M2: Lecture 9

Models of Consumption: Marginal Propensity to Consume

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Outline of today's lecture

1. Closed-form examples continued
2. GHH preferences
3. Return risk
4. Income fluctuations and precautionary savings
5. Linearization of consumption Euler equation: deterministic and stochastic
6. Interest rate risk (Gabaix linearization)

1.1. Example: no risk, no income (eat-the-pie)

- Time is continuous, $t \in [0, \infty)$
- Infinitely-lived household's wealth evolves according to

$$\frac{da}{dt} = ra - c$$

given initial wealth position a_0, constant interest rate r

- The HJB equation is given by

$$\rho V(a) = \frac{1}{\gamma} c(a)^{1-\gamma} + V_a(ra - c(a))$$

where $c(a)$ solves first-order condition $c(a)^{-\gamma} = V_a$

- Consider Ansatz $V = Aa^B$, so that $V_a = ABa^{B-1}$ and $c = [ABa^{B-1}]^{-\frac{1}{\gamma}}$
- Then HJB becomes

$$\rho A a^B = \frac{1}{\gamma}\left([ABa^{B-1}]^{-\frac{1}{\gamma}}\right)^{1-\gamma} + ABa^{B-1}\left(ra - [ABa^{B-1}]^{-\frac{1}{\gamma}}\right)$$
$$\rho A a^B - rABa^B = \frac{\gamma}{1-\gamma}[ABa^{B-1}]^{\frac{\gamma-1}{\gamma}}$$

Plug in $B = 1-\gamma$ and solve for (do this yourself!):

$$A = \frac{1}{1-\gamma}\left[\frac{1}{\gamma}\rho - \frac{1-\gamma}{\gamma}r\right]^{-\gamma}$$

- Solution of this model:

$$V(a) = \frac{1}{1-\gamma}\kappa^{-\gamma}a^{1-\gamma}$$
$$V_a(a) = \kappa^{-\gamma}a^{-\gamma}$$
$$c(a) = \kappa a,$$

- Marginal propensity to consume (MPC) is defined as

$$c'(a) = \kappa \equiv \frac{1}{\gamma}\rho - \frac{1-\gamma}{\gamma}r$$

- Consider $u(c) = \log(c)$, with $\gamma \to 1$ (income and substitution effects offset each other), then

$$c'(a) = \kappa = \rho \approx 0.5\% \text{ annually}$$

- The standard CRRA calibration with $\gamma = 2$ yields

$$c'(a) = \kappa \equiv \frac{1}{2}(\rho + r).$$

In models with uninsurable risk and incomplete financial markets, $r < \rho$

1.2. Example: unearned labor income

- Infinitely-lived household's wealth now evolves according to

$$\frac{da}{dt} = ra + w - c$$

- The HJB equation now given by

$$\rho V(a) = \frac{1}{\gamma} c(a)^{1-\gamma} + V_a(ra + w - c(a))$$

- Solution of this model (work this out yourself):

$$V(a) = \frac{1}{1-\gamma} \kappa^{-\gamma} \left(a + \frac{w}{r}\right)^{1-\gamma}$$
$$c(a) = \kappa \left(a + \frac{w}{r}\right)$$

  with MPC given by: $c'(a) = \kappa$ (as before)

- Intuition? Human capital affects lifetime wealth but not MPC

2. Example: labor supply with GHH preferences

- What happens when households can decide how much to work?

- Households solve: max $\int_0^\infty e^{-\rho t} u(c_t, h_t) dt$.

$$u(c,h) = \frac{1}{1-\gamma}\left(c - \frac{h^{1+\eta}}{1+\eta}\right)^{1-\gamma}$$
$$\rho V(a) = u(c,h) + (ra + wh - c)\partial_a V(a)$$

where FOCs are $u_c = V_a$ and $u_h = -wV_a$, so

$$\left(c - \frac{h^{1+\eta}}{1+\eta}\right)^{-\gamma} = V_a$$
$$\left(c - \frac{h^{1+\eta}}{1+\eta}\right)^{-\gamma} h^\eta = wV_a$$

- Putting FOCs together, $c = V_a^{-\frac{1}{\gamma}} + \frac{1}{1+\eta} w^{\frac{1+\eta}{\eta}}$ and $h = w^{\frac{1}{\eta}}$

- HJB becomes

$$
\begin{aligned}
\rho V &= \frac{1}{\gamma} V_a^{\frac{\gamma-1}{\gamma}} + V_a \Big(ra + wh - c \Big) \\
&= \frac{1}{\gamma} V_a^{\frac{\gamma-1}{\gamma}} + V_a \left(ra + \frac{\eta}{1+\eta} w^{\frac{1+\eta}{\eta}} - V_a^{-\frac{1}{\gamma}} \right)
\end{aligned}
$$

- Solution of this model given by (work this out yourself):

$$
V(a) = \frac{1}{1-\gamma} \kappa^{-\gamma} \left(a + \frac{\eta}{1+\eta} \frac{w^{\frac{1+\eta}{\eta}}}{r} \right)^{1-\gamma}
$$

implying

$$
\begin{aligned}
V'(a) &= \kappa^{-\gamma} \left(a + \frac{\eta}{1+\eta} \frac{w^{\frac{1+\eta}{\eta}}}{r} \right)^{-\gamma} \\
c(a) &= \kappa a + \left(\frac{\eta}{1+\eta} \frac{\tilde{\kappa}}{r} + \frac{1}{1+\eta} \right) w^{\frac{1+\eta}{\eta}}
\end{aligned}
$$

- The household's MPC (out of wealth) is still constant and given by $c'(a) = \text{MPC} = \kappa$ (same κ as before)
- Intuition: we correct for an *effective wage adjustment* in human capital / lifetime wealth but consumption is unaffected by labor supply under GHH
- This is because GHH shuts down income effects on labor supply!

3. Example: Return risk

- In the data, we see (a) much higher MPCs and (b) MPCs are higher at low income / wealth

- So far: we considered deterministic consumption-savings problems. They all yielded (roughly) MPC $\approx \rho \approx 5\%$ annually.

- We now start exploring theories of consumption that can break this and match the data much better

- Let's start with a simple example of return risk: You can only trade stocks (no bonds) and stocks trade at a stochastic price

$$Qdk = Dk - c$$

  where D is the dividend

- Define net worth as $a = Qk$, so $da = kdQ + Qdk$ by Ito's product rule, noting that $(dk)(dQ) = 0$, so

$$da = \frac{D}{Q} - c + a\frac{dQ}{Q}$$

- Assume stock prices follow a diffusion process (geometric Brownian):

$$\frac{dQ}{Q} = \mu dt + \sigma dB$$

- Rewrite wealth as: $da = (\mu_R - c)dt + a\sigma dB$, where $\mu_R = \mu_Q + \frac{D}{Q}$ (dividend + capital gains yield)

- HJB:

$$\rho V(a) = u(c(a) + V_a(\mu_R a - c) + \frac{1}{2}(a\sigma)^2 V_{aa}$$

- Solution of this model is (work this out yourself):

$$V(a) = \frac{1}{1-\gamma}\tilde{\kappa}^{-\gamma}a^{1-\gamma}$$
$$c(a) = \tilde{\kappa}a,$$

where

$$\kappa \equiv \frac{1}{\gamma}\left[\rho - (1-\gamma)\mu_R + \underbrace{\gamma(1-\gamma)\frac{\sigma^2}{2}}_{\text{Precautionary savings}}\right]$$

- Households' MPC now has a precautionary savings term $\frac{1}{2}(1-\gamma)\sigma^2$
- The consumption Euler equation for this model (work this out yourself) is:

$$\mathbb{E}\left(\frac{dc}{c}\right) = \frac{r-\rho}{\gamma}dt - \frac{1-\gamma}{2}\sigma^2 dt$$

- For $\gamma = 2$, precautionary term is negative, so households tilt consumption profile towards the future (hence, precautionary savings)

4. Income fluctuations

- We now work through arguably *the* benchmark model of consumption in macro

- Households face uninsurable income risk dz_t. They can trade a bond, a_t allowing for partial self-insurance, but face borrowing constraint $a_t \geq 0$.

- Sequence problem:

$$V_0 = \max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\begin{aligned} da_t &= ra_t + e^{z_t} - c_t \\ dz_t &= -\theta z_t dt + \sigma dB_t \end{aligned}$$

- We will use isoelastic (CRRA) preferences with $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$

- Household states are (a,z), so recursive representation is

$$\rho V = u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz}$$

where $c(a,z) = V_a(a,z)^{-\frac{1}{\gamma}}$

- First step in deriving Euler equation: envelope condition

$$(\rho - r)V_a = (ra + e^z - c)V_{aa} - \theta z V_{za} + \frac{\sigma^2}{2} V_{zza}$$

- Using Ito's lemma, noting V_a is function of (a_t, z_t),

$$\begin{aligned} dV_a &= V_{aa} da + V_{az} dz + \frac{1}{2} V_{azz} (dz)^2 \\ &= V_{aa}(ra + e^z - c)dt + V_{az}(-\theta z dt + \sigma dB) + \frac{\sigma^2}{2} V_{azz} dt \\ &= (\rho - r)V_a dt + \sigma V_{az} dB \end{aligned}$$

- Using $u_c = V_a$ and $V_{az} = u_{cc} c_z$, Euler equation for marginal utility is

$$\frac{du_c}{u_c} = (\rho - r)dt + \sigma \frac{u_{cc} c_z}{u_c} dB$$

- For isoelastic (CRRA), we have $\frac{u_{cc}}{u_c} = -\frac{\gamma}{c}$
- Consumption Euler equation (derivation is on PSET):

$$\frac{dc}{c} = \frac{r - \rho}{\gamma} dt + \frac{1 + \gamma}{2} \left(\frac{\sigma c_z}{c} \right)^2 dt + \frac{c_z}{c} \sigma dB,$$

  where c_z is $\approx$ the marginal propensity out of income shocks

- The term $\frac{1+\gamma}{2} \left(\frac{\sigma c_z}{c} \right)^2$ captures a precautionary savings motive due to uncertainty about future income fluctuations (that are not insurable)
- As always: Euler equation doesn't hold at the borrowing constraint