Applied Optimization
Exercise 1 - Convex Sets

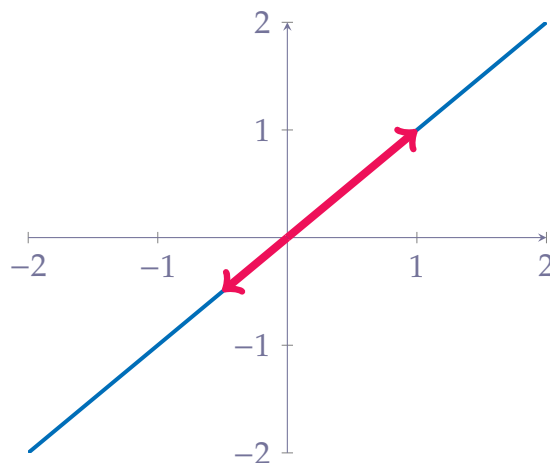V. Carrel & J. Fontana & A. Schaller
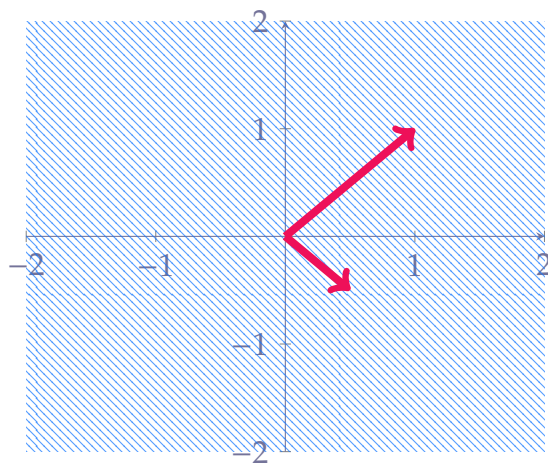Prof.: Prof. David Bommes

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1 Convex Sets

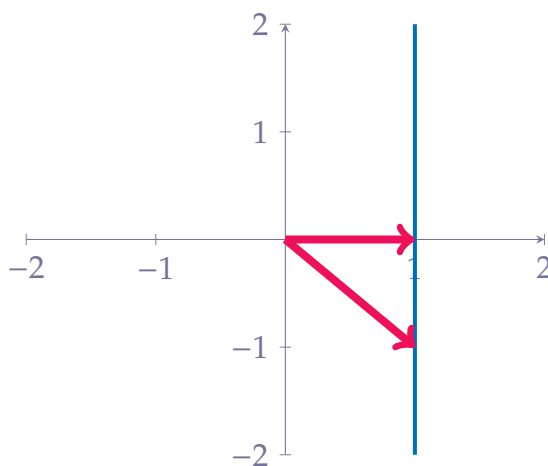**1. Example sets**

Sketch the following sets in $\mathbb{R}^2$

1. span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \right\}$
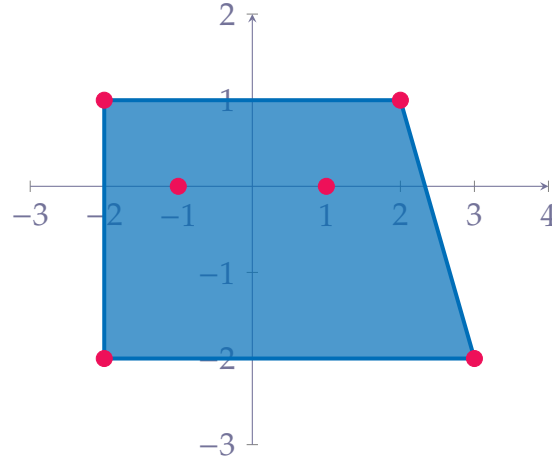
2. span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$

3. aff $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

4. $\mathrm{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}-2\\1\end{pmatrix},\begin{pmatrix}-2\\-2\end{pmatrix}\right\}$

2. Convexity

Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Show. that $\theta_1 + ... + \theta_k \in C$

Proof. <u>Base Case $k = 2$</u> By definition, it holds for $k = 2$.

<u>Inductive Step</u> Assume it is correct for $k = n$, then for $k = n + 1$, we have:

$$x_1, ..., x_{n+1} \in C$$

$$\theta_1, ..., \theta_{n+1} \in C$$

with $\theta_i \geq 0$ and $\sum_{i=0}^{n+1} \theta_i = 1 \Rightarrow \theta_{n+1} = 1 - \sum_{i=1}^{n} \theta_i$

$$\Rightarrow \sum_{i=1}^{n+1} \theta_i x_i = \sum_{i=1}^{n} \theta_i x_i + \theta_{n+1} x_{n+1} = \sum_{i=1}^{n} \theta_i x_i + x_{n+1} = \sum_{i=1}^{n} \theta_i x_{n+1}$$

$$= \underbrace{x_{n+1}}_{\in C} \left. \underbrace{\sum_{i=1}^{n} \theta_i (x_i - x_{n+1})}_{\substack{\in C \text{ by hypothesis on } k = n \\ \text{because } \sum_{i=1}^{n} \theta_i \in [0,1]}} \right\} \Rightarrow \in C$$

$\square$

3. Linear Equations

Show that the solution set of linear equations $\{x|Ax = b\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ is an affine set.

Proof. Let x_0, x_1 be solutions of linear equations, i.e. $Ax_0 = b$ and $Ax_1 = b$

Then

$$A((1 - \beta)x_0 + \beta x_1) = (1 - \beta)Ax_0 + \beta Ax_1$$
$$= (1 - \beta)b + \beta b = b$$

$\Rightarrow$ The set is affine. $\qquad\qquad\square$

4. Linear Inequations

1. Show that the solution set of linear inequations $\{x|Ax \leq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$ and $d \in \mathbb{R}^k$ is a convex set. Here $\leq$ means componentwise less or equal.

   Proof. Let x_0, x_1 be solutions of linear inequations, i.e. $Ax_0 \leq b, Cx_0 = d$ and $Ax_1 \leq b, Cx_1 = d$, for $\beta \in [0, 1]$

   Then

   $$A((1 - \beta)x_0 + \beta x_1) = \underbrace{(1 - \beta)}_{\geq 0} \underbrace{Ax_0}_{\leq b} + \underbrace{\beta}_{\geq 0} \underbrace{Ax_1}_{\leq b} \leq (1 - \beta)b + \beta b = b \qquad (1)$$

   and

   $$C((1 - \beta)x_0 + \beta x_1) = (1 - \beta)Cx_0 + \beta Cx_1 = (1 - \beta)d + \beta d = d$$

   $\Rightarrow$ convex set. $\qquad\qquad\square$

2. Is it an affine set?

   No, cf Equation 1, if we don't have the $a_i \geq 0$ for $\forall i$ constraint, it removes the constraint on the $\beta \in [0, 1]$, which then means we could have $(1 - \beta) < 0$. Therefore the step $(1 - \beta)Ax_0 \leq (1 - \beta)b$ is not doable, we might have $(1 - \beta)Ax_0 \geq (1 - \beta)b$. $\qquad\blacksquare$

5. Voronoi description of halfspace

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x\,|\,\|x-a\|^2 \leq \|x-b\|^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Proof. Be $c \in \mathbb{R}^n$ a vector, $c = b - a$ be $d \in \mathbb{R}$, $d = c^T x_0 = c^T \frac{a+b}{2}$
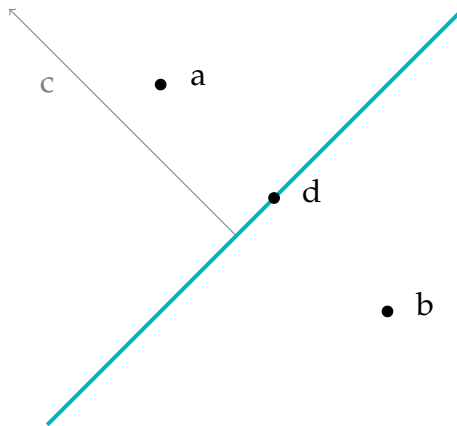
then we have the halfspace

$$\left\{x \,\middle|\, (b-a)^T x \leq \frac{(b-a)^T(b-a)}{2}\right\}$$

$\square$

Check.

$$\|x-a\|^2 \leq \|x-b\|^2$$

$$\Leftrightarrow (x-a)^T(x-a) \leq (x-b)^T(x-b)$$

$$\Leftrightarrow x^T x - 2a^T x + a^T a \leq x^T x - 2b^T x + b^T b$$

$$\Leftrightarrow (b-a)^T x \leq (b^T b - a^T a)\,\frac{1}{2}$$

2 Convex Illumination Problem

Show that the solution $p^* = (p_1^*, p_2^*, ..., p_*^n)^T \in \mathbb{R}^n$ of the non-convex illumination problem from the lecture

$$\text{minimize} \quad \max_{k=1...m} \left| \log I_k - \log I_{des} \right|$$
$$\text{subject to} \quad 0 \le p_j \le p_{max}, \quad j = 1...n$$

with $I_k = \sum_{j=1}^{n} a_{kj} p_j$ for geometric constants $a_{jk} \in \mathbb{R}$, a constant desired illumination $I_{des} \in \mathbb{R}$ and an upper bound $p_{max} \in \mathbb{R}$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\text{minimize} \quad \max_{k=1...m} h(I_k/I_{des})$$
$$\text{subject to} \quad 0 \le p_j \le p_{max}, \quad j = 1...n$$

with $h(u) = max\{u, \frac{1}{u}\}$.

Proof.

$$\max_{k=1,...,m} \left| \log(I_k) - \log(I_{des}) \right| \qquad \Big| \text{log rules}$$

$$= \max_{k=1,...,m} \left| \log\left(\frac{I_k}{I_{des}}\right) \right| \qquad \Big| \text{remove } |.|$$

$$= \max_{k=1,...,m} \max\left\{ \log\left(\frac{I_k}{I_{des}}\right), \log\left(\frac{I_{des}}{I_k}\right) \right\} \quad \Big| \max \log = \log \max$$

$$= \log \max_{k=1,...,m} \max\left\{ \frac{I_k}{I_{des}}, \frac{I_{des}}{I_k} \right\} \qquad \Big| \text{with } h = (u) = max\left\{u, \frac{1}{u}\right\}$$

$$= \log \max_{k=1,...,m} h\left(\frac{I_k}{I_{des}}\right)$$

The log is a monotonically increasing function, i.e. $\forall x, y$ with $x \le y$ we have $\log(x) \le \log(y)$. So when minimizing, we can remove it without affecting the solution.

$$\Rightarrow = \max_{k=1,...,m} h\left(\frac{I_k}{I_{des}}\right) \text{ with } h(u) = \max\left\{u, \frac{1}{u}\right\}. \qquad \square$$