M2: Lecture 4

Dynamic Programming in Discrete Time (II)

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Outline of today's lecture

1. Example: growth model with labor supply
2. Stochastic dynamic programming
3. History notation
4. Stochastic growth model

1. Example: growth model with labor supply

- Time is discrete with $t = 0, 1, \ldots$ and there is no uncertainty
- Representative household consumes (c_t), saves and works (l_t)
- Preferences over consumption and work given by

$$V(k_0) = \max_{\{c_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

- We will use constant relative risk aversion (CRRA) preferences:

$$u(c, l) = \frac{1}{1-\gamma} c^{1-\gamma} - \frac{1}{1+\phi} l^{1+\phi}$$

- Budget constraint: household owns capital and rents it at rate r_t

$$k_{t+1} = (1 + r_t - \delta) k_t + w_t l_t - c_t$$

Definition (household problem): The representative household chooses sequences $\{c_t, l_t\}_{t=0}^{\infty}$ to maximize lifetime value $V(k_0)$ subject to the budget constraint, given initial capital k_0 and taking as given paths of prices $\{r_t, w_t\}_{t=0}^{\infty}$

- Household problem in sequence form associated with Lagrangian:

$$L(k_0) = \sum_{t=0}^{\infty} \beta^t \left[u(c_t, l_t) + \lambda_t \Big((1 + r_t - \delta) k_t + w_t l_t - c_t - k_{t+1} \Big) \right]$$

- FOCs for c_t, l_t, and k_{t+1}:

$$\begin{aligned} 0 &= u_c(c_t, l_t) - \lambda_t \\ 0 &= u_l(c_t, l_t) + w_t \lambda_t \\ 0 &= -\lambda_t + \beta \lambda_{t+1} (1 + r_t - \delta) \end{aligned}$$

- Euler equation for consumption: $u_c(c_t, l_t) = \beta(1 + r_t - \delta) u_c(c_{t+1}, l_{t+1})$

- Optimal labor supply: $MRS = -\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = w_t$

- Recursive representation (show this):

$$V(k) = \max_{c,l} \left\{ u(c,l) + \beta V((1+r-\delta)k + wl - c) \right\}$$

- FOCs:

$$u_c = \beta V_k(k')$$
$$u_l = -\beta w V_k(k')$$

  which again implies $-u_l / u_c = w$

- Use the envelope condition to derive the consumption Euler equation

- Plug in for CRRA utility and interpret Euler equation and optimal labor supply condition

2. Stochastic dynamic programming

- Follow Ljungqvist-Sargent notation, Chapter 3.2
- Under uncertainty, household problem takes the form

$$\max_{\{c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = g(k_t, c_t, \epsilon_{t+1})$ (*first-order stochastic difference equation*)

- $\{\epsilon_t\}_{t=0}^{\infty}$ is sequence of iid random variables (*stochastic process*)
- Initial condition x_0 given

- Dynamic programming approach: we again look for recursive representation on state space $k_t \in \mathcal{X}$

- The problem is to look for a *policy function* $c(k)$ that solves

$$V(k) = \max_c \left\{ u(c) + \beta \mathbb{E}\left[V\Big(g(k, c, \epsilon) \Big) \,\Big|\, k \right] \right\}$$

where $\mathbb{E}[V(\cdot) \mid k] = \int V(\cdot) dF(\epsilon)$

- $V(k)$ is the (lifetime) value that an agent obtains from solving this problem starting from k

- FOC that characterizes the consumption policy function $c(k)$ is

$$0 = u'(c(k)) + \beta \mathbb{E}\left\{ \partial_k V\Big(g(k, c(k), \epsilon) \Big) \cdot \partial_c g(k, c(k), \epsilon) \,\Big|\, k \right\} = 0$$

3. History notation

- A very popular approach to deal with uncertainty in macro is to use history notation (Ljungqvist-Sarget, e.g., chapters 8, 12)

- Economy populated by individuals, indexed by $i \in I$

- Time is discrete and indexed by $t = 0, 1, \ldots$

- At every t, there is a realization of a stochastic event $s_t \in \mathcal{S}$

- We denote the **history** of such events up to t by $s^t = \{s_0, s_1, \ldots, s_t\}$

- The unconditional probability of history s^t is given by $\pi_t(s^t \mid s_0)$

- If Markov, $\pi_t\left(s^t \mid s_0\right) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \ldots \pi(s_0)$

- Single consumption good (dollars) as numeraire

- The **lifetime value** of individual i is then defined as

$$V_i(s_0) = \sum_{t=0}^{T} \left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \mid s_0\right) u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)$$

- *Generalizations*: heterogeneous beliefs, general preferences (Epstein-Zin), recursive formulation, multiple commodities, intergenerational considerations

- Suppose $c_t^i(\cdot)$ and $l_t^i(\cdot)$ are functions of some primitive (policy) θ

- Consider policy experiment $d\theta$, then i's **private welfare assessment** is

$$\frac{dV_i(s_0)}{d\theta} = \sum_{t=0}^{T} \left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \mid s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} \frac{du_{i|c}(s^t)}{d\theta}$$

Instantaneous consumption-equivalent effect of policy $d\theta$ at date t, history s^t on individual i:

$$\frac{du_{i|c}(s^t)}{d\theta} \equiv \frac{\frac{du_i(c_t^i(s^t),n_t^i(s^t))}{d\theta}}{\frac{\partial u_i(s^t)}{\partial c_t^i}} = \underbrace{\frac{dc_t^i(s^t)}{d\theta} + \frac{\frac{\partial u_i(s^t)}{\partial n_t^i}}{\frac{\partial u_i(s^t)}{\partial c_t^i}} \frac{dn_t^i(s^t)}{d\theta}}_{\text{in } \textit{consumption} \text{ units}}$$

- Example: $d\theta$ gives i marginal dollar at s^t, then

$$\frac{dV_i(s_0)}{d\theta} = (\beta_i)^t \pi_t(s^t \mid s_0) \frac{\partial u_i(s^t)}{\partial c_t^i}$$

Conventional benchmarks:

- Policy $d\theta$ is Pareto improving if $\frac{dV_i(s_0)}{d\theta} \geq 0$ for all i, strictly for some i

- Welfarist planners: SWF given by $\mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right)$
  Utilitarian, Isoelastic, Rawlsian, Nash, Dictator, ...

- Policy $d\theta$ is desirable for a welfarist planner if

$$\int \lambda_i\left(s_0\right)\frac{dV_i(s_0)}{d\theta}di > 0\,, \qquad \text{where} \quad \lambda_i\left(s_0\right) = \frac{\partial \mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right)}{\partial V_i}$$

Remarks:

- All welfarist planners agree when individuals ex-ante (s_0) homogeneous

- How to make welfare assessments with *heterogeneous* individuals?

- If you're interested, see Dávila-Schaab (2022)

4. Stochastic Growth Model

- Discrete time: $t \in \{0, 1, \ldots, T\}$, where $T \leq \infty$
- At t, event $s_t \in \mathcal{S}$ is realized; history $s^t = (s_0, \ldots, s_t)$ has probability $\pi_t(s^t)$
- Representative household has preferences of paths of consumption $c_t(s^t)$ and labor $l_t(s^t)$

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t\left(s^t\right) u\left(c_t(s^t), l_t(s^t)\right)$$

- Inada conditions $\lim_{c \to 0} u_c(c, l) = \lim_{l \to 0} u_l(c, l) = \infty$
- At $t = 0$, household endowed with k_0

- Technology, capital accumulation, and budget / resource constraint:

$$c_t(s^t) + \iota_t(s^t) \leq A_t(s^t)F(k_t(s^{t-1}), l_t(s^t))$$
$$k_{t+1}(s^t) = (1-\delta)k_t(s^{t-1}) + \iota_t(s^t)$$

- $F(\cdot)$ is twice continuously differentiable and constant returns to scale
- Source of uncertainty is stochastic process for TFP $A_t(s^t)$
- Standard regularity conditions on $F(\cdot)$ (see LS)

Lagrangian approach to sequence problem

- Form Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \Big\{ u(c_t(s^t), l_t(s^t))$$
$$+ \lambda_t(s^t) \Big[A_t(s^t) F(k_t(s^{t-1}), l_t(s^t)) - c_t(s^t) + (1-\delta) k_t(s^{t-1}) - k_{t+1}(s^t) \Big] \Big\}$$

- FOCs for $c_t(s^t)$, $l_t(s^t)$ and $k_{t+1}(s^t)$ are given by

$$\begin{aligned} u_c(s^t) &= \lambda_t(s^t) \\ u_l(s^t) &= u_c(s^t) A_t(s^t) F_n(s^t) \\ u_c(s^t)\pi_t(s^t) &= \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1}) \pi_{t+1}(s^{t+1}) \Big[A_{t+1}(s^{t+1}) F_k(s^{t+1}) + (1-\delta) \Big] \end{aligned}$$

- Summation over $(s^{t+1} \mid s^t)$ is like conditional expectation (summing over histories that branch out from s^t)

Recursive representation: dynamic programming

- Assume time-homogeneous Markov process:

$$\mathbb{E}_t(A_{t+1}) = \mathbb{E}\Big[A(s^{t+1}) \mid s^t\Big] = \mathbb{E}\Big[A(s_{t+1}) \mid s_t\Big] = \sum_{s'} \pi(s' \mid s_t) A(s')$$

- Drop t subscripts: s is current state, s' denotes next period's draw

- Denote by X_t the *endogenous state* of the problem: for now, assume there is such a representation (Fabrice's part will be all about this)

- Intuitively: s is the exogenous state and X is the endogenous state

- Bellman equation can be written:

$$V(X, s) = \max_{c,l} \left\{ u(c, l) + \beta \sum_{s'} \pi(s' \mid s) V(X', s') \right\}$$

subject to $X' = g(X, c, l, s, s')$