

## Exercise 4 - Duality

Swiss Joint Master of Science in Computer Science - Applied Optimization

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```
%matplotlib inline
%load_ext autoreload
%autoreload 2

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches
from scipy import optimize
from matplotlib import patheffects
```

### Lagrange Duality (3 pts)

Consider the optimization problem

$$\begin{aligned} &\text{minimize} && x^2 + 1 && (1) \\ &\text{subject to} && (x - 2)(x - 4) \leq 0 && (2) \end{aligned}$$

with variable  $x \in \mathbb{R}$ .

- (a) Analysis of primal problem: give the feasible set, the optimal value, and the optimal solution.
- (b) Lagrangian and dual function: plot the objective  $x^2 + 1$  versus  $x$ . On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian  $L(x, \lambda)$  versus  $x$  for a few positive values of  $\lambda$ . Verify the lower bound property ( $p^* \geq \inf_x L(x, \lambda)$  for  $\lambda \geq 0$ ). Compute and sketch the Lagrange dual function  $g$ .
- (c) Lagrange dual problem: state the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution  $\lambda$ . Does strong duality hold?

(a)

The feasible set is  $[2, 4]$

The optimal solution is  $x = 2$

The optimal value is  $y = 5$

(b)

**Lagrangian and dual function: plot the objective  $x^2 + 1$ . On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian  $L(x, \lambda)$  for a few positive values of  $\lambda$**

The function to optimize is below the dashed white one. Play with the lambda value to make it visible again.

```
f = lambda x: x**2 + 1
lagrangian = lambda x,l: x**2 + 1 + l*(x**2 - 6*x + 8)

plt.figure(figsize=(15, 10))

x = np.linspace(0, 6, 200)
y = f(x)

#Plot function to minimize
plt.plot(x, y, color='black', linewidth=5, label="f(x) = $x^2 + 1$")
```

```

#Add constraints
plt.fill_between(x, y.min(), y.max(), where=x > 4,
                 facecolor='grey', alpha=0.5)
plt.fill_between(x, y.min(), y.max(), where=x < 2,
                 facecolor='grey', alpha=0.5)

#Add solution and value
plt.vlines(2, 0, f(2), linestyle="dashed", linewidth=3, label="optimal solution", colors="green")
plt.hlines(f(2), 0, 2, linestyle="dashed", linewidth=3, label="optimal value", colors="blue")

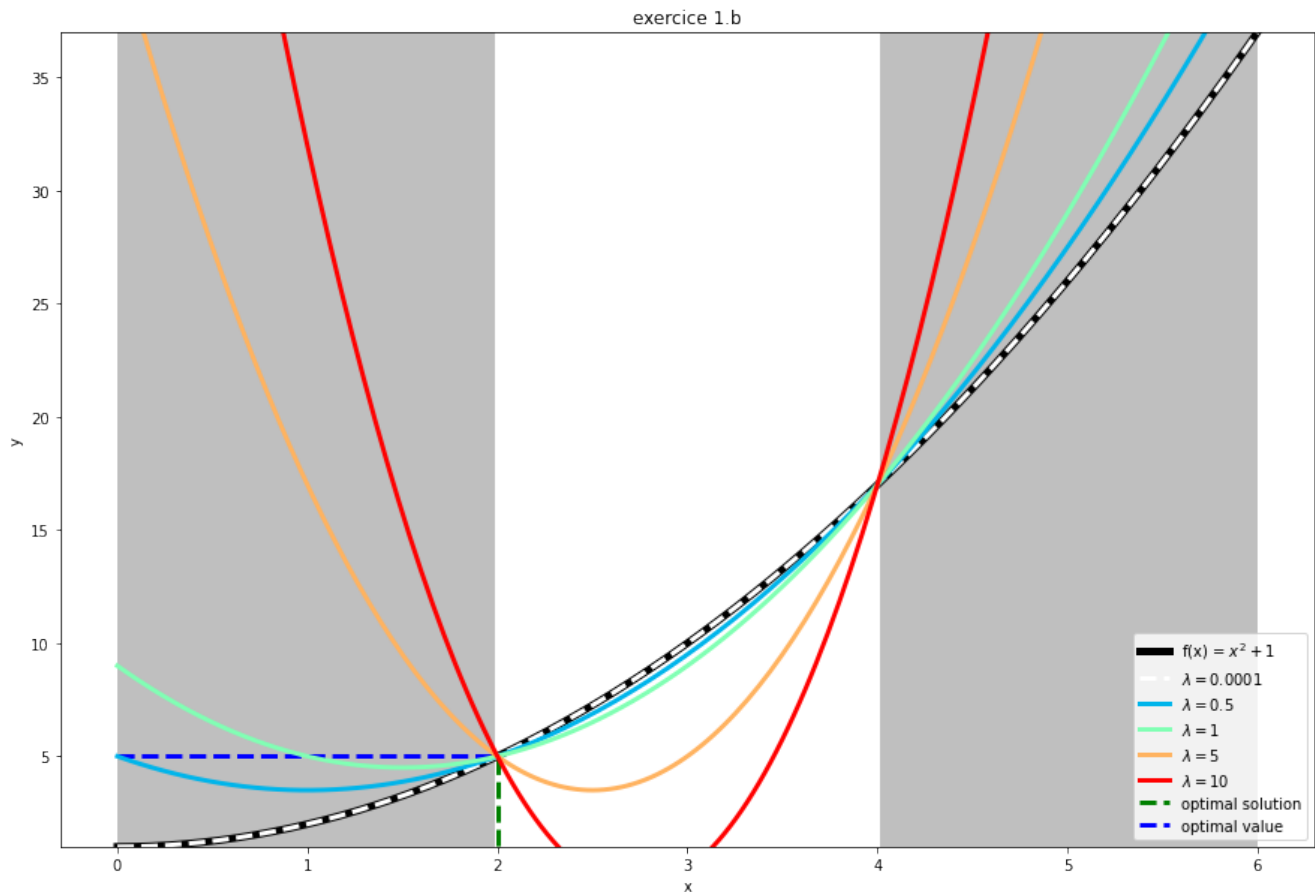
#Add different lagrangian
lambda_value = [0.0001, 0.5, 1, 5, 10]
color = plt.cm.rainbow(np.linspace(0, 1, len(lambda_value)))

for i, (l, c) in enumerate(zip(lambda_value, color)):
    c, ls = ("white", "--") if i == 0 else (c, "-")
    plt.plot(x, lagrangian(x, l), color=c, ls=ls, linewidth=3, label=f"$\lambda = {l}$")

plt.ylim(y.min(), y.max())
plt.xlabel("x")
plt.ylabel("y")
plt.title("exercice 1.b")
plt.legend()

plt.show()

```



Verify the lower bound property ( $p^* \geq \inf_x L(x, \lambda)$  for  $\lambda \geq 0$ )

We can see it graphically. We can play with the lambda value

By calcul:

Be  $x$  in our feasible set  $[2, 4]$ ,  $\lambda$  set.

Then  $\lambda(x^2 - 6x + 8) \leq 0$

$\rightarrow L(x, \lambda) \leq f(x)$

$\rightarrow \inf_x L(x, \lambda) \leq \inf_x f(x)$

$\rightarrow \inf_x L(x, \lambda) \leq p^*$

**Compute and sketch the Lagrange dual function  $g$**

$g(\lambda) = \inf_x L(x, \lambda) = \inf_x x^2 + 1 + \lambda(x^2 - 6x + 8)$

$\rightarrow g(\lambda) = \inf_x (1 + \lambda)x^2 - 6\lambda x + 9\lambda$

$\frac{\partial L(x, \lambda)}{\partial x} = 2(1 + \lambda)x - 6\lambda = 0$

$\rightarrow x = \frac{3\lambda}{(1+\lambda)}$

$\rightarrow g(\lambda) = \frac{-\lambda^2 + 9\lambda + 1}{(\lambda + 1)}$

`g = lambda 1: (1+9*1-1**2)/(1+1)`

`lambdas = np.linspace(0,8,200)`

`plt.plot(lambdas,g(lambdas), color='black',label="$g(\lambda)$")`

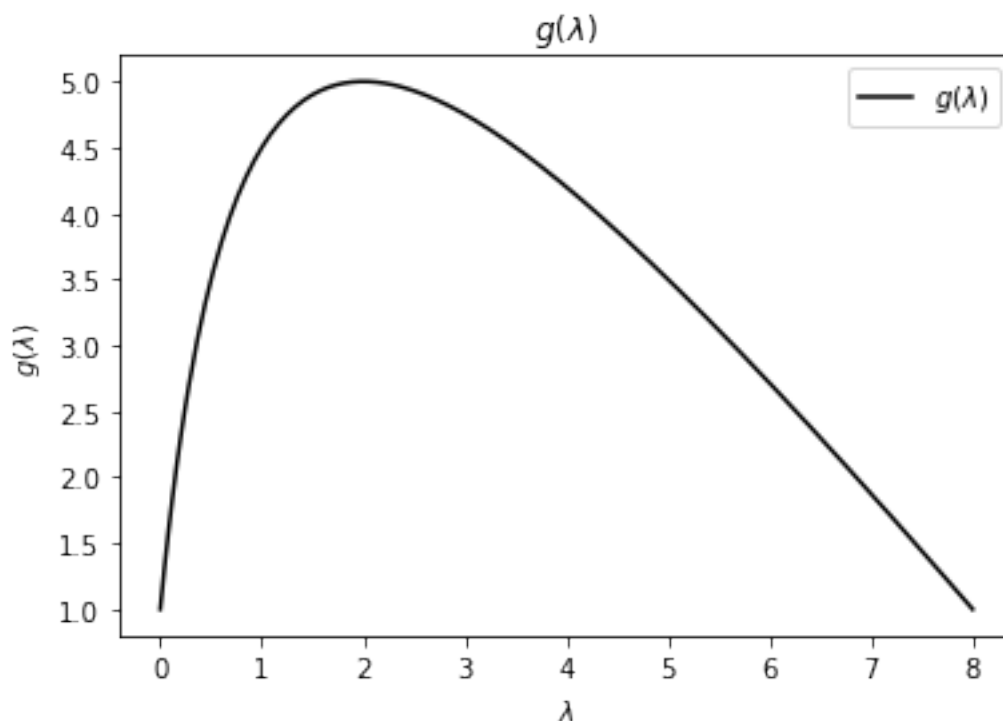
`plt.xlabel("$\lambda$")`

`plt.ylabel("$g(\lambda)$")`

`plt.title("$g(\lambda)$")`

`plt.legend()`

`plt.show()`



(c)

**Lagrange dual problem:** state the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution  $\lambda$ . Does strong duality hold?

The Lagrange dual problem is basically:

$$\text{maximize } g(\lambda) = \frac{-\lambda^2 + 9\lambda + 1}{(\lambda + 1)}$$

subject to  $\lambda \geq 0$

We know that it is a maximization problem, because  $\frac{p(x)}{q(x)}$  with  $\deg(p) = 2$  and  $\deg(q) = 1$  has a max.

It is also concave because  $-g(\lambda) = \frac{p(x)}{q(x)}$  with  $p(x)$  convex (polynome of degree 2) and  $q(x)$  affine non-decreasing. So  $-g(\lambda)$  is convex, therefore  $g(\lambda)$  is concave

```
solution_l = optimize.fmin(lambda x: -g(x), 1)[0]
```

```
print("the dual optimal value is ", g(solution_l), " for the dual optimal solution lambda = ", solution_l)
```

Optimization terminated successfully.

Current function value: -5.000000

Iterations: 17

Function evaluations: 34

```
the dual optimal value is 5.0 for the dual optimal solution lambda = 2.0000000000000002
```

The strong duality hold:

we see that  $p^* = 5 = d^*$

As well as  $f(x^*) = f(2) = 5 = g(2) = g(\lambda^*)$

## KKT Condition (2 pts)

Consider the following optimization problem:

$$\text{minimize} \quad x_1^2 - 2x_2^2 \quad (3)$$

$$\text{subject to} \quad (x_1 + 4)^2 - 2 \leq x_2 \quad (4)$$

$$x_1 - x_2 + 4 = 0 \quad (5)$$

$$x_1 \geq -10 \quad (6)$$

(1) sketch the problem and graphically determine the primal solution  $x^*$ .

(2) verify your  $x^*$  by determining suitable  $\lambda^*$  and  $v^*$  such that the KKT conditions are satisfied for  $(x^*, \lambda^*, v^*)$ .

### (1) Sketch the problem and determine $x^*$

```
fig, ax = plt.subplots(figsize=(15, 15))

x = np.linspace(-12.0, 12.0, 500)
y = np.linspace(-12.0, 12.0, 500)

X, Y = np.meshgrid(x, y)

# Objective function
obj = X**2 - 2 * Y**2

#Constraint 1: (x+4)^2 - 2 <= y
g1 = (X + 4)**2 - 2 - Y

#Constraint 2: x - y + 4 = 0
g2 = X - Y + 4

#Constraint 3: x >= -10
g3 = -X - 10

#Plot the function to optimize, we want the min so a Red value
im = plt.imshow(obj, cmap=plt.cm.RdBu, extent=[-12,12,-12,12], origin="lower")

#Plot constraint 1
cg1 = ax.contour(X, Y, g1, [0], colors='sandybrown')
plt.setp(cg1.collections,
         path_effects=[patheffects.withTickedStroke(angle=135,length=3)])
cg1_legend = mpatches.Patch(color='sandybrown', label=r"$(x_1+4)^2 - 2 \leq x_2$")

#constraint 2, remember we must be on this line as it is an equality constraint
cg2 = ax.contour(X, Y, g2, [0], colors='mediumblue')
cg2_legend = mpatches.Patch(color='mediumblue', label=r"$x_1 - x_2 + 4 = 0$")

#plot constraint 3
cg3 = ax.contour(X, Y, g3, 0, colors='green')
plt.setp(cg3.collections,
         path_effects=[patheffects.withTickedStroke(angle=60, length=3)])
cg3_legend = mpatches.Patch(color='green', label=r"$x_1 \geq -10$")

fig.colorbar(im, shrink=0.5, aspect=5)
```

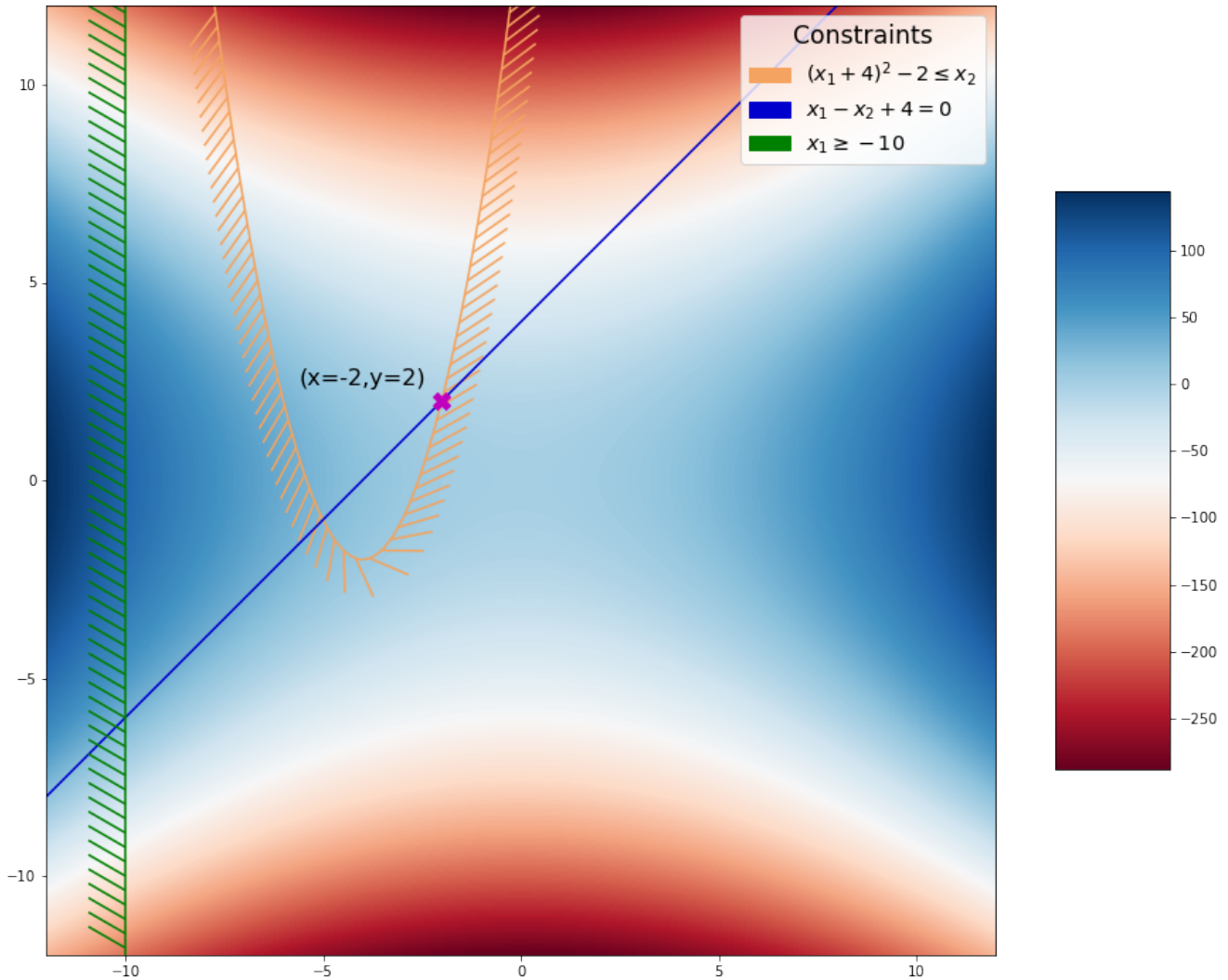
```

ax.plot(-2, 2, 'mX', markersize=12)
ax.text(-2.4, 2.4, f"(x=-2,y=2)", ha="right", fontsize=16)

plt.legend(handles=[cg1_legend, cg2_legend, cg3_legend], title="Constraints",
           title_fontsize="xx-large", fontsize="x-large")

plt.show()

```



We can see that the solution will be approximately  $x^* = (x, y) = (-2, 2)$

We must follow the blue line (constraint 2: equality constraint, but we are blocked by constraint 1). We can rapidly calculate it by taking the root of:

$$(x + 4)^2 - 2 - (x + 4)$$

which are  $-5$  and  $-2$ . We keep  $-2$  according to the graphic, and compute  $x + 4 = y$  for  $x = -2 \rightarrow y = 2 \rightarrow x^* = (x, y) = (-2, 2)$

and  $f(x^*) = -4$

**(2) Verify  $x^*$  by determining suitable  $\lambda^*$  and  $v^*$  st. KKT conditions are satisfied**

We have:

**1) Primal constraints:**

- $(x + 4)^2 - 2 - y \leq 0$
- $-x - 10 \leq 0$
- $x - y + 4 = 0$

**2) Dual constraints:**

- $\lambda_1 \geq 0$
- $\lambda_2 \geq 0$

**3) Complementary slackness:**

- $\lambda_1((x + 4)^2 - 2 - y) = 0$
- $\lambda_2(-x - 10) = 0$

**4) Vanishing gradient of Lagrangian:**

$$\begin{aligned} & \bullet \nabla_x L(x, y, \lambda_1, \lambda_2, v) = \nabla f_0(x, y) + \lambda_1 \nabla f_1(x, y) + \lambda_2 \nabla f_2(x, y) + v \nabla h_1(x, y) = 0 \\ \rightarrow & \begin{pmatrix} 2x \\ -4y \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x + 8 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

The slackness

- $\lambda_2(-x - 10) = 0$

tells us that  $\lambda_2 = 0$ , because clearly  $x \geq -10$ , gradient of objective is annihilated by **active** constraint grad

We can then compute  $\lambda_1$  and  $v$

With  $(x, y) = (-2, 2)$  the point to explore according to our graphical exploration, we have for the gradient:

- $-4 + 4\lambda_1 + v = 0$
- $-8 - \lambda_1 - v = 0$

And we get directly by resolution

- $\lambda_1 = 4$
- $v = -12$

**So we have  $\lambda^* = (4, 0)$  and  $\nu^* = -12$  which respect the KKT conditions for  $x^* = (-2, 2)$**