Deterministic Modelling: Exercise 5

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We consider the map $x_{n+1} = x_n e^{\mu(1-x_n)}$ for $0 < \mu$.

Fixed Points

We want to find the points with the property $x^* = x^* e^{\mu(1-x^*)}$. The first obvious fixed point is $x_1^* = 0$. We get the second fixed point with:

$$\begin{aligned} x^* = x^* e^{\mu(1-x^*)} &\overset{:x}{\Leftrightarrow} 1 = e^{\mu(1-x^*)} \\ &\overset{\ln}{\Leftrightarrow} 0 = \mu(1-x^*) \\ &\overset{:\mu}{\Leftrightarrow} x^* = 1 \end{aligned}$$

and hence $x_2^* = 1$. Note that we can divide by μ because we assumed μ to be strictly greater than 0. Furthermore one can see in Figure 1 a general plot of the considered function for three different μ together with the diagonal $f(x) = x$. There are no more fixed points.

Stability

To check stability of a fixed point we have to examine the first derivative

$$|f'(x^*)| < 1$$

at a fixed point x^*.

By applying the product rule we get for the first derivative of $f(x) = xe^{\mu(1-x)}$:

$$f'(x) = (1 - \mu x)e^{\mu(1-x)}$$

Is $x_1^* = 0$ stable?

$$f'(x_1^*) = e^{\mu} \Rightarrow f'(x_1^*) < 1 \Leftrightarrow \mu < 0$$

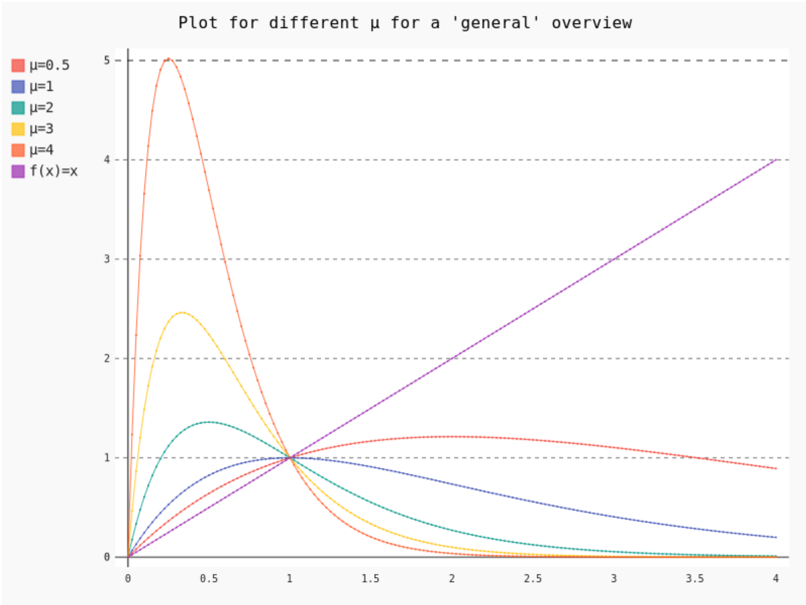

Figure 1: General Plot for different μ. Additionally the diagonal $f(x) = x$ is plotted for a better visualization of the fixed points.

Thus $x_1^* = 0$ is not stable for all (considered) $\mu > 0$.

Is $x_2^* = 1$ stable?

$$\begin{aligned} f'(x_2^*) = (1-\mu)e^0 = 1-\mu &\Rightarrow |f'(x_2^*)| < 1 \\ &\Leftrightarrow |1-\mu| < 1 \\ &\Leftrightarrow 0 < \mu < 2 \end{aligned}$$

Therefore $x_2^* = 1$ is stable for all $0 < \mu < 2$. Otherwise it is unstable.

Bifurcation Diagrams

In Figure 2 and 3 are the plots of the respective maps.

Note: In the zip file one can find each plot as a .png and .svg file. Especially the latter file format offers a better interactivity if its opened with a suitable program (like a web browser).

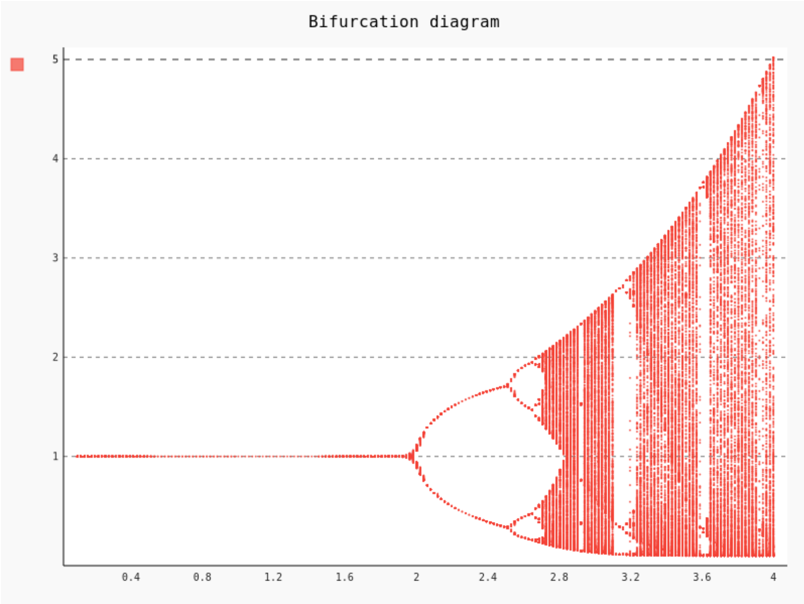

Figure 2: Orbit Diagram of $x_{n+1} = x_n e^{\mu(1-x_n)}$ for different $0 < \mu \leq 4$.

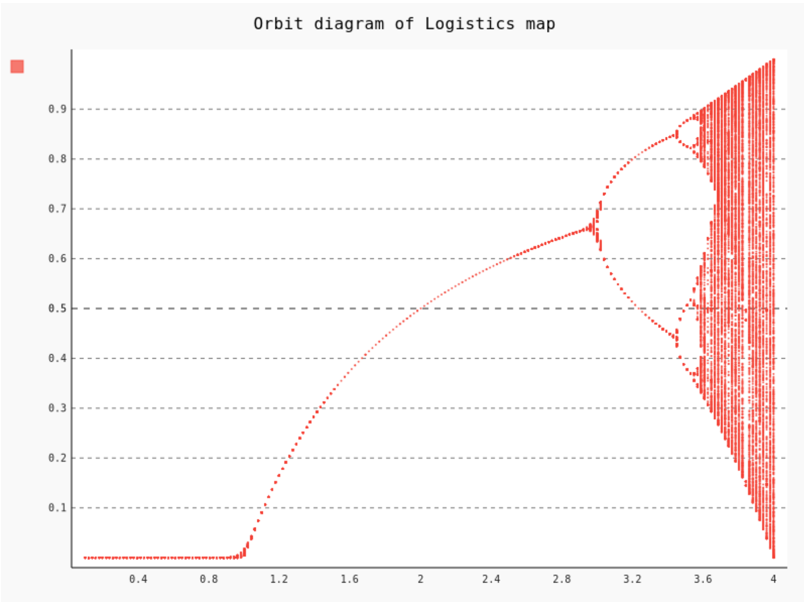

Figure 3: Orbit Diagram of a logistics map $x_{n+1} = rx_n(1-x_n)$ for $0 < r \leq 4$.