Exercise 4 - Duality

Swiss Joint Master of Science in Computer Science - Applied Optimization

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```
%matplotlib inline
%load_ext autoreload
%autoreload 2

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches
from scipy import optimize
from matplotlib import patheffects
```

Lagrange Duality (3 pts)

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & x^2 + 1 & (1) \\ \text{subject to} \quad & (x-2)(x-4) \leq 0 & (2) \end{aligned}$$

with variable $x \in \mathbb{R}$.

(a) Analysis of primal problem: give the feasible set, the optimal value, and the optimal solution.

(b) Lagrangian and dual function: plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq inf_x L(x, \lambda)$ for $\lambda \geq 0$). Compute and sketch the Lagrange dual function g.

(c) Lagrange dual problem: state the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ. Does strong duality hold?

(a)

The feasible set is $[2, 4]$

The optimal solution is $x = 2$

The optimal value is $y = 5$

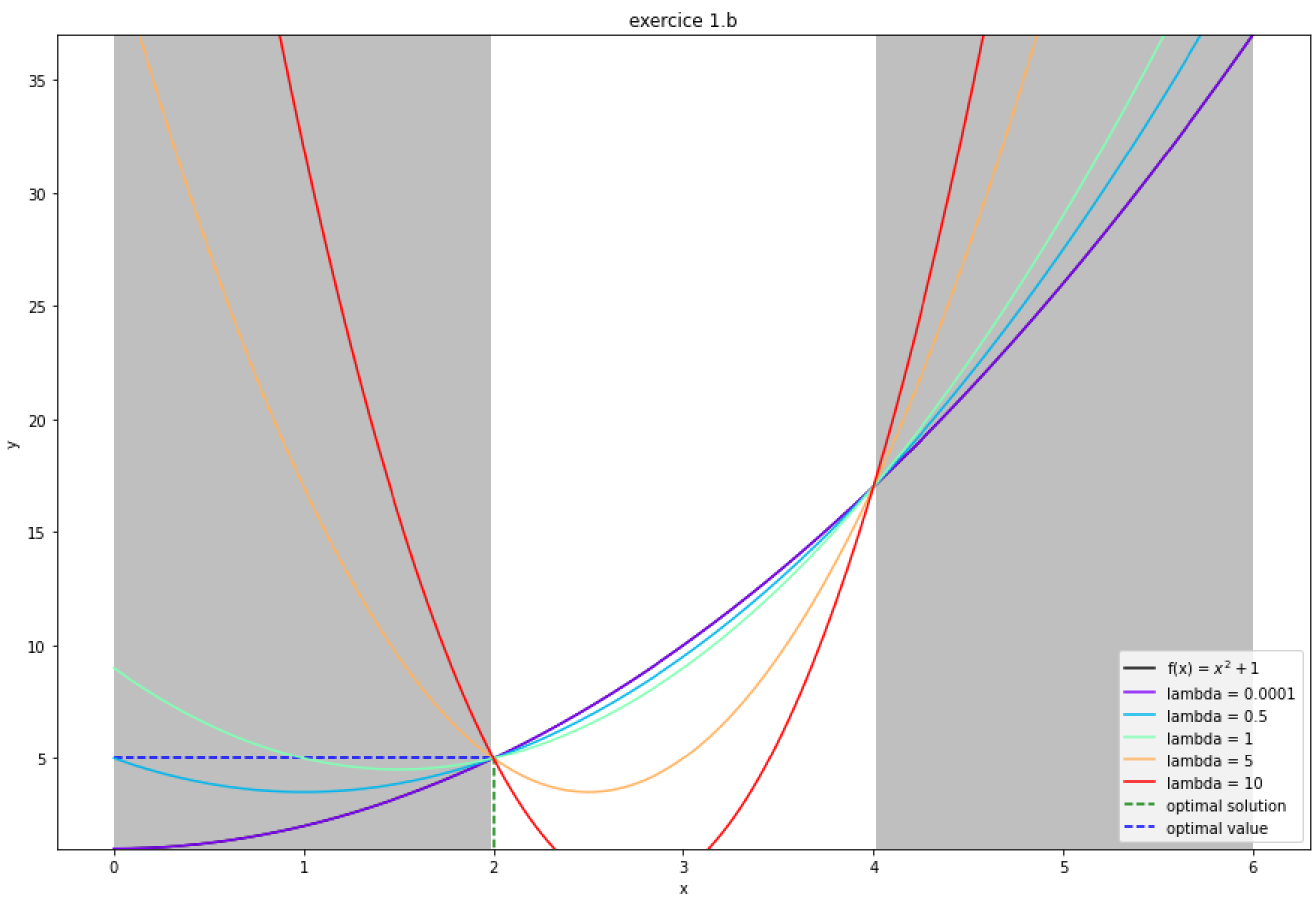
(b)

Lagrangian and dual function: plot the objective $x^2 + 1$. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ for a few positive values of λ

The function to optimize is below the violet one. Play with the lambda value to make it visible again.

```
f = lambda x: x**2 + 1
lagrangian = lambda x,l: x**2 + 1 + l*(x**2 - 6*x + 8)

plt.figure(figsize=(15, 10))

x = np.linspace(0, 6, 200)
y = f(x)

#Plot function to minimize
plt.plot(x, y, color='black',label="f(x) = $x^2 + 1$")
```

```python
#Add constraints
plt.fill_between(x, y.min(), y.max(), where=x > 4,
                 facecolor='grey', alpha=0.5)
plt.fill_between(x, y.min(), y.max(), where=x < 2,
                 facecolor='grey', alpha=0.5)

#Add solution and value
plt.vlines(2, 0, f(2), linestyle="dashed",label="optimal solution",colors="green")
plt.hlines(f(2), 0, 2, linestyle="dashed",label="optimal value",colors="blue")

#Add different lagrangian
lambda_value = [0.0001, 0.5,1,5,10]
color = plt.cm.rainbow(np.linspace(0, 1, len(lambda_value)))

for l,c in zip(lambda_value,color):
    plt.plot(x,lagrangian(x,l),color=c,label=f"lambda = {l}")

plt.ylim(y.min(),y.max())
plt.xlabel("x")
plt.ylabel("y")
plt.title("exercice 1.b")
plt.legend()

plt.show()
```



Verify the lower bound property ($p^* \geq inf_x L(x, \lambda)$ for $\lambda \geq 0$)

We can see it graphically. We can play with the lambda value

By calcul:

Be x in our feasible set $[2, 4]$, λ set.

Then $\lambda(x^2 - 6x + 8) \leq 0$

$\rightarrow L(x, \lambda) \leq f(x)$

$\rightarrow inf_x L(x, \lambda) \leq inf_x f(x)$

$\rightarrow inf_x L(x, \lambda) \leq p^*$

Compute and sketch the Lagrange dual function g

$g(\lambda) = inf_x L(x, \lambda) = inf_x x^2 + 1 + \lambda(x^2 - 6x + 8)$

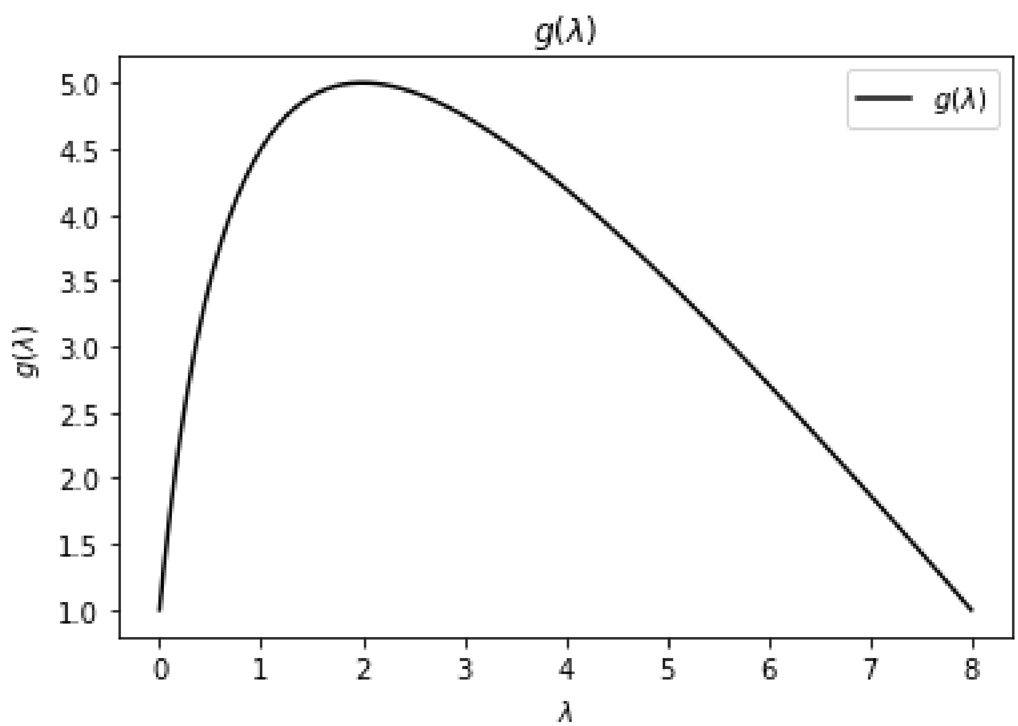
$\rightarrow g(\lambda) = inf_x (1 + \lambda)x^2 - 6\lambda x + 9\lambda$

$\frac{\partial L(x,\lambda)}{\partial x} = 2(1 + \lambda)x - 6\lambda = 0$

$\rightarrow x = \frac{3\lambda}{(1+\lambda)}$

$\rightarrow g(\lambda) = \frac{-\lambda^2+9*\lambda+1}{(\lambda+1)}$

```
g = lambda l: (1+9*l-l**2)/(1+l)

lambdas = np.linspace(0,8,200)

plt.plot(lambdas,g(lambdas), color='black',label="$g(\lambda)$")

plt.xlabel("$\lambda$")
plt.ylabel("$g(\lambda)$")
plt.title("$g(\lambda)$")
plt.legend()

plt.show()
```



(c)

Lagrange dual problem: state the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ. Does strong duality hold?

The Lagrange dual problem is basically:

maximize $g(\lambda) = \frac{-\lambda^2+9*\lambda+1}{(\lambda+1)}$

subject to $\lambda \geq 0$

We know that it is a maximization problem, because $\frac{p(x)}{q(x)}$ with deg(p) = 2 and deg(q) = 1 has a max.

It is also concave because $-g(\lambda) = \frac{p(x)}{q(x)}$ with p(x) convexe (polynome of degree 2) and q(x) affine non-decreasing. So $-g(\lambda)$ is convexe, therefore $g(\lambda)$ is concave

```
solution_l = optimize.fmin(lambda x: -g(x),1)[0]

print("the dual optimal value is ", g(solution_l), " for the dual optimal solution lambda = ", solution_l)
```

```
Optimization terminated successfully.
         Current function value: -5.000000
         Iterations: 17
         Function evaluations: 34
the dual optimal value is  5.0  for the dual optimal solution lambda =  2.000000000000002
```

The strong duality hold:

we see that $p* = 5 = d*$

As well as $f(x^*) = f(2) = 5 = g(2) = g(\lambda^*)$

KKT Condition (2 pts)

Consider the following optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & x_1^2 - 2x_2^2 && (3)\\
\text{subject to} \quad & (x_1+4)^2 - 2 \leq x_2 && (4)\\
& x_1 - x_2 + 4 = 0 && (5)\\
& x_1 \geq -10 && (6)
\end{aligned}
$$

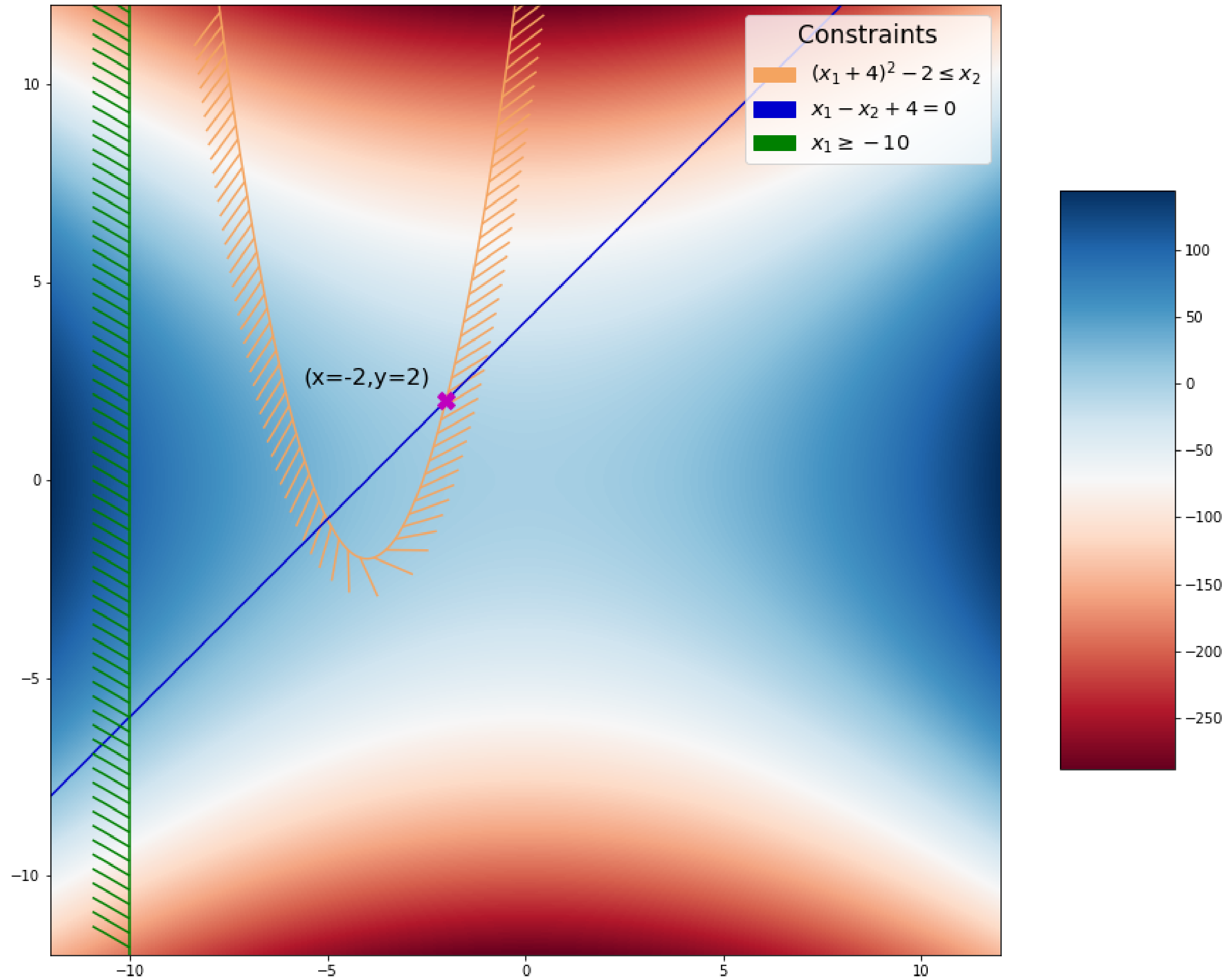
(1) sketch the problem and graphically determine the primal solution x^*.

(2) verify your x^* by determining suitable λ^* and v^* such that the KKT conditions are satisfied for (x^*, λ^*, v^*).

(1) Sketch the problem and determine x^*

```
fig, ax = plt.subplots(figsize=(15, 15))

x = np.linspace(-12.0, 12.0, 500)
y = np.linspace(-12.0, 12.0, 500)

X, Y = np.meshgrid(x, y)

# Objective function
obj = X**2 - 2 * Y**2

#Constraint 1: (x+4)^2 - 2 <= y
g1 = (X + 4)**2 - 2 - Y

#Constraint 2: x - y + 4 = 0
g2 = X - Y + 4

#Constraint 3: x >= -10
g3 = -X - 10

#Plot the function to optimize, we want the min so a Red value
im = plt.imshow(obj,cmap=plt.cm.RdBu,extent=[-12,12,-12,12],origin="lower")

#Plot constraint 1
cg1 = ax.contour(X, Y, g1, [0], colors='sandybrown')
plt.setp(cg1.collections,
         path_effects=[patheffects.withTickedStroke(angle=135,length=3)])
cg1_legend = mpatches.Patch(color='sandybrown', label=r"$(x_1+4)^2 - 2 \leq x_2$")

#contraint 2, remember we must be on this line as it is an equality constraint
cg2 = ax.contour(X, Y, g2, [0], colors='mediumblue')
cg2_legend = mpatches.Patch(color='mediumblue', label=r"$x_1 - x_2 + 4 = 0$")

#plot constraint 3
cg3 = ax.contour(X, Y, g3, 0, colors='green')
plt.setp(cg3.collections,
         path_effects=[patheffects.withTickedStroke(angle=60, length=3)])
cg3_legend = mpatches.Patch(color='green', label=r"$x_1 \geq -10$")

fig.colorbar(im, shrink=0.5, aspect=5)
```

```
ax.plot(-2, 2, 'mX', markersize=12)
ax.text(-2.4, 2.4, f"(x=-2,y=2)", ha="right", fontsize=16)

plt.legend(handles=[cg1_legend, cg2_legend, cg3_legend], title="Constraints",
           title_fontsize="xx-large", fontsize="x-large")

plt.show()
```



We can see that the solution will be approximately $x^* = (x, y) = (-2, 2)$

We must follow the blue line (constraint 2: equality constraint, but we are blocked by constraint 1). We can rapidly calculate it by taking the root of:

$$(x+4)^2 - 2 - (x+4)$$

which are -5 and -2. We keep -2 according to the graphic, and compute $x + 4 = y$ for $x = -2 \rightarrow y = 2 \rightarrow x^* = (x, y) = (-2, 2)$

(2) Verify x^* by determining suitable λ^* and v^* *st.* KKT conditions are satisfied

We have:

1) Primal constraints:

- $(x+4)^2 - 2 - y \leq 0$
- $-x - 10 \leq 0$
- $x - y + 4 = 0$

2) Dual constraints:

- $\lambda_1 \geq 0$
- $\lambda_2 \geq 0$

3) Complementary slackness:

- $\lambda_1((x+4)^2 - 2 - y) = 0$
- $\lambda_2(-x - 10) = 0$

4) Vanishing gradient of Lagrangian:

- $\nabla_x L(x, y, \lambda_1, \lambda_2, \upsilon) = \nabla f_0(x, y) + \lambda_1 \nabla f_1(x, y) + \lambda_2 \nabla f_2(x, y) + \upsilon \nabla h_1(x, y) = 0$

$\rightarrow \begin{pmatrix} 2x \\ -2y \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x+8 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \upsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The slackness

- $\lambda_2(-x - 10) = 0$

tells us that $\lambda_2 = 0$, because clearly $x \geq -10$, gradient of objective is annihilated by **active** constraint grad

We can then compute λ_1 and υ

With (x,y) = (-2,2) the point to explore according to our graphical exploration, we have for the gradient:

- $-4 + 12\lambda_1 + \upsilon = 0$
- $-4 - \lambda_1 - \upsilon = 0$

And we get directly by resolution

- $\lambda_1 = 8/11$
- $\upsilon = -52/11$

So we have $\lambda^* = (\frac{8}{11}, 0)$ and $\nu = \frac{-52}{11}$ which respect the KKT conditions for $x^* = (-2, 2)$