Applied Optimization
Exercise 1 - Convex Sets

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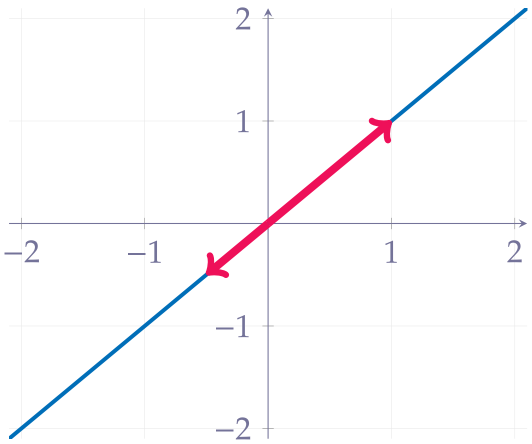
1 Convex Sets

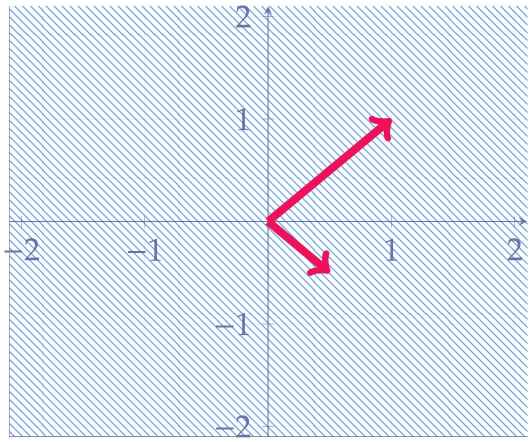
1. Example sets

Sketch the following sets in $\mathbb{R}^2$

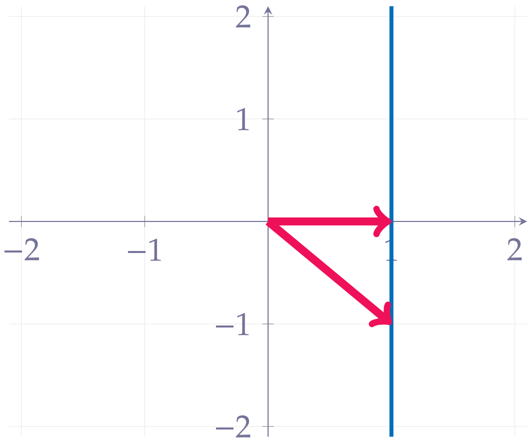
1. $\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-0.5\\-0.5\end{pmatrix}\right\}$



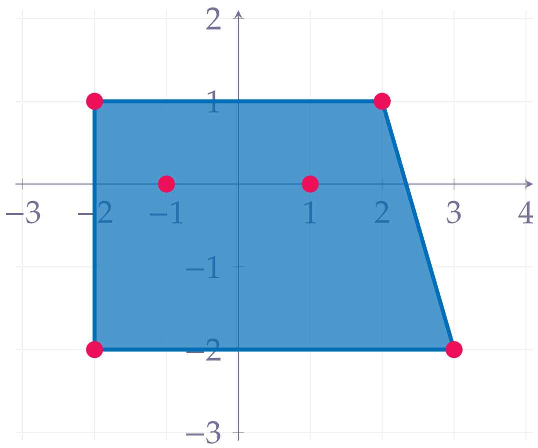
2. $\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0.5\\-0.5\end{pmatrix}\right\}$



3. $\operatorname{aff}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$



4. $\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}-2\\1\end{pmatrix},\begin{pmatrix}-2\\-2\end{pmatrix}\right\}$



2. Convexity

Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + \ldots + \theta_k = 1$. Show that $\theta_1 x_1 + \ldots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Show. that $\theta_1 + \ldots + \theta_k \in C$

Proof. Base Case $k = 2$ By definition, it holds for $k = 2$.

Inductive Step Assume it is correct for $k = n$, then for $k = n + 1$, we have:

$$x_1, \ldots, x_{n+1} \in C$$

$$\theta_1, \ldots, \theta_{n+1} \in C$$

with $\theta_i \geq 0$ and $\sum_{i=0}^{n+1} \theta_i = 1 \Rightarrow \theta_{n+1} = 1 - \sum_{i=1}^{n} \theta_i$

$$\Rightarrow \sum_{i=1}^{n+1} \theta_i x_i = \sum_{i=1}^{n} \theta_i x_i + \theta_{n+1} x_{n+1} = \sum_{i=1}^{n} \theta_i x_i + x_{n+1} = \sum_{i=1}^{n} \theta_i x_{n+1}$$

$$= \underbrace{x_{n+1}}_{\in C} \underbrace{\sum_{i=1}^{n} \theta_i (x_i - x_{n+1})}_{\substack{\in C \text{ by hypothesis on } k=n \\ \text{because } \sum_{i=1}^{n} \theta_i \in [0,1]}} \Bigg\} \Rightarrow \in C$$

□

3. Linear Equations

Show that the solution set of linear equations $\{x|Ax = b\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is an affine set.

Proof. Let x_0, x_1 be solutions of linear equations, i.e. $Ax_0 = b$ and $Ax_1 = b$

Then

$$\begin{aligned} A((1-\beta)x_0 + \beta x_1) &= (1-\beta)Ax_0 + \beta Ax_1 \\ &= (1-\beta)b + \beta b = b \end{aligned}$$

$\Rightarrow$ The set is affine. □

4. Linear Inequations

1. Show that the solution set of linear inequations $\{x|Ax \preceq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k\times n}$ and $d \in \mathbb{R}^k$ is a convex set. Here $\preceq$ means componentwise less or equal.

   Proof. Let x_0, x_1 be solutions of linear inequations, i.e. $Ax_0 \preceq b, Cx_0 = d$ and $Ax_1 \preceq b, Cx_1 = d$, for $\beta \in [0,1]$

   Then

   $$A((1-\beta)x_0 + \beta x_1) = \underbrace{(1-\beta)}_{\geq 0}\underbrace{Ax_0}_{\preceq b} + \underbrace{\beta}_{\geq 0}\underbrace{Ax_1}_{\preceq b} \preceq (1-\beta)b + \beta b = b \tag{1}$$

   and

   $$C((1-\beta)x_0 + \beta x_1) = (1-\beta)Cx_0 + \beta Cx_1 = (1-\beta)d + \beta d = d$$

   $\Rightarrow$ convex set. □

2. Is it an affine set?

   No, cf Equation 1, if we don't have the $a_i \geq 0$ for $\forall i$ constraint, it removes the constraint on the $\beta \in [0,1]$, which then means we could have $(1-\beta) < 0$. Therefore the step $(1-\beta)Ax_0 \preceq (1-\beta)b$ is not doable, we might have $(1-\beta)Ax_0 \succeq (1-\beta)b$. ■

5. Voronoi description of halfspace

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x| \, ||x-a||^2 \leq ||x-b||^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Proof. Be $c \in \mathbb{R}^n$ a vector, $c = b - a$ be $d \in \mathbb{R}$, $d = c^T x_0 = c^T \frac{a+b}{2}$

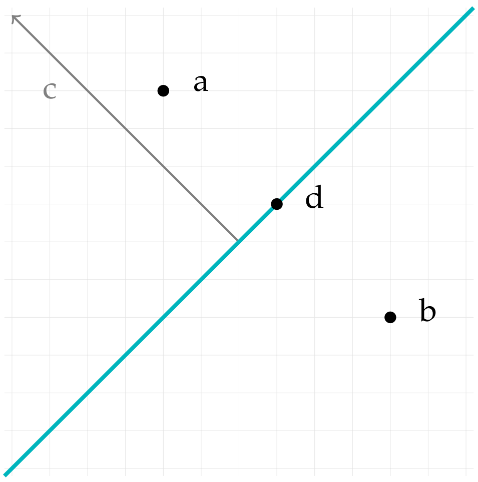
then we have the halfspace

$$\left\{ x \middle| (b-a)^T x \leq \frac{(b-a)^T(b-a)}{2} \right\}$$

□

Check.

$$||x-a||^2 \leq ||x-b||^2$$
$$\Leftrightarrow (x-a)^T(x-a) \leq (x-b)^T(x-b)$$
$$\Leftrightarrow \cancel{x^T x} - 2a^T x + a^T a \leq \cancel{x^T x} - 2b^T x + b^T b$$
$$\Leftrightarrow (b-a)^T x \leq (b^T b - a^T a)\, \frac{1}{2}$$



2 Convex Illumination Problem

Show that the solution $p^* = (p_1^*, p_2^*, ..., p_*^n)^T \in \mathbb{R}^n$ of the non-convex illumination problem from the lecture

$$\begin{aligned} &\text{minimize} && \max_{k=1...m} \left|\log I_k - \log I_{des}\right| \\ &\text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1...n \end{aligned}$$

with $I_k = \sum_{j=1}^{n} a_{kj} p_j$ for geometric constants $a_{jk} \in \mathbb{R}$, a constant desired illumination $I_{des} \in \mathbb{R}$ and an upper bound $p_{max} \in \mathbb{R}$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{aligned} &\text{minimize} && \max_{k=1...m} h(I_k/I_{des}) \\ &\text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1...n \end{aligned}$$

with $h(u) = max\{u, \frac{1}{u}\}$.

Proof.

$$\begin{aligned} &\max_{k=1,...,m} \left|\log(I_k) - \log(I_{des})\right| && \Big| \text{log rules} \\ &= \max_{k=1,...,m} \left|\log\left(\frac{I_k}{I_{des}}\right)\right| && \Big| \text{remove } |\,.\,| \\ &= \max_{k=1,...,m} \max\left\{\log\left(\frac{I_k}{I_{des}}\right), \log\left(\frac{I_{des}}{I_k}\right)\right\} && \Big| \text{max log} = \text{log max} \\ &= \log \max_{k=1,...,m} \max\left\{\frac{I_k}{I_{des}}, \frac{I_{des}}{I_k}\right\} && \Big| \text{with } h = (u) = max\left\{u, \frac{1}{u}\right\} \\ &= \log \max_{k=1,...,m} h\left(\frac{I_k}{I_{des}}\right) \end{aligned}$$

The log is a monotonically increasing function, i.e. $\forall x, y$ with $x \leq y$ we have $\log(x) \leq \log(y)$. So when minimizing, we can remove it without affecting the solution.

$\Rightarrow = \max_{k=1,...,m} h\left(\frac{I_k}{I_{des}}\right)$ with $h(u) = \max\left\{u, \frac{1}{u}\right\}$. □