Topics in Heterogeneous Agent Macro: Dynamics in Discrete and Continuous Time

Lecture 1

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Introduction

- Goals of this course: hard skills + topics in heterogeneous agent macro
- The more you can "tool up" during your first 3 years, the better

Acknowledgements

- Course builds on excellent material developed by others
- Huge thanks to Benjamin Moll
  - His material is what I used to learn all this
  - First half of course draws on: `https://benjaminmoll.com/lectures/`
  - As well as code repository: `https://benjaminmoll.com/codes/`
- Second half of the course draws on material developed by Auclert-Rognlie-Straub
- Textbooks for continuous time methods: LeVeque + Oksendal

Outline

Goal for today: review deterministic and stochastic dynamics

1. Discrete time dynamics
2. Continuous time dynamics

Discrete Time

Stochastic processes

- Let X_t be a random variable that is time t adapted
- Discrete time: We index time discretely $t = 0, 1, 2, \ldots, T \leq \infty$
- Stochastic process in discrete time: a sequence of random variables indexed by t, $\{X_t\}_{t=0}^{T}$
- Continuous time: We index time continuously $t \in [0, T]$ with $T \leq \infty$
- Stochastic process in continuous time: a sequence of random variables indexed by t, $\{X_t\}_{t \geq 0}$

Markov chains

- A stochastic process $\{X_t\}$ has the *Markov property* if for all $k \geq 1$ and all t:

$$\mathbb{P}(X_{t+1} = x \mid X_t, X_{t-1}, \ldots, X_{t-k}) = \mathbb{P}(X_{t+1} = x \mid X_t)$$

- *State space* of the Markov process = set of events or states that it visits
- A Markov chain is a Markov process (stochastic process with Markov property) that visits a finite number of states (*discrete state space*)
- Simplest example: Individual i is randomly hit by earnings (employment) shocks and switches between $X_t \in \{X^L, X^H\}$

Difference equations

- We start with deterministic (non-random) dynamics and then conclude with stochastic (random) dynamics
- The *first-order linear difference equation* is defined by

$$x_{t+1} = bx_t + cz_t \tag{1}$$

  where $\{z_t\}$ is an exogenously given, bounded sequence
- For now, all objects are (real) scalars (easy to extend to vectors and matrices)
- Suppose we have an *initial condition* (i.e., given initial value) x_0
- When $c = 0$, (1) is a *time-homogeneous* difference equation
- When cz_t is constant for all t, (1) is an *autonomous* difference equation

Autonomous equations

- Consider the autonomous equation with $z_t = 1$
- A particular solution is the constant solution with $x_t = \frac{c}{1-b}$ when $b \neq 1$
- Such a point is called a *stationary point* or *steady state*
- General solution of the autonomous equation (for some constant x):

$$x_t = (x_0 - x)b^t + x \tag{2}$$

- Important question is long-run behavior (stability / convergence)
- When $|b| < 1$, (2) converges asymptotically to steady state x for any initial value x_0 (steady state x is globally stable)
- If $|b| > 1$, (2) explodes and is not stable (except when $x_0 = x$)

Difference equations: examples in macro

Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

- δ is depreciation and I_t is investment
- This is a *forward equation* and requires an initial condition K_0
- If $I_t = 0$ and $0 < \delta < 1$, $K_t \to 0$
- If $I_t = c$ constant, then K_t converges to $\frac{c}{\delta}$: $K_{t+1} = (1 - \delta)\frac{c}{\delta} + c = \frac{c}{\delta}$

Wealth dynamics:

$$a_{t+1} = R_t a_t + y_t - c_t$$

- R_t is the gross real interest rate, y_t is income, c_t is consumption
- This is a *forward equation* and requires an initial condition a_0
- We will study this as a *controlled* process because c_t will be chosen optimally
- Work out the following: $R_t = R$ and $y_t = y$ constant, and

$$c_t = \left(1 - \frac{1}{R}\right)\left(a_t + \sum_{s=t}^{\infty} R^{-(s-t)} y\right)$$

What are the dynamics of a_t?

Consumption Euler equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., c_T must converge to something
- Suppose there exists time T s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

New Keynesian Phillips curve:

$$\pi_t = \beta\pi_{t+1} + \kappa x_t$$

- π_t is inflation, κ is the slope of the PC, x_t is output gap
- This is a *backward equation* and requires a terminal condition
- NK analysis often studies the case $\lim_{T\to\infty} \pi_T = 0$ (0 inflation steady state)
- Suppose output gap $\{x_t\}$ exogenously given and there exists T s.t. for $t \geq T$, $\pi_t = 0$ and $x_t = 0$
- Then we solve backwards: $\pi_{T-1} = \beta\pi_T + \kappa x_{T-1}$
- The *initial value* π_0 is *endogenous*: backward equations solve for initial value π_0, forward equations solve for long run (e.g., K_T)

Stochastic difference equations

- Consider the process $\{X_t\}$ with

$$X_{t+1} = AX_t + Cw_{t+1} \tag{3}$$

  where w_{t+1} is an iid. process with $w_{t+1} \sim \mathcal{N}(0,1)$

- Equation (3) is a *first-order, linear stochastic difference equation*

- Let $\mathbb{E}_t$ the *conditional expectation* operator (conditional on time t information)

- For example:

$$\begin{aligned}\mathbb{E}_t(X_{t+1}) &= \mathbb{E}(X_{t+1} \mid X_t) = \mathbb{E}(AX_t + Cw_{t+1} \mid X_t) \\ &= AX_t + C\mathbb{E}(w_{t+1} \mid X_t) = AX_t + C\mathbb{E}(w_{t+1}) = AX_t\end{aligned}$$

- Rational expectations: agents' beliefs about stochastic processes are consistent with the true distribution of the process

- Key equation: wealth dynamics with income fluctuations:

$$a_{t+1} = R_t a_t + y_t - c_t,$$

  where y_t is a stochastic process

- Consumption Euler equation with uncertainty (e.g., stochastic income):

$$u'(C_t) = \beta R \mathbb{E}_t \Big[u'(C_{t+1}) \Big]$$

- New Keynesian Phillips curve with uncertainty (e.g., demand shocks):

$$\pi_t = \beta \mathbb{E}_t \Big[\pi_{t+1} \Big] + \kappa x_t$$

Continuous Time

Ordinary differential equations

- Consider the "discrete-time" equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- *Continuous-time limit*: consider the limit as $\Delta t \to 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$ is *autonomous* and dropping subscripts: $\dot{X} = g(X)$

- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

Boundary conditions

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval $t \in [0, 1]$. We call $[0, 1]$ the *state space*. $(0, 1)$ is the *interior of the state space* and $\{0, 1\}$ is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

Boundary conditions

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions
- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for X_t with some *initial condition* X_0
  - *Terminal value problems* instead specify X_T
- More broadly: We need sufficient information to characterize the function of interest along the boundary
- Types of boundary conditions: Dirichlet ($X_0 = c$), von-Neumann ($\frac{dX_0}{dt} = c$), reflecting boundaries, ...
- Boundary conditions are very important and can be very subtle (especially for PDEs)

Linear first-order ODEs

- Consider the equation:

$$\dot{X}(t) = a(t)X(t) + b(t) \tag{4}$$

- If $b(t) = 0$, (4) is a *homogeneous* equation, if $a(t) = a$ and $b(t) = b$ we say (4) has *constant coefficients*

- Start with $\dot{X}(t) = aX(t)$, divide by $X(t)$ and integrate with respect to t

$$\begin{aligned}\int \frac{\dot{X}(t)}{X(t)} dt &= \int a dt \\ \log X(t) + c_0 &= at + c_1 \\ X(t) &= Ce^{at}\end{aligned}$$

  where $C = e^{c_1 - c_0}$

- Pin down constant C by using the boundary condition (we need 1)

- Consider time-varying coefficient with $\dot{X}(t) = a(t)X(t)$ with initial condition $X(0) = \bar{x}$
- Dividing by $X(t)$, integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition: $C = \bar{x}$
- Finally, for $\dot{X}(t) = aX(t) + b$, we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

  after using change of variables $Y(t) = X(t) + \frac{b}{a}$
- Many results for systems of linear differential equations: $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}\boldsymbol{X}(t)$

Examples of differential equations in macro

Capital accumulation:

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps, $K_{t+1} = I_t + (1-\delta)K_t$
- With arbitrary Δ time step, $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned} K_{t+\Delta} &= K_t + \Delta(I_t - \delta K_t) \\ \frac{K_{t+\Delta} - K_t}{\Delta} &= I_t - \delta K_t \\ \dot{K}_t &= I_t - \delta K_t \end{aligned}$$

- Suppose $\{I_t\}_{t\geq 0}$ exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\dot{K}_t + \delta K_t = I_t$$
$$e^{\int_0^t \delta ds}\dot{K}_t + e^{\int_0^t \delta ds}\delta K_t = e^{\int_0^t \delta ds} I_t$$

- Notice that $\int_0^t \delta ds = \delta \int_0^t ds = \delta [s]_0^t = \delta(t-0) = \delta t$, so

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

- We have $e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = \frac{d}{dt}(K_t e^{\delta t})$, integrating:

$$K_t e^{\delta t} = \tilde{C} + \int_0^t e^{\delta s} I_s ds$$
$$K_t = C + \int_0^t e^{-\delta(t-s)} I_s ds$$

- Integrating constant solves initial condition: $C = K_0$

Wealth dynamics (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- r_t is the real rate of return on wealth, y_t is income, and c_t is consumption
- Structure of the equation similar to capital accumulation equation

Consumption Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

- The Euler equation typically takes the form of a *backward equation* and comes with a terminal condition (C_T) or transversality condition ($\lim_{T\to\infty} C_T$)
- Stationary point only if $r_t = \rho$
- Suppose we are at $r_t = r = \rho$ and a shock is realized. $r_0 > r$ what happens? $r_0 < r$ what happens?

New Keynesian Phillips curve:

$$\dot{\pi}_t = \rho\pi_t + \kappa x_t$$

- This is a backward equation that requires a terminal condition
- As in discrete time, we often consider the 0 inflation steady state with $\pi_T \to 0$
- Then we can solve (work this out yourselves):

$$\pi_t = -\kappa \int_t^{\infty} x_s ds$$

What are partial differential equations?

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...
  $\Longrightarrow$ increasingly used in economics

- Consider a function $u(x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$
- Partial derivatives of $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in u and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_{x_1 x_1} u, \ldots)$$

- The *order* of the PDE, is the order of the highest partial derivative
- Examples from physics
  - Heat equation: $\partial_t u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Wave equation: $\partial_{tt} u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Transport equation: $\partial_t u = \partial_x u$ (first-order, linear, homogeneous)
- Income distribution “solves heat equation”, wealth dynamics “solve transport equations”, dynamic programming often transport + heat

Solow growth model

- As before, $Y_t = C_t + I_t$ and
$$\dot{K}_t = Y_t - C_t - \delta K_t$$
- Representative firms operates neoclassical production function
$$Y_t = F(K_t, L_t, A_t)$$
- Normalize labor to $L_t = 1$ and hold TFP constant $A_t = A$
- We again assume constant savings rate: $Y_t - C_t = I_t = sY_t$
- Assume Cobb-Douglas $Y_t = AK_t^\alpha$ so equilibrium allocation
$$\dot{K}_t = sAK_t^\alpha - \delta K_t$$

- Steady state is given by

$$K_{ss} = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- Key equilibrium condition in $\dot{K}_t$ is *non-linear* — how to proceed?

- Let $X_t = K_t^{1-\alpha}$, then

$$\begin{aligned}
\dot{X}_t &= (1-\alpha)K_t^{-\alpha}\dot{K}_t \\
&= (1-\alpha)K_t^{-\alpha}(sAK_t^{\alpha} - \delta K_t) \\
&= (1-\alpha)sA - (1-\alpha)K_t^{1-\alpha}\delta \\
&= (1-\alpha)sA - (1-\alpha)\delta X_t
\end{aligned}$$

- Solution with initial condition X_0 (work this out):

$$X_t = X_{ss} + e^{-(1-\alpha)\delta t}\Big[X_0 - X_{ss}\Big], \quad \text{where } X_{ss} = \frac{sA}{\delta}$$

- Transition dynamics (rate of convergence) governed by $-(1-\alpha)\delta$

Continuous-time Markov chains

- **Definition.** Let $X = \{X_t\}_{t \geq 0}$ be a sequence of random variables taking values in a finite or countable state space $\mathcal{X}$. Then X is a *continuous-time Markov chain* if it satisfies the *Markov property*: For any sequence $0 \leq t_1 < t_2 < \ldots < t_n$ of times

$$\mathbb{P}(X_{t_n} = x \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \mathbb{P}(X_{t_n} = x \mid X_{t_{n-1}})$$

- Process X is *time-homogeneous* if the conditional probability does not depend on the current time, i.e., for $x, y \in \mathcal{X}$:

$$\mathbb{P}(X_{t+s} = x \mid X_s = y) = \mathbb{P}(X_t = x \mid X_0 = y)$$

- The *transition density* of process X is denoted $p(t, x \mid s, y)$ and is defined as

$$\mathbb{P}(X_t \in A \mid Y_s = y) = \int_A p(t, x \mid s, y) dx$$

  for any (Borel) set $A \subset \mathcal{X}$. In words: $p(t, x \mid s, y)$ is the probability (density) that process X_t ends up at $X_t = x$ at time t if it started at $X_s = y$ at time s

- *Conditional expectation* can be written as: $\mathbb{E}[f(X_t) \mid X_0 = y] = \int p(t, x \mid 0, y) f(x) dx$

Example:

- Consider the two-state employment process $z_t \in \{z^L, z^H\}$ with transition rates λ^{LH} (from L to H) and λ^{HL} (from H to L)

- The associated transition matrix (*generator*) is

$$\mathcal{A}^z = \begin{pmatrix} -\lambda^{LH} & \lambda^{LH} \\ \lambda^{HL} & -\lambda^{HL} \end{pmatrix}$$

- Interpretation: households transition *out of* state i at rate λ^{ij}

- Notice: In discrete time, Markov transition matrix rows sum to 1. Here, rows sum to 0 (*mass preservation*)

Brownian motion and SDEs

Definition. Brownian motion $\{B_t\}_{t \geq 0}$ is a stochastic process with properties:

(i) $B_0 = 0$

(ii) (*Independent increments*) For non-overlapping $0 \leq t_1 < t_2 < t_3 < t_4$, we have $B_{t_2} - B_{t_1}$ independent from $B_{t_4} - B_{t_3}$

(iii) (*Normal, stationary increments*) $B_t - B_s \sim \mathcal{N}(0, t-s)$ for any $0 \leq s < t$

(iv) (*Continuity of paths*) The sample paths of B_t are continuous

- Brownian motion is the only stochastic process with stationary and independent increments that's also continuous
- Einstein (1905) uses Brownian motion to model motion of particles
- Brownian motion is a Markov process
- $B_t \sim \mathcal{N}(0, t)$
- Brownian motion is nowhere differentiable

- Stochastic differential equations (SDEs) add noise / uncertainty to ordinary differential equations (ODEs)

- Start with $\dot{X}_t = \mu X_t$ with solution $X_t = X_0 e^{\mu t}$

- Rewrite as $dX_t = \mu X_t dt$ and "add noise" (using Brownian motion):

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

- Important: $dB_t \sim \mathcal{N}(0, dt)$ because

$$dB_t \approx B_{t+\Delta} - B_t \sim \mathcal{N}(0, t + \Delta - t = \mathcal{N}(0, \Delta)$$

  and now take $\Delta \to dt$ (continuous-time limit)

- Alternatively: $B_{t+\Delta} - B_t \sim \mathcal{N}(0, \Delta) \sim \epsilon_t \sqrt{\Delta}$ where $\epsilon_t \sim \mathcal{N}(0, 1)$. So as $\Delta \to dt$,

$$\mathbb{E}(dB_t) = \mathbb{E}(\epsilon_t \sqrt{dt}) = 0$$
$$\mathbb{E}[(dB_t)^2] = \mathbb{E}[(\epsilon_t \sqrt{dt})^2] = dt$$

- Suppose we have a function of Brownian motion, $X_t = f(t, B_t)$
- We know how Brownian motion dB_t evolves, what about dX_t? (That's like $\dot{X}_t$)
- Answer: **Ito's lemma** (core building block of stochastic calculus)

$$dX_t = df(t, B_t) = \partial_t f(t, B_t)dt + \frac{1}{2}\partial_{xx} f(t, B_t)dt + \partial_x f(t, B_t)dB_t$$

- Will not prove this, but heuristically: $(dt)^2 \to 0$ and $(dB_t)^2 \to dt$
- For example from previous slide, $dX_t = \mu X_t dt + \sigma X_t dB_t$:

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

- This is called *geometric Brownian motion* (used to model stock prices)
- *Ornstein-Uhlenbeck (OU) process* is a popular model for earnings risk and income fluctuations (think: continuous-time AR(1) process):

$$dz_t = \theta(\bar{z} - z_t)dt + \sigma dB_t$$

- Very important class is the **diffusion process**

- They take the form (not the formal definition)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

where $\mu(\cdot)$ is the *drift* and $\sigma(t, X_t)$ the *diffusion* (volatility) parameter of the process

- This is a shorthand for the (stochastic) integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$