UNIVERSITÄT BERN

Applied Optimization
Exercise 1 - Convex Sets

September 15, 2021

Hand-in instructions:

Please hand-in **only one** compressed file named after the following convention: `Exercisen-GroupMemberNames.zip`, where n is the number of the current exercise sheet. This file should contain:

- The complete code folder except for the `build` subfolder. Make sure that your code submission compiles on your machine. Zip the code folder.
- A `readme.txt` file containing a description on how you solved each exercise (use the same numbers and titles) and the encountered problems.
- Other files that are required by your `readme.txt` file. For example, if you mention some screenshot images in `readme.txt`, these images need to be submitted too.
- Submit your solutions to ILIAS before the submission deadline.

Convex Sets (7 pts)

Example sets (2 pt)

Sketch the following sets in $\mathbb{R}^2$

1. $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}\right\}$
2. $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}\right\}$
3. $\operatorname{aff}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$
4. $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}\right\}$

Convexity (1 pt)

Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + \ldots + \theta_k = 1$. Show that $\theta_1 x_1 + \ldots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Linear Equations (1 pt)

Show that the solution set of linear equations $\{x | Ax = b\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ is an affine set.

Linear Inequations (1 pt)

1. Show that the solution set of linear inequations $\{x | Ax \preceq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$ and $d \in \mathbb{R}^k$ is a convex set. Here $\preceq$ means componentwise less or equal.
2. Is it an affine set?

Voronoi description of halfspace (2 pt)

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x | \|x - a\|^2 \leq \|x - b\|^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Convex Illumination Problem (3 pts)

Show that the solution $p^* = (p_1^*, p_2^*, \ldots, p_n^*)^T \in \mathbb{R}^n$ of the non-convex illumination problem from the lecture

$$\begin{array}{ll} \text{minimize} & \max_{k=1\ldots m} |\log I_k - log I_{des}| \\ \text{subject to} & 0 \leq p_j \leq p_{max}, \quad j = 1\ldots n \end{array}$$

with $I_k = \sum_{j=1}^{n} a_{kj} p_j$ for geometric constants $a_{jk} \in \mathbb{R}$, a constant desired illumination $I_{des} \in \mathbb{R}$ and an upper bound $p_{max} \in \mathbb{R}$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{array}{ll} \text{minimize} & \max_{k=1\ldots m} h(I_k / I_{des}) \\ \text{subject to} & 0 \leq p_j \leq p_{max}, \quad j = 1\ldots n \end{array}$$

with $h(u) = \max\{u, 1/u\}$.