M2: Lecture 2

Continuous Time Dynamics

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Outline of today's lecture

1. Hamilton-Jacobi-Bellman (HJB) equation
2. First-order condition for consumption
3. Envelope condition and Euler equation
4. Connection between calculus of variations / optimal control and HJBs
5. Boundary conditions: no-borrowing in the wealth / capital dimension
6. Example: solving the growth model in closed form

1. Hamilton-Jacobi-Bellman equation

- Recall the neoclassical growth model in continuous time

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\begin{aligned} \dot{k}_t &= F(k_t) - \delta k_t - c_t \\ k_0 &\text{ given }, \end{aligned}$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now

- This is the infinite-horizon sequence problem, $t \in [0, \infty)$

- A function $v(\cdot)$ that solves this problem is a solution to the neoclassical growth model

- We will now work towards a recursive representation (good reference: Stokey textbook)

- The discrete-time Bellman equation would be

$$v(k_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} v(k_{t+\Delta}) \right\}$$

where $\beta = \frac{1}{1+\rho\Delta t}$

- Next: multiply by $1+\rho\Delta t$ and note that $(\Delta t)^2 \approx 0$

$$(1+\rho\Delta t)v(k_t) = \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + v(k_{t+\Delta}) \right\}$$

$$\rho\Delta t v(k_t) = \max_c \left\{ u(c)\Delta t + v(k_{t+\Delta}) - v(k_t) \right\}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta t} \right\}$$

- Finally: take limit $\Delta t \to 0$ and drop t subscripts

$$\rho v(k) = \max_c \left\{ u(c) + dv \right\}$$

- We want to express dv in terms of $v'(\cdot)$ and dk
- Different ways to think about this: chain rule, Ito's lemma (though no uncertainty here), generator
- Recall generator of (stochastic) process dk_t: For any $f(\cdot)$

$$\mathcal{A}f(k_t) = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(k_{t+\Delta t}) - f(k_t)}{\Delta t}$$

- For simple ODE (no uncertainty) $dk = (F(k) - \delta k - c)dt$, we have

$$\mathcal{A}f(k) = (F(k) - \delta k - c)f'(k)$$

- Therefore, we arrive at the **Hamilton-Jacobi-Bellman equation**:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c\Big) v'(k) \right\}$$

- Notice: We conjectured a stationary value function (what does this mean?)

2. First-order condition for consumption

- HJB still has "max" operator:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c\Big) v'(k) \right\}$$

- To get rid of this, we have to resolve optimal consumption choice
- First-order condition:

$$u'(c(k)) = v'(k)$$

- This defines the **consumption policy function**
- We can now plug back in, obtaining an ODE in $v'(k)$

$$\rho v(k) = u(c(k)) + \Big(F(k) - \delta k - c(k)\Big) v'(k)$$

- Why is this a "stationary" value function and ODE? What would a time-dependent ODE look like? When would we get one?

3. Envelope condition and Euler equation

- We now derive the Euler equation in continuous time

- We start with the **HJB envelope condition**. Differentiating in k:

$$\rho v'(k) = u'(c(k))c'(k) + \Big(F'(k) - \delta - c'(k)\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)$$

$$\rho v'(k) = \Big(\underbrace{F'(k) - \delta}_{\text{interest rate } r}\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

$$(\rho - r)v'(k) = \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

- Next, we characterize *process* $dv'(k)$. Using Ito's lemma (even though no uncertainty):

$$\begin{aligned} dv'(k) &= v''(k)dk \\ &= v''(k)(F(k) - \delta k - c(k))dt \\ &= (\rho - r)v'(k)dt. \end{aligned}$$

- Recall first-order condition $u'(c(k)) = v'(k)$.
- The **Euler equation for marginal utility** is given by

$$\frac{du'(c)}{u'(c)} = (\rho - r)dt.$$

- To go from marginal utility to consumption, we use CRRA utility: $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. $u'(c) = c^{-\gamma}$ is a function of *process* c, so by Ito's lemma:

$$\begin{aligned} du'(c) &= -\gamma c^{-\gamma-1}dc \\ &= -\gamma u'(c)\frac{dc}{c} \end{aligned}$$

- Plugging in yields **Euler equation for consumption** in continuous time:

$$\frac{dc}{c} = \frac{r-\rho}{\gamma}dt$$

or (you'll often see this notation when no uncertainty): $\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}$

4. Connection between calculus of variations and HJB

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5. Boundary conditions

- This is really important: everything we have done so far is only valid in the **interior of the state space**
- What's the state space of a model?
- For the neoclassical growth model without uncertainty, state space is $k \in [0, \infty)$, or
$$\mathcal{X} = \left\{ k \mid k \in [0, \bar{k}] \right\}$$
where we impose an upper boundary $\bar{k}$
- This is like the domain of the function $v(k)$ that will be valid
- We say $\partial\mathcal{X} = \{0, \bar{k}\}$ is the **boundary** of the state space and $\mathcal{X} \backslash \partial\mathcal{X} = (0, \bar{k})$ is the **interior**
- As is the case **for all differential equations**, the HJB holds on the interior and we need **boundary conditions** to characterize $v(k)$ along the boundary

- What kind of differential equation is the HJB in this model?
- So how many boundary conditions do we need?
- In terms of the economics, what is the correct boundary condition? I.e., what is the correct economic behavior at the boundary $k \in \{0, \bar{k}\}$?
- Nice intuition: 2 boundary inequalities do same job as 1 boundary equality
- We want households to not leave the state space, so we impose that they do not dissave / borrow as $k \to 0$ and save as $k \to \bar{k}$
- This implies: (why?)

$$
\begin{aligned}
u'(c(0)) &\geq v'(0) \\
u'(c(\bar{k})) &\leq v'(\bar{k})
\end{aligned}
$$

- If households ever hit the boundaries (in the neoclassical growth model, this doesn't really happen), then consumption behavior is no longer determined by the Euler equations but rather by the boundary conditions

6. Example

- Closed-form solution of neoclassical growth model