Topics in Heterogeneous Agent Macro: Dynamic Programming in Continuous Time

Lecture 2

Andreas Schaab

Outline

Part 1: deterministic dynamics

1. Neoclassical growth model in continuous time
2. Calculus of variations
3. Optimal control theory
4. Simple example
5. Hamilton-Jacobi-Bellman (HJB) equation
6. First-order condition for consumption
7. Envelope condition and Euler equation
8. Connection between calculus of variations / optimal control and HJBs
9. Boundary conditions: no-borrowing in the wealth / capital dimension

Outline

Part 2: stochastic dynamics

1. What is the generator of a stochastic process?
2. Stochastic neoclassical growth model
3. Stochastic neoclassical growth with diffusion process
4. Stochastic neoclassical growth with Poisson process
5. Many examples

Part 1: deterministic dynamics

1. Neoclassical growth model in continuous time

- The lifetime value of the representative household is

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now
- This is the **sequence problem** in continuous time

2. Calculus of variations [*skip*]

- Resources:
  - LeVeque: Finite Difference Methods for Ordinary and Partial Differential Equations
  - Kamien and Schwartz: Dynamic Optimization
  - Gelfand and Fomin: Calculus of Variations

- This dynamic optimization problem is associated with the Lagrangian

$$L = \int_0^\infty e^{-\rho t} \left[u(c_t) + \mu_t \left(F(k_t) - \delta k_t - c_t - \dot{k}_t \right) \right] dt$$

- μ_t is the Lagrange multiplier on the capital accumulation ODE

- What do we do with $\dot{k}_t$??

- Integrate by parts:

$$\int_0^\infty e^{-\rho t}\mu_t \dot{k}_t dt = e^{-\rho t}\mu_t k_t\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(e^{-\rho t}\mu_t\right)k_t dt$$
$$= -\mu_0 k_0 + \int_0^\infty e^{-\rho t}\rho\mu_t k_t dt - \int_0^\infty e^{-\rho t}\dot{\mu}_t k_t dt$$

- Plugging into Lagrangian:

$$L = \int_0^\infty e^{-\rho t}\left[u(c_t) + \mu_t\left(F(k_t) - \delta k_t - c_t\right) - \rho\mu_t k_t + \dot{\mu}_t k_t\right]dt + \mu_0 k_0$$

- What have we accomplished?

- Notice $\mu_0 k_0$, this is crucial. What's intuition?

$$L = \int_0^\infty e^{-\rho t}\Big[u(c_t) + \mu_t\Big(F(k_t) - \delta k_t - c_t\Big) - \rho\mu_t k_t + \dot{\mu}_t k_t\Big]dt + \mu_0 k_0$$

- The planner optimizes over paths $\{c_t\}$ and $\{k_t\}$

- At an optimum, there cannot be *any* small perturbation in these paths that the planner finds preferable

- Let $\{c_t\}$ and $\{k_t\}$ be *candidate* optimal paths. Consider $\hat{c}_t = c_t + \alpha h_t^c$ and $\hat{k}_t = k_t + \alpha h_t^k$ for arbitrary functions h_t^c and h_t^k

$$L(\alpha) = \int_0^\infty e^{-\rho t}\Big[u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta\alpha h_t^k - c_t - \alpha h_t^c\Big) \\ - \rho\mu_t(k_t + \alpha h_t^k) + \dot{\mu}_t(k_t + \alpha h_t^k)\Big]dt + \mu_0(k_0 + \alpha h_0^k)$$

- What about *boundary conditions*? At $t = 0$, capital stock is fixed (k_0 given) while consumption is free. So must have: $h_0^k = 0$ while h_0^c is free

Necessary condition for optimality: $\frac{d}{d\alpha}L(0) = 0$

$$L(\alpha) = \int_0^\infty e^{-\rho t}\Big[u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta\alpha h_t^k - c_t - \alpha h_t^c\Big)$$
$$- \rho\mu_t(k_t + \alpha h_t^k) + \dot{\mu}_t(k_t + \alpha h_t^k)\Big]dt + \mu_0(k_0 + \alpha h_0^k)$$

Work this out yourselves (many times, in many applications!)

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\Big[u'(c_t)h_t^c + \mu_t\Big(F'(k_t)h_t^k - \delta h_t^k - h_t^c\Big)$$
$$- \rho\mu_t h_t^k + \dot{\mu}_t h_t^k\Big]dt + \mu_0 h_0^k$$

where $h_0^k = 0$ because k_0 is fixed

Group terms:

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\left[\Big(u'(c_t) - \mu_t\Big)h_t^c + \Big(\mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t\Big)h_t^k\right]dt$$

Fundamental Theorem of the Calculus of Variations: Since h_t^c and h_t^k were arbitrary, we must have *pointwise*

$$0 = u'(c_t) - \mu_t$$
$$0 = \mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t$$

Proposition. (Euler equation for marginal utility)

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dot{u}_{c,t}}{u_{c,t}} = \rho - F'(k_t) + \delta = \rho - r_t$$

- We have now solved the neoclassical growth model in continuous time. Its solution is given by a system of two ODEs.

- Suppose $u(c) = \log(c)$ and $F(k) = k^\alpha$, then:

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \delta - \rho$$
$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t$$

  with k_0 given

- Derive the consumption Euler equation yourselves!

- What are the boundary conditions? (Always ask about BCs!)
  - Initial condition on capital: k_0 given
  - Terminal condition on consumption : $\lim_{T\to\infty} c_T = c_{\text{ss}}$

3. Optimal control theory

- Optimal control theory emerged from the calculus of variations

- Applies to dynamic optimization problems in continuous time that feature (ordinary) differential equations as constraints

- Again the neoclassical growth model:

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

  subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t, \quad k_0 \text{ given}$$

- Three new terms:
  - **State variable**: k_t
  - **Control variable**: c_t
  - **Hamiltonian**: $H(c_t, k_t, \mu_t) = u(c_t) + \mu_t \left[F(k_t) - \delta k_t - c_t \right]$

- With Hamiltonian in hand, *copy-paste* formula that we can always use:
  - **Optimality condition**: $\frac{\partial}{\partial c} H = 0$
  - **Multiplier condition**: $\rho \mu_t - \dot{\mu}_t = \frac{\partial}{\partial k} H$
  - **State condition**: $\dot{k}_t = \frac{\partial}{\partial \mu} H$

- This gives us the same equations that we derived using calc of variations:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \rho \mu_t - \dot{\mu}_t &= \mu_t (F'(k_t) - \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

- We again get system of Euler equation and capital accumulation:

$$\begin{aligned} \dot{c}_t &= \frac{u'(c_t)}{u''(c_t)} \Big(\rho - F'(k_t) + \delta \Big) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

4. Simple example [*skip*]

- Credit: Kamien-Schwartz p. 129

- Simple problem: not much intuition, but illustrates mechanics

$$\max \int_0^1 (x+u)dt$$

  subject to $\dot{x} = 1 - u^2$ and initial condition $x_0 = 1$

- Step 1: form Hamiltonian $H(t, x, u, \lambda) = x + u + \lambda(1 - u^2)$

- Step 2: necessary conditions (note: no discounting here)

$$\begin{aligned} 0 &= H_u = 1 - 2\lambda u \\ -\dot{\lambda} &= H_x = 1 \end{aligned}$$

  and terminal condition $\lambda_1 = 0$ (because u_1 is *free*)

- Step 3: manipulate necessary conditions:

$$\lambda = 1 - t$$
$$u = \frac{1}{2\lambda}$$

and therefore: $u = \frac{1}{2}(1 - t)$

- Finally: solve for all paths (control, state, multiplier)

$$x_t = t - \frac{1}{4}(1 - t) + \frac{5}{4}$$
$$\lambda_t = 1 - t$$
$$u_t = \frac{1}{2}(1 - t)$$

5. Hamilton-Jacobi-Bellman equation

- Recall the neoclassical growth model in continuous time

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

  subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

  where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now

- This is the infinite-horizon sequence problem, $t \in [0, \infty)$

- A function $v(\cdot)$ that solves this problem is a solution to the neoclassical growth model

- We will now work towards a recursive representation (good reference: Stokey textbook)
- The discrete-time Bellman equation would be

$$v(k_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} v(k_{t+\Delta}) \right\}$$

where $\beta = \frac{1}{1+\rho\Delta t}$

- Next: multiply by $1+\rho\Delta t$ and note that $(\Delta t)^2 \approx 0$

$$(1+\rho\Delta t)v(k_t) = \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + v(k_{t+\Delta}) \right\}$$

$$\rho\Delta t v(k_t) = \max_c \left\{ u(c)\Delta t + v(k_{t+\Delta}) - v(k_t) \right\}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta t} \right\}$$

- Finally: take limit $\Delta t \to 0$ and drop t subscripts

$$\rho v(k) = \max_c \left\{ u(c) + dv \right\}$$

- We want to express dv in terms of $v'(\cdot)$ and dk
- Different ways to think about this: chain rule, Ito's lemma (though no uncertainty here), generator
- Recall generator of (stochastic) process dk_t: For any $f(\cdot)$

$$\mathcal{A}f(k_t) = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(k_{t+\Delta t}) - f(k_t)}{\Delta t}$$

- For simple ODE (no uncertainty) $dk = (F(k) - \delta k - c)dt$, we have

$$\mathcal{A}f(k) = (F(k) - \delta k - c)f'(k)$$

- Therefore, we arrive at the **Hamilton-Jacobi-Bellman equation**:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c\Big) v'(k) \right\}$$

- Notice: We conjectured a stationary value function (what does this mean?)

6. First-order condition for consumption

- HJB still has "max" operator:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c\Big) v'(k) \right\}$$

- To get rid of this, we have to resolve optimal consumption choice
- First-order condition:

$$u'(c(k)) = v'(k)$$

- This defines the **consumption policy function**
- We can now plug back in, obtaining an ODE in $v'(k)$

$$\rho v(k) = u(c(k)) + \Big(F(k) - \delta k - c(k)\Big) v'(k)$$

- Why is this a "stationary" value function and ODE? What would a time-dependent ODE look like? When would we get one?

7. Envelope condition and Euler equation

- We now derive the Euler equation in continuous time
- We start with the **HJB envelope condition**. Differentiating in k:

$$\rho v'(k) = u'(c(k))c'(k) + \Big(F'(k) - \delta - c'(k)\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

$$\rho v'(k) = \Big(\underbrace{F'(k) - \delta}_{\text{interest rate } r}\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

$$(\rho - r)v'(k) = \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

- Next, we characterize *process* $dv'(k)$. Using Ito's lemma (even though no uncertainty):

$$\begin{aligned} dv'(k) &= v''(k)dk \\ &= v''(k)(F(k) - \delta k - c(k))dt \\ &= (\rho - r)v'(k)dt. \end{aligned}$$

- Recall first-order condition $u'(c(k)) = v'(k)$.
- The **Euler equation for marginal utility** is given by

$$\frac{du'(c)}{u'(c)} = (\rho - r)dt.$$

- To go from marginal utility to consumption, we use CRRA utility: $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. $u'(c) = c^{-\gamma}$ is a function of *process* c, so by Ito's lemma:

$$\begin{aligned} du'(c) &= -\gamma c^{-\gamma-1} dc \\ &= -\gamma u'(c) \frac{dc}{c} \end{aligned}$$

- Plugging in yields **Euler equation for consumption** in continuous time:

$$\frac{dc}{c} = \frac{r-\rho}{\gamma} dt$$

or (you'll often see this notation when no uncertainty): $\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}$

Connection between calculus of variations and HJB:

- What is the connection between costate / multiplier μ_t and marginal value of wealth $V'(k)$?

- What is the connection between multiplier equation and envelope condition?

9. Boundary conditions

- This is really important: everything we have done so far is only valid in the **interior of the state space**
- What's the state space of a model?
- For the neoclassical growth model without uncertainty, state space is $k \in [0, \infty)$, or

$$\mathcal{X} = \left\{ k \mid k \in [0, \bar{k}] \right\}$$

  where we impose an upper boundary $\bar{k}$
- This is like the domain of the function $v(k)$ that will be valid
- We say $\partial\mathcal{X} = \{0, \bar{k}\}$ is the **boundary** of the state space and $\mathcal{X} \backslash \partial\mathcal{X} = (0, \bar{k})$ is the **interior**
- As is the case **for all differential equations**, the HJB holds on the interior and we need **boundary conditions** to characterize $v(k)$ along the boundary

- What kind of differential equation is the HJB in this model?

- So how many boundary conditions do we need?

- In terms of the economics, what is the correct boundary condition? I.e., what is the correct economic behavior at the boundary $k \in \{0, \bar{k}\}$?

- Nice intuition: 2 boundary inequalities do same job as 1 boundary equality

- We want households to not leave the state space, so we impose that they do not dissave / borrow as $k \to 0$ and save as $k \to \bar{k}$

- This implies: (why?)

$$u'(c(0)) \geq v'(0)$$
$$u'(c(\bar{k})) \leq v'(\bar{k})$$

- If households ever hit the boundaries (in the neoclassical growth model, this doesn't really happen), then consumption behavior is no longer determined by the Euler equations but rather by the boundary conditions

Part 2: stochastic dynamics

1. The generator of a stochastic process

- We start with the diffusion process

$$dX = \mu(t, X)dt + \sigma(t, X)dB$$

  where dB is a standard Brownian motion

- The generator $\mathcal{A}$ tells us how the stochastic process is *expected* to evolve

- The generator $\mathcal{A}$ is a functional operator

- Formally, for $f(t, X)$, we have

$$\mathcal{A}f = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(t + \Delta t, X(t + \Delta t)) - f(t, X(t))}{\Delta t}$$

- We will now show that:

$$\mathcal{A}f = \partial_t f(t, X) + \mu(t, X)\partial_X f(t, X) + \frac{1}{2}\sigma(t, X)^2 \partial_{XX} f(t, X)$$

- For the general / multi-dimensional version see Oksendal

- Next, we consider the poisson process $\{Y_t\}$ where $Y_t \in \{Y^1, Y^2\}$. This is a two-state Markov chain in continuous time.

- We assume that the Poisson intensity / arrival rate / hazard rate is λ

- The generator is now given by

$$\mathcal{A}f(Y^j) = \lambda\Big[f(Y^{-j}) - f(Y^j)\Big]$$

- Intuition: at rate λ you transition, so you lose the value of your current state, $f(Y^j)$, and obtain the value of the new state, $f(Y^{-j})$

- Again see Oksendal for general version of this and more details

2. Neoclassical stochastic growth

- Time is continuous and the horizon is infinite, $t \in [0, \infty)$

- Economy populated by representative household that operates production technology $F(k_t, z_t)$ where z_t is exogenous productivity

- Assume $F(\cdot)$ is well behaved: $F(0, z) = 0$, as well as F_k, $F_z > 0$, and $F_{kk} < 0$

- At time $t = 0$, economy's initial state is (k_0, z_0)

- Lifetime value of household is given by

$$V(k_0, z_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

  where $u(\cdot)$ is instantaneous utility flow and $\{c_t\}_{t\geq 0}$ is stochastic consumption process. $\mathbb{E}_0$ denotes expectation over future productivity realizations. We assume labor supply is inelastic and normalized to 1

- Capital evolves as before: $\frac{d}{dk} k_t = F(k_t, z_t) - \delta k_t - c_t$

3. Productivity as a diffusion process

- We start with diffusion process:

$$dz_t = -\theta z_t dt + \sigma dB_t,$$

where θ and σ are constants

- This is a continuous-time, mean-reverting AR(1) process called the Ornstein-Uhlenbeck process

- State space is now given by

$$\left\{ (k, z) \mid k \in [0, \bar{k}] \text{ and } z \in [\underline{z}, \bar{z}] \right\}$$

- In discrete time, we would have

$$V(k_t, z_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\}$$

- Difference from previous lecture: $\mathbb{E}$ because there is uncertainty

$$\begin{aligned}
(1+\rho\Delta t)V(k_t, z_t) &= \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\} \\
\rho\Delta t V(k_t, z_t) &= \max_c \left\{ u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t) \right\} \\
\rho V(k_t, z_t) &= \max_c \left\{ u(c) + \mathbb{E}_t \frac{V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t)}{\Delta t} \right\}
\end{aligned}$$

- Take limit $\Delta t \to 0$ and drop time subscripts:

$$\rho V(k, z) = \max_c \left\{ u(c) + \mathbb{E} \frac{dV(k, z)}{dt} \right\}$$

- What remains? Characterizing continuation value $\frac{d}{dt} V(k, z)$ (i.e., characterizing how process dV evolves)

- The generator $\mathcal{A}$ is exactly the answer to this question! I.e.,

$$\begin{aligned}\mathbb{E}\frac{dV(k,z)}{dt} &= \mathcal{A}V(k,z) \\ &= \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\end{aligned}$$

- Therefore, we arrive at the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\rho V(k,z) = \max_c \Big\{ u(c) + \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) \\ - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\Big\}\end{aligned}$$

with first-order condition

$$u'(c(k,z)) = \partial_k V(k,z)$$

4. Productivity as a Poisson process

- Next, consider Poisson process for $\{z_t\}$ with $z_t \in \{z^L, z^H\}$

- Generator now given by

$$\mathcal{A}V(k,z^j) = \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) + \lambda\Big[V(k,z^{-j}) - V(k,z^j)\Big]$$

- Note: derivation of HJB exactly as before *up to* characterizing $\mathbb{E}[dV]$

- With Poisson process, HJB becomes

$$\rho V(k,z^j) = \max_c \Big\{u(c) + \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) + \lambda\Big[V(k,z^{-j}) - V(k,z^j)\Big]\Big\}$$

with first-order condition

$$u'(c(k,z^j)) = \partial_k V(k,z^j)$$

5.1. Example: income fluctuations

- Economy is populated by representative household that faces income risk

- Household accumulates wealth according to

$$\dot{a}_t = ra_t + e^{z_t} - c_t$$

subject to borrowing constraint $a_t \geq 0$

- Preferences again: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$

- Income follows diffusion process: $dy_t = -\theta y_t dt + \sigma dB_t$

- Away from borrowing constraint, HJB given by

$$\rho V = \max_c \left\{ u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz} \right\}$$

with $V_a = \partial_a V(a, z)$ (you'll see this often)

5.2. Example: firm profit maximization

- Firm maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$

- For now, profit given by: $\pi_t = A_t n_t^\alpha - w_t n_t$ where firm chooses labor n_t
  Assume $\alpha < 1$, so this is a decreasing-returns production function

- Firm is small and takes wage $\{w_t\}$ as given (wages determined in general equilibrium)

- Productivity follows two-state high-low process, with $A_t \in \{A^{\text{rec}}, A^{\text{boom}}\}$

- Recursive representation: A is only state variable, $w_t = w(A_t)$

$$rV(A^{\text{boom}}) = \max_n \left\{ A^{\text{boom}} n^\alpha - w(A^{\text{boom}}) n + \lambda \left[V(A^{\text{rec}}) - V(A^{\text{boom}}) \right] \right\}$$

  with first-order condition

$$n = \left(\frac{\alpha A^j}{w(A^j)} \right)^{\frac{1}{1-\alpha}}$$

5.3. Example: capital investment with adjustment cost

- Firm again maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$

- Now: let $\psi(\cdot)$ denote an adjustment cost

$$
\begin{aligned}
\pi_t &= e^{A_t} k_t^\alpha - Q_t \iota_t - \psi(\iota_t, k_t) \\
dk_t &= (\iota_t - \delta k_t) dt \\
dA_t &= -\theta A_t dt + \sigma dB_t
\end{aligned}
$$

- Firm is small and takes capital price as given

- Recursive representation in terms of (k, A), i.e., $Q_t = Q(k_t, A_t)$

$$
\begin{aligned}
rV(k, A) = \max_\iota \Big\{ & e^{A_t} k_t^\alpha - Q(A) \iota_t - \psi(\iota_t, k_t) + (\iota - \delta k) \partial_k V(k, A) \\
& - \theta A \partial_A V(k, A) + \frac{\sigma^2}{2} \partial_{AA} V(k, A) \Big\}
\end{aligned}
$$

with first-order condition: $Q(k, A) + \partial_\iota \psi(\iota(k, A), k) = \partial_k V(k, A)$

5.4. Example: investing in stocks

- Suppose you optimize lifetime utility $V_0 = \mathbb{E}_0 \int_0^\infty u(c_t)dt$

- You can trade two assets: riskfree bond (return rdt), and risky stock

$$dR = (r + \pi)dt + \sigma dB, \text{ where } \pi \text{ is the equity premium}$$

- You have wealth a_t and invest a share θ_t in stocks, thus,

$$da_t = \theta_t a_t dR_t + (1 - \theta_t) a_t r_t dt + y - c_t$$

or, rearranging, and dropping t subscripts

$$da = ra + \theta a \pi dt + y - c + \theta a \sigma dB$$

- HJB becomes:

$$\rho V(a) = \max_{c,\theta} \left\{ u(c) + (ra + \theta a \pi dt + y - c)V'(a) + \frac{1}{2}(\sigma\theta a)^2 V''(a) \right\}$$

with FOCs: (i) $u'(c) = V'(a)$ and (ii) $\theta = -\frac{\pi}{\sigma^2}\frac{V'(a)}{aV''(a)}$

5.5. Example: tax competition

- Two countries, $i \in \{A, B\}$, setting corporate tax rates τ_t^i on firms operating / headquartered in country i

- Mass of multinational firms j, with μ_t denoting $\%$ in country A at time t

- Firms relocate activity / headquarters at rate θ towards low-tax country:
$$d\mu_t = \theta \mu_t (\tau_t^B - \tau_t^A)^\gamma dt$$

- Country A maximizes tax revenue: $\max \int_0^\infty e^{-\rho t} \tau_t^A \mu_t dt$. Countries compete over taxes $\{\tau_{it}\}$

- Dynamic Nash: country A sets τ_t^A as best response taking τ_t^B as given

- Recursive representation: the only state variable is μ_t
$$\rho V^A(\mu) = \max_{\tau^A} \left\{ \tau^A \mu + \theta \mu \left(\tau^B(\mu) - \tau^A \right)^\gamma \partial_\mu V^A(\mu) \right\}$$
Best response strategies: $0 = \mu + \gamma \theta \mu (\tau^B(\mu) - \tau^A)^{\gamma - 1} V_\mu^A(\mu)$