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Exercise 8

Trust Region Subproblem:

Consider the following trust region Subproblem in two dim, $v \in \mathbb{R}^2$

$$\text{minimize} \quad m(v) = \frac{1}{2} v^T B v + g^T v$$
$$\text{subject to} \quad \|v\| \leq \Delta$$

Problem 1:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Delta = \frac{1}{2}$$

Compute an approximation of v^* with Cauchy point method:

$$g^T B g = (1 \ 1) \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6 > 0 \qquad \|g\| = \sqrt{2}$$

$\Rightarrow$ case 2

$$v^* = -t \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} = \frac{-t}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{with } t = \min\left\{ \frac{\|g\|^3}{\Delta g^T B g}, 1 \right\}$$

$$\Rightarrow t = \min\left\{ \frac{2\sqrt{2}}{3}, 1 \right\} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow v^* = -\frac{2\sqrt{2}}{3} \cdot \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} \qquad \underline{\underline{v^* = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}}}$$

Problem 2:

Compute v^* with Dogleg method:

$$v^N = -\nabla^2 f(x)^{-1} \nabla f(x) = -B^{-1} g = -\begin{pmatrix} 1/4 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ -1/2 \end{pmatrix}$$

$$v^G = -\left(\frac{g^T g}{g^T B g} \right) g = \frac{-2}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}$$

test 1: $\|v^N\|^2 \leq \Delta^2$

$$\|v^N\|^2 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \overset{?}{\leq} \frac{2}{8} = \Delta^2 \ \lightning$$

test 2: $\|v^G\|^2 > \Delta^2$

$$\|v^G\|^2 = \frac{1}{9} + \frac{1}{9} = \frac{2}{9} \overset{?}{>} \frac{2}{8} = \Delta^2 \ \lightning$$

Case 3:

$v^* = v^G + (\tau - 1)(v^N - v^G)$ with $\|v\|^2 = \Delta^2 = \frac{1}{4}$, $\tau \in [0, 2]$

$$\left\| \begin{pmatrix} -5/12 \\ -2/12 \end{pmatrix} + \tau \begin{pmatrix} 1/12 \\ -2/12 \end{pmatrix} \right\|^2 = \frac{1}{4}$$

$$\Rightarrow \left(\frac{\tau}{12} - \frac{5}{12}\right)^2 + \left(-\frac{2\tau}{12} - \frac{2}{12}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \frac{\tau^2}{144} + \frac{25}{144} - \frac{10\tau}{144} + \frac{4\tau^2}{144} + \frac{4}{144} + \frac{8\tau}{144} = \frac{1}{4}$$

$$\Rightarrow \tau^2 + 25 - 10\tau + 4\tau^2 + 4 + 8\tau = 36$$

$$\Rightarrow 5\tau^2 - 2\tau - 7 = 0$$

$$\Rightarrow \tau_{1,2} = \frac{7}{5}, -1 \quad , \quad \tau \in [0, 2] \Rightarrow \tau = \frac{7}{5}$$

$$\Rightarrow v = \begin{pmatrix} -25/60 \\ -10/60 \end{pmatrix} + \begin{pmatrix} 7/60 \\ -14/60 \end{pmatrix} = \begin{pmatrix} -18/60 \\ -24/60 \end{pmatrix} = \begin{pmatrix} -3/10 \\ -4/10 \end{pmatrix}$$

$$\Rightarrow \underline{\underline{v^* = \begin{pmatrix} -3/10 \\ -4/10 \end{pmatrix}}}$$

Problem 3:

$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\Delta = 1$, compute λ^*, v^* for KKT:

Test:

Suppose $\lambda^* = 0$

$$\Rightarrow Bv = -g \Rightarrow v = -B^{-1} g = \begin{pmatrix} -1/4 \\ -1/2 \end{pmatrix}$$

Check:

1) $\|v\|^2 = \frac{3}{8} \leq 1 = \Delta^2$ ✓

2) $\lambda^* = 0 \geq 0$ ✓

3) $\lambda^*(v^T v - \Delta^2) = 0 (v^T v - 1) = 0$ ✓

4) $(B + \lambda^* I) v = Bv = -g$ ✓

5) $(B + \lambda^* I) = B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ positive semi-definite ✓

$$\Rightarrow \underline{\underline{\lambda^* = 0}} \text{ and } \underline{\underline{v^* = \begin{pmatrix} -1/4 \\ -1/2 \end{pmatrix}}}$$

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Problem 4:

$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\Delta = \frac{1}{2}$, compute λ^*, v^* for KKT:

We already know from Problem 3 that we can't suppose $\lambda^* = 0$, we would fail the condition $\|v\|^2 \leq \Delta$

$$\Rightarrow B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\lambda_1, \lambda_2 = 2, 4$, eigenvalue

$q_1, q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding eigenvectors

$$\Rightarrow \|v(\lambda)\|^2 = \frac{\left((0\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2}{(2+\lambda)^2} + \frac{\left((1\ 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2}{(4+\lambda)^2}$$

$$= \frac{1}{(2+\lambda)^2} + \frac{1}{(4+\lambda)^2}$$

$\Rightarrow$ Find $\|v(\lambda)\|^2 \overset{!}{=} \Delta^2$ with $\lambda \in (-2, \infty)$

$$\Rightarrow (\lambda+4)^2 + (\lambda+2)^2 = \frac{(\lambda+4)^2(\lambda+2)^2}{4}$$

$\Rightarrow \lambda = 0{,}26448$ or $\lambda = -6{,}26448$, but $\lambda^* \in (-2, \infty)$

$$\Rightarrow \lambda^* = 0{,}26448$$

$$\Rightarrow v^* = -(B + \lambda^* I)^{-1} g = -\begin{pmatrix} 4{,}26 & 0 \\ 0 & 2{,}26 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -\begin{pmatrix} 1/4{,}26 & 0 \\ 0 & 1/2{,}26 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4{,}26 \\ -1/2{,}26 \end{pmatrix}$$

conditions 1 - 5 by construction or trivial

$$\Rightarrow \lambda^* = 0{,}26, \quad v^* = \begin{pmatrix} -1/4{,}26 \\ -1/2{,}26 \end{pmatrix}$$