Aopt

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Exercise 9

Problem 1:

Solve the optimization problem in $\mathbb{R}^3$

$$\min \quad f(x) = \exp(1^T x) + \exp(-1^T x)$$

$$\text{subject to} \quad \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

which is equivalent to

$$\min (\text{over } z) \quad \hat{f}(z) = f(Fz + x_0)$$

with F and x_0 such that $Ax = b \Leftrightarrow x = Fz + x_0$ for some $z \in \mathbb{R}^{3-2=1}$

$k = n - \text{rank}(A) = 3 - 2 = 1$

$\Rightarrow F \in \mathbb{R}^{n \times k} = \mathbb{R}^{3 \times 1}$ $\quad Q = [Q_1 | Q_2] \in \mathbb{R}^{n \times n} = \mathbb{R}^{3 \times 3}$, $Q_1 \in \mathbb{R}^{3 \times 1}$, $Q_2 \in \mathbb{R}^{3 \times 2}$

$R = \begin{bmatrix} \bar{R} \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times k} = \mathbb{R}^{3 \times 1}$, with $\bar{R} \in \mathbb{R}^{1 \times 1}$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & z \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & z \end{array}\right)$$

$$\Rightarrow x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{x_0} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{F} z$$

$$\Rightarrow \hat{f}(z) = f(Fz + x_0) = f(2, 1, z) = \exp(z+3) + \exp(-z-3)$$

$$f'(2,1,z) = e^{z+3} - e^{-3-z} \overset{!}{=} 0 \quad \Rightarrow \underline{\underline{z^* = -3}}$$

$$\Rightarrow x^* = Fz^* + x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot (-3) + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}}}$$

$$\underline{\underline{x^* = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}}}$$

Problem 2:

Solve the following problem in $\mathbb{R}^3$

$$\min \quad f(x) = \frac{1}{2} x^T \begin{pmatrix} 6 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix} x + \begin{pmatrix} -3 & -2 & 7 \end{pmatrix} x$$

$$\text{subject to} \quad \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad Q = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix} \quad q = \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix}$$

$$\Rightarrow Q^{-1} = \frac{1}{16} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow A Q^{-1} A^T = \frac{1}{16} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{16} \begin{pmatrix} 1 & 4 & 5 \\ -1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 15 & 9 \\ 9 & 7 \end{pmatrix}$$

$$\Rightarrow (A Q^{-1} A^T)^{-1} = \frac{2}{3} \begin{pmatrix} 7 & -9 \\ -9 & 15 \end{pmatrix}$$

$$\Rightarrow b + A Q^{-1} q = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{16} \begin{pmatrix} 1 & 4 & 5 \\ -1 & 4 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \frac{1}{16} \begin{pmatrix} 24 \\ 16 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\Rightarrow v^* = -(A Q^{-1} A^T)^{-1} (b + A Q^{-1} q)$$

$$= -\frac{2}{3} \begin{pmatrix} 7 & -9 \\ -9 & 15 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = -\begin{pmatrix} 7 & -9 \\ -9 & 15 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\begin{pmatrix} -11 \\ 21 \end{pmatrix} = \begin{pmatrix} 11 \\ -21 \end{pmatrix}$$

$$\underline{\underline{v^* = \begin{pmatrix} 11 \\ -21 \end{pmatrix}}}$$

$$\Rightarrow x = -Q^{-1}(A^T v + q) = -\frac{1}{16} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ -21 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix} \right)$$

$$= -\frac{1}{16} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ -12 \\ 8 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 4 \\ -12 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \quad \Rightarrow \underline{\underline{x^* = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}}}$$