M2: Lecture 3

Dynamic Programming in Discrete Time (I)

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Outline of today's lecture

1. Neoclassical growth model (no uncertainty)
2. Sequence problem
3. Dynamic programming: the principle of optimality
4. Bellman equation
5. Solving the Bellman equation via guess-and-verify
6. Example: optimal stopping

1. Neoclassical growth model

- Time is discrete with $t = 0, 1, \ldots$ and there is no uncertainty
- Consider a **representative household**
- Preferences over consumption c_t given by

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- $u(\cdot)$ is **flow utility**: units of c_t “dollars” or “apples”, units of $u(c_t)$ “utils”
- Subject to capital accumulation equation

$$k_{t+1} = f(k_t) - c_t = k_t^{\alpha} - c_t,$$

for $\alpha \in (0, 1)$, $c_t, k_t \geq 0$, and k_0 given (**initial condition**)

2. Sequence problem

- We define the **lifetime value** of the representative household as

$$V(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$k_{t+1} = k_t^{\alpha} - c_t.$$

- This is the **sequence problem** (or problem in sequence form)
- We can substitute in:

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(k_t^{\alpha} - k_{t+1})$$

- Given initial condition k_0

- We can tackle the sequence problem directly using tools from constrained dynamic optimization

- Lagrangian after substituting with FOC for k_{t+1}:

$$L(k_0) = \sum_{t=0}^{\infty} \beta^t u(k_t^\alpha - k_{t+1})$$

$$0 = -\beta^t u'(c_t) + \beta^{t+1} u'(c_{t+1}) \alpha k_{t+1}^{\alpha-1}$$

- Lagrangian with multiplier before substituting with FOCs for c_t and k_{t+1}:

$$L(k_0) = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(k_t^\alpha - c_t - k_{t+1} \right) \right]$$

$$\begin{aligned} 0 &= \beta^t u'(c_t) - \beta^t \lambda_t \\ 0 &= -\beta^t \lambda_t + \alpha \beta^{t+1} \lambda_{t+1} k_{t+1}^{\alpha-1} \end{aligned}$$

3. Bellman equation

- **Definition:** The **Bellman equation** characterizes the value function as the sum of the **flow payoff** and the discounted **continuation value**

$$V(k) = \max_{k'} \left\{ u(k^\alpha - k') + \beta V(k') \right\} \quad \text{for all } k$$

- We call $V(k)$ the value function and $u(k^\alpha - k')$ the flow payoff or (instantaneous) utility flow

- **Recursive representation** (not sequence form)

- We say that $\mathcal{X} = [0, \bar{k}]$ is the **state space** of the neoclassical growth model. The Bellman equation must hold for all feasible levels of capital $k \in \mathcal{X}$.

- If $V(k)$ solves the above equation, then it is a solution to the Bellman equation

- We now show that the unique value function that solves the sequence problem also solves the Bellman equation

- Sequence problem $\longrightarrow$ recursive representation (Bellman equation)

$$
\begin{aligned}
V(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(k_t^\alpha - k_{t+1}) \right\} \\
&= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(k_0^\alpha - k_1) + \sum_{t=1}^{\infty} \beta^t u(k_t^\alpha - k_{t+1}) \right\} \\
&= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(k_0^\alpha - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(k_t^\alpha - k_{t+1}) \right\} \\
&= \max_{k_1} \left\{ u(k_0^\alpha - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(k_{t+1}^\alpha - k_{t+2}) \right\} \right\} \\
&= \max_{k_1} \left\{ u(k_0^\alpha - k_1) + \beta V(k_1) \right\}
\end{aligned}
$$

- Recursively, with k capital "today" and k' capital "tomorrow"

$$
V(k) = \max_{k'} \left\{ u(k^\alpha - k') + \beta V(k') \right\}
$$

- A solution to the Bellman equation also solves Sequence Problem

$$
\begin{aligned}
V(k_0) &= \max_{k_1} \left\{ u(k_0^\alpha - k_1) + \beta V(k_1) \right\} \\
&= \max_{k_1} \left\{ u(k_0^\alpha - k_1) + \beta \left(\max_{k_2} \left\{ u(k_1^\alpha - k_2) + \beta V(k_2) \right\} \right) \right\} \\
&= \max_{k_1, k_2} \left\{ u(k_0^\alpha - k_1) + \beta u(k_1^\alpha - k_2) + \beta^2 V(k_2) \right\} \\
&= \max_{k_1, k_2, k_3} \left\{ u(k_0^\alpha - k_1) + \beta u(k_1^\alpha - k_2) + \beta^2 u(k_2^\alpha - k_3) + \beta^3 V(k_3) \right\} \\
&\vdots \\
&= \max_{\{k_{t+1}\}_{t=0}^\infty} \left\{ \sum_{t=0}^\infty \beta^t u(k_t^\alpha - k_{t+1}) \right\}
\end{aligned}
$$

- Stokey and Lucas Thm 4.3: Sufficient condition is that $\lim_{n\to\infty} \beta^n V(k_n) = 0$ for all feasible sequences of $\{k_t\}$

- How do we find optimal behavior?
- First-order condition for k' is

$$\frac{\partial u(k^\alpha - k')}{\partial k'} = \beta \frac{\partial V(k')}{\partial k'}$$

- FOC defines **policy function** $k'(k)$, use to plug back into Bellman:

$$V(k) = u(k^\alpha - k'(k)) + \beta V(k'(k))$$

- The implied **consumption policy function** is: $c(k) = k^\alpha - k'(k)$
- Envelope theorem (in discrete time):

$$\frac{\partial V(k)}{\partial k} = \frac{\partial u(k^\alpha - k')}{\partial k}$$

- Work out at home: (i) prove envelope condition (ii) show resulting Euler equation coincides with solving sequence problem using Lagrangian

4. Solving the Bellman equation with guess-and-verify

- Assume the functional form $u(c_t) = \log(c_t)$
- Guess that the value function takes the form

$$V(k) = A + B\log(k)$$

- Strategy: Plug into Bellman equation, match coefficients
- Oftentimes, value function inherits functional form / shape of utility function

- First-order condition for capital with log utility:

$$\frac{1}{k^{\alpha} - k'} = \beta B \frac{1}{k'} \implies k' = \frac{\beta B}{1 + \beta B} k^{\alpha}$$

- Plug guess into Bellman:

$$\begin{aligned} V(k) &= \max_{k'} \left\{ \log(k^{\alpha} - k') + \beta V(k') \right\} \\ A + B \log(k) &= \max_{k'} \left\{ \log(k^{\alpha} - k') + \beta A + \beta B \log(k') \right\} \\ A + B \log(k) &= \log\left(k^{\alpha} - \frac{\beta B}{1 + \beta B} k^{\alpha} \right) + \beta A + \beta B \log\left(\frac{\beta B}{1 + \beta B} k^{\alpha} \right) \end{aligned}$$

- Solve for A and B so that coefficients of $\log(k)$ and constant terms cancel

More general Stokey-Lucas notation: Let $F(x_t, x_{t+1})$ be flow payoff

Sequence problem: Find $V(x)$ such that

$$V(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to x_{t+1} in some feasible set $\Gamma(x_t)$, with x_0 given

Bellman equation: Find $V(x)$ such that

$$V(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V(x') \right\}$$

for all x in the state space

5. Bellman operator

- In discrete time, Bellman equation is a **functional equation**

- Define the **Bellman operator** B, operating on function $w(\cdot)$, as

$$(Bw)(x) \equiv \max_{x'} \left\{ F(x, x') + \beta w(x') \right\} \quad \text{for all } x$$

- **Operators** are maps from one function space to another (learn functional analysis!)

- The value function $V(\cdot)$ is a fixed point of the operator B:

$$\begin{aligned}(BV)(x) &= \max_{x'} \left\{ F(x, x') + \beta w(x') \right\} \\ &= V(x)\end{aligned}$$

- This idea is useful in numerical analysis: Suppose we guess V^0 and look for fixed point
$$\lim_{n\to\infty} B^n V^0 = V$$
- The Bellman operator is a **contraction mapping** under some conditions
- This tells us that the Bellman operator converges (and we can use this to construct numerical fixed point algorithms)
- For example: If an operator B maps a complete metric space into itself and is a contraction mapping, then it has a unique fixed point
- This is a very important idea. And you should read about this on your own.

6. Example: Optimal stopping

Problem: Every period t, an agent draws an offer x from the unit interval $[0, 1]$. The agent can accept the offer, in which case her payoff is x, and the game ends. Draws are independent. The agent discounts the future at β. The game continues until the agent receives an offer she accepts.

- Agent's dynamic optimization problem given recursively by Bellman equation
$$V(x) = \max\left\{x,\, \beta \mathbb{E} V(x')\right\}$$
where the expectation (operator) $\mathbb{E}$ is taken over the next draw x'

- There will be a threshold $x^* \in [0, 1]$ such that agent accepts for $x \geq x^*$

- This is also called a **free boundary problem** because we have to look for the endogenous boundary of the problem x^*

- Many applications (problems in life) look like this:
buying a house, searching for a partner, closing a production plant, exercising an option, adopting a new technology, ...

- The value function will look like:

$$V(x) = \begin{cases} x & \text{if} \quad x \geq x^* \\ x^* & \text{if} \quad x < x^* \end{cases}$$

- Find the value x^* such that this function satisfies the Bellman equation

- At $x = x^*$, indifferent between accepting and stopping:

$$\begin{aligned} V(x^*) &= x^* \\ &= \beta \mathbb{E} V(x') \\ &= \beta \int_0^{x^*} x^* f(x) dx + \beta \int_{x^*}^1 x f(x) dx \\ &= \beta x^* [x]_0^{x^*} + \beta \frac{1}{2} [x^2]_{x^*}^1 \end{aligned}$$

  where $f(x) = \frac{1}{1-0}$ is the uniform density

- Solution: $x^* = \beta (x^*)^2 + \beta \frac{1}{2} [1 - (x^*)^2]$ or $x^* = \frac{1}{\beta}(1 - \sqrt{1 - \beta^2})$