Topics in Heterogeneous Agent Macro: Heterogeneous-Agent Models in Continuous Time

Lecture 5

Andreas Schaab

Outline

Paper: *Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach.*

Slides based on Ben's: `https://benjaminmoll.com/lectures/`

1. Textbook heterogeneous agent model (no aggregate shocks)
   Aiyagari-Bewley-Huggett model
2. Some theoretical results
3. Computations

What this lecture is about

- Many interesting questions require thinking about distributions
  - Why are income and wealth so unequally distributed?
  - Is there a trade-off between inequality and economic growth?
  - What are the forces that lead to the concentration of economic activity in a few very large firms?

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  - computations are challenging

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- Modeling distributions is hard
  - closed-form solutions are rare
  - computations are challenging
- Main idea: solving heterogeneous agent model = solving PDEs
  - main difference to existing continuos-time literature:
    handle models for which closed-form solutions do not exist

Solving het. agent model = solving PDEs

- More precisely: a system of two PDEs
    1. Hamilton-Jacobi-Bellman equation for individual choices
    2. Kolmogorov Forward equation for evolution of distribution

- Many well-developed methods for analyzing and solving these

  `https://benjaminmoll.com/codes/`

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- Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents
  1. heterogeneous households (Aiyagari, Bewley, Huggett,...)
  2. heterogeneous producers (Hopenhayn,...)

- can be extended to handle aggregate shocks (Krusell-Smith,...)

Computational Advantages relative to Discrete Time

1. Borrowing constraints only show up in boundary conditions
   - FOCs always hold with "$=$"

2. "Tomorrow is today"
   - FOCs are "static", compute by hand: $c^{-\gamma} = v_a(a, y)$

3. Sparsity
   - solving Bellman, distribution = inverting matrix
   - but matrices very sparse ("tridiagonal")
   - reason: continuous time $\Rightarrow$ one step left or one step right

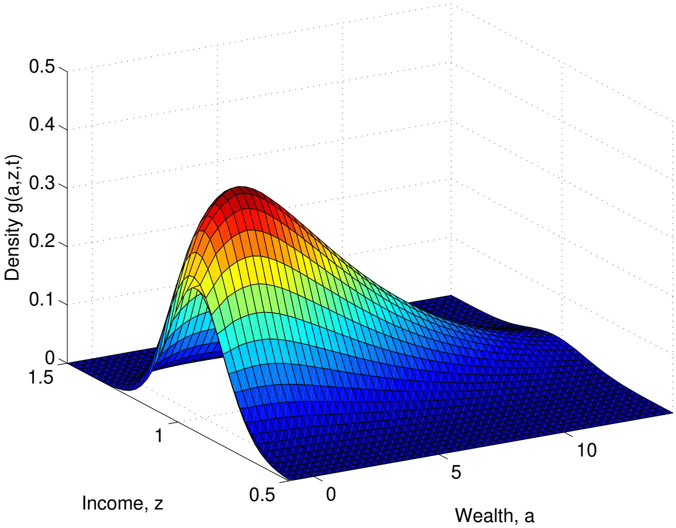
4. Two birds with one stone
   - tight link between solving (HJB) and (KF) for distribution
   - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
   - reason: diff. operator in (KF) is adjoint of operator in (HJB)

Real Payoff: extends to more general setups

- non-convexities
- stopping time problems
- multiple assets
- aggregate shocks

What you'll be able to do at end of this lecture

- Joint distribution of income and wealth in Aiyagari model



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- Experiment: effect of one-time redistribution of wealth

What you'll be able to do at end of this lecture

Video of convergence back to steady state

https://www.dropbox.com/s/op5u2nlifmmer2o/distribution_tax.mp4?dl=0

Model

Workhorse Model of Income and Wealth Distribution

Households are heterogeneous in their wealth a and income y, solve

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$\dot{a}_t = y_t + r a_t - c_t$$

$$y_t \in \{y_1, y_2\} \text{ Poisson with intensities } \lambda_1, \lambda_2$$

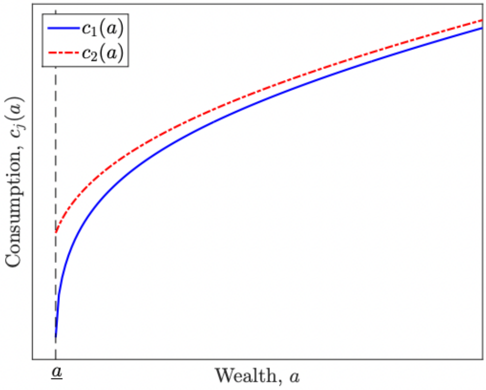
$$a_t \geq \underline{a}$$

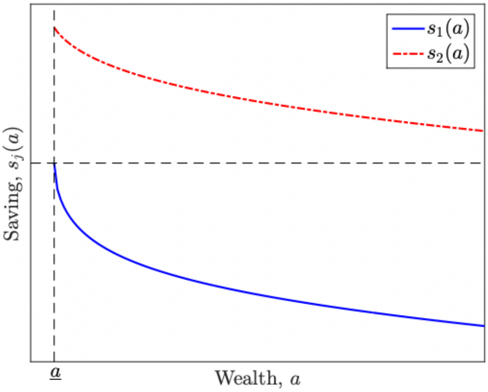
- c_t: consumption
- u: utility function, $u' > 0, u'' < 0$
- ρ: discount rate
- r_t : interest rate
- $\underline{a} \geq -y_1/r$: borrowing limit e.g. if $\underline{a} = 0$, can only save

Later: carries over to y_t = more general processes, e.g. diffusion

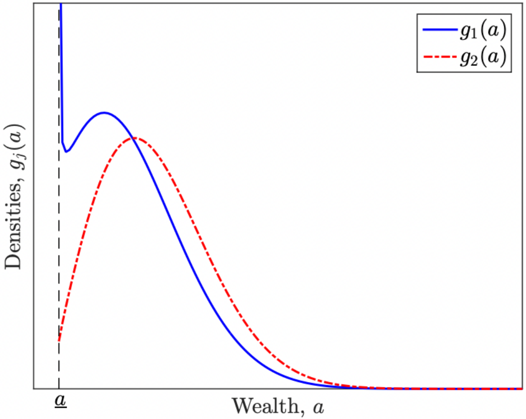
Equilibrium (Huggett): bonds in fixed supply, i.e. aggregate a_t = fixed

Typical Consumption and Saving Policy Functions





Typical Stationary Distribution



Equations for Stationary Equilibrium

$$\rho v_j(a) = \max_c \; u(c) + v_j'(a)(y_j + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)) \tag{HJB}$$

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$$s_j(a) = y_j + ra - c_j(a) = \text{saving policy function from (HJB)},$$

$$\int_{\underline{a}}^{\infty} (g_1(a) + g_2(a))da = 1, \quad g_1, g_2 \geq 0$$

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$$S(r) := \int_{\underline{a}}^{\infty} a g_1(a)da + \int_{\underline{a}}^{\infty} a g_2(a)da = B, \qquad B \geq 0 \tag{EQ}$$

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- The two PDEs (HJB) and (KF) together with (EQ) fully characterize stationary equilibrium ‣ Derivation of (HJB) ‣ (KF)

Transition Dynamics

- Needed whenever initial condition $\neq$ stationary distribution
- Equilibrium still coupled systems of HJB and KF equations...
- ... but now time-dependent: $v_j(a, t)$ and $g_j(a, t)$
- See next slides for equations
- Difficulty: the two PDEs run in opposite directions in time
  - HJB looks forward, runs backwards from terminal condition
  - KF looks backward, runs forward from initial condition

Transition Dynamics

$$B = \int_{\underline{a}}^{\infty} a g_1(a,t)da + \int_{\underline{a}}^{\infty} a g_2(a,t)da \tag{EQ}$$

$$\begin{aligned}\rho v_j(a,t) = &\max_c \ u(c) + \partial_a v_j(a,t)(y_j + r(t)a - c) \\ &+ \lambda_j(v_{-j}(a,t) - v_j(a,t)) + \partial_t v_j(a,t),\end{aligned} \tag{HJB}$$

$$\partial_t g_j(a,t) = -\partial_a[s_j(a,t)g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t), \tag{KF}$$

$$s_j(a,t) = y_j + r(t)a - c_j(a,t), \quad c_j(a,t) = (u')^{-1}(\partial_a v_j(a,t)),$$

$$\int_{\underline{a}}^{\infty} (g_1(a,t) + g_2(a,t))da = 1, \quad g_1, g_2 \geq 0$$

- Given initial condition $g_{j,0}(a)$, the two PDEs (HJB) and (KF) together with (EQ) fully characterize equilibrium.
- Notation: for any function f, $\partial_x f$ means $\frac{\partial f}{\partial x}$

Borrowing Constraints?

- Q: where is borrowing constraint $a \geq \underline{a}$ in (HJB)?
- A: "in" boundary condition

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- Result: v_j must satisfy
$$v_j'(\underline{a}) \geq u'(y_j + r\underline{a}), \quad j = 1,2 \tag{BC}$$
- Derivation:
  - the FOC still holds at the borrowing constraint
$$u'(c_j(\underline{a})) = v_j'(\underline{a}) \tag{FOC}$$
  - for borrowing constraint not to be violated, need
$$s_j(\underline{a}) = y_j + r\underline{a} - c_j(\underline{a}) \geq 0 \tag{$*$}$$
  - (FOC) and $(*)$ $\Rightarrow$ (BC).

Plan

- New theoretical results:

    1. analytics: consumption, saving, MPCs of the poor
    2. closed-form for wealth distribution with 2 income types
    3. unique stationary equilibrium if IES ≥ 1 (sufficient condition)
    4. “soft" borrowing constraints

    Note: for 1., 2. and 4. analyze partial equilibrium with $r < \rho$

- Computational algorithm:

    - problems with non-convexities
    - transition dynamics

Theoretical Results

Result 1: Consumption, Saving Behavior of the Poor

Consumption/saving behavior near borrowing constraint depends on:

1. tightness of constraint
2. properties of u as $c \to 0$

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Assumption 1:

As $a \to \underline{a}$, coefficient of absolute risk aversion $R(c) := -u''(c)/u'(c)$ remains finite

$$-\frac{u''(y_1 + r\underline{a})}{u'(y_1 + r\underline{a})} < \infty$$

- will show: A1 $\Rightarrow$ borrowing constraint “matters" (in fact, it’s an $\Leftrightarrow$)

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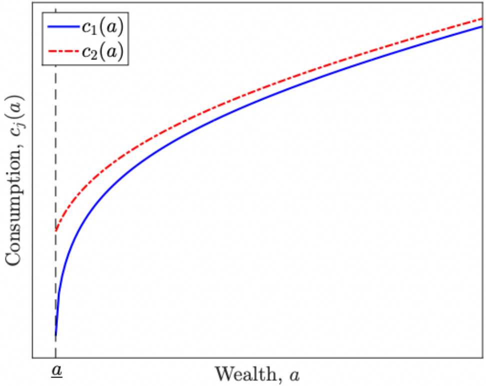
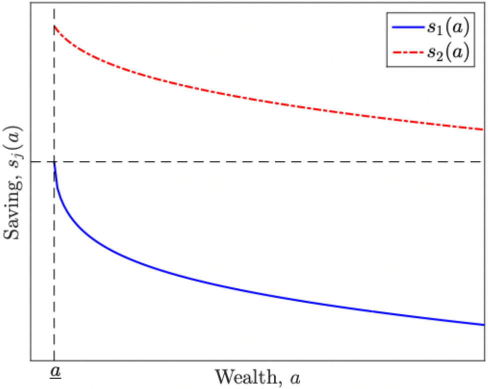
How to read A1?

- “standard" utility functions, e.g. CRRA, satisfy $-\frac{u''(0)}{u'(0)} = \infty$
- hence for standard utility functions A1 equivalent to $\underline{a} > -y_1/r$, i.e. constraint matters if it is tighter than “natural borrowing constraint"
- but weaker: e.g. if $u'(c) = e^{-\theta c}$, constraint matters even if $\underline{a} = -\frac{y_1}{r}$

Result 1: Consumption, Saving Behavior of the Poor

Rough version of Proposition: under A1 policy functions look like this





Result 1: Consumption, Saving Behavior of the Poor

Proposition: Assume $r < \rho, y_1 < y_2$ and that A1 holds.
Then saving and consumption policy functions close to $a = \underline{a}$ satisfy

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1(a) \sim y_1 + ra + \sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1'(a) \sim r + \frac{1}{2}\sqrt{\frac{\nu_1}{2(a - \underline{a})}}$$

where $\nu_1 =$ constant that depends on $r, \rho, \lambda_1, \lambda_2$ etc – see next slide

Note: "$f(a) \sim g(a)$" means $\lim_{a \to \underline{a}} f(a)/g(a) = 1$, "$f$ behaves like g close to $\underline{a}$"

Result 1: Consumption, Saving Behavior of the Poor

Corollary: The wealth of worker who keeps y_1 converges to borrowing constraint in finite time at speed governed by ν_1:

$$a(t) - \underline{a} \sim \frac{\nu_1}{2}\left(T - t\right)^2, \quad T := \text{“hitting time"} = \sqrt{\frac{2(a_0 - \underline{a})}{\nu_1}}, \quad 0 \leq t \leq T$$

Proof: integrate $\dot{a}(t) = -\sqrt{2\nu_1}\sqrt{a(t) - \underline{a}}$

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Proof: integrate $\dot{a}(t) = -\sqrt{2\nu_1}\sqrt{a(t) - \underline{a}}$

And have analytic solution for speed

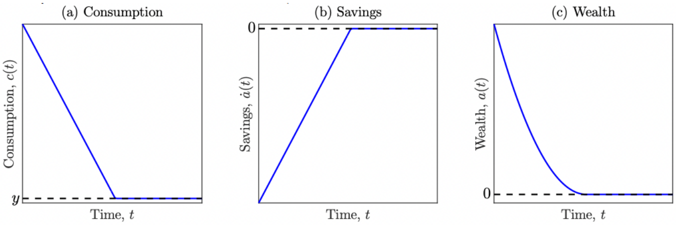
$$\nu_1 = \frac{(\rho - r)u'(\underline{c}_1) + \lambda_1(u'(\underline{c}_1) - u'(\underline{c}_2))}{-u''(\underline{c}_1)}$$
$$\approx (\rho - r)\mathrm{IES}(\underline{c}_1)\underline{c}_1 + \lambda_1(\underline{c}_2 - \underline{c}_1)$$

Intuition for Result 1: Two Special Cases

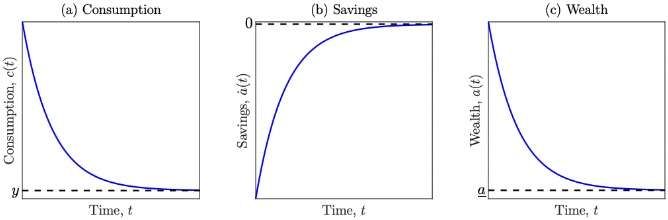
- What's the role of A1? And why the square root?
- Explain using two special cases with analytic solution
- Both cases: no income uncertainty

Intuition for Result 1: Two Special Cases

- Special case 1: A1 holds, hit constraint



- Special case 2: A1 violated, approach constraint asymptotically



Special case 1: hit constraint

- exponential utility $u'(c) = e^{-\theta c}$, tight constraint

$$\dot{c} = \frac{1}{\theta}(r - \rho), \qquad \dot{a} = y + ra - c, \qquad a \geq 0$$

- satisfies A1: $-\frac{u''(y)}{u'(y)} = \theta < \infty$.

Special case 2: only approach constraint asymptotically

- CRRA utility $u'(c) = c^{-\gamma}$, loose constraint

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(r - \rho), \qquad \dot{a} = y + ra - c, \qquad a \geq \underline{a} = -\frac{y}{r}$$

- violates A1: $-\frac{u''(y+ra)}{u'(y+ra)} \to \infty$ as $a \to \underline{a}$.

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$$c(t) = y + (r + \eta)a(t), \quad a(t) - \underline{a} = (a_0 - \underline{a})e^{-\eta t}, \quad \eta := \frac{\rho - r}{\gamma}$$

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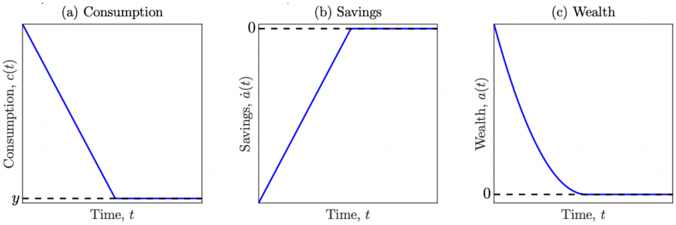

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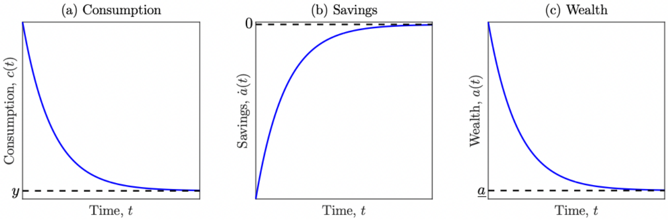

Consumption, Saving Behavior of the Rich

- Skip this today. See paper.

Marginal Propensities to Consume and Save

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- **Lemma:** If τ sufficiently small so that no income switches, then

$$\mathrm{MPC}_{1,\tau}(a) \sim \min\{\tau c_1'(a), 1 + \tau r\}$$

Note: $\mathrm{MPC}_{1,\tau}(a)$ bounded above even though $c_1'(a) \to \infty$ as $a \downarrow \underline{a}$

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- If new income draws before τ, no more analytic solution
- But straightforward computation using **Feynman-Kac formula**

Using the Formula for ν_1 to Better Understand MPCs

- Consider dependence of low-income type's $\mathrm{MPC}_{1,\tau}(a)$ on y_1

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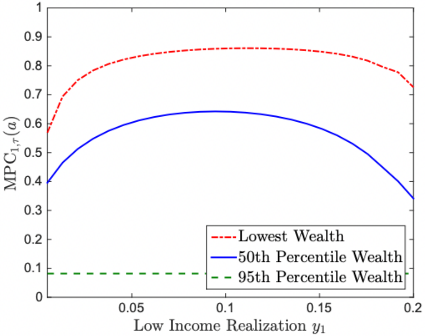

- Why hump-shaped?!?

Using the Formula for ν_1 to Better Understand MPCs

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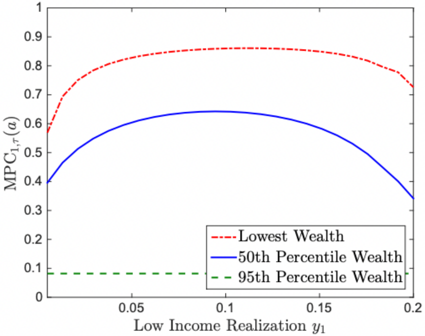


- Why hump-shaped?!? Answer: $\mathrm{MPC}_{1,\tau}(a)$ proportional to

$$c_1'(a) \sim r + \frac{1}{2}\sqrt{\frac{\nu_1}{2(a-\underline{a})}}, \quad \nu_1 \approx (\rho - r)\frac{1}{\gamma}\underline{c}_1 + \lambda_1(\underline{c}_2 - \underline{c}_1)$$

and note that $\underline{c}_1 = y_1 + r\underline{a}$

- Can see: increase in y_1 has two offsetting effects

Result 2: Stationary Wealth Distribution

- Recall equation for stationary distribution

$$0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a) \tag{KF}$$

- **Lemma:** the solution to (KF) is

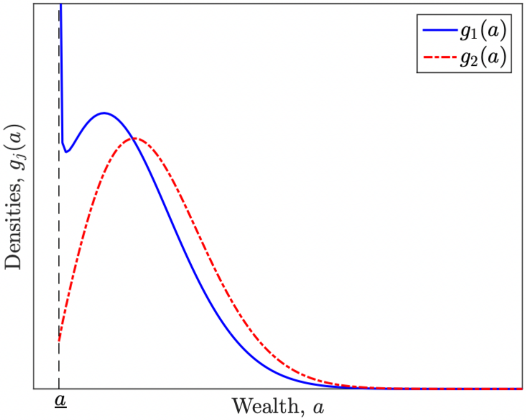
$$g_i(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}dx\right)\right)$$

  with κ_1, κ_2 pinned down by g_j's integrating to one

- Features of wealth distribution:
  - Dirac point mass of type y_1 individuals at constraint $G_1(\underline{a}) > 0$
  - thin right tail: $g(a) \sim \xi(a_{\max} - a)^{\lambda_2/\zeta_2 - 1}$, i.e. not Pareto
  - see paper for more

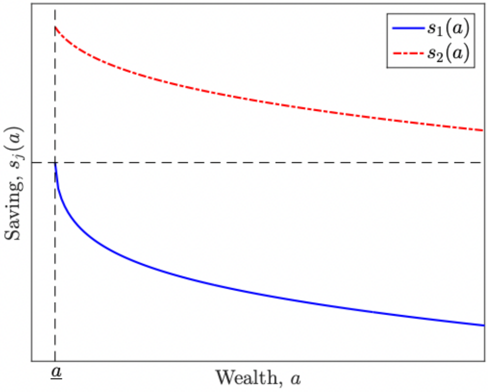
- Later in paper: extension with Pareto tail (Benhabib-Bisin-Zhu)

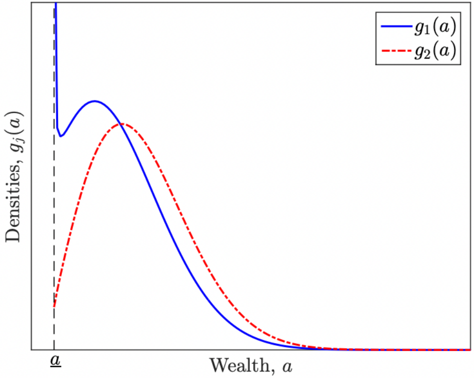
Result 2: Stationary Wealth Distribution



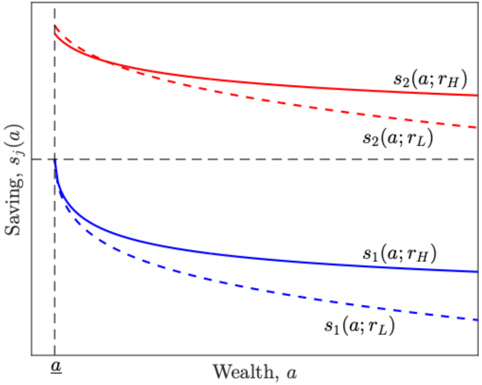
Note: in numerical solution, Dirac mass = finite spike in density

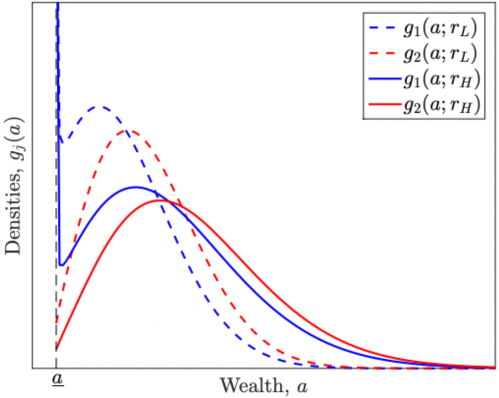
General Equilibrium: Existence and Uniqueness



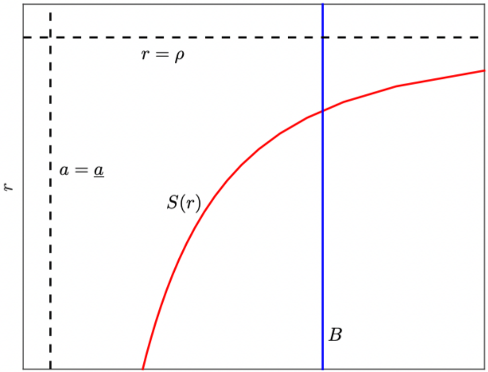


Increase in r from r_L to $r_H > r_L$



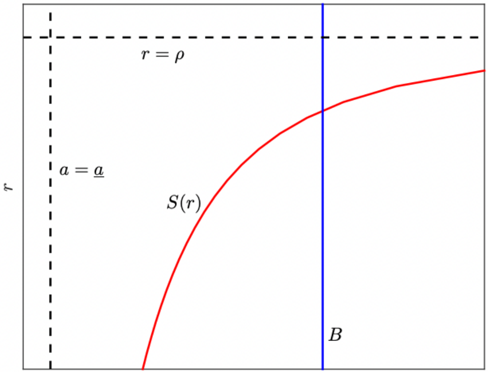


Stationary Equilibrium



Asset Supply $S(r) = \int_{\underline{a}}^{\infty} a g_1(a;r)da + \int_{\underline{a}}^{\infty} a g_2(a;r)da$

Stationary Equilibrium



Asset Supply $\quad S(r) = \int_{\underline{a}}^{\infty} a g_1(a;r)da + \int_{\underline{a}}^{\infty} a g_2(a;r)da$

- **Proposition:** a stationary equilibrium exists

Result 3: Uniqueness of Stationary Equilibrium

Proposition: Assume that the IES is weakly greater than one

$$\text{IES}(c) := -\frac{u'(c)}{u''(c)c} \geq 1 \quad \text{for all } c \geq 0,$$

and that there is no borrowing $a \geq 0$. Then:

1. Individual consumption $c_j(a;r)$ is strictly decreasing in r
2. Individual saving $s_j(a;r)$ is strictly increasing in r
3. $r \uparrow \Rightarrow$ CDF $G_j(a;r)$ shifts right in FOSD sense
4. Aggregate saving $S(r)$ is strictly increasing $\Rightarrow$ **uniqueness**

Note: holds for any labor income process, not just two-state Poisson

Uniqueness: Proof Sketch

- Parts 2 to 4 direct consequences of part 1 ($c_j(a;r)$ decreasing in r)

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Uniqueness: Proof Sketch

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- **Lemma** (Olivi, 2017): c response to change in r is

$$\frac{\partial c_j(a)}{\partial r} = \underbrace{\frac{1}{u''(c_0)} \mathbb{E}_0 \int_0^T e^{-\int_0^t \xi_s ds} u'(c_t) dt}_{\text{substitution effect} < 0} + \underbrace{\frac{1}{u''(c_0)} \mathbb{E}_0 \int_0^T e^{-\int_0^t \xi_s ds} u''(c_t) a_t \partial_a c_t dt}_{\text{income effect} > 0}$$

where $\xi_t := \rho - r + \partial_a c_t$ and $T := \inf\{t \geq 0 | a_t = 0\} =$ time at which hit 0

Uniqueness: Proof Sketch

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- **Lemma** (Olivi, 2017): c response to change in r is

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where $\xi_t := \rho - r + \partial_a c_t$ and $T := \inf\{t \geq 0 | a_t = 0\}$ = time at which hit 0

- We show: IES$(c) := -\frac{u'(c)}{u''(c)c} \geq 1 \Rightarrow$ substitution effect dominates
  $\Rightarrow \partial c_j(a)/\partial r < 0$, i.e. consumption decreasing in r

Result 4: “Soft" Borrowing Constraints

- Empirical wealth distributions:
  1. individuals with positive wealth
  2. individuals with negative wealth
  3. spike at close to zero net worth

- Does not square well with Aiyagari-Bewley-Huggett model

Result 4: “Soft" Borrowing Constraints

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- Simple solution: “soft" borrowing constraint = wedge between borrowing and saving r

- Paper: first theoretical characterization of “soft" constraint
  - square root formulas
  - Dirac mass at zero net worth

Computations

Computational Advantages relative to Discrete Time

1. Borrowing constraints only show up in boundary conditions
   - FOCs always hold with "$=$"
2. "Tomorrow is today"
   - FOCs are "static", compute by hand: $c^{-\gamma} = v_j'(a)$
3. Sparsity
   - solving Bellman, distribution = inverting matrix
   - but matrices very sparse ("tridiagonal")
   - reason: continuous time $\Rightarrow$ one step left or one step right
4. Two birds with one stone
   - tight link between solving (HJB) and (KF) for distribution
   - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
   - reason: diff. operator in (KF) is adjoint of operator in (HJB)

Computations for Heterogeneous Agent Model

- Hard part: HJB equation
- Easy part: KF equation. Once you solved HJB equation, get KF equation “for free"

Computations for Heterogeneous Agent Model

- Hard part: HJB equation
- Easy part: KF equation. Once you solved HJB equation, get KF equation “for free"
- System to be solved

$$
\begin{aligned}
\rho v_1(a) &= \max_c \; u(c) + v_1'(a)(y_1 + ra - c) + \lambda_1(v_2(a) - v_1(a)) \\
\rho v_2(a) &= \max_c \; u(c) + v_2'(a)(y_2 + ra - c) + \lambda_2(v_1(a) - v_2(a)) \\
0 &= -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a) \\
0 &= -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_2 g_2(a) + \lambda_1 g_1(a) \\
1 &= \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da \\
B &= \int_{\underline{a}}^{\infty} a g_1(a)da + \int_{\underline{a}}^{\infty} a g_2(a)da := S(r)
\end{aligned}
$$

Bird's Eye View of Algorithm for Stationary Equilibria

- Use finite difference method
- Discretize state space $a_i, i = 1, ..., I$ with step size Δa

$$v_j'(a_i) \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta a} \quad \text{or} \quad \frac{v_{i,j} - v_{i-1,j}}{\Delta a}$$

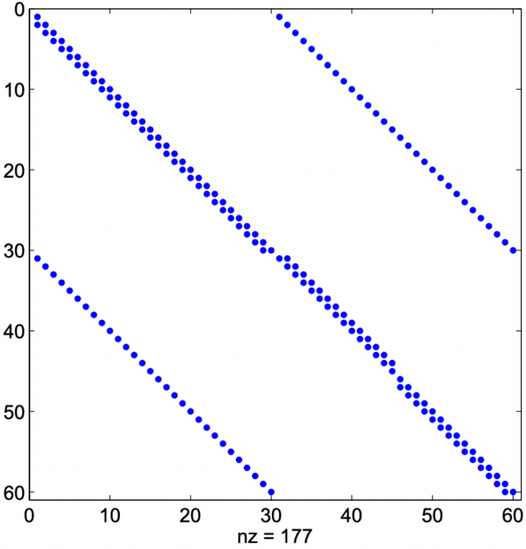
$$\text{Denote} \quad \mathbf{v} = \begin{bmatrix} v_1(a_1) \\ \vdots \\ v_2(a_I) \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1(a_1) \\ \vdots \\ g_2(a_I) \end{bmatrix}, \quad \text{dimension} = 2I \times 1$$

- End product of FD method: system of sparse matrix equations

$$\begin{aligned} \rho \mathbf{v} &= \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r)\mathbf{v} \\ \mathbf{0} &= \mathbf{A}(\mathbf{v}; r)^{\mathrm{T}} \mathbf{g} \\ B &= S(\mathbf{g}; r) \end{aligned}$$

which is easy to solve on computer

Visualization of A (output of `spy(A)` in Matlab)



Transition Dynamics: Intuition in Growth Model

- Next two slides: intuition for algorithm in rep agent growth model
- In three slides: solve Huggett model in exactly analogous fashion
- Equilibrium in growth model is solution to:

$$
\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= \frac{1}{\gamma}(r(t) - \rho) \\
\dot{K}(t) &= w(t) + r(t)K(t) - C(t) \\
w(t) &= (1-\alpha)K(t)^{\alpha}, \quad r(t) = \alpha K(t)^{\alpha-1} \\
K(0) &= K_0, \quad \lim_{T\to\infty} C(T) = C_{\infty}
\end{aligned}
$$

- For numerical solution, solve on $[0, T]$ for large T with $C(T) = C_{\infty}$
- Define $w(r) = (1-\alpha)(\alpha/r)^{\frac{\alpha}{1-\alpha}} \Rightarrow$ only one price, $r(t)$

Transition Dynamics: Intuition in Growth Model

Equilibrium is therefore solution to

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\gamma}(r(t) - \rho), \quad C(T) = C_\infty \tag{1}$$

$$\dot{K}(t) = w(r(t)) + r(t)K(t) - C(t), \quad K(0) = K_0 \tag{2}$$

$$r(t) = \alpha K(t)^{\alpha - 1}$$

Define excess capital demand $D_t(\{r(s)\}_{s\geq 0})$ as follows:

1. given $\{r(s)\}_{s\geq 0}$, solve (1) backward in time
2. given $\{C(s)\}_{s\geq 0}$, solve (2) forward in time
3. given $\{K(s)\}_{s\geq 0}$, compute $D_t(\{r(s)\}_{s\geq 0}) = \alpha K(t)^{\alpha-1} - r(t)$

Then find $\{r(s)\}_{s\geq 0}$ such that

$$D_t(\{r(s)\}_{s\geq 0}) = 0 \quad \text{all } t$$

Different options for solving this: (i) ad hoc, (ii) Newton-based methods

Transition Dynamics in Huggett Model

- Natural generalization of algorithm for stationary equilibrium
  - denote $v_{i,j}^n = v_i(a_j, t^n)$ and stack into $\mathbf{v}^n$
  - denote $g_{i,j}^n = g_i(a_j, t^n)$ and stack into $\mathbf{g}^n$

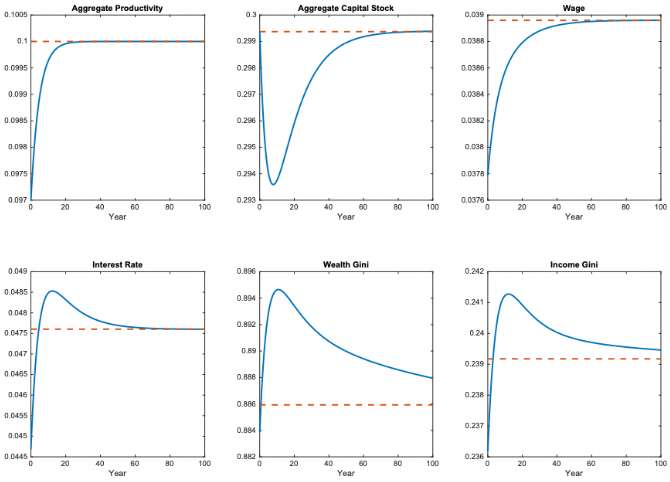
- System of sparse matrix equations for transition dynamics:

$$\begin{aligned}
\rho \mathbf{v}^n &= \mathbf{u}(\mathbf{v}^{n+1}) + \mathbf{A}(\mathbf{v}^{n+1}; r^n)\mathbf{v}^n + \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t}, \\
\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} &= \mathbf{A}(\mathbf{v}^n; r^n)^{\mathrm{T}} \mathbf{g}^{n+1}, \\
B &= S(\mathbf{g}^n; r^n),
\end{aligned}$$

- Terminal condition for $\mathbf{v}$: $\mathbf{v}^N = \mathbf{v}_\infty$ (steady state)
- Initial condition for $\mathbf{g}$: $\mathbf{g}^1 = \mathbf{g}_0$.

An MIT Shock in the Aiyagari Model

- Production: $Y_t = F_t(K,L) = A_t K^{\alpha} L^{1-\alpha}, dA_t = \nu(\bar{A} - A_t)dt$



Generalizations and Applications

A Model with a Continuum of Income Types

- Assume idiosyncratic income follows diffusion process

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$$

- Reflecting barriers at $\underline{y}$ and $\bar{y}$
- Value function, distribution are now functions of 2 variables:

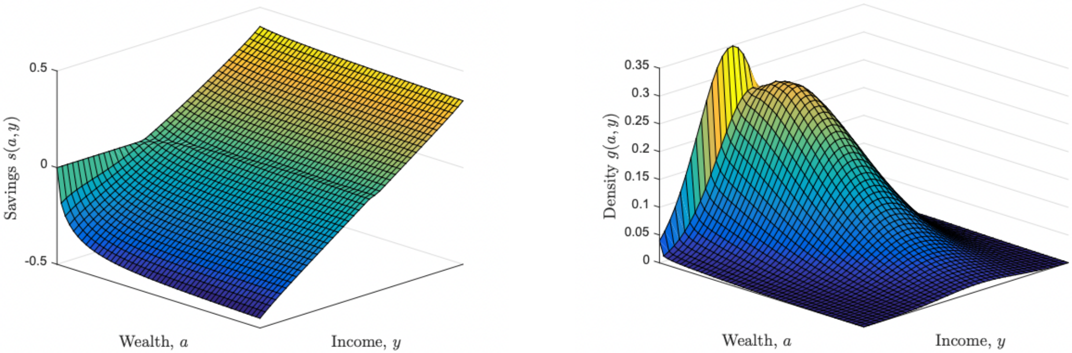
$$v(a, y) \quad \text{and} \quad g(a, y)$$

- $\Rightarrow$ HJB and KF equations are now PDEs in (a, y)-space

It doesn’t matter whether you solve ODEs or PDEs

$\Rightarrow$ everything generalizes

Saving Policy Function and Stationary Distribution



- Analytic characterization of MPCs: $c(a,y) \sim \sqrt{2\nu(y)}\sqrt{a-\underline{a}}$ with

$$\nu(y) = (\rho - r)\text{IES}(\underline{c}(y))\underline{c}(y) + \left(\mu(y) - \frac{\sigma^2(y)}{2}\mathcal{P}(\underline{c}(y))\right)\underline{c}'(y) + \frac{\sigma^2(y)}{2}\underline{c}''(y)$$

where $\mathcal{P}(c) := -u'''(c)/u''(c) =$ absolute prudence, and $\underline{c}(y) = c(\underline{a},y)$

Other Applications – see Paper

- Non-convexities: indivisible housing, mortgages, poverty traps
- Fat-tailed wealth distribution
- Multiple assets with adjustment costs (Kaplan-Moll-Violante)
- Stopping time problems

Appendix

Derivation of Poisson KF Equation ‣ Back

- Work with CDF (in wealth dimension)

$$G_j(a,t) := \Pr(\tilde{a}_t \leq a, \tilde{y}_t = y_i)$$

- Income switches from y_j to y_{-j} with probability $\Delta\lambda_j$
- Over period of length Δ, wealth evolves as $\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_j(\tilde{a}_t)$
- Similarly, answer to question “where did $\tilde{a}_{t+\Delta}$ come from?" is

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_j(\tilde{a}_{t+\Delta})$$

- Momentarily ignoring income switches and assuming $s_j(a) < 0$

$$\Pr(\tilde{a}_{t+\Delta} \leq a) = \underbrace{\Pr(\tilde{a}_t \leq a)}_{\text{already below } a} + \underbrace{\Pr(a \leq \tilde{a}_t \leq a - \Delta s_j(a))}_{\text{cross threshold } a} = \Pr(\tilde{a}_t \leq a - \Delta s_j(a))$$

- Fraction of people with wealth below a evolves as

$$\begin{aligned}\Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j) = (1 - \Delta\lambda_j) \Pr(\tilde{a}_t \leq a - \Delta s_j(a), \tilde{y}_t = y_j) \\ + \Delta\lambda_j \Pr(\tilde{a}_t \leq a - \Delta s_{-j}(a), \tilde{y}_t = y_{-j})\end{aligned}$$

- Intuition: if have wealth $< a - \Delta s_j(a)$ at t, have wealth $< a$ at $t + \Delta$

Derivation of Poisson KF Equation

- Subtracting $G_j(a,t)$ from both sides and dividing by Δ

$$\frac{G_j(a,t+\Delta)-G_j(a,t)}{\Delta} = \frac{G_j(a-\Delta s_i(a),t)-G_j(a,t)}{\Delta} - \lambda_j G_j(a-\Delta s_j(a),t) + \lambda_{-j} G_{-j}(a-\Delta s_{-j}(a),t)$$

- Taking the limit as $\Delta \to 0$

$$\partial_t G_j(a,t) = -s_j(a)\partial_a G_j(a,t) - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t)$$

where we have used that

$$\begin{aligned}\lim_{\Delta\to 0}\frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{\Delta} &= \lim_{x\to 0}\frac{G_j(a-x,t)-G_j(a,t)}{x}s_j(a)\\ &= -s_j(a)\partial_a G_j(a,t)\end{aligned}$$

- Intuition: if $s_j(a)<0, \Pr(\tilde{a}_t \le a, \tilde{y}_t = y_j)$ increases at rate $g_j(a,t)$
- Differentiate w.r.t. a and use $g_j(a,t) = \partial_a G_j(a,t) \Rightarrow$

$$\partial_t g_j(a,t) = -\partial_a[s_j(a,t)g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t)$$

Accuracy of Finite Difference Method?

Two experiments:

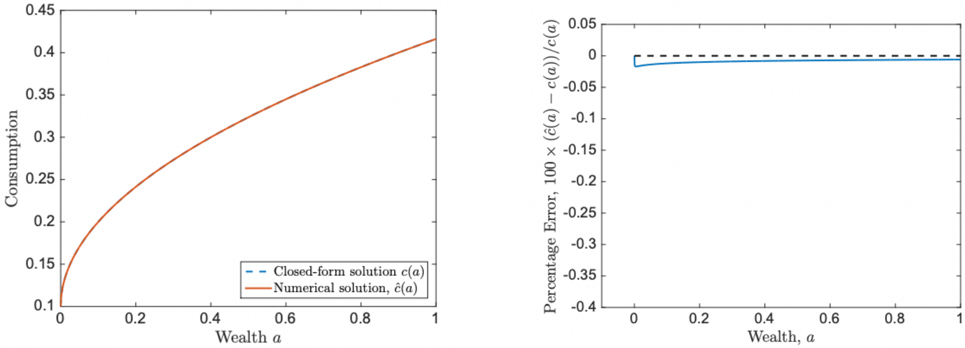
1. special case: comparison with closed-form solution
2. general case: comparison with numerical solution computed using very fine grid

Accuracy of Finite Difference Method, Experiment 1

- Recall: get closed-form solution if
  - exponential utility $u'(c) = c^{-\theta c}$
  - no income risk and $r = 0$ so that $\dot{a} = y - c$ (and $a \geq 0$)

$$\Rightarrow \qquad s(a) = -\sqrt{2\nu a}, \qquad c(a) = y + \sqrt{2\nu a}, \qquad \nu := \frac{\rho}{\theta}$$

- Accuracy with $I = 1000$ grid points ($\widehat{c}(a)$ = numerical solution)



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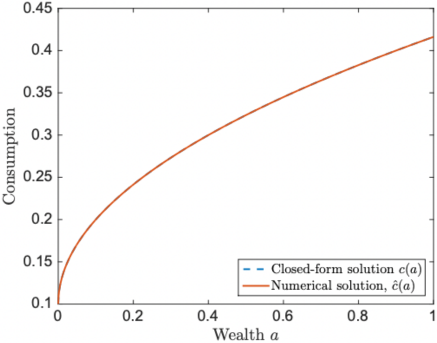
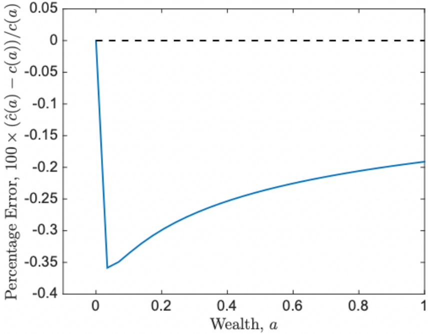
$$\Rightarrow \qquad s(a) = -\sqrt{2\nu a}, \qquad c(a) = y + \sqrt{2\nu a}, \qquad \nu := \frac{\rho}{\theta}$$

- Accuracy with $I = 30$ grid points ($\widehat{c}(a)$ = numerical solution)





Accuracy of Finite Difference Method, Experiment 2

- Consider HJB equation with continuum of income types

$$\rho v(a,y) = \max_c \; u(c) + \partial_a v(a,y)(y + ra - c) + \mu(y)\partial_y v(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy} v(a,y)$$

- Compute twice:
  1. with very fine grid: $I = 3000$ wealth grid points
  2. with coarse grid: $I = 300$ wealth grid points

  then examine speed-accuracy tradeoff (accuracy = error in agg C)

	Speed (in secs)	Aggregate C
$I = 3000$	0.916	1.1541
$I = 300$	0.076	1.1606
row 2/row 1	0.0876	1.005629

- i.e. going from $I = 3000$ to $I = 300$ yields $> 10\times$ speed gain and 0.5% reduction in accuracy (but note: even $I = 3000$ very fast)
- Other comparisons? Feel free to play around with `HJB_accuracy2.m`