Dynamic Optimization: Problem Set #1

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DIFFICULTY: * easy, ** medium, *** hard

Problem 1 * (Canonical)

Credit: QuantEcon https://python.quantecon.org/finite_markov.html#exercises

Let y_t denote the employment / earnings state of an individual. Consider the state space $y_t \in \Upsilon = \{y^U, y^E\}$, where y^U corresponds to unemployment and y^E corresponds to employment. Let $\boldsymbol{y}$ denote the column vector $(y^U, y^E)'$ representing this state space (this is the grid you would construct on a computer). Suppose the employment dynamics of the individual are characterized by the invariant transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

We interpret a time period as a quarter.

(a) Give economic interpretations of $\alpha = P_{11}$ and $\beta = P_{22}$

(b) Why do the rows of P sum to 1?

(c) Is there an absorbing state in this model?

(d) Compute the probability of being unemployed two quarters after being employed.

(e) Denote the *marginal (probability) distribution* of y_t at time t by ψ_t. $\psi_t(y^L)$ is the probability that process y_t is in state y^L at time t. It is easiest to think of ψ_t as a time-varying row vector. Use the law of total probability to decompose $y_{t+1} = y^L$, accounting for all the possible ways in which state y^L can be reached at time $t+1$.

(f) Show that the resulting equation can be written as the vector-matrix product

$$\psi_{t+1} = \psi_t P.$$

Therefore: The evolution of the marginal distribution of a Markov chain is obtained by post-multiplying by the transition matrix.

(g) Show that

$$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t,$$

where $\sim$ reads "is distributed according to".

(h) We call ψ^* a *stationary distribution* of the Markov chain if it satisfies

$$\psi^* = \psi^* P.$$

Compute the probability of being unemployed n quarters after being employed. Take $n \to \infty$ and find the stationary distribution of this Markov chain. Find the stationary distribution by alternatively plugging into the above equation for ψ^*.

(i) Suppose $y_0 = y^H$. Solve for $\mathbb{E}_0(y_t)$. Use the law of total / iterated expectation to relate expectation to probabilities. Then use the formulas for marginal (probability) distributions derived above.

Problem 2 *

Consider the first-order linear *homogeneous* difference equation

$$x_{t+1} = \rho x_t.$$

We parameterize the initial condition by $x_0 = c$.

(a) Prove by induction that the *general solution* (for arbitrary c) is given by

$$x_t = \rho^t c.$$

(b) Show that the *particular solution* for initial value $x_0 = x$ is given by $x_t = \rho^t x$.

(c) Show that this also implies

$$x_t = \rho^{t-s} x_s$$

for $t > s$.

(d) Prove that this difference equation satisfies the Markov property.

Problem 3 *

Credit: Klaus Neusser `http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf` on p. 19 (of the PDF)

We study the dynamics of loan amortization. Denote by D_t the amount of debt owed at time t. The debt contract is serviced by paying an amount Z_t each period. Z_t is given exogenously for this problem (you could imagine some agent optimizing in the background).

(a) Explain why debt dynamics are characterized by the equation

$$D_{t+1} = RD_t - Z_t$$

where R is the constant gross interest rate associated with the loan. What kind of difference equation is this? Is this a forward or a backward equation?

(b) Suppose we start with an initial loan D_0. Solve iteratively (by induction) for D_t. You should get two terms — explain the economics for both terms.

(c) Suppose the loan needs to be repaid at time T. Solve for the constant repayment schedule $Z_t = Z$ such that the loan is repaid in period T.

(d) What is the condition on constant repayment rate Z relative to D_0 such that the loan is repaid in finite time?

Problem 4 *

Credit: Klaus Neusser `http://www2.econ.iastate.edu/classes/econ600/rksingh/fall16/TA/DifferenceEquations.pdf` on p. 38 (of the PDF)

We study Cagan (1956)'s model for hyperinflation. The model is summarized by the three equations

$$m_t^d - p_t = \alpha(p_{t+1}^e - p_t)$$

$$m_t^s = m_t^d$$

$$p_{t+1}^e - p_t = \gamma(p_t - p_{t_1})$$

where (all in logs) m_t^d is money demand, m_t^s is money supply, p_t is the price level and p_{t+1}^e is private agents' expectations for the price level in period $t+1$. The first equation of the above system characterizes money demand and the third equation characterizes *adaptive* inflation expectations. Assume that $\alpha < 0$ and $\gamma > 0$.

(a) Characterize a first-order difference equation that solves for p_t as a function of p_{t-1} and m_{t-1}. What kind of difference equation is this?

(b) Using the tools already developed, solve for p_t in terms of some initial conditions on the system.

(c) Characterize the stability condition such that if $m_t \to m$ in the long run, p_t converges to a steady state p. Interpret the economics of this stability condition.

(d) Explain why this equation should be thought of as a *forward* equation.

(e) Now assume that agents form expectations rationally instead of adaptively. That is, replace the third equation above by

$$p^e_{t+1} = p_{t+1}.$$

Simplify the model equations again to obtain a difference equation for p_t in terms of m_t. What kind of difference equation is this?

(f) Argue that we should think of this equation now as a *backward* equation. Solve again for the *general* solution of this difference equation (backwards), i.e., express p_t in terms of p_{t+s} and m_{t+s}.

(g) Solve for a *particular* solution by imposing some transversality (terminal) condition on $\lim_{T\to\infty} p_T$.

Problem 5 * (** the Euler) (Canonical)

Take the continuous-time limit of the following equations:

(a) $a_{t+1} = R_t a_t + y_t - c_t$

(b) $u'(c_t) = \beta R_t u'(c_{t+1})$

(c) $\pi_t = \beta \pi_{t+1} + \kappa x_t$

Problem 6 * (** the Euler)

Revert back from continuous to discrete time by plugging in for the definition of first-order derivative

$$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

for small Δ.

(a) $\dot{K}_t = I_t - \delta K_t$

(b) $\dot{a}_t = r_t a_t + y_t - c_t$

(c) $\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho}{\gamma}$

(d) $\dot{\pi}_t = \rho \pi_t - \kappa x_t$

Problem 7 ** (Canonical)

Consider the equation for wealth dynamics

$$\dot{a}_t = r_t a_t + y_t - c_t.$$

We take $\{r_t\}$ and $\{y_t - c_t\}$ as exogenously given.

(a) Solve for the lifetime budget constraint.

(b) Solve the ODE for its general solution using the integrating factor method introduced in class, i.e., find an expression for a_t in terms of the exogenous processes $\{r_t, y_t, c_t\}$ and some arbitrary initial condition $a_0 = c$.

Problem 8 **

Credit: Miranda Holmes-Cerfon https://cims.nyu.edu/~holmes/teaching/asa19/handout_Lecture4_2019.pdf

In this problem, we will prove the Chapman-Kolmogorov Equation for a *time-homogeneous* continuous-time Markov chains. Denote the *transition probability* as

$$P_{ij}(t+s) = \mathbb{P}(X_{t+s} = j \mid X_t = i)$$

where i and j should be thought of as indices on the state space, i.e., the ith value of the finite state space $\mathcal{X}$ of the Markov chain.

Denote I the indices associated with the state space $\mathcal{X}$. The Chapman-Kolmogorov equation is:

$$P_{ij}(t+s) = \sum_{k \in I} P_{ik}(t) P_{kj}(s).$$

(a) Use the law of total probability to show that

$$\mathbb{P}(X_{t+s} = j \mid X_0 = i) = \sum_k \mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i)\mathbb{P}(X_t = k \mid X_0 = i)$$

(b) Use the Markov property

(c) Invoke time homogeneity to arrive at the result

Problem 9 *** (the (a) **)

This problem collects several exercises on Brownian motion and stochastic calculus.

(a) Show that $\mathbb{C}ov(B_s, B_t) = \min\{s, t\}$ for two times $0 \leq s < t$. Use the following tricks: Use the covariance formula $\mathbb{C}ov(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B)$. Use $B_t \sim \mathcal{N}(0, t)$ as well as $B_t - B_s \sim \mathcal{N}(0, t-s)$. And use $B_t = B_s + (B_t - B_s)$.

(b) Geometric Brownian motion evolves as: $dX_t = \mu X_t dt + \sigma X_t dB_t$. Show that

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

for a given initial value X_0.

(c) For Geometric Brownian motion as defined above, show that $\mathbb{E} = X_0 e^{\mu t}$.

(d) The Ornstein-Uhlenbeck (OU) process is like a continuous-time variant of the AR(1) process. It evolves as $dX_t = -\mu X_t dt + \sigma dB_t$ for drift parameter μ, diffusion parameter σ, and some X. Show that it solves

$$X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s.$$