r_t \Delta - \rho \Delta - r_t \rho \Delta^2) \gamma \frac{c_{t+\Delta} - c_t}{c_t} = r_t \Delta - \rho \Delta - r_t \rho \Delta^2$$

Divide by Δ each side

$$(1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2)\gamma\frac{(c_{t+\Delta} - c_t)/\Delta}{c_t} = (r_t\Delta - \rho\Delta - r_t\rho\Delta^2)/\Delta$$

Taking $\Delta \rightarrow 0$ we get

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\gamma}$$

(c) $\pi_t = \beta\pi_{t+1} + \kappa x_t$

As before, let $\beta = 1 - \rho\Delta$ then the NKPC with time step Δ is

$$\pi_{t+\Delta} - \pi_t = \rho\Delta\pi_{t+\Delta} - \Delta\kappa x_t$$

Divide by Δ and take $\Delta \rightarrow 0$

$$\dot{\pi}_t = \rho\pi_t - \kappa x_t$$

Problem 6 * (** the Euler)

Revert back from continuous to discrete time by plugging in for the definition of first-order derivative

$$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

for small Δ .

(a) $\dot{K}_t = I_t - \delta K_t$

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t. \text{ Take } \Delta \text{ to 1 and rearrange.}$$

(b) $\dot{a}_t = r_t a_t + y_t - c_t$

Same as a)

(c) $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\gamma}$

Basically we reverse the order of the steps in 5.b). We have

$$\frac{\dot{c}_t}{c_t} = \lim_{\Delta \rightarrow 0} (1 + r_t\Delta - \rho\Delta - r_t\rho\Delta^2) \frac{(c_{t+\Delta} - c_t)/\Delta}{c_t}$$

$$\frac{r_t - \rho}{\gamma} = \lim_{\Delta \rightarrow 0} \frac{(r_t \Delta - \rho \Delta - r_t \rho \Delta^2) / \Delta}{\gamma}$$

Combining the two and substituting for gamma

$$(1 + r_t \Delta - \rho \Delta - r_t \rho \Delta^2)(c_{t+\Delta} - c_t)u''(c_t) = -(r_t \Delta - \rho \Delta - r_t \rho \Delta^2)u'(c_t)$$

$$(1 + r_t \Delta)(1 - \rho \Delta)(c_{t+\Delta} - c_t)u''(c_t) = (1 - (1 + r_t \Delta)(1 - \rho \Delta))u'(c_t)$$

Using the first order approximation

$$(1 + r_t \Delta)(1 - \rho \Delta)u'(c_{t+\Delta}) = u'(c_t)$$

Taking $\Delta \rightarrow 1$ we get back the discrete time Euler equation

$$(d) \quad \dot{\pi}_t = \rho \pi_t - \kappa x_t$$

We have

$$\frac{\pi_{t+\Delta} - \pi_t}{\Delta} = \rho \pi_{t+1} - \kappa x_t$$

setting $\Delta = 1$ and rearranging

$$\pi_t = (1 - \rho)\pi_{t+1} + \kappa x_t = \beta \pi_{t+1} + \kappa x_t$$

Problem 7 ** (Canonical)

Consider the equation for wealth dynamics

$$\dot{a}_t = r_t a_t + y_t - c_t.$$

We take $\{r_t\}$ and $\{y_t - c_t\}$ as exogenously given.

- Solve for the lifetime budget constraint.
- Solve the ODE for its general solution using the integrating factor method introduced in class, i.e., find an expression for a_t in terms of the exogenous processes $\{r_t, y_t, c_t\}$ and some arbitrary initial condition $a_0 = c$.

We do the same steps as in class only with the difference that r (or δ) is not constant:

$$e^{-\int_0^t r_s ds} \dot{a}_t - e^{-\int_0^t r_s ds} a_t = e^{-\int_0^t r_s ds} y_t - e^{-\int_0^t r_s ds} c_t$$

Notice the LHS is equal to $\frac{d}{dt}(a_t e^{-\int_0^t r_s ds})$. Integrate

$$a_t e^{-\int_0^t r_s ds} = C + \int_0^t e^{-\int_0^s r_s ds} (y_s - c_s) ds$$

Setting $t = 0$ we find $a_0 = C$. For the lifetime budget, take $t \rightarrow \infty$ to get

$$a_0 = \int_0^\infty e^{-\int_0^s r_s ds} (c_s - y_s) ds$$

Problem 8 **

Credit: Miranda Holmes-Cerfon https://cims.nyu.edu/~holmes/teaching/asa19/handout_Lecture4_2019.pdf

In this problem, we will prove the Chapman-Kolmogorov Equation for a *time-homogeneous* continuous-time Markov chains. Denote the *transition probability* as

$$P_{ij}(t+s) = \mathbb{P}(X_{t+s} = j \mid X_t = i)$$

where i and j should be thought of as indices on the state space, i.e., the i th value of the finite state space \mathcal{X} of the Markov chain.

Denote I the indices associated with the state space \mathcal{X} . The Chapman-Kolmogorov equation is:

$$P_{ij}(t+s) = \sum_{k \in I} P_{ik}(t) P_{kj}(s).$$

(a) Use the law of total probability to show that

$$\mathbb{P}(X_{t+s} = j \mid X_0 = i) = \sum_k \mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i) \mathbb{P}(X_t = k \mid X_0 = i)$$

This is a direct application of the law of total probability, i.e. $P(A) = \sum_n P(A \mid B_n) P(B_n)$, applied with the probability mass function $\mathbb{P}(\cdot \mid X_0 = i)$.

(b) Use the Markov property

$$\text{By the Markov property } \mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i) = \mathbb{P}(X_{t+s} = j \mid X_t = k)$$

(c) Invoke time homogeneity to arrive at the result

$$\text{The time-homogeneity implies that } \mathbb{P}(X_{t+s} = j \mid X_t = k) = P(X_s = j \mid X_0 = k) = P_{kj}(s)$$

Problem 9 *** (the (a) **)

This problem collects several exercises on Brownian motion and stochastic calculus.

- (a) Show that $\text{Cov}(B_s, B_t) = \min\{s, t\}$ for two times $0 \leq s < t$. Use the following tricks: Use the covariance formula $\text{Cov}(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B)$. Use $B_t \sim \mathcal{N}(0, t)$ as well as $B_t - B_s \sim \mathcal{N}(0, t - s)$. And use $B_t = B_s + (B_t - B_s)$.

First we have $\text{Cov}(B_s, B_t) = \mathbb{E}(B_s B_t) - \mathbb{E}(B_s)\mathbb{E}(B_t) = \mathbb{E}(B_s B_t)$. Then

$$\mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + (B_t - B_s))) = \mathbb{E}(B_s^2) + \mathbb{E}(B_s(B_t - B_s))$$

Notice that B_s and $(B_t - B_s)$ are two independent random variables, hence

$$\text{Cov}(B_s, B_t) = s + \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) = s$$

- (b) Geometric Brownian motion evolves as: $dX_t = \mu X_t dt + \sigma X_t dB_t$. Show that

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

for a given initial value X_0 .

To derive the solution it will be useful to apply Ito's lemma to the function $f = \log(X_t)$. Recall with a function of stochastic process we cannot use standard calculus, instead we can use Ito's lemma as a stochastic version of the chain rule. Applying Ito's lemma to f

$$df = d\log(X_t) = \partial_t f dt + \partial_X f \mu X_t dt + \frac{1}{2} \partial_{XX} f \sigma^2 X_t^2 dt + \sigma X_t \partial_X f dB_t$$

The partial derivatives are: $\partial_t f = \frac{\partial \log(X_t)}{\partial t} = 0$, $\partial_X f = \frac{1}{X_t}$ and $\partial_{XX} f = -\frac{1}{X_t^2}$. Substituting

$$d\log(X_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$$

Integrate (and use $B_0 = 0$)

$$\log(X_t) - \log(X_0) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t$$

$$e^{\log(\frac{X_t}{X_0})} = e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

Rearranging we get the result.

- (c) For Geometric Brownian motion as defined above, show that $\mathbb{E} = X_0 e^{\mu t}$.

Taking expectations

$$\mathbb{E}(X_t) = X_0 e^{\mu t - \frac{\sigma^2}{2} t} \mathbb{E}(e^{\sigma B_t})$$

Recall $B_t \sim \mathcal{N}(0, t)$, it is useful to substitute $\sigma B_t = \sigma\sqrt{t}Z$ where $Z \sim \mathcal{N}(0, 1)$. Then

$$\mathbb{E}(e^{\sigma B_t}) = \int e^{\sigma\sqrt{t}Z} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

For the term in the exponential we have

$$\sigma\sqrt{t}Z - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z) = -\frac{1}{2}(z - \sigma\sqrt{t})^2 + \frac{\sigma^2 t}{2}$$

Substitute back

$$\mathbb{E}(e^{\sigma B_t}) = e^{\frac{\sigma^2 t}{2}} \int \frac{e^{-\frac{(z - \sigma\sqrt{t})^2}{2}}}{\sqrt{2\pi}} dz = e^{\frac{\sigma^2 t}{2}}$$

Because the second term is the density of a $\mathcal{N}(\sigma\sqrt{t}, 1)$ which integrates to one. Substituting into the first equation we get the result.

- (d) The Ornstein-Uhlenbeck (OU) process is like a continuous-time variant of the AR(1) process. It evolves as $dX_t = -\mu X_t dt + \sigma dB_t$ for drift parameter μ , diffusion parameter σ , and some X . Show that it solves

$$X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s.$$

As before we derive the solution by applying the Ito's lemma to a properly chosen function.

Let $f = e^{\mu t} X_t$, applying Ito's lemma

$$df = de^{\mu t} X_t = \partial_t f dt + \partial_X f (-\mu X_t) dt + \frac{1}{2} \partial_{XX} f \sigma^2 dt + \sigma \partial_X f dB_t$$

Substitute the partial derivatives

$$de^{\mu t} X_t = \mu e^{\mu t} X_t dt + -\mu X_t e^{\mu t} dt + 0 + \sigma e^{\mu t} dB_t$$

Integrate:

$$e^{\mu t} X_t - X_0 = \sigma \int_0^t e^{\mu s} dB_s$$

Multiplying by $e^{-\mu t}$ we get the result.