Dynamic Optimization and Applications in Macroeconomics

**Instructors**:

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**Prerequisites**: If you have had no exposure to any of the following, you should try to brush up

* Constrained optimization: Lagrange multipliers, necessary and sufficient conditions for optimality, Karush-Kuhn-Tucker conditions
* Difference equations
* Ordinary differential equations (we will also encounter partial differential equations)
* Probability theory: Markov chains, stochastic processes, Brownian motion, stochastic differential equations (we will also encounter stochastic calculus and Ito’s lemma)

**Textbooks and References**: No required textbooks. But if you plan on doing research in macro, I’d strongly encourage you to get the following:

* Stokey and Lucas [**SL**]: Recursive methods in economic dynamics
* Ljungqvist and Sargent [**LS**]: Recursive macroeconomic theory
* Acemoglu [**A**]: Introduction to modern economic growth
* Barro and Sala-i-Martin [**BS**]: Economic growth
* Romer: Advanced macro

Other useful references:

* Blitzstein: Introduction to probability
* Stokey: The economics of inaction
* Oksendal: Stochastic differential equations
* Steele: Stochastic calculus and financial applications
* Durrett: Probability: theory and examples
* Graduate-level probability theory (Harvard Stats 210):

https://www.ekzhang.com/assets/pdf/Stat\_210\_Notes.pdf

* LeVeque: Finite difference methods for ordinary and partial differential equations
* Kamien and Schwartz [**KS**]: Dynamic optimization: the calculus of variations and optimal control
* Stachurski: Economic dynamics
* Miao: Economic dynamics in discrete time

*Discrete-time Markov chains and (stochastic) difference equations*:

* Stachurski chapters 4, 5, 6, 8, 11
* Miao chapters 1, 2, 3
* LS chapters 2
* SL chapters 6, 8

*Differential equations*:

* A appendix B
* BS appendix A.1 through A.3
* BS chapters 2, 3
* LeVeque (entire book)
* KS (entire book on calc of variations and optimal control theory)

*Integration, measure, and probability theory*:

* Blitzstein (this is a fantastic undergraduate-level intro to probability theory)
* Durrett (entire book on probability theory)
* Stachurski chapters 7, 9
* SL chapter 7
* Miao appendix D

*Stochastic processes, continuous-time Markov chains, stochastic differential equations*:

* Durrett (entire book)
* Steele (entire book)
* Oksendal (entire book)
* Stokey chapters 2, 3

**Grading and Homework**: The final exam will account for 100% of your grade. There will be a homework every week. The solutions will be made available, and you are strongly encouraged but not required to complete each homework.

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PART (1): Dynamics (**2 lectures**)

**Lecture 1**: discrete time – difference equations

* Introduction to difference equations
* The Solow growth model
* How to solve difference equations
* Stochastic difference equations
* Taking the continuous-time limit

**Lecture 2**: continuous time – differential equations

* Ordinary differential equations (ODEs)
* Boundary conditions
* ODEs with closed-form solutions: population growth, half-life formula, AR(1) process, general homogeneous solution, …
* Partial differential equations (PDEs)
* Stochastic processes
* Brownian motion
* Stochastic differential equations (SDEs)

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PART (2): Constrained Dynamic Optimization and Dynamic Programming (**5 lectures**)

**Lecture 3**: dynamic programming in discrete time (I)

* Neoclassical growth model
* Sequence problem
* Dynamic programming: the principle of optimality
* Bellman equation
* Solving the Bellman equation via guess-and-verify (closed-form solution of growth model)

**Lecture 4**: dynamic programming in discrete time (II)

* Bellman operator
* Contraction mapping
* Gentle introduction to measure theory: Expectation, conditional expectation, law of iterated expectations,
* Adding uncertainty: the Bellman equation with stochastic dynamics

**Lecture 5**: calculus of variations and optimal control theory

* Neoclassical growth model in continuous time
* Sequence problem
* Calculus of variations
* Optimal control theory

**Lecture 6**: deterministic dynamic programming in continuous time

* Deriving the Hamilton-Jacobi-Bellman (HJB) equation (Stokey derivation)
* First-order condition for consumption
* Envelope condition and Euler equation
* Connection between calculus of variations / optimal control and HJBs
* Boundary conditions: no-borrowing in the wealth / capital dimension

**Lecture 7**: stochastic dynamic programming in continuous time

* What is the generator of a stochastic process?
* Neoclassical growth model with diffusion process
* Neoclassical growth model with Poisson process
* Examples

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PART (3): Applications (**6 lectures**)

**Lecture 8**: consumption (I) – the permanent income hypothesis

* Permanent income hypothesis
* Lifetime budget constraint
* Certainty equivalence and consumption as a martingale (quadratic preferences)
* Eat-the-pie problem (CRRA preferences)
* Closed-form examples

**Lecture 9**: consumption (II) – what determines MPCs

* Closed-form examples continued
* GHH preferences
* Income fluctuations and precautionary savings
* Linearization of consumption Euler equation: deterministic and stochastic
* What about a case with time-varying stochastic interest rate + linearization (Gabaix example)?

**Lecture 10**: consumption (III) – the “standard incomplete markets” model and life-cycle

* Liquidity constraints: the buffer stock model
* Price-theory decompositions
* Consumption over the life-cycle

**Lecture 11**: portfolio choice and asset pricing

* Consumption, savings, and portfolio choice with a risky asset
* Merton asset pricing model and equity premium

**Lecture 12**: optimization problems of the firm

* When to shut down a plant?
* Optimal stopping problems
* Option value
* Tobin’s Q