Dynamic Optimization: Problem Set #2

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Problem 1: Going on the job market

Credit: David Laibson

Consider the following "job market" optimization problem. A graduate student works on a project with current market value x, where x is an Ito process with drift α and standard deviation σ :

$$dx = \alpha dt + \sigma dz$$

The student works on the project until she chooses to either (i) take the paper to the job market, generating termination payoff x, or (ii) abandon the project and start a new project that has current market value $x = 0$. This replacement of projects is made at zero cost. (We assume that the student can only work on one project at a time.)

The optimal policy is a two-sided threshold rule. The student stops when (i) $x \geq M$, or (ii) $x \leq 0$. Here M is an endogenous boundary representing the job M arket. The zero bound follows from the economics of the problem (free substitution).

Assume that time has opportunity cost w (annual wage) and the student has an annual discount rate ρ.

(a) Show that in the continuation region the HJB satisfies

$$\rho V = -w + \alpha V' + \left(\sigma^2/2\right) V''$$

Solution: In this region, the flow utility is $-w$. And the only state variable of the student's problem is x

$$\begin{aligned}
\rho V &= -w + EdV \\
&= -w + \alpha V' + \left(\sigma^2/2\right) V''
\end{aligned}$$

(b) This is the so-called complete equation. The associated reduced equation is

$$\left(\sigma^2/2\right) V'' + \alpha V' - \rho V = 0$$

Guess the solution of the reduced form is $V = \exp(rx)$ and verify, show there are two solutions.

Solution:

$$\begin{aligned}\left(\sigma^2/2\right) r^2 \exp(rx) + \alpha r \exp(rx) - \rho \exp(rx) &= 0\\ \left(\sigma^2/2\right) r^2 + \alpha r - \rho &= 0\end{aligned}$$

The two roots that solve the reduced equation are

$$r^{+,-} = \frac{-\alpha \pm \sqrt{\alpha^2 + 2\sigma^2\rho}}{\sigma^2}.$$

A particular solution of the complete equation is generated by the policy of never stopping:

$$V(x) = -w/\rho.$$

The general solution of the complete equation is:

$$V(x) = C^+ \exp\left(r^+ x\right) + C^- \exp\left(r^- x\right) - w/\rho$$

Remember that the superscripts $+$ and $-$ are not arithmetic operators.

(c) We have three boundary conditions (one value matching condition and two smooth pasting conditions; why aren't we using the other value matching condition at 0?).

$$V(M) = M$$

$$V'(M) = 1$$

$$V'(0) = 0$$

Derive the two smooth pasting conditions (with the V'), interpret each of the boundary conditions.

Solution: Using the boundary condition

$$(V(M) = M)$$

differentiate w.r.t x and evaluate at $x = M$. For $x = 0$, because at this boundary start a new project, the value function does not depend on the current x.

(d) We can now solve for our three free variables: C^+, C^-, M. Show the solution must satisfy

$$C^- = -r^+ C^+ / r^-$$

$$C^+ = \frac{1}{r^+ \left[\exp\left(r^+ M\right) - \exp\left(r^- M\right)\right]}$$

$$C^- = \frac{1}{r^- \left[\exp\left(r^- M\right) - \exp\left(r^+ M\right)\right]}$$

$$M = \frac{r^+ \exp\left(r^+ M\right) - r^- \exp\left(r^- M\right)}{\exp\left(r^+ M\right) - \exp\left(r^- M\right)} - w/\rho$$

$$C^+ + C^- = \frac{1/r^+ - 1/r^-}{\exp\left(r^+ M\right) - \exp\left(r^- M\right)}$$

Solution: Using the boundary conditions

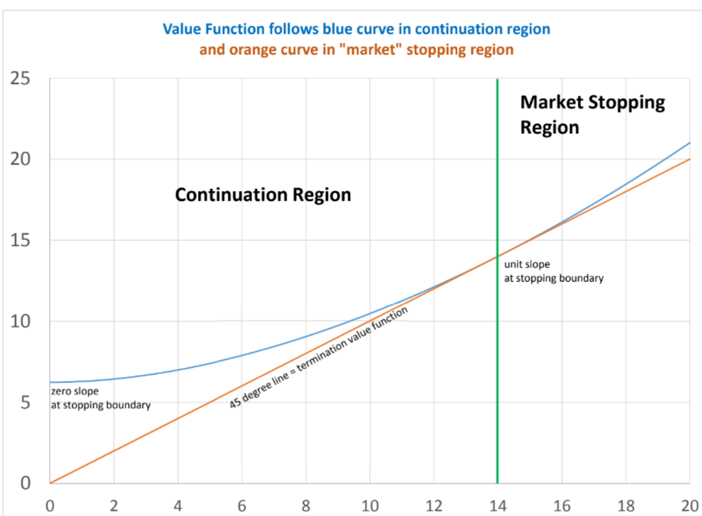
$$V(M) = M = C^+ \exp\left(r^+ M\right) + C^- \exp\left(r^- M\right) - w/\rho$$

$$V'(M) = 1 = r^+ C^+ \exp\left(r^+ M\right) + r^- C^- \exp\left(r^- M\right)$$

$$V'(0) = 0 = r^+ C^+ + r^+ C^-$$

After some algebra, we can then get the equations above.

The solution of the value function looks as follows (for parameters $\alpha = 1, \sigma = 5, w = 1$, and $\rho = 0.05$)



(e) If you could pick projects with less positive drift (smaller α) and more Brownian noise (larger σ) would you? Why or why not?

Solution: Depends on parameters. But the main idea is that because only care about getting a very high draw to go on the market, a high σ may maximize the probability of getting into the region $x > M$ fast. That is, it is a good idea to try risky projects!

Problem 2: The Equity Premium

Credit: Pablo Kurlat

Suppose that consumption growth is:

$$\frac{c_{t+1}}{c_t} = \begin{cases} 1+g & \text{with probability } 1-\mu \\ 1-D & \text{with probability } \mu \end{cases}$$

and the (gross) return on the stock market is:

$$R_{t+1} = \begin{cases} 1+y & \text{with probability } 1-\mu \\ 1-F & \text{with probability } \mu \end{cases}$$

Furthermore, suppose the event $\frac{c_{t+1}}{c_t} = 1-D$ and the event $R_{t+1} = 1-F$ always coincide, i.e. $\frac{c_{t+1}}{c_t}$ and R_t are perfectly correlated. The risk-free (gross) real interest rate is $1+r$. There is a representative household, with preferences given by:

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t u(c_t)\right)$$

with

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

(a) Write down the Euler equations for risk-free assets and for the stock market.

Solution: For risk-free assets:

$$1 = \beta(1+r)\mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]$$

For the stock market:

$$1 = \beta\mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]\mathbb{E}_t(R_{t+1}) + \beta\,\mathrm{Cov}\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right)$$

(b) Find explicit expressions in terms of parameters for the following quantities:

a) $\mathbb{E}\left(\frac{u'(c_{t+1})}{u'(ct)}\right)$

Solution:

$$\begin{aligned}\frac{u'(c_{t+1})}{u'(c_t)} &= \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \\ &= \begin{cases} (1+g)^{-\gamma} & \text{with probability } 1-\mu \\ (1-D)^{-\gamma} & \text{with probability } \mu \end{cases} \\ \mathbb{E}\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) &= (1-\mu)(1+g)^{-\gamma} + \mu(1-D)^{-\gamma}\end{aligned}$$

b)$\mathbb{E}(R_{t+1})$

Solution:

$$\mathbb{E}\left(R_{t+1}\right)=(1-\mu)(1+y)+\mu(1-F)$$

c) $Cov\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right)$

Solution:

$$Cov\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{t+1}\right)=\mathbb{E}\left[\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}-\mathbb{E}\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right)\right)\left(R_{t+1}-\mathbb{E}\left(R_{t+1}\right)\right)\right]$$

$$=(1-\mu)\,\mathbb{E}\left[\left((1+g)^{-\gamma}-(1-\mu)\,(1+g)^{-\gamma}-\mu\,(1-D)^{-\gamma}\right)\left((1+y)-(1-\mu)\,(1+y)-\mu\,(1-F)\right)\right]$$

$$+\mu\mathbb{E}\left[\left((1-D)^{-\gamma}-(1-\mu)\,(1+g)^{-\gamma}-\mu\,(1-D)^{-\gamma}\right)\left((1-F)-(1-\mu)\,(1+y)-\mu\,(1-F)\right)\right]$$

$$=(1-\mu)\,\mu\left((1+g)^{-\gamma}-(1-D)^{-\gamma}\right)(y+F)-\mu\,(1-\mu)\left((1-D)^{-\gamma}-(1+g)^{-\gamma}\right)(y+F)$$

$$=2\,(1-\mu)\,\mu\,(y+F)\left[(1+g)^{-\gamma}-(1-D)^{-\gamma}\right]$$

(c) 3. Find an explicit expression in terms of parameters for the value of β that is consistent with the household's Euler equation for risk-free assets, given that the risk-free rate is $1+r$.

Solution:

$$\beta=\frac{1}{(1+r)\mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]}$$

$$=\frac{1}{(1+r)\left[(1-\mu)(1+g)^{-\gamma}+\mu(1-D)^{-\gamma}\right]}$$

(d) Derive a joint restriction on the values of β,γ,μ,g,D,y and F that needs to be satisfied for the household's Euler equation for the stock market to hold. **Solution:** Using the Euler equation:

$$1=\beta\mathbb{E}_t\left[\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right]\mathbb{E}_t\left(R_{t+1}\right)+\beta\,\mathrm{Cov}\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{t+1}\right)$$

$$=\beta\left[(1-\mu)(1+g)^{-\gamma}+\mu(1-D)^{-\gamma}\right]\left[(1-\mu)(1+y)+\mu(1-F)\right]$$

$$+2\beta(1-\mu)\mu(y+F)\left[(1+g)^{-\gamma}-(1-D)^{-\gamma}\right]$$

(e) Set the following parameter values:

r	γ	μ	g	D	F
0.01	3	0.02	0.015	0.3	0.45

What must be the values of β and y for the household's Euler equations to hold?

Solution: Replacing:

$$\beta=0.9946$$

$$y=0.0595$$

(f) What is the value of $\mathbb{E}(R_{t+1})$? What is the equity premium?

Solution: Replacing:

$$\mathbb{E}(R_{t+1}) = 1.0493$$

so the premium is

$$\mathbb{E}(R_{t+1}) - (1+r) = 0.0393$$

(g) Suppose a researcher is trying to measure the equity premium empirically in a sample the event "$\frac{c_{t+1}}{c_t} = 1 - D$ and $R_{t+1} = 1 - F$" has never been realized. What would the researcher's estimates be for:

(a) $\mathbb{E}\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)$

Solution:

$$\mathbb{E}\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) = (1+g)^{-\gamma} = 0.956$$

(b) $\mathbb{E}(R_{t+1})$

Solution:

$$\mathbb{E}(R_{t+1}) = 1 + y = 1.0595$$

(c) $\mathrm{Cov}\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right)$

Solution:

$$\mathrm{Cov}\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}\right) = 0$$

Would the researcher conclude that there is an equity premium “puzzle”? Explain.

Problem 3

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Please solve Problem set 4 of David.

Problem 4

Credit: David Laibson (https://projects.iq.harvard.edu/econ2010c/problem-sets-david-laibson)

Please solve Problem 1 of David's problem set (PSET) #5.