Arnold diffusion using several combinations of Scattering maps

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Abstract

In this work we illustrate the Arnold diffusion in a concrete example—the a priori unstable Hamiltonian system of $2+1/2$ degrees of freedom $H(p,q,I,\varphi,s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + h(q,\varphi,s;\varepsilon)$—proving that for any small periodic perturbation of the form $h(q,\varphi,s;\varepsilon) = \varepsilon \cos q\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$ ($a_{10}a_{01} \neq 0$ and $\varepsilon \neq 0$ small enough) there is global instability for the action, i.e., $I(0) \leqslant -I(\varepsilon) < I(\varepsilon) \leqslant I(T)$ for some T and for any positive $I(\varepsilon) \leqslant C \log \frac{1}{\varepsilon}$ for some constant C. For this, we apply a geometrical mechanism based in the so-called Scattering map.

This work has the following structure: In a first stage, for a more restricted case ($I(\varepsilon) \sim \pi/2\mu$, $\mu = \frac{a_{10}}{a_{01}}$), we use only one Scattering map. Later, in the general case we combine a Scattering map and the inner map (inner dynamics) to prove the main result (the existence of the instability for any μ). Finally, we consider multiple combination of several scattering maps and we show different "ways of diffusion" .

Introduction

Consider an *a priori unstable* Hamiltonian with $2+\frac{1}{2}$ degrees of freedom with 2π-periodic time dependence:

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon f(q) g(\varphi,s), \tag{1}$$

where p, $I \in \mathbb{R}$, q, φ, $s \in \mathbb{T}$ and ε small enough.

In the unperturbed case, that is, $\varepsilon = 0$, the Hamiltonian H_0 represents the standard pendulum plus a rotor:

$$H_0(p,q,I,\varphi,s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2},$$

with associated equations:

$$\begin{aligned} \dot q &= \frac{\partial H_0}{\partial p} = p & \dot p &= -\frac{\partial H_0}{\partial q} = \sin q \\ \dot\varphi &= \frac{\partial H_0}{\partial I} = I & \dot I &= -\frac{\partial H_0}{\partial \varphi} = 0. \\ \dot s &= 1. \end{aligned} \tag{2}$$

and flow

$$\phi_t(p,q,I,\varphi,s) = (p(t), q(t), I, It+\varphi, t+s)\,.$$

In this case, $(0,0)$ is a saddle point on the variables (p,q), i.e., it has an unstable and a stable invariant manifold, denoted by $W^u(0)$ and $W^s(0)$ respectively. Considering $P(p,q) = \frac{p^2}{2} + \cos q - 1$, we have that $P^{-1}(0)$ divides the phase space, concerning the behavior of orbits. The branches of $P^{-1}(0)$ are called *separatrices* and take the form

$$(p_0(t), q_0(t)) = \left(\frac{2}{\cosh t}, 4\arctan e^{\pm t}\right).$$

These orbits are homoclinic orbits. They are at the same time stable and unstable manifolds of the saddle point $(0,0)$.

For an initial condition $(0,0,I,\varphi,s)$, the unperturbed flow is $\phi_t(0,0,I,\varphi,s) = (0,0,I,It+\varphi,t+s)$, that is, the torus $\tau_I^0 = \{(0,0,I,\varphi,s);\,(\varphi,s) \in \mathbb{T}^2\}$ is an invariant set for the flow. τ_I^0 is called *whiskered torus*. And we called *whiskers* the set

$$W^0\tau_I^0 = \{(p_0(\tau), q_0(\tau), I, \varphi, s);\, \tau \in \mathbb{R}, (\varphi,s) \in \mathbb{T}^2)\}.$$

Consider now any interval $[-I^*, I^*]$, I^* any positive value, and an uncountable family of tori

$$\tilde\Lambda = \{\tau_I^0\}_{I\in[-I^*,I^*]}.$$

The set $\tilde\Lambda$ is a normally hyperbolic invariant manifold (NHIM) with coincident stable and unstable invariant manifolds:

$$W^0\tilde\Lambda = \left\{(p_0(\tau), q_0(\tau), I, \varphi, s);\, \tau \in \mathbb{R},\, I \in [-I^*, I^*]\,,\, (\varphi,s) \in \mathbb{T}^2\right\}.$$

We now come back to the perturbed case, that is, $\varepsilon \neq 0$. By the theory of NHIM, if $f(q)g(\varphi,s)$ is smooth enough, there exists $\tilde\Lambda_\varepsilon$ close to $\tilde\Lambda$ and the local invariant manifolds $W^u_{loc}(\Lambda_\varepsilon)$ and $W^s_{loc}(\tilde\Lambda_\varepsilon)$ are ε-close to $W^0(\tilde\Lambda)$. But now, in general, $W^u(\tilde\Lambda_\varepsilon)$ and $W^s(\tilde\Lambda_\varepsilon)$ do not coincide, because the *separatrices* split.

Therefore, considering $f(q)g(\varphi,s) = \cos q(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$, there exists a normally hiperbolic invariant manifold $\tilde\Lambda_\varepsilon$ in the dynamics associated to the Hamiltonian

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon f(q) g(\varphi,s). \tag{3}$$

The Inner and the Outer dynamics

We have two dynamics associated to $\tilde\Lambda_\varepsilon$, the inner and outer dynamics. For the study of the inner dynamics we use the *Inner map* and for the outer we use the *Scattering map*.

Inner map

The inner dynamics is the dynamics **on** the NHIM. Being $\tilde\Lambda_\varepsilon \approx \tilde\Lambda$, the Hamiltonian H_ε restricted to $\tilde\Lambda_\varepsilon$ is

$$K(I,\varphi,s;\varepsilon) = \frac{I^2}{2} + \varepsilon\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)\,. \tag{4}$$

The Hamiltonian equations are:

$$\dot\varphi = I \qquad \dot I = \varepsilon\, a_{10}\sin\varphi \qquad \dot s = 1.$$

The inner map is the solution of this system (figure 5). In our case, this system is integrable. Therefore it does not require the KAM theory.

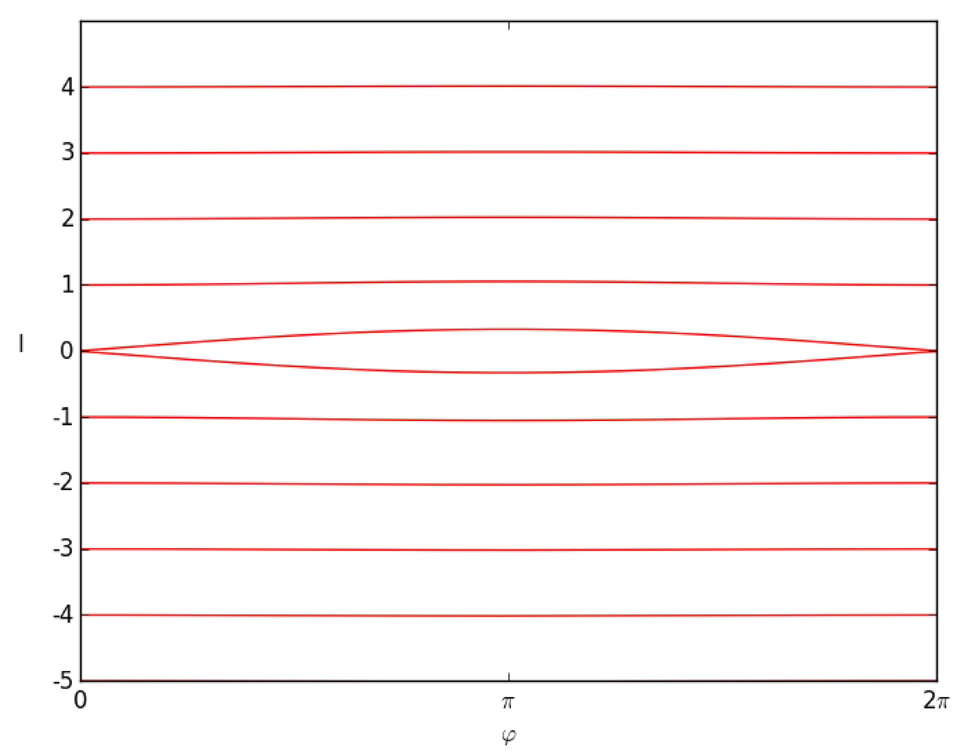

Figure 1: Inner dynamics for $a_{10}/a_{01} = 0.6$ and $\varepsilon = 0.01$

Scattering map

Let $\tilde\Lambda$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold $\tilde\Gamma$. A scattering map is a map $S : \tilde\Lambda \longrightarrow \tilde\Lambda$, such that $S(\tilde x_-) = \tilde x_+$ if there exists $\tilde z \in \tilde\Gamma$ satisfying

$$\begin{aligned} \phi_t(\tilde z) &\longrightarrow \phi_t(\tilde x_-) \text{ as } t \longrightarrow -\infty \\ \phi_t(\tilde z) &\longrightarrow \phi_t(\tilde x_+) \text{ as } t \longrightarrow +\infty, \end{aligned}$$

In order to define the *Scattering map* $S_\varepsilon : \tilde\Lambda_\varepsilon \to \tilde\Lambda_\varepsilon$ we have to look for homoclinic points, that is, $\tilde z \in W^u(\tilde\Lambda_\varepsilon) \pitchfork W^s(\tilde\Lambda_\varepsilon)$. We are going to use the Melnikov potential to find these points. When the intersection is not transversal, we can have problems.

Scattering map: Melnikov potential and crests

Melnikov potential

We have the following preposition

Proposition 1. *Given $(I,\varphi,s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function*

$$\tau \in \mathbb{R} \longmapsto \mathfrak{L}(I, \varphi - I\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I,\varphi,s)$, where

$$\mathfrak{L}(I,\varphi,s) = \int_{-\infty}^{+\infty} \left(f(q_0(\sigma))g(\varphi + I\sigma, s+\sigma; 0) - f(0)g(\varphi + I\sigma, s+\sigma; 0)\right) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde z$ to $\tilde\Lambda_\varepsilon$, which is ε-close to the point $\tilde z^*(I,\varphi,s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\tilde\Lambda)$:

$$\tilde z = \tilde z(I,\varphi,s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\tilde\Lambda_\varepsilon) \pitchfork W^s(\tilde\Lambda_\varepsilon).$$

The function $\mathfrak{L}$ is called the *Melnikov potential*. In our case

$$\mathfrak{L}(I,\varphi,s) = A_{00} + A_{10}(I)\cos\varphi + A_{01}\cos s,$$

where $A_{00} = 4\,a_{00}$, $A_{10}(I) = \dfrac{2\,\pi\, I\, a_{10}}{\sinh(\frac{I\pi}{2})}$ and $A_{01} = \dfrac{2\,\pi\, a_{01}}{\sinh(\frac{\pi}{2})}$.

Then, the critical points of $\mathfrak{L}(I, \varphi - I\tau, s - \tau)$ are τ^* such that

$$I\,A_{10}(I)\sin(\varphi - I\,\tau^*) + A_{10}\sin(s - \tau^*) = 0.$$

We consider the following change of variables

$$\theta = \varphi - I\,s.$$

The *scattering map* has the explicit form

$$S_\varepsilon(I,\theta) = (I + \varepsilon\frac{\partial}{\partial\theta}\mathfrak{L}^*(I,\theta) + O(\varepsilon^2), \theta - \varepsilon\frac{\partial}{\partial I}\mathfrak{L}^*(I,\theta) + O(\varepsilon^2)).$$

where

$$\mathfrak{L}^*(I,\theta) = \mathfrak{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)).$$

$\mathfrak{L}^*$ is called *the reduced Poincaré function*.

Crests and several Scattering maps

Define by *crests* $\mathrm{C}(I)$, the curves on (φ,s) such that satisfy

$$I\,A_{10}(I)\sin\varphi + A_{01}\sin s = 0. \tag{5}$$

Remark 1. We have two curves satisfying the equation above, the *maximum* crest, $\mathrm{C}_M(I)$, and the *minimum* crest, $\mathrm{C}_m(I)$. The maximum crest contains the point $\varphi = 0$ and $s = 0$, and the minimum the point $\varphi = \pi$ and $s = \pi$.

Then the function $\tau^*(I,\varphi,s)$ is determined by the value τ where the straight lines $R_\theta = \{(\varphi - I\tau, s-\tau);\, \tau \in \mathbb{R}\}$, where $\theta = \varphi - Is$, intersects $\mathrm{C}(I)$.

Therefore, we have to study the crests. Rewrite the equation (5) as

$$\mu\alpha(I)\,\sin\varphi + \sin s = 0, \tag{6}$$

where

$$\alpha(I) = \frac{\sinh(\frac{\pi}{2})\,I^2}{\sinh(\frac{\pi\,I}{2})}$$

and

$$\mu = \frac{a_{10}}{a_{01}}.$$

We have to consider two cases, depending on the values of I:

- For $|\alpha(I)| \le \frac{1}{|\mu|}$, the crests are horizontal, see figure 2, $\mathrm{C}_{M,m}(I) = \{(\varphi, \xi_{M,m}(I,\varphi)), \varphi \in \mathbb{T}\}$

$$\begin{aligned} \xi_M(\varphi, I) &= -\arcsin(\mu\alpha(I)\sin\varphi) && \mod 2\pi \\ \xi_m(\varphi, I) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi && \mod 2\pi \end{aligned} \tag{7}$$

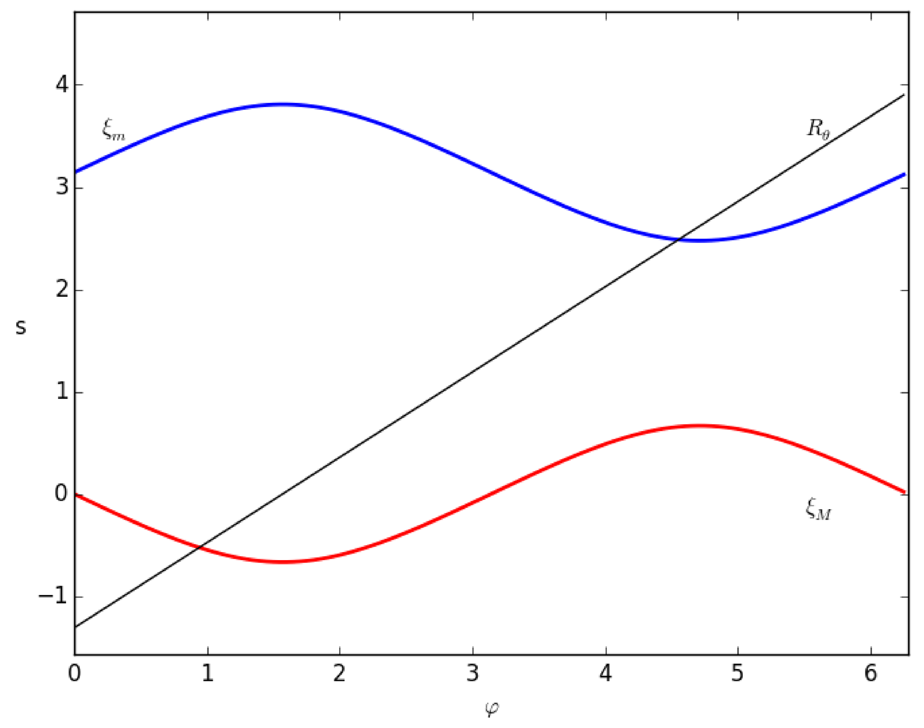

Figure 2: *Horizontal* crests for $\mu := a_{10}/a_{01} = 0.6$ and $I = 1.2$

- For $|\alpha(I)| \ge \frac{1}{|\mu|}$, the crests are vertical, see figure 3, $\mathrm{C}_{M,m}(I) = \{(\eta_{M,m}(I,s), s); s \in \mathbb{T}\}$

$$\begin{aligned} \eta_M(s, I) &= -\arcsin(\sin s/(\mu\alpha(I))) && \mod 2\pi \\ \eta_m(s, I) &= \arcsin(\sin s/(\mu\alpha(I))) + \pi && \mod 2\pi. \end{aligned} \tag{8}$$

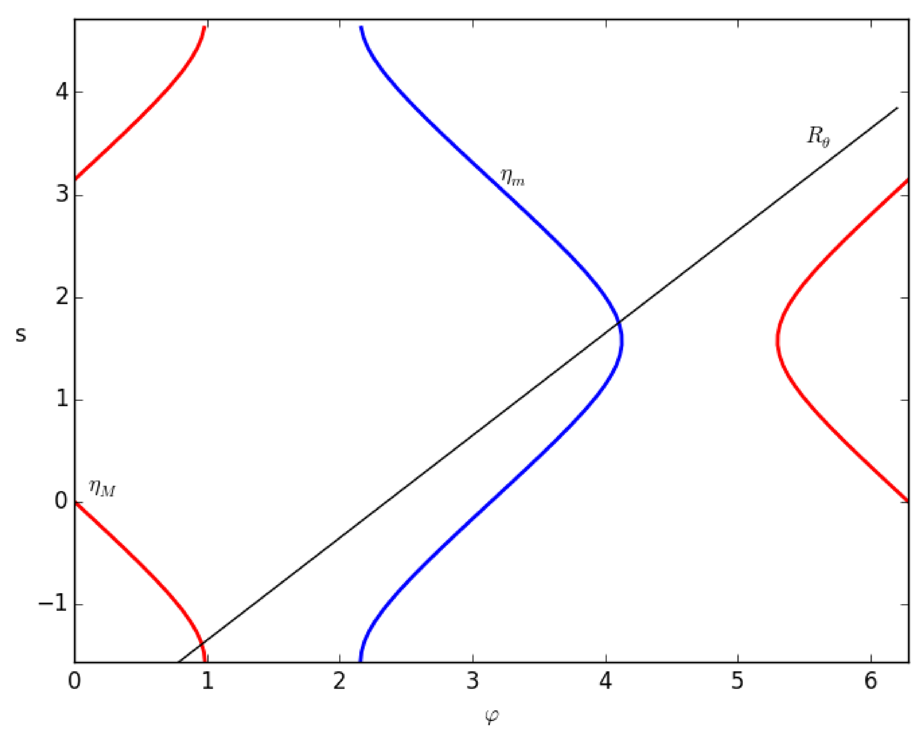

Figure 3: *Vertical crests* for $\mu = 1.2$ and $I = 1$

Remark 2. The crest for $|\alpha(I)| = \frac{1}{|\mu|}$ are not displayed (they are straight lines).

In the figure 2 and 3 the straight line R_θ intersects each crest $\mathrm{C}_M(I)$ and $\mathrm{C}_m(I)$ transversally, giving rise to two values τ_M^* and τ_m^* , therefore to two different scattering maps. However, tangencies between R_θ and two crests can appear some values of I, originating new scattering maps.

The level curves of $\mathfrak{L}^*(I,\theta)$ and the Arnold diffusion

Proposition 2. *Consider the function*

$$\mathfrak{L}^*(I,\theta) = A_{00} + A_{10}(I)\cos(\theta - I\tau^*(I,\theta,0)) + A_{01}\cos(-\tau^*(I,\theta)),$$

where $A_{10}(I) = \frac{2\pi I a_{10}}{\sinh(I\pi/2)}$, $A_{01} = \frac{2\pi a_{01}}{\sinh(\pi/2)}$, A_{00} *a constant and* $\tau^*(I,\theta)$ *satisfies the equation:*

$$IA_{10}(I)\sin(\theta - I\tau^*(I,\theta)) + A_{01}\sin(\xi_M(I, \theta - I\tau^*(I,\theta))) = 0,$$

ξ_M *a horizontal parametrization of the maximum crest* $C_M(I)$.
Assume that

$$a_{10}\, a_{01} \neq 0.$$

Then there exists a level curve defined, at least, in $(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$ *. where* $I_+ \sim \sqrt{\frac{\pi}{2\mu\sinh(\pi/2)}}$ *and* $I_{++} \approx \frac{4\sinh(\pi/2)\log(|\mu|)}{\pi}$.

Definition 1. We call *highways* the curves in the cylinder $(I,\theta) \in \mathbb{R}\times\mathbb{T}$ such that $\mathfrak{L}^*(I,\theta) = A_{00} + A_{01}$.

These *highways* are "vertical" and exist, at least, for $(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$.
The existence of the highways gives us an "easy" way for the diffusion:

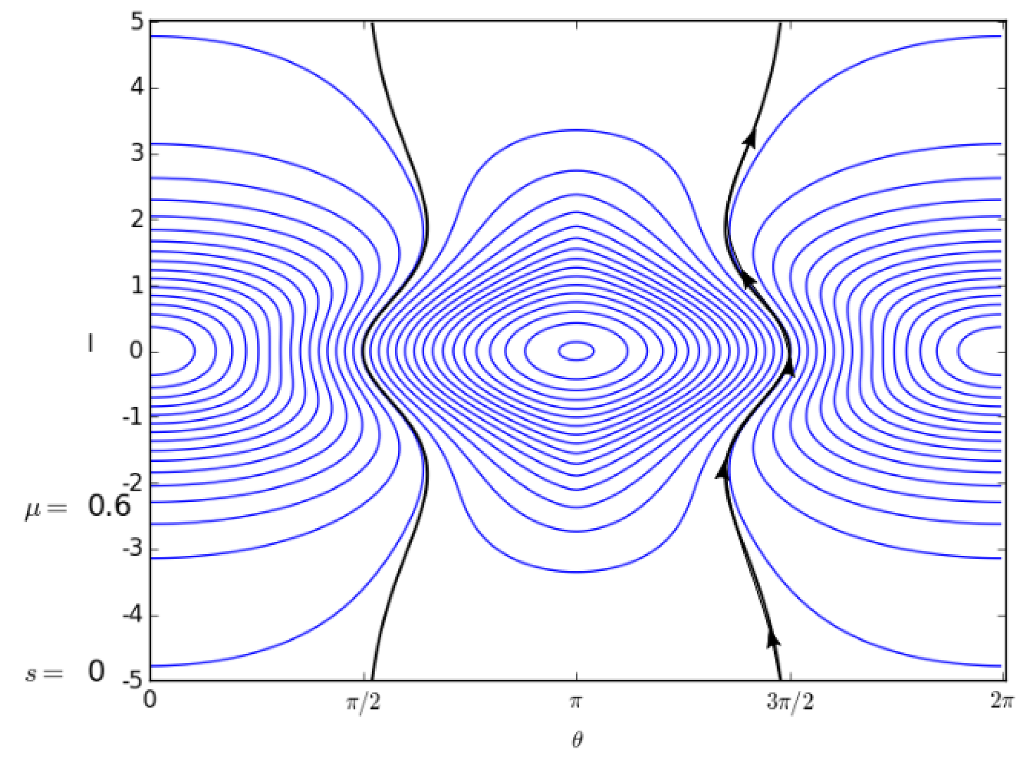

Figure 4: The level curves of $\mathfrak{L}^*$ and the *highways* in black, for $a_{10}/a_{01} = 0.6$

Theorem 1. *Consider a Hamiltonian of the form* $H_\varepsilon(p,q,I,\varphi,t) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + \varepsilon f(q)\, g(\varphi, t)$, *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\cos\varphi + a_{01}\cos t$. *Assume that*

$$a_{10}\, a_{01} \neq 0$$

Then, there exists $\varepsilon^* = \varepsilon^*(I^*) > 0$ *such that for*

$$I_+ \sim \sqrt{\frac{\pi}{2\mu\sinh(\pi/2)}} \tag{9}$$

and $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \le 0; \qquad I(T) \ge I_+.$$

But if we want to guarantee the diffusion for a more general case, we have to combine different Scattering maps or a Scattering map with the inner map.

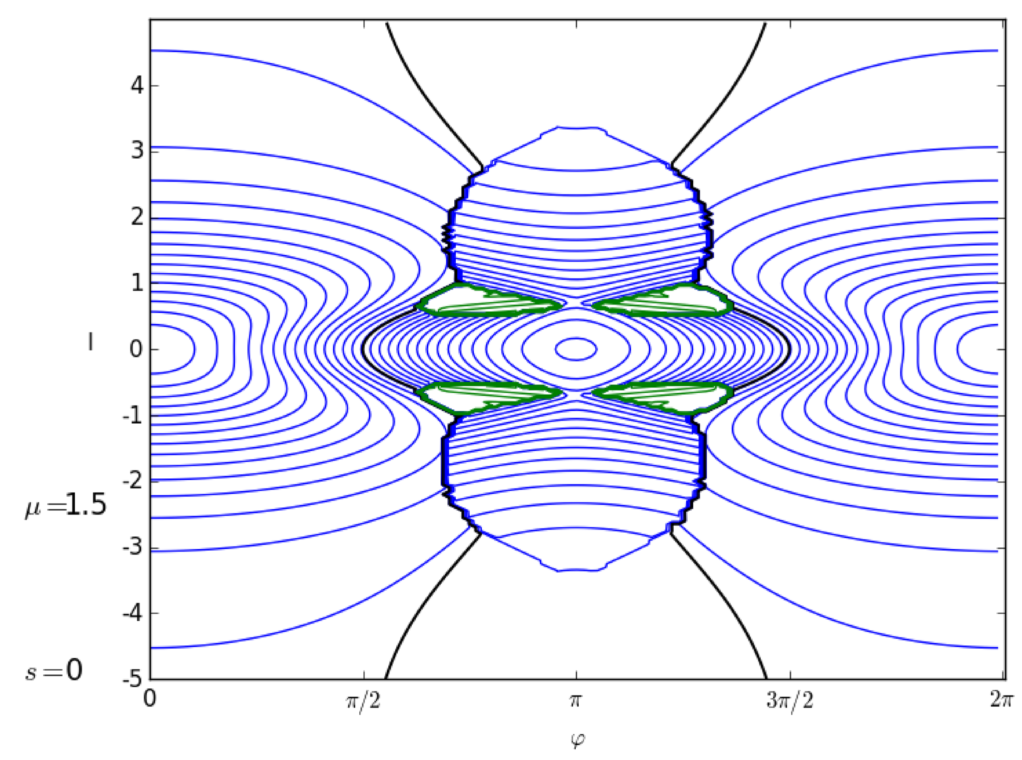

Figure 5: The level curves of $\mathfrak{L}^*(I,\theta)$ for $\mu = a_{10}/a_{01} = 1.7$. Notice that the highways are not defined for all I

Theorem 2. *Consider a Hamiltonian of the form* $H_\varepsilon(p,q,I,\varphi,t) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + \varepsilon f(q)\, g(\varphi, t)$, *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\cos\varphi + a_{01}\cos t$. *Assume that*

$$a_{10}\, a_{01} \neq 0$$

Then, for any $I^* > 0$, *there exists* $\varepsilon^* = \varepsilon^*(I^*) > 0$ *such that for any* $-I^* < I_- < I_+ \le I^*$ *and* $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \le I_-; \qquad I(T) \ge I_+.$$

References

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Acknowledgements

This work was supported by the Catalan Grant 2014SGR504. Rodrigo Schaefer is supported by CNPq/CsF GDE 246579/2013-7.