

# Applied Optimization

## Exercise 1 - Convex Sets

V. Carrel & J. Fontana & A. Schaller  
Prof.: Prof. David Bommes

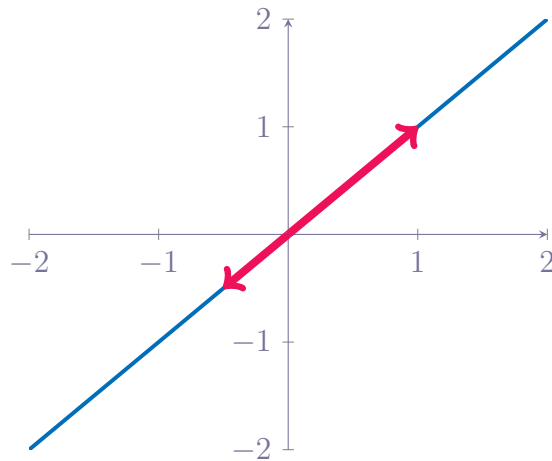
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### 1 Convex Sets

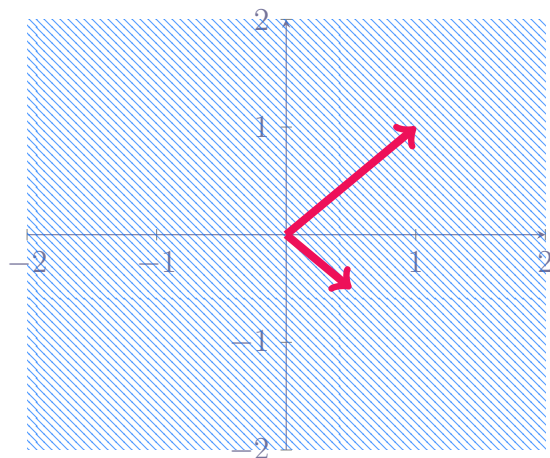
#### 1. Example sets

Sketch the following sets in  $R^2$

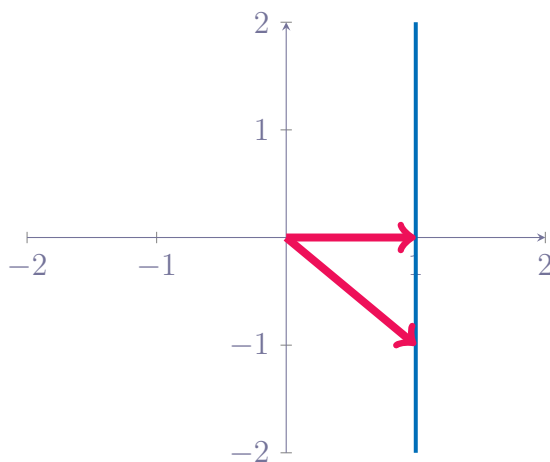
1.  $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \right\}$



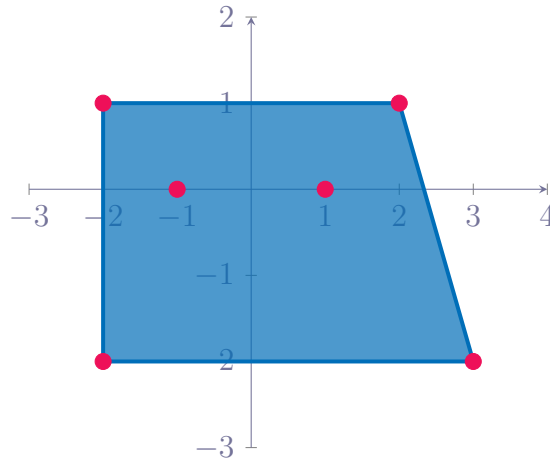
2.  $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$



3.  $\text{aff} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$



4.  $\text{conv} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}$



## 2. Convexity

Let  $C \in \mathbb{R}^n$  be a convex set, with  $x_1, \dots, x_k \in C$ , and let  $\theta_1, \dots, \theta_k \in \mathbb{R}$  satisfy  $\theta_i \geq 0, \theta_1 + \dots + \theta_k = 1$ . Show that  $\theta_1 x_1 + \dots + \theta_k x_k \in C$ . (The definition of convexity is that this holds for  $k = 2$ ; you must show it for arbitrary  $k$ .) Hint. Use induction on  $k$ .

*Show.* that  $\theta_1 + \dots + \theta_k \in C$

*Proof.* Base Case  $k = 2$  By definition, it holds for  $k = 2$ .

Inductive Step Assume it is correct for  $k = n$ , then for  $k = n + 1$ , we have:

$$x_1, \dots, x_{n+1} \in C$$

$$\theta_1, \dots, \theta_{n+1} \in C$$

with  $\theta_i \geq 0$  and  $\sum_{i=1}^{n+1} \theta_i = 1 \Rightarrow \theta_{n+1} = 1 - \sum_{i=1}^n \theta_i$

$$\Rightarrow \sum_{i=1}^{n+1} \theta_i x_i = \sum_{i=1}^n \theta_i x_i + \theta_{n+1} x_{n+1} = \sum_{i=1}^n \theta_i x_i + x_{n+1} = \sum_{i=1}^n \theta_i x_{n+1}$$

$$= \underbrace{x_{n+1}}_{\in C} \underbrace{\sum_{i=1}^n \theta_i (x_i - x_{n+1})}_{\substack{\in C \text{ by hypothesis on } k=n \\ \text{because } \sum_{i=1}^n \theta_i \in [0, 1]}} \Rightarrow \in C$$

□

## 3. Linear Equations

Show that the solution set of linear equations  $\{x | Ax = b\}$  with  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$  is an affine set.

*Proof.* Let  $x_0, x_1$  be solutions of linear equations, i.e.  $Ax_0 = b$  and  $Ax_1 = b$

Then

$$\begin{aligned} A((1 - \beta)x_0 + \beta x_1) &= (1 - \beta)Ax_0 + \beta Ax_1 \\ &= (1 - \beta)b + \beta b = b \end{aligned}$$

$\Rightarrow$  The set is affine. □

#### 4. Linear Inequations

1. Show that the solution set of linear inequations  $\{x | Ax \preceq b, Cx = d\}$  with  $x \in R^n$ ,  $A \in R^{m \times n}$  and  $b \in R^m$ ,  $C \in R^{k \times n}$  and  $d \in R^k$  is a convex set. Here  $\preceq$  means componentwise less or equal.

*Proof.* Let  $x_0, x_1$  be solutions of linear inequations, i.e.  $Ax_0 \preceq b, Cx_0 = d$   
TODO what is written on sheet? "an"  $Ax_1 \preceq b, Cx_1 = d$ , for  $\beta \in [0, 1]$

TODO Beta? Not mentioned until here

Then

$$A((1 - \beta)x_0 + \beta x_1) = \underbrace{(1 - \beta)}_{\geq 0} \underbrace{Ax_0}_{\preceq b} + \underbrace{\beta}_{\geq 0} \underbrace{Ax_1}_{\preceq b} \preceq (1 - \beta)b + \beta b = b$$

and

$$C((1 - \beta)x_0 + \beta x_1) = (1 - \beta)Cx_0 + \beta Cx_1 = (1 - \beta)d + \beta d = d$$

$\Rightarrow$  convex set. □

2. Is it an affine set?

No,  $C \in$  TODO ...in what?, if we don't have the  $a_i \geq 0$  for  $\forall i$  constraint, we could have  $(1 - \beta) < 0$ , and therefore  $(1 - \beta)Ax_0 \not\preceq b$  ■

#### 5. Voronoi description of halfspace

Let  $a$  and  $b$  be distinct points in  $R^n$ . Show that the set of all points that are closer (in Euclidean norm) to  $a$  than  $b$ , i.e.,  $\{x | \|x - a\|^2 \leq \|x - b\|^2\}$ , is a halfspace. Describe it explicitly as an inequality of the form  $c^T x \leq d$ . Draw a picture.

*Proof.* Be  $c \in R^n$  a vector,  $c = b - a$  be  $d \in R$ ,  $d = c^T x_0 = c^T \frac{a+b}{2}$

then we have the halfspace

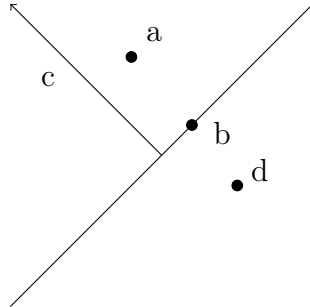
$$\left\{ x \mid (b - a)^T x \leq \frac{(b - a)^T (b - a)}{2} \right\}$$

□

*Check.*

$$\begin{aligned} \|x - a\|^2 &\leq \|x - b\|^2 \\ \Leftrightarrow (x - a)^T (x - a) &\leq (x - b)^T (x - b) \\ \Leftrightarrow \cancel{x^T x} - 2a^T x + a^T a &\leq \cancel{x^T x} - 2b^T x + b^T b \end{aligned}$$

$$\Leftrightarrow (b-a)^T x \leq (b^T b - a^T a) \frac{1}{2}$$



TODO ok placement of points?

## 2 Convex Illumination Problem

Show that the solution  $p^* = (p_1^*, p_2^*, \dots, p_n^*)^T \in R^n$  of the non-convex illumination problem from the lecture

$$\begin{aligned} & \text{minimize} && \max_{k=1\dots m} |\log I_k - \log I_{des}| \\ & \text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1\dots n \end{aligned}$$

with  $I_k = \sum_{j=1}^n a_{kj} p_j$  for geometric constants  $a_{kj} \in R$ , a constant desired illumination  $I_{des} \in R$  and an upper bound  $p_{max} \in R$  on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{aligned} & \text{minimize} && \max_{k=1\dots m} h(I_k/I_{des}) \\ & \text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1\dots n \end{aligned}$$

with  $h(u) = \max\{u, \frac{1}{u}\}$ .

*Proof.*

$$\begin{aligned} & \max_{k=1,\dots,m} |\log(I_k) - \log(I_{des})| && \left| \begin{array}{l} \text{log rules} \end{array} \right. \\ &= \max_{k=1,\dots,m} \left| \log\left(\frac{I_k}{I_{des}}\right) \right| && \left| \begin{array}{l} \text{remove } | \cdot | \end{array} \right. \\ &= \max_{k=1,\dots,m} \max \left\{ \log\left(\frac{I_k}{I_{des}}\right), \log\left(\frac{I_{des}}{I_k}\right) \right\} && \left| \begin{array}{l} \text{max log} = \text{log max} \end{array} \right. \\ &= \log \max_{k=1,\dots,m} \max \left\{ \frac{I_k}{I_{des}}, \frac{I_{des}}{I_k} \right\} && \left| \begin{array}{l} \text{with } h(u) = \max\left\{u, \frac{1}{u}\right\} \end{array} \right. \\ &= \log \max_{k=1,\dots,m} h\left(\frac{I_k}{I_{des}}\right) \end{aligned}$$

The log is a monotonically increasing function, i.e.  $\forall x, y$  with  $x \leq y$  we have  $\log(x) \leq \log(y)$ .  
 So when minimizing, we can remove it without affecting the solution.

$$\Rightarrow = \max_{k=1, \dots, m} h\left(\frac{I_k}{I_{des}}\right) \text{ with } h(u) = \max\left\{u, \frac{1}{u}\right\}. \quad \square$$