1 The system

We consider an *a priori unstable* Hamiltonian with $3+1/2$ degrees of freedom

$$H_\varepsilon(p,q,I_1,I_2,\varphi_1,\varphi_2,s) = \pm\left(\frac{p^2}{2}+\cos q-1\right)+h(I_1,I_2)+\varepsilon f(q)\,g(\varphi_1,\varphi_2,s), \tag{1}$$

where $f(q)=\cos q$, $h(I_1,I_2)=\Omega_1 I_1^2/2+\Omega_2 I_2^2/2$ and

$$g(\varphi_1,\varphi_2,s) = a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos s. \tag{2}$$

From now on, for a simpler notation, we denote $I=(I_1,I_2)$ and $\varphi=(\varphi_1,\varphi_2)$.

Remark 1. A similar, but more complicated, case appears in [DLS16]. In such paper g is taken as

$$g(\varphi_1,\varphi_2,s) = a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos(\varphi_1+\varphi_2-s).$$

2 Unperturbed case

In the unperturbed case ($\varepsilon=0$), such system is the Hamiltonian system with Hamiltonian

$$H_0(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2}+\cos q-1\right)+h(I),$$

and equations

$$\begin{aligned} \dot q &= p & \dot p &= \sin q \\ \dot\varphi_1 &= \omega_1 & \dot I_1 &= 0 \\ \dot\varphi_2 &= \omega_2 & \dot I_2 &= 0 \\ \dot s &= 1, \end{aligned} \tag{3}$$

where $\omega_i=\Omega_i I_i.$, $i=1,2$. This system consists of a pendulum plus two rotors. From the equations above, I_1 and I_2 are constants and the flow has the form

$$\Phi_t(p,q,I,\varphi) = (p(t),\, q(t), I, \varphi+t\omega),$$

where $\omega=(\omega_1,\omega_2)$. And we have an invariant set (on the extend phase space)

$$\mathcal{T}_I = \{(0,0,I,\varphi,s);\varphi,s\in\mathbb{T}^3\}.$$

In this case, the NHIM is

$$\tilde\Lambda = \{(0,0,I,\varphi,s):(I,\varphi,s)\,\in\,\mathbb{R}^2\times\mathbb{T}^3\}, \tag{4}$$

3 Inner dynamics

The inner dynamics is derived form the restriction of the Hamiltonian (1) and its equations to $\tilde\Lambda$, given in (4), i.e.,

$$K_\varepsilon(I,\varphi,s) = h(I)+\varepsilon\,(a_1\cos\varphi_1+a_2\cos\varphi_2+a_3\cos s)$$

and its equations

$$\begin{aligned} \dot\varphi_1 &= \omega_1 & \dot I_1 &= \varepsilon a_1\sin\varphi_1 \\ \dot\varphi_2 &= \omega_2 & \dot I_2 &= \varepsilon a_2\sin\varphi_2 \\ \dot s &= 1. \end{aligned}$$

Remark 2. Note that the inner dynamics is integrable.

To describe the resonant regions we consider the autonomous extended Hamiltonian

$$\overline{K}_\varepsilon = A + h(I) + \varepsilon\left(a_1 \cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos s\right),$$

$$\begin{aligned} \dot\varphi_1 &= \omega_1 & \dot I_1 &= \varepsilon a_1 \sin\varphi_1 \\ \dot\varphi_2 &= \omega_2 & \dot I_2 &= \varepsilon a_2 \sin\varphi_2 \\ \dot s &= 1 & \dot A &= \varepsilon a_3 \sin s. \end{aligned}$$

Consider a change of variables ε-close to the identity

$$(I, A, \varphi, s) = g(J, B, \phi, \sigma) = (J, B, \phi, \sigma) + \mathcal{O}(\varepsilon)$$

such that it is one-time flow for a Hamiltonian εG, i.e., $g = g_{t=1}$, where g_t is solution of

$$\frac{dg_t}{dt} = J_0 \nabla \varepsilon G \circ g_t, \qquad J_0 \text{ is the symplectic matrix } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Composing $\overline{K}$ with g and expanding in a Taylor series around $t = 0$, one obtains

$$\overline{K} \circ g = \overline{K} + \left\{\overline{K}, \varepsilon G\right\} + \frac{1}{2}\left\{\left\{\overline{K}, \varepsilon G\right\}, \varepsilon G\right\} + \dots,$$

where $\{\cdot\}$ is the Poisson bracket.

$$\begin{aligned} \overline{K}_\varepsilon \circ g = {} & B + h(J) + \varepsilon\left(a_1\cos\phi_1 + a_2\cos\phi_2 + a_3\cos s + \{B + h(J), G\}\right) \\ & + \varepsilon^2\left(\left\{a_1\cos\phi_1 + a_2\cos\phi_2 + a_3\cos s + \frac{1}{2}\{B + h(J), G\}, G\right\}\right) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

We want to find G satisfying

$$a_1\cos\phi_1 + a_2\cos\phi_2 + a_3\cos\sigma - \omega_1\frac{\partial G}{\partial\varphi_1} - \omega_2\frac{\partial G}{\partial\varphi_2} - \frac{\partial G}{\partial\sigma} = 0. \tag{5}$$

Given $a < b < 1$, consider any function $\Psi \in C^\infty(\mathbb{R})$ satisfying $\Psi(x) = 1$ for $x \in [-a, a]$ and $\Psi(x) = 0$ for $x \notin [-b, b]$, and introduce

$$G(J, \phi, \sigma) = \frac{a_1(1 - \Psi(I_1))\sin\phi_1}{\omega_1} + \frac{a_2(1 - \Psi(I_2)\sin\phi_2}{\omega_2} + a_3\sin\sigma.$$

For J_1 and $J_2 \notin [-a, a]$,

$$\overline{K} \circ g = h(J) + B + \mathcal{O}(\varepsilon^2). \tag{6}$$

For $J_i \in [-a, a]$ and J_j and $\notin [-b, b]$, $i, j = \{1, 2\}$ and $i \neq j$,

$$\overline{K} \circ g = h(J) + B + \varepsilon a_i \cos\phi_i + \mathcal{O}(\varepsilon^2). \tag{7}$$

From (6) and (7), there are just two resonances, on $J_1 = 0$ and $J_2 = 0$. Therefore, $J = (J_1, J_2) = 0$ is a double resonance. Moreover, there are no resonances of second order.

Coming back to the original variables (I, φ, s), the model just has two resonant sets at first order:

$$\begin{aligned} \mathcal{R}_1 &= \{(I_1, I_2), I_1 = 0\} \\ \mathcal{R}_2 &= \{(I_1, I_2), I_2 = 0\} \end{aligned} \tag{8}$$

and there is no resonance of second order. And, $\mathcal{R}_1 \cap \mathcal{R}_2$ is the double resonant set.

4 Scattering map

4.1 Definition of scattering map

We are going to explore the properties of the scattering maps of Hamiltonian (1). The notion of scattering map on a NHIM was introduced in [DLS00]. Let W be an open set of $[-I_1^*, I_1^*] \times [-I_2^*, I_2^*] \times \mathbb{T}^3$ such that the invariant manifolds of the NHIM $\tilde{\Lambda}$ introduced in (4) intersect transversally along a homoclinic manifold $\Gamma = \{\tilde{z}(I, \varphi, s; \varepsilon), (I, \varphi, s) \in W\}$ and for any $\tilde{z} \in \Gamma$ there exists an unique $\tilde{x}_{+,-} = \tilde{x}_{+,-}(I, \varphi, s; \varepsilon) \in \tilde{\Lambda}$ such that $\tilde{z} \in W^s_\varepsilon(x_-) \cap W^u_\varepsilon(\tilde{x}_+)$. Let

$$H_{+,-} = \bigcup \{\tilde{x}_{+,-}(I, \varphi, s; \varepsilon) : (I, \varphi, s) \in W\}.$$

The scattering map associated to Γ is the map

$$\begin{aligned} S : H_- &\longrightarrow H_+ \\ \tilde{x}_- &\longmapsto S(\tilde{x}_-) = \tilde{x}_+. \end{aligned}$$

For the characterization of the scattering maps, it is required to select the homoclinic manifold Γ and this be done using the Poincaré-Melnikov theory. From [DLS06, DH11], we have the following proposition

Proposition 3. *Given $(I, \varphi, s) \in [-I_1^*, I_1^*] \times [-I_1^*, I_1^*] \times \mathbb{T}^3$, assume that the real function*

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - \tau\omega,\ s - \tau) \in \mathbb{R} \tag{9}$$

has a non degenerate critical point $\tau^* = \tau^*(I, \varphi, s)$, where $\omega = (\omega_1, \omega_2)$ and

$$\mathcal{L}(I, \varphi, s) := \int_{-\infty}^{+\infty} \left(f(q_0(\rho)) - f(0)\right) g(\varphi + \rho\omega, s + \rho; 0) d\rho.$$

Then, for $0 < \varepsilon$ small enough, there exists a unique transversal homoclinic point $\tilde{z}$ to $\tilde{\Lambda}_\varepsilon$ of Hamiltonian (1), which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\tilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\tilde{\Lambda_\varepsilon}) \pitchfork W^s(\tilde{\Lambda_\varepsilon}). \tag{10}$$

The function $\mathcal{L}$ is called the *Melnikov potential* of Hamiltonian (1). The Melnikov potential takes the form

$$\mathcal{L}(I, \varphi, s) = A_1 \cos\varphi_1 + A_2 \cos\varphi_2 + A_3 \cos s, \tag{11}$$

where and

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$$A_i := A(\omega_i) = \frac{2\pi\omega_i a_i}{\sinh(\pi\omega_i/2)}, \quad i = 1, 2 \quad \text{and } A_3 = \frac{2\pi a_3}{\sinh(\pi/2)}$$

The homoclinic manifold Γ is characterized by the function $\tau^*(I, \varphi, s)$. Once a $\tau^*(I, \varphi, s)$ is chosen, by the geometric properties of the scattering map, see [**?**, DH09, DH11], the scattering map has the explicit form

$$S(I, \varphi, s) = \left(I \ + \varepsilon\nabla_\varphi L^* + (\mathcal{O}(\varepsilon^2), \mathcal{O}(\varepsilon^2)), \varphi - \varepsilon\nabla_I L^* + (\mathcal{O}(\varepsilon^2), \mathcal{O}(\varepsilon^2)), s\right),$$

where

$$L^* = L^*(I, \varphi, s) = \mathcal{L}(I, \varphi - \tau^*(I, \varphi, s)\omega, s - \tau^*(I, \varphi, s)). \tag{12}$$

Notice that the variable s is fixed under the scattering map. As a consequence [DH11], introducing the variable

$$\theta = \varphi - s\,\omega \tag{13}$$

and defining the *reduced Poincaré function*

$$\mathcal{L}^*(I,\theta) := L^*(I,\varphi - s\omega, 0) = L^*(I,\varphi,s), \tag{14}$$

in the variables (I,θ) the scattering map has the simple form

$$\mathcal{S}(I,\theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathbf{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathbf{O}(\varepsilon^2) \right),$$

where $\mathbf{O}(\varepsilon^2) = (\mathcal{O}(\varepsilon^2), \mathcal{O}(\varepsilon^2))$. So up to $\mathcal{O}(\varepsilon^2)$ terms, $\mathcal{S}(I,\theta)$ is the ε times flow of the *autonomous* Hamiltonian $-\mathcal{L}^*(I,\theta)$. In particular, the iterates under the scattering map follow the level curves of $\mathcal{L}^*$ up to $\mathcal{O}(\varepsilon^2)$.

4.2 Crests and NHIM lines

We have seen that the function τ^* plays a central role in our study. Therefore, we are interested in finding the critical points $\tau^* = \tau^*(I,\varphi,s)$ of function (9) or, for our concrete case (11), τ^* solution of

$$\frac{\partial \mathcal{L}}{\partial \tau}(I,\varphi - \omega\tau, s - \tau) = \omega_1 A_1 \sin(\varphi_1 - \omega_1 \tau) + \omega_2 A_2 \sin(\varphi_2 - \omega_2 \tau) + A_3 \sin(s - \tau). \tag{15}$$

This equation can be viewed from two equivalently geometrical viewpoints. The first one is that to find $\tau^* = \tau^*(I,\varphi,s)$ satisfying (15) for any $(I,\varphi,s) \in [-I_1^*, I_1^*] \times [-I_2^*, I_2^*] \times \mathbb{T}^3$ is the same as to look for the extrema of $\mathcal{L}$ on the *NHIM line*

$$R(I,\varphi,s) = \{(I,\varphi - \tau\omega, s - \tau) : \tau \in \mathbb{R}\}. \tag{16}$$

The other viewpoint is that, fixing (I,φ,s), a solution τ^* of (15) is equivalent to finding intersections between a NHIM line (16) and a surface defined by

$$\omega_1 A_1 \sin\varphi_1 + \omega_2 A_2 \sin\varphi_2 + A_3 \sin s = 0. \tag{17}$$

These surfaces are called *crests*, and in a general way can be defined by

Definition 4. [DH11] We define by *Crests* $\mathcal{C}(I)$ the surfaces on (I,φ,s), $(\varphi,s) \in \mathbb{T}^3$, such that

$$\frac{\partial \mathcal{L}}{\partial \tau}(I,\varphi - \tau\omega, s - \tau)|_{\tau=0} = 0, \tag{18}$$

or equivalently,

$$\omega \cdot \frac{\partial \mathcal{L}}{\partial \varphi}(I,\varphi,s) + \frac{\partial \mathcal{L}}{\partial s}(I,\varphi,s) = 0.$$

Note that equation (17) can be rewritten as

$$\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2 + \sin s = 0 \tag{19}$$

where, for $i = 1,2$,

$$\mu_i = \frac{a_i}{a_3} \quad \text{and} \quad \alpha_i(I_i) = (\omega_i)^2 \, \frac{\sinh(\pi/2)}{\sinh(\omega_i \pi/2)}. \tag{20}$$

Observe that α_i is well defined for any value of I_i. To understand the intersection between NHIM lines and the crests $\mathcal{C}(I)$, first we need to study how these surfaces look like for different values of μ_i and ω_i, for $i = 1,2$.

Remark 5. With Eq. (19) we wish to emphasize the similarity between such crests with the crests studied in [DS17a, DS17b].

As explained before, when we have introduced the crests, we are interested in their geometrical behavior. For this purpose, we study their possible parameterization. One can see from (19) that if

$$|\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha(I_2)\mu_2 \sin\varphi_2| \leq 1 \tag{21}$$

we can write s as a function of φ_1 and φ_2, more exactly

$$s = \begin{cases} \xi_{\mathrm{M}}(I,\varphi) = \arcsin\left(\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2\right) \mod 2\pi \\ \xi_{\mathrm{m}}(I,\varphi) = -\arcsin\left(\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2\right) + \pi \mod 2\pi. \end{cases}$$

In accordance with the notation used in [DS17a, DS17b], the crests $\mathcal{C}(I)$ parameterized by $\xi_{\mathrm{M}}(I,\varphi)$ and $\xi_{\mathrm{m}}(I,\varphi)$ are called *horizontal* crests. From expression of the function $\alpha_i(I_i)$ given in (20), we have $|\alpha_i(I_i)| < 1.03$. This implies

$$|\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2| \leq 1.03(|\mu_1| + |\mu_2|).$$

Therefore, if

$$|\mu_1| + |\mu_2| \leq 1/1.03 \approx 0.97, \tag{22}$$

the crests $\mathcal{C}(I)$ are horizontal for any value of I_1 and I_2.

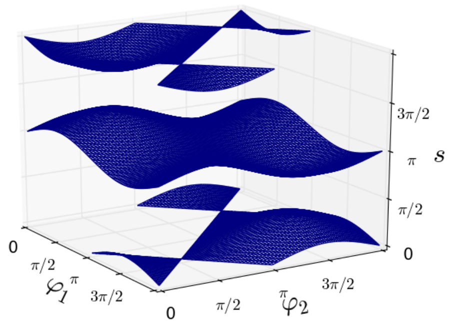


Fig. 1: *Horizontal Crests* $\mathcal{C}(I)$: $\mu_1 = \mu_2 = 0.4$ and $\omega_1 = \omega_2 = 1$.

If condition (21) is not satisfied, s cannot be written as a function of φ_1 and φ_2, then we have two possibilities: a) we can write φ_i as a function of φ_j and s, or b) the projection of the crests $\mathcal{C}(I)$ on each plane ((φ_1,φ_2) , (φ_1,s) and (φ_2,s)) has holes.

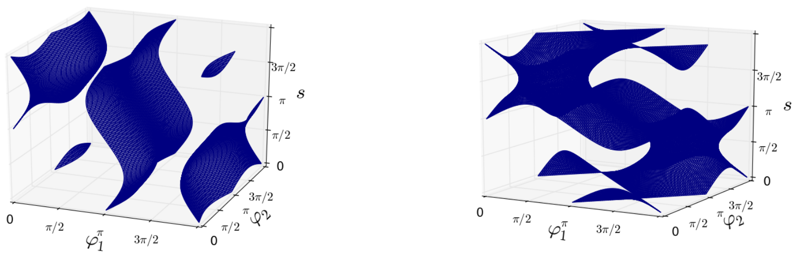
The fist case is only possible if $\left|\frac{\alpha(I_j)\mu_j}{\alpha_i(I_i)\mu_i}\sin\varphi_j + \frac{\sin s}{\alpha_i(I_i)\mu_i}\right| \leq 1$. Then the crests $\mathcal{C}(I)$ are called *vertical* crests and can be parameterized by

$$\varphi_i = \begin{cases} \eta_{\mathrm{M},i}(I,\varphi_j,s) = \arcsin\left(\frac{1}{\alpha_i(I_i)\mu_i}\left(\sin s - \alpha_j(I_j)\mu_j \sin\varphi_j\right)\right) \\ \eta_{\mathrm{m},i}(I,\varphi_j,s) = -\arcsin\left(\frac{1}{\alpha_i(I_i)\mu_i}\left(\sin s - \alpha_j(I_j)\mu_j \sin\varphi_j\right)\right) + \pi. \end{cases}$$

In the second case, Eq. (4) defines a unique surface, see Fig. 2(b). Note that for horizontal and vertical crests $\mathcal{C}(I)$ is formed by two surfaces that can be parameterized separately. Then, in such case $\mathcal{C}(I)$ is called *unseparated* crests.

To write φ_i as a function of φ_j and s $|\mu_i|$ is needed to be greater than 0.97. In fact, suppose that there exists an I such that $\varphi_i(\varphi_j,s)$. From Eq. (17), we have

$$\sin\varphi_i = -\left(\frac{A_j(I_j)\omega_j \sin\varphi_j}{A_i(I_i)\omega_i} + \frac{A_3 \sin s}{A_i(I_i)\omega_i}\right),$$



(a) *Vertical Crests* $\mathcal{C}(I)$: $\mu_1 = 1.7$, $\mu_2 = 0.4$ and $\omega_1 = \omega_2 = 1$.

(b) *Unseparated Crests* $\mathcal{C}(I)$: $\mu_1 = 0.7$, $\mu_2 = 0.7$ and $\omega_1 = \omega_2 = 1$.

Fig. 2: Different kinds of Crests

for any φ_j and s with $\left|\frac{A_j(I_j)\omega_j \sin\varphi_j}{A_i(I_i)\omega_i} + \frac{A_3 \sin s}{A_i(I_i)\omega_i}\right| \leq 1$. In particular for $\varphi_j = 0$ and $s = \pi/2$, so

$$\left|\frac{A_3}{A_i(I_i)\omega_i}\right| \leq 1, \quad \text{or equivalentely} \quad \left|\frac{1}{\alpha_i(I_i)\mu_i}\right| \leq 1.$$

Therefore, we obtain

$$0.97 \approx \frac{1}{1.03} < \left|\frac{1}{\alpha_i(I_i)}\right| \leq |\mu_i|\,.$$

As a consequence, if $|\mu_1| + |\mu_2| > 0.97$, but $|\mu_1|, |\mu_2| < 0.97$ then there are no vertical crests, only horizontal or unseparated crests.

4.2.1 Tangency condition

We now explore the existence of tangency between the crests $\mathcal{C}(I)$ and the lines $R(I, \varphi, s)$. The crests are a family of surfaces, so there exists a tangency such tangecy if a tangent vector of the straight line $R(I, \varphi, s)$ lies on the bundle tangent of one of these surfaces.

The vector tangent of $R(I, \varphi, s)$ at any point is $v = -(\omega_1, \omega_2, 1)$. Consider the function $F_I : \mathbb{T}^3 \mapsto \mathbb{R}$,

$$F_I(\varphi, s) = \alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2 + \sin s,$$

we note that the crests $\mathcal{C}(I)$ can be defined as $(\varphi, s) \in \mathbb{T}^3$ such that $F_I(\varphi, s) = 0$. Fixing a point $\mathbf{p} = (\varphi, s)$ in $\mathcal{C}(I)$, the normal vector of $\mathcal{C}(I)$ at the point $\mathbf{p}$ is

$$\nabla F_I(\mathbf{p}) = (\alpha_1(I_1)\mu_1 \cos\varphi_1, \alpha_2(I_2)\mu_2 \cos\varphi_2, \cos s).$$

The vector v lies on the tangent space of the crests at the point $\mathbf{p}$ if, and only if $\nabla F(\mathbf{p}) \cdot v = 0$. This condition is equivalent to

$$\alpha_1(I_1)\omega_1\mu_1 \cos\varphi_1 + \alpha_2(I_2)\omega_2\mu_2 \cos\varphi_2 + \cos s = 0. \tag{23}$$

From (19) and (23), there is tangency between a horizontal crest $\mathcal{C}(I)$ and the NHIM lines $R(I, \varphi, s)$ for φ_1 and φ_2 satisfying

$$(\omega_1\alpha_1(I_1)\mu_1 \cos\varphi_1 + \omega_2\alpha_2(I_2)\mu_2 \cos\varphi_2)^2 + (\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2)^2 = 1$$

Denote by $f_I(\varphi) = (\omega_1\alpha_1(I_1)\mu_1 \cos\varphi_1 + \omega_2\alpha_2(I_2)\mu_2 \cos\varphi_2)^2 + (\alpha_1(I_1)\mu_1 \sin\varphi_1 + \alpha_2(I_2)\mu_2 \sin\varphi_2)^2$ Note that, if there are values of μ_1 and μ_2 such that $f_I(\varphi) < 1$ for any I, there is no tangency. As commented before, $|\alpha_i(I_i)| < 1.03$ From (20), we obtain $|\alpha_i(I_i)\omega_i| < 1.6$. This implies

$$\begin{aligned} f_I(\varphi) &< (1.6)^2 \left(|\mu_1 \cos\varphi_1| + |\mu_2 \cos\varphi_2|\right)^2 + (1.03)^2 \left(|\mu_1 \sin\varphi_1| + |\mu_2 \sin\varphi_2|\right)^2 \\ &< 1.6^2 \left|\mu_1\right|^2 + 1.6^2 \left|\mu_2\right|^2 + 2\left|\mu_1\right|\left|\mu_2\right| \left(1.6^2 \left|\cos\varphi_1\right|\left|\cos\varphi_2\right| + 1.03^2 \left|\sin\varphi_1\right|\left|\sin\varphi_2\right|\right) \\ &< 1.6^2(|\mu_1| + |\mu_2|)^2. \end{aligned}$$

It is enough to require $|\mu_1| + |\mu_2| < 1/1.6 = 0.625$ to ensure $f_I(\varphi) < 1$ for any value of I. It is easy to verify that if $|\mu_1| + |\mu_2| > 1/1.6$ there are a I and a φ such that $f_I(\varphi) > 1$.

Proposition 6. *Consider the crest $\mathcal{C}(I)$ defined by* (4) *and the NHIM lines $R(I, \varphi, s)$ defined in* (16)*.*

a) For $|\mu_1| + |\mu_2| > 0.625$ the crests are horizontal and the intersections between any crest and any NHIM line are transversal.

b) For $0.625 \le |\mu_1| + |\mu_2| \le 0.97$ the two crests $\mathcal{C}(I)$ are still horizontal, but for some value of I there are NHIM lines which are tangent to the crests.

c) For $0.97 < |\mu_1| + |\mu_2|$ and $|\mu_1|, |\mu_2| < 0,97$ The crests $\mathcal{C}(I)$ are horizontal or unseparated and for some value of I there are NHIM lines which are tangent to the crests.

4.3 Diffusion

$$\dot{I}_i = -A_i \sin(\theta_i - \omega_i\tau^*) \qquad \dot{\theta}_i = -\Omega_i \left(\frac{dA_i}{d\omega_i}\cos(\theta_i - \omega_i\tau^*) + \tau^* A_i \sin(\theta_i - \omega_i\tau^*)\right), \tag{24}$$

$i = 1, 2$.

Note we are interested in the diffusion in the variables $I = (I_1, I_2)$. The worst situation for the diffusion is I constant, i.e., $\dot{I} = 0$. From (24), it happens for

$$A_i \sin(\theta_i - \omega_i\tau^*(I, \theta)) = 0, \tag{25}$$

where $i = 1, 2$, $I = (I_1, I_2)$ and $\theta = (\theta_1, \theta_2)$. It is easy to check that for $\Omega_i \neq 0$, $A_i \neq 0$ for any I_i. Therefore, (25) is equivalento to

$$\sin(\theta_i - \omega_i\tau^*(I, \theta)) = 0.$$

This is satisfied for $\theta_i - \omega_i\tau^*(I, \theta) = 0, \pi$, $i = 1, 2$.

4.3.1 Horizontal Crests

Now, we are assuming that for any value of μ_1, μ_2 and I the crests are "horizontal". So, $\psi = \theta - \tau^*(I, \theta)\omega$ is always defined, and we can write the inverse function as

$$\theta = \psi - \xi_j(I, \psi)\omega, \tag{26}$$

where if $j \bmod 2 = 0$, $\xi_j(I, \psi) = -\arcsin\left(\alpha_1(I_1)\mu_1 \sin\psi_1 + \alpha_2(I_2)\mu_2 \sin\psi_2\right) + 2j\pi$. Otherwise, $\xi_j(I, \psi) = \arcsin\left(\alpha_1(I_1)\mu_1 \sin\psi_1 + \alpha_2(I_2)\mu_2 \sin\psi_2\right) + \pi + 2j\pi$.

We begin for the maximum crest $\mathcal{C}_{\mathrm{M}}(I)$ parameterized by $\xi_j(\, j \bmod 2 = 0)$. From (25) and the definition of ψ, $\dot{I} = 0$ if, and only if $\psi_i = 0$, π, for $i = 1$, 2. From (13) and the expression of ξ_j, $\theta_i = -2\pi\omega_i j$ for $\psi_i = 0$, and $\theta_i = (1 - 2j\omega_i)\pi$ for $\psi_i = \pi$. This implies that (I, θ) such that $\dot{I} = 0$ lie on the union of 4 planes, namely $\mathbf{P}_j$, $(0, 0, 0, \pi) + \mathbf{P}_j$, $(0, 0, \pi, 0) + \mathbf{P}_j$, $(0, 0, \pi, \pi) + \mathbf{P}_j$, where

$$\mathbf{P}_j = \langle(1, 0, -2\pi\Omega_1 j, 0), (0, 1, 0, -2\pi\Omega_2 j)\rangle.$$

For the minimal crest $\mathcal{C}_{\rm m}(I)$ parameterized by ξ_j with $j \bmod 2 = 1$. In this case, $\theta_i = -(1+2j)\omega_i\pi$ for $\psi_i = 0$ and $\theta_i = \pi + (1+2j)\omega_i\pi$ for $\psi_i = \pi$. As before, $\dot{I} = 0$ on 4 planes: $\mathbf{P}_j$, $(0,0,\pi,0)+\mathbf{P}_j$, $(0,0,0,\pi)+\mathbf{P}_j$ and $(0,0,\pi,\pi)+\mathbf{P}_j$, where

$$\mathbf{P}_j = \langle (1,0,-(1+2j)\Omega_1\pi,0),(0,1,0,-(1+2j)\Omega_2\pi)\rangle.$$

As an example we study two concrete cases: First we are going to take the scattering maps $\mathcal{S}_0(I,\theta)$ and $\mathcal{S}_1(I,\theta)$ associated to ξ_0 and ξ_1 respectively, and to look for the intersection set between two planes of the eight planes associated to $\mathcal{C}_0(I)$ and $\mathcal{C}_1(I)$ such that the value of I is constant. Later, we do the same study for $\mathcal{S}_0(I,\theta)$ and $\mathcal{S}_2(I,\theta)$.

For $\mathcal{S}_0(I,\theta)$ and $\mathcal{S}_1(I,\theta)$, we have

$$\mathbf{P}_0 = \langle (1,0,0,0),(0,1,0,0)\rangle \quad \text{and} \quad \mathbf{P}_1 = \langle (1,0,-3\Omega_1\pi,0),(0,1,0,-3\Omega_2\pi)\rangle.$$

The intersection set between two planes of the eight planes associated to $\mathcal{C}_0(I)$ and $\mathcal{C}_1(I)$ has 16 points, more specifically, is the set

$$\begin{aligned}\mathfrak{I}_{0,1} = \Big\{&(0,0,0,0), \left(0,\frac{1}{3\Omega_2},0,0\right), \left(\frac{1}{3\Omega_1},0,0,0\right), \left(\frac{1}{3\Omega_1},\frac{1}{3\Omega_2},0,0\right),\\ &\left(0,\frac{-1}{3\Omega_2},0,\pi\right), (0,0,0,\pi), \left(\frac{1}{3\Omega_1},\frac{-1}{3\Omega_2},0,\pi\right), \left(\frac{1}{3\Omega_1},0,0,\pi\right),\\ &\left(\frac{-1}{3\Omega_1},0,\pi,0\right), \left(\frac{-1}{3\Omega_1},\frac{1}{3\Omega_2},\pi,0\right), (0,0,\pi,0), \left(0,\frac{1}{3\Omega_2},\pi,0\right),\\ &\left(\frac{-1}{3\Omega_1},\frac{-1}{3\Omega_2},\pi,\pi\right), \left(\frac{-1}{3\Omega_1},0,\pi,\pi\right), \left(0,\frac{-1}{3\Omega_2}.\pi,\pi\right), (0,0,\pi,\pi)\Big\}\end{aligned}$$

For $\mathcal{S}_0(I,\theta)$ and $\mathcal{S}_2(I,\theta)$, since

$$\mathbf{P}_2 = \langle (1,0,-4\Omega_1\pi,0),(0,1,0,-4\Omega_2\pi)\rangle.$$

The intersection set between two planes of the eight planes associated to $\mathcal{C}_0(I)$ and $\mathcal{C}_2(I)$ is the set

$$\begin{aligned}\mathfrak{I}_{0,2} = \Big\{&(0,0,0,0), \left(0,\frac{1}{4\Omega_2},0,0\right), \left(\frac{1}{4\Omega_1},0,0,0\right), \left(\frac{1}{4\Omega_1},\frac{1}{4\Omega_2},0,0\right),\\ &\left(0,\frac{-1}{4\Omega_2},0,\pi\right), (0,0,0,\pi), \left(\frac{1}{4\Omega_1},\frac{-1}{4\Omega_2},0,\pi\right), \left(\frac{1}{4\Omega_1},0,0,\pi\right),\\ &\left(\frac{-1}{4\Omega_1},0,\pi,0\right), \left(\frac{-1}{4\Omega_1},\frac{1}{4\Omega_2},\pi,0\right), (0,0,\pi,0), \left(0,\frac{1}{4\Omega_2},\pi,0\right),\\ &\left(\frac{-1}{4\Omega_1},\frac{-1}{4\Omega_2},\pi,\pi\right), \left(\frac{-1}{4\Omega_1},0,\pi,\pi\right), \left(0,\frac{-1}{4\Omega_2}.\pi,\pi\right), (0,0,\pi,\pi)\Big\}\end{aligned}$$

Observe these two sets imply that in a combination of these three scattering maps $\mathcal{S}_0(I,\theta)$, $\mathcal{S}_1(I,\theta)$ and $\mathcal{S}_2(I,\theta)$, the value of I is constant just in $(0,0,0,0)$, $(0,0,0,\pi)$, $(0,0,\pi,0)$ and $(0,0,\pi,\pi)$, this mean such combination provide us a great freedom to construct of diffusion. So, it is enough to study the regions where the value of I for each scattering maps is increase and ways to avoid the points cited above.

5 Theorems

Theorem 7 (More general result). *Consider the Hamiltonian* (1)+(2). *Assume* $a_1a_2a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$. *Then, for every* δ *there exists* $0 < \varepsilon_0$ *such that for every* $0 < |\varepsilon| < \varepsilon_0$, *given* $I_\pm \in \mathcal{I}^* \setminus \{(0,0)\}$, *there exists an orbit* $\tilde{x}(t)$ *and* $T > 0$, *such that*

$$\begin{aligned}|I(\tilde{x}(0)) - I_-| &\leq C\delta \\ |I(\tilde{x}(T)) - I_-| &\leq C\delta\end{aligned} \tag{27}$$

Actually, we will prove that given $I_\pm \in \mathcal{I}^* \setminus \{(0,0)\}$, we are able to build a path $\gamma(s) \subset \mathcal{I}^* \setminus \{(0,0)\}$ such that there exist an orbit $\tilde{x}(t)$ where $I(\tilde{x}(t))$ is δ-close to $\gamma(\Psi(t))$ for some parameterization Ψ.

Proof. $\square$

Definition 8 ([GM17]). Let A be some set and consider a set $f = \{f_i | i \in I\}$ of locally defined maps $f_i : \mathit{Dom}\, f_i \to A$. A finite sequence $(x_n)_{0 \le n \le N-1}$ of points of A is an *orbit of length N of the polysystem f*, when there exists a sequence $(i_n)_{0 \le n \le N-1} \in I^N$ such that for $0 \le n < N$,

$$x_{n+1} = f_{i_n}(x_n).$$

Theorem 9 (Diffusion paths using only Scattering maps). *Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian* (1)+(2). *Given any two $(I_\pm, \theta_\pm) \in \tilde{\mathcal{I}}$, where*

$$\tilde{\mathcal{I}} = \mathbb{R}^2 \times \mathbb{T}^2 \setminus \{(0,0,0,0), (0,0,\pi,0), (0,0,0,\pi), (0,0,\pi,\pi)\},$$

and any δ there exists $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$ there is an orbit $(I^i, \theta^i)_{0 \le i < N}$ of the polyscattering map (S_0, S_1, S_2)*:*

$$(I^{i+1}, \theta^{i+1}) = S_\ell(I^i, \theta^i), \text{ where } \ell \in \{0,1,2\},$$

such that

$$\left|(I^0, \theta^0) - (I_-, \theta_-)\right| < \delta \text{ and } \left|(I^N, \theta^N) - (I_+, \theta_+)\right| < \delta.$$

Proof. $\square$

Theorem 10 (Existence of Highways). *Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian* (1)+(2). *Given any $0 < c_j < C_j$, $j = 1, 2$, there is an orbit $(I^i, \theta^i)_{0 \le i < N}$ of the scattering map S_0 such that*

$$\left|I_j^0\right| < c_j \quad \text{and} \quad \left|I_j^N\right| > C_j, \qquad j = 1, 2.$$

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