Arnold diffusion for several examples of perturbation using Scattering maps

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Abstract

In this work we illustrate the Arnold diffusion for several examples of the a priori unstable Hamiltonian system of $2+1/2$ degrees of freedom $H(p,q,I,\varphi,s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + h(q,\varphi,s;\varepsilon)$—proving that for any small periodic perturbation of the form $h(q,\varphi,s;\varepsilon)$, where $h(q,\varphi;\varepsilon) = \varepsilon \cos q\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$ or $h(q,\varphi;\varepsilon) = \varepsilon\cos q\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos(\varphi - s))$

($a_{10}a_{01} \neq 0$ and $\varepsilon \neq 0$ small enough) there is global instability for the action, i.e., $I(0) \leqslant -I(\varepsilon) < I(\varepsilon) \leqslant I(T)$ for some T and for any positive $I(\varepsilon) \leqslant C\log\frac{1}{\varepsilon}$ for some constant C.

For this, we apply a geometrical mechanism based in the so-called Scattering map.

We present some similarities and differences between these cases. Besides, we study present a case with $3+1/2$ degrees of freedom, represented by Hamiltonian

$H(p,q,I_1,I_2,\varphi_1,\varphi_2,s) = \frac{p^2}{2} + \cos q - 1 + \frac{I_1^2}{2} + \frac{I_1^2}{2} + \varepsilon\cos q\,(a_{00} + a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos s)$.

Introduction

Consider an *a priori unstable* Hamiltonian with $2+\frac{1}{2}$ degrees of freedom with 2π-periodic time dependence:

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon f(q)g(\varphi,s), \tag{1}$$

where $p,\, I \in \mathbb{R}$, $q,\, \varphi,\, s \in \mathbb{T}$ and ε small enough. In the unperturbed case ($\varepsilon = 0$), the Hamiltonian H_0 represents the standard pendulum plus a rotor:

$$H_0(p,q,I,\varphi,s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2},$$

with associated equations:

$$\begin{aligned} \dot q &= \frac{\partial H_0}{\partial p} = p & \dot p &= -\frac{\partial H_0}{\partial q} = \sin q \\ \dot\varphi &= \frac{\partial H_0}{\partial I} = I & \dot I &= -\frac{\partial H_0}{\partial \varphi} = 0. \\ \dot s &= 1. \end{aligned} \tag{2}$$

and flow $\phi_t(p,q,I,\varphi,s) = (p(t),q(t),I,It+\varphi,t+s)$.

In this case, $(0,0)$ is a saddle point on the variables (p,q), i.e., it has an unstable and a stable invariant manifold, denoted by $W^u(0)$ and $W^s(0)$ respectively. Considering $P(p,q) = \frac{p^2}{2} + \cos q - 1$, we have that $P^{-1}(0)$ divides the phase space, concerning the behavior of orbits. The branches of $P^{-1}(0)$ are called *separatrices* and take the form

$$(p_0(t), q_0(t)) = \left(\frac{2}{\cosh t}, 4\arctan e^{\pm t}\right).$$

These orbits are homoclinic orbits. They are at the same time stable and unstable manifolds of the saddle point $(0,0)$.

For an initial condition $(0,0,I,\varphi,s)$, the unperturbed flow is $\phi_t(0,0,I,\varphi,s) = (0,0,I,It+\varphi,t+s)$, that is, the torus $\tau_I^0 = \{(0,0,I,\varphi,s);\ (\varphi,s) \in \mathbb{T}^2\}$ is an invariant set for the flow. τ_I^0 is called *whiskered torus*. And we called *whiskers* the set

$$W^0\tau_I^0 = \{(p_0(\tau), q_0(\tau), I, \varphi, s); \tau \in \mathbb{R}, (\varphi,s) \in \mathbb{T}^2)\}.$$

Consider now any interval $[-I^*, I^*]$, I^* any positive value, and an uncountable family of tori

$$\tilde\Lambda = \{\tau_I^0\}_{I\in[-I^*,I^*]}.$$

The set $\tilde\Lambda$ is a normally hyperbolic invariant manifold (NHIM) with coincident stable and unstable invariant manifolds:

$$W^0\tilde\Lambda = \left\{(p_0(\tau), q_0(\tau), I, \varphi, s); \tau \in \mathbb{R},\ I \in [-I^*, I^*],\ (\varphi,s) \in \mathbb{T}^2\right\}.$$

We now come back to the perturbed case, that is, $\varepsilon \neq 0$. By the theory of NHIM, if $f(q)g(\varphi,s)$ is smooth enough, there exists $\tilde\Lambda_\varepsilon$ close to $\tilde\Lambda$ and the local invariant manifolds $W^u_{loc}(\Lambda_\varepsilon)$ and $W^s_{loc}(\tilde\Lambda_\varepsilon)$ are ε-close to $W^0(\tilde\Lambda)$. But now, in general, $W^u(\tilde\Lambda_\varepsilon)$ and $W^s(\tilde\Lambda_\varepsilon)$ do not coincide, because the *separatrices* split.

The Inner and the Outer dynamics

We have two dynamics associated to $\tilde\Lambda_\varepsilon$, the inner and outer dynamics. For the study of the inner dynamics we use the *Inner map* and for the outer we use the *Scattering map*.

Inner map

The inner dynamics is the dynamics **on** the NHIM. Being $\tilde\Lambda_\varepsilon \approx \tilde\Lambda$, the Hamiltonian H_ε restricted to $\tilde\Lambda_\varepsilon$ is

$$K(I,\varphi,s;\varepsilon) = \frac{I^2}{2} + \varepsilon\, f(0)g(\varphi,s).$$

The Hamiltonian equations are:

$$\dot\varphi = I \qquad \dot I = \varepsilon f(0)\frac{\partial g}{\partial\varphi}(\varphi,s) \qquad \dot s = 1. \tag{3}$$

Scattering map

Let $\tilde\Lambda$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold $\tilde\Gamma$. A scattering map is a map $S : \tilde\Lambda \longrightarrow \tilde\Lambda$, such that $S(\tilde x_-) = \tilde x_+$ if there exists $\tilde z \in \tilde\Gamma$ satisfying

$$\begin{aligned} \phi_t(\tilde z) &\longrightarrow \phi_t(\tilde x_-) \text{ as } t \longrightarrow -\infty \\ \phi_t(\tilde z) &\longrightarrow \phi_t(\tilde x_+) \text{ as } t \longrightarrow +\infty, \end{aligned}$$

In order to define the *Scattering map* $S_\varepsilon : \tilde\Lambda_\varepsilon \to \tilde\Lambda_\varepsilon$ we have to look for homoclinic points, that is, $\tilde z \in W^u(\tilde\Lambda_\varepsilon) \pitchfork W^s(\tilde\Lambda_\varepsilon)$. We are going to use the Melnikov potential to find these points. When the intersection is not transversal, we can have problems.

Scattering map: Melnikov potential and crests We have the following preposition

Proposition 1. *Given $(I,\varphi,s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function*

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I,\varphi,s)$, where

$$\mathcal{L}(I,\varphi,s) = \int_{-\infty}^{+\infty} \left(f(q_0(\sigma))g(\varphi + I\sigma, s+\sigma; 0) - f(0)g(\varphi + I\sigma, s + \sigma; 0)\right) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde z$ to $\tilde\Lambda_\varepsilon$, which is ε-close to the point $\tilde z^*(I,\varphi,s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\tilde\Lambda)$:

$$\tilde z = \tilde z(I,\varphi,s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\tilde\Lambda_\varepsilon) \pitchfork W^s(\tilde\Lambda_\varepsilon).$$

The function $\mathcal{L}$ is called the *Melnikov potential*. We consider the following change of variables

$$\theta = \varphi - I\,s.$$

The *scattering map* has the explicit form

$$S_\varepsilon(I,\theta) = \left(I + \varepsilon\frac{\partial\mathcal{L}^*}{\partial\theta}(I,\theta) + O(\varepsilon^2), \theta - \varepsilon\frac{\partial\mathcal{L}^*}{\partial I}(I,\theta) + O(\varepsilon^2)\right).$$

where

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)).$$

$\mathcal{L}^*$ is called *the reduced Poincaré function*.

Remark 1. The level curves of the Reduced Poincaré function is very useful to understand the behavior of the scattering map.

Objective

We want to build pseudo-orbits using inner and scattering maps that is increase on the variable I. It holds the existence of a real orbit close to that pseudo-orbit, i. e., it is increase on the variable I too.

1 Case: $f(q)g(\varphi,s) = \cos q(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$

Inner dynamics

In this case, this system is integrable. Equations (3) turn out

$$\dot\varphi = I \qquad \dot I = \varepsilon a_{10}\sin\varphi \qquad \dot s = 1$$

It does not require the KAM theory.

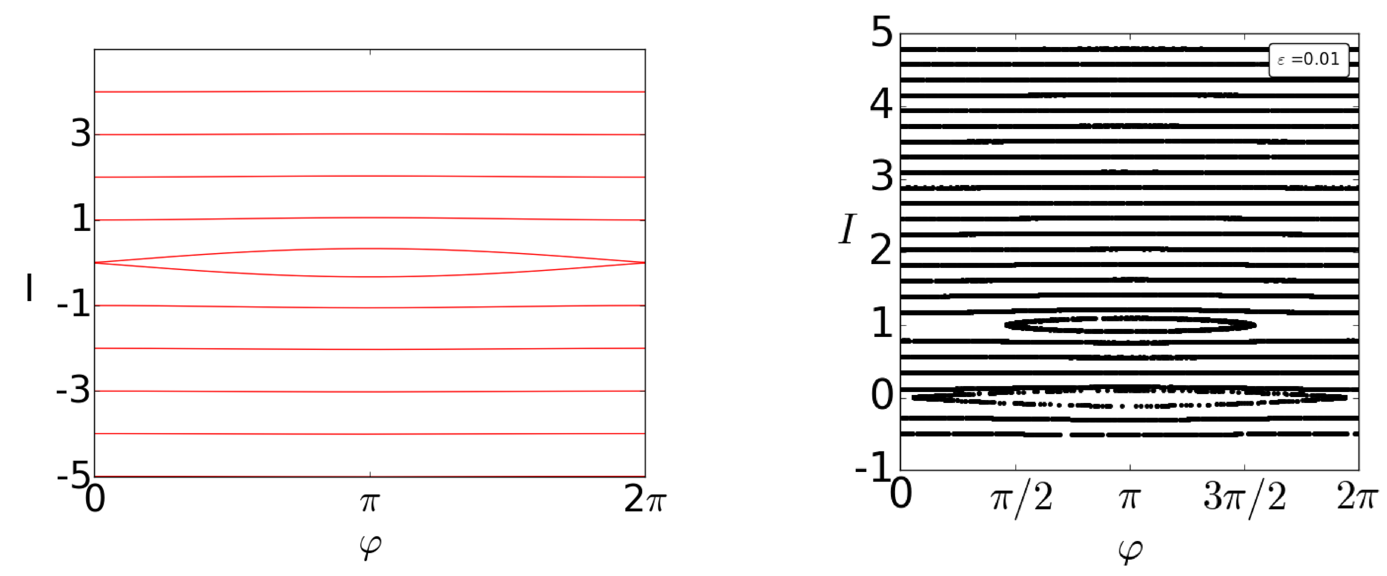

Figure 1: (a) An exemple of inner dynamics for the first case (b)Inner dynamics for the second case.

Scattering map

The Melnikov Potential is given by

$$\mathcal{L}(I,\varphi,s) = A_{00} + A_{10}(I)\cos\varphi + A_{01}\cos s,$$

where $A_{00} = 4\,a_{00}$, $A_{10}(I) = \dfrac{2\,\pi\, I\, a_{10}}{\sinh(\frac{I\,\pi}{2})}$ and $A_{01} = \dfrac{2\,\pi\, a_{01}}{\sinh(\frac{\pi}{2})}$.

Then, the critical points of $\mathcal{L}(I, \varphi - I\,\tau, s - \tau)$ are τ^* such that

$$I\,A_{10}(I)\sin(\varphi - I\,\tau^*) + A_{10}\sin(s - \tau^*) = 0.$$

Crests and several Scattering maps

Define by *crests* $\mathrm{C}(I)$, the curves on (φ, s) such that satisfy

$$I\,A_{10}(I)\sin\varphi + A_{01}\sin s = 0. \tag{4}$$

Remark 2. We have two curves satisfying the equation above, the *maximum* crest, $\mathrm{C}_M(I)$, and the *minimum* crest, $\mathrm{C}_m(I)$. The maximum crest contains the point $\varphi = 0$ and $s = 0$, and the minimum the point $\varphi = \pi$ and $s = \pi$.

Then the function $\tau^*(I,\varphi,s)$ is determined by the value τ where the straight lines $R_\theta = \{(\varphi - I\tau, s - \tau); \tau \in \mathbb{R}\}$, where $\theta = \varphi - Is$, intersects $\mathrm{C}(I)$.

Therefore, we have to study the crests. Rewrite the equation (4) as

$$\mu\alpha(I)\,\sin\varphi + \sin s = 0, \tag{5}$$

where

$$\alpha(I) = \frac{\sinh(\frac{\pi}{2})\,I^2}{\sinh(\frac{\pi I}{2})} \quad \text{and} \quad \mu = \frac{a_{10}}{a_{01}}.$$

We have to consider two cases, depending on the values of I:

- For $|\alpha(I)| < \frac{1}{|\mu|}$, the crests are horizontal, see figure 2, $\mathrm{C}_{M,m}(I) = \{(\varphi, \xi_{M,m}(I,\varphi)), \varphi \in \mathbb{T}\}$

$$\begin{aligned} \xi_M(\varphi, I) &= -\arcsin(\mu\alpha(I)\sin\varphi) && \mod 2\pi \\ \xi_m(\varphi, I) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi && \mod 2\pi \end{aligned} \tag{6}$$

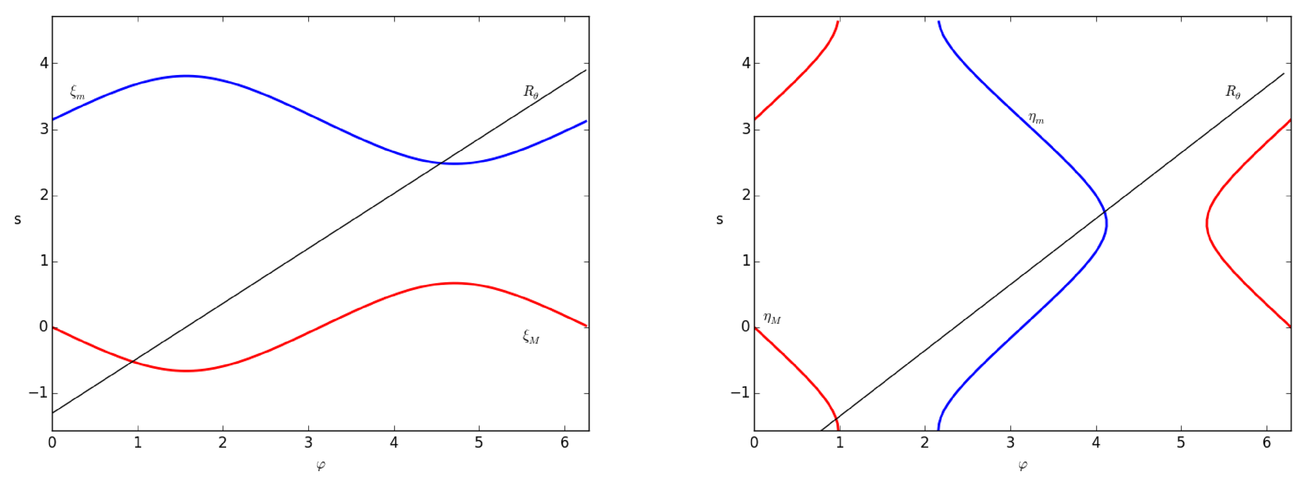

Figure 2: (a)*Horizontal* crests for $\mu := a_{10}/a_{01} = 0.6$ and $I = 1.2$ (b) *Vertical crests* for $\mu = 1.2$ and $I = 1$

- For $|\alpha(I)| > \frac{1}{|\mu|}$, the crests are vertical, see figure 2 (b), $\mathrm{C}_{M,m}(I) = \{(\eta_{M,m}(I,s), s); s \in \mathbb{T}\}$

$$\begin{aligned} \eta_M(s,I) &= -\arcsin(\sin s/(\mu\alpha(I))) \qquad \mod 2\pi \\ \eta_m(s,I) &= \arcsin(\sin s/(\mu\alpha(I))) + \pi \qquad \mod 2\pi. \end{aligned} \tag{7}$$

Remark 3. The crests for $|\alpha(I)| = \frac{1}{|\mu|}$ are not displayed (they are straight lines).

In the figure 2 (a) and (b) the straight line R_θ intersects each crest $\mathrm{C}_M(I)$ and $\mathrm{C}_m(I)$ transversally, giving rise to two values τ_M^* and τ_m^*, therefore to two different scattering maps. However, tangencies between R_θ and two crests can appear some values of I, originating new scattering maps.

The level curves of $\mathcal{L}^*(I,\theta)$ and the Arnold diffusion

Proposition 2. *Consider the function*

$$\mathcal{L}^*(I,\theta) = A_{00} + A_{10}(I)\cos(\theta - I\tau^*(I,\theta,0)) + A_{01}\cos(-\tau^*(I,\theta)),$$

where $A_{10}(I) = \frac{2\pi I a_{10}}{\sinh(I\pi/2)}$, $A_{01} = \frac{2\pi a_{01}}{\sinh(\pi/2)}$, A_{00} *a constant and* $\tau^*(I,\theta)$ *satisfies the equation:*

$$IA_{10}(I)\sin(\theta - I\tau^*(I,\theta)) + A_{01}\sin(\xi_M(I,\theta - I\tau^*(I,\theta))) = 0,$$

ξ_M *a horizontal parametrization of the maximum crest* $C_M(I)$.

Assume that

$$a_{10}\,a_{01} \neq 0.$$

Then there exists a level curve defined, at least, in $(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$ *. where* $I_+ \sim \sqrt{\frac{\pi}{2\mu\sinh(\pi/2)}}$ *and* $I_{++} \approx \frac{4\sinh(\pi/2)\log(|\mu|)}{\pi}$.

Definition 1. We call *highways* the curves in the cylinder $(I,\theta) \in \mathbb{R}\times\mathbb{T}$ such that $\mathcal{L}^*(I,\theta) = A_{00} + A_{01}$.

These *highways* are "vertical" and exist, at least, for $(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$.

The existence of the highways gives us an "easy" way for the diffusion: the pseudo-orbit is built along the highway.

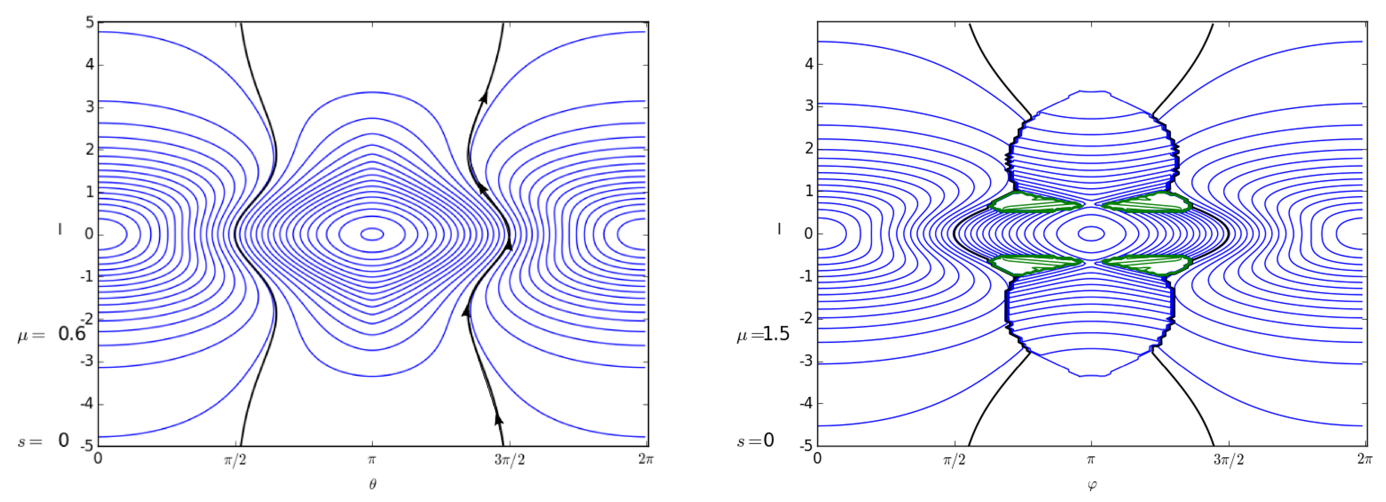

Figure 3: The level curves of $\mathcal{L}^*$ and the *highways* in black, for $a_{10}/a_{01} = 0.6$

For a general case, we can combine different Scattering maps or a Scattering map with the inner map to ensure the Arnold diffusion.

Theorem 1. *Consider a Hamiltonian of the form* (1) *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\cos\varphi + a_{01}\cos t$*. Assume that*

$$a_{10}\,a_{01} \neq 0.$$

Then, for any $I^* > 0$*, there exists* $\varepsilon^* = \varepsilon^*(I^*) > 0$ *such that for any* ε*,* $0 < \varepsilon < \varepsilon^*$*, there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

A complete study of this case can be found in [4].

2 Case: $f(q)g(\varphi,s) = \cos q(a_{00} + a_{10}\cos\varphi + a_{01}\cos(\varphi - s))$

This case is given as an example in [1].

Inner dynamics

We have that the inner dynamics is described by a Hamiltonian system with the Hamiltonian

$$K(I,\varphi,s) = \frac{I^2}{2} + \varepsilon\left(a_{1,0}\cos\varphi + a_{1,-1}\cos(\varphi - s)\right), \tag{8}$$

and differential equations

$$\dot{I} = \varepsilon\left(a_{1,0}\sin\varphi + a_{1,-1}\sin(\varphi - s)\right) \qquad \dot{\varphi} = I \qquad \dot{s} = 1. \tag{9}$$

We have two resonances $I = 0$ and $I = 1$, so it is necessary to be careful. In general our method is to restrict the inner map in such way that the inner dynamics has horizontal invariant tori.

Scattering map

Now, the *Melnikov Potential* takes the form

$$\mathcal{L}(I,\varphi,s) = A_{1,0}(I)\cos\varphi + A_{1,-1}(I)\cos(\varphi - s),$$

where

$$A_{1,0}(I) = \frac{2\pi I a_{1,0}}{\sinh(\pi/2I)} \quad \text{and} \quad A_{1,-1}(I) = \frac{2\pi(I-1)a_{1,-1}}{\sinh(\pi/2(I-1))}. \tag{10}$$

Crests $\mathcal{C}(I)$ are defined as the curves on $(I, \varphi, \varphi - s = \sigma)$, $(\varphi,\sigma) \in \mathbb{T}^2$, satisfying

$$IA_{1,0}(I)\sin\varphi + (I-1)A_{1,-1}(I)\sin\sigma = 0. \tag{11}$$

This equation can be rewritten, for $I \neq 1$ as

$$\mu\alpha(I)\sin\varphi + \sin\sigma = 0, \tag{12}$$

where

$$\mu = \frac{a_{1,0}}{a_{1,-1}} \quad \text{and} \quad \alpha(I) = \frac{I^2\sinh(\frac{\pi}{2}(I-1))}{(I-1)^2\sinh(\frac{\pi I}{2})}.$$

Remark 4. In this case, the crests have the same parameterization that in the previous case, but we use the variables I and σ.

Now, the function $\alpha(I)$ is not bounded, not symmetric to $I = 0$ and not defined in $I = 1$. It has some consequences on the scattering map:

- There is no scattering map defined on whole $\mathbb{T}\times\mathbb{R}$;
- Scattering maps are not symmetric to $I = 0$;
- Large highways do not exist.

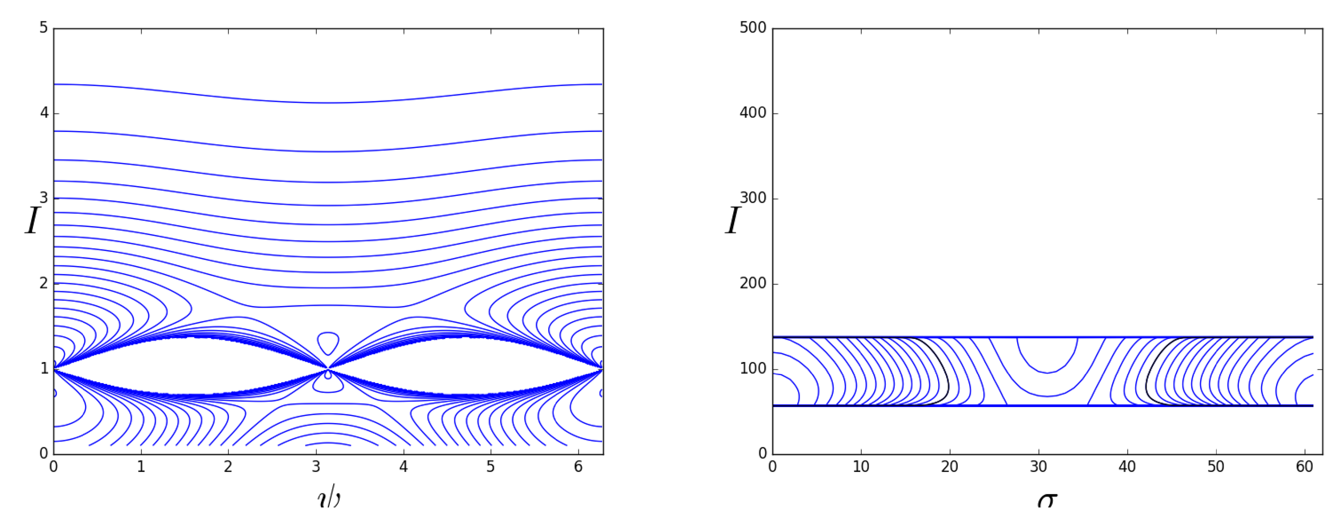

Figure 4: (a)Scattering map associated to a horizontal crest. (b)Scattering map associated to a vertical crest.

3 An example for $3 + 1/2$ degrees of freedom

We consider a Hmiltonian system represented by:

$$H_\varepsilon(I_1, I_2, \varphi_1, \varphi_2, p, q, t) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + h(I_1, I_2) + \varepsilon\cos q\, g(\varphi_1, \varphi_2, s), \tag{13}$$

where

$$h(I_1, I_2) = \Omega_1\frac{I_1^2}{2} + \Omega_2\frac{I_2^2}{2}, \text{ and } g(\varphi_1,\varphi_2,s) = a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos s.$$

Crests equation:

$$\sum_{i=1}^{3} A_i\omega_i\sin(\varphi_i) = 0, \tag{14}$$

where $\omega_i = \Omega_i I_i$ for $i = 1, 2$, $\omega_3 = -1$ and

$$A_i = A_i(\omega_i) = \frac{2\pi\omega_i}{\sinh(\pi\omega_i/2)}a_i, \qquad i = 1, 2.$$

That equation can be rewritten as

$$\sin s = \alpha_1(\omega_1)\mu_1\sin\varphi_1 + \alpha_2(\omega_2)\mu_2\sin\varphi_2, \tag{15}$$

where $\alpha_i(\omega_i) = -\omega_i^2\sinh(\pi/2)/\sinh(\omega_i\pi/2)$, $i = 1, 2$,and $\mu_i = a_i/a_3$, $i = 1, 2$.

Remark 5. It is well defined if, only if, $-1 \leq \alpha_1(\omega_1)\mu_1\sin\varphi_1 + \alpha_2(\omega_2)\mu_2\sin\varphi_2 \leq 1$. So, Besides dynamics variables φ_1 and φ_2, the 4 constants ω_1, ω_2, μ_1 and μ_2 have an important role.

This system has 3 different kinds of crests (see Fig. 5), and they appear in the following way:

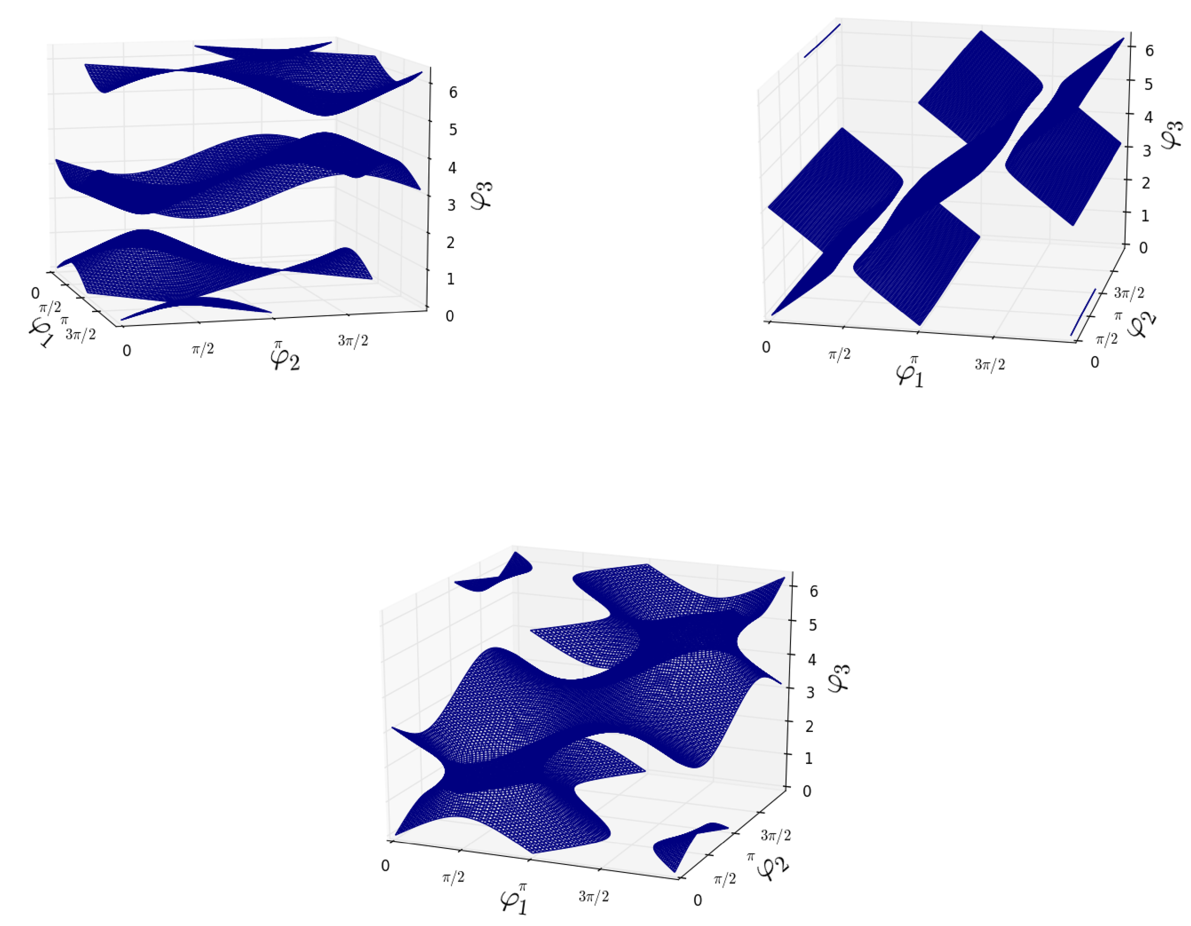

Figure 5: (a)Horizontal crests (b)Vertical crests (c)Inseparable crests.

- For $|\mu_1| + |\mu_2| \leq 0.97$ the crests are horizontal;
- For $0 < \mu_1, \mu_2 \leq 0.97$, but $|\mu_1| + |\mu_2| > 0.97$ the crests are horizontal or inseparable.
- For $\mu_i > 0.97$ $(i, j = 1, 2, i \neq j)$. The crests can be horizontal, inseparable or vertical.

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