Exercise 4 - Duality

Swiss Joint Master of Science in Computer Science - Applied Optimization

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```
%matplotlib inline
%load_ext autoreload
%autoreload 2

import matplotlib.pyplot as plt
import numpy as np
from sympy import solve
from sympy.abc import x, y

from helpers import center_axis, center_right_axis

plt.rcParams["figure.figsize"] = (8,8)
```

```
The autoreload extension is already loaded. To reload it, use:
  %reload_ext autoreload
```

Lagrange Duality (3 pts)

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & x^2 + 1 & (1) \\ \text{subject to} \quad & (x-2)(x-4) \leq 0 & (2) \end{aligned}$$

with variable $x \in \mathbb{R}$.

(a) Analysis of primal problem: give the feasible set, the optimal value, and the optimal solution.

(b) Lagrangian and dual function: plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq inf_x L(x, \lambda)$ for $\lambda \geq 0$). Compute and sketch the Lagrange dual function g.

(c) Lagrange dual problem: state the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ. Does strong duality hold?

(a)

Feasible set

With subject to inequality $x = [2, 4]$

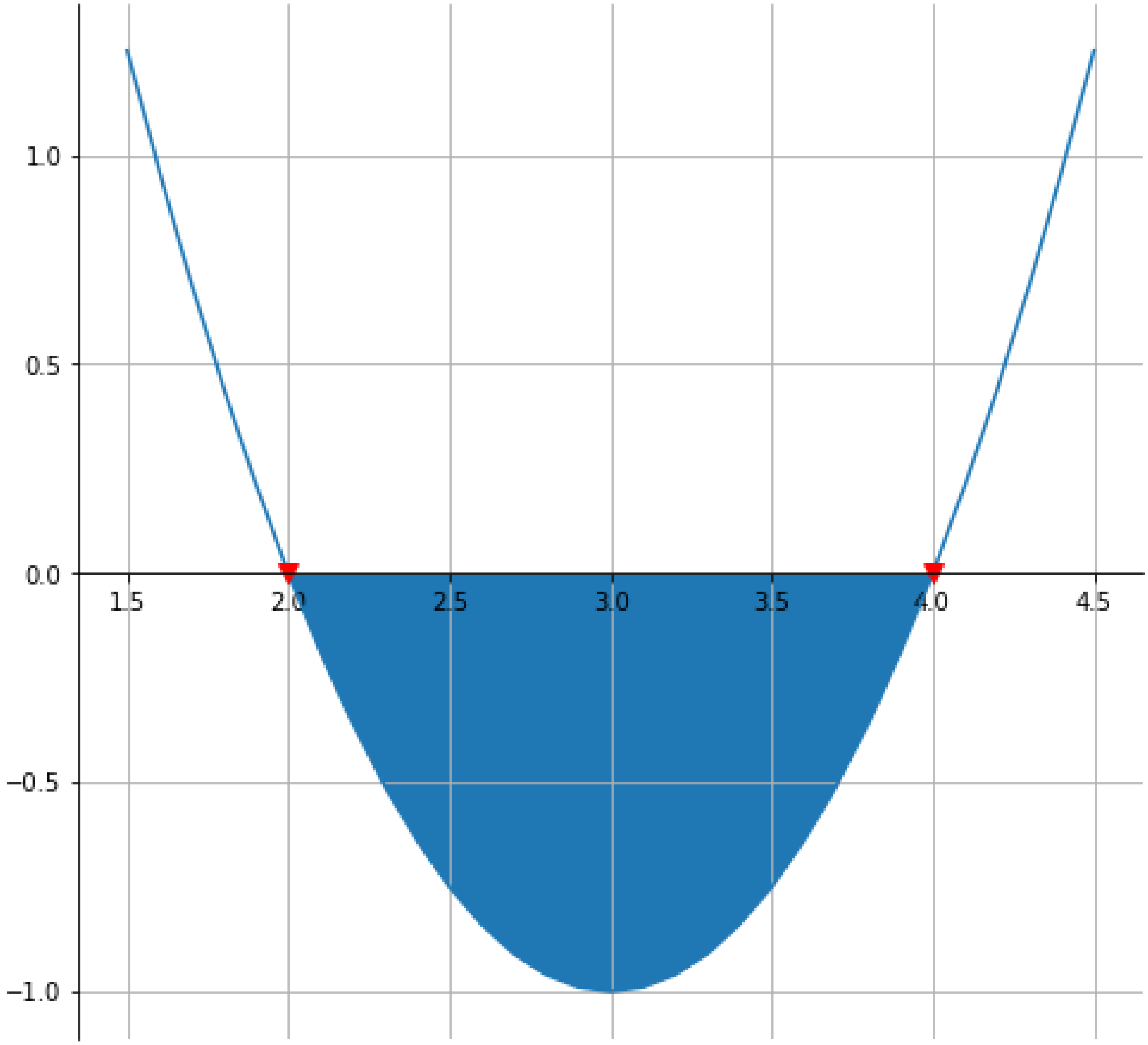
```
roots = solve((x-2)*(x-4))
root_low, root_high = roots

def ineq(X):
    return (X-2)*(X-4)

ineq_X = np.linspace(int(root_low) - 0.5, int(root_high) + 0.5, 31)
ineq_Y = ineq(ineq_X)

fig, ax = plt.subplots()
ax.plot(ineq_X, ineq_Y,
        roots, [0, 0], 'rv', markersize=8)
```

```python
ax.fill_between(ineq_X, 0, ineq_Y, where=(ineq_Y <= 0))
center_axis(ax, left_spine=False)
```



Optimal value

$y = 5$

TODO add any comment? What? I followed the plot to see which x^* to choose and evaluate the function (trivial)

Optimal solution

$x = 2$

(b)

Plot the objective function with feasible set, optimal point and value

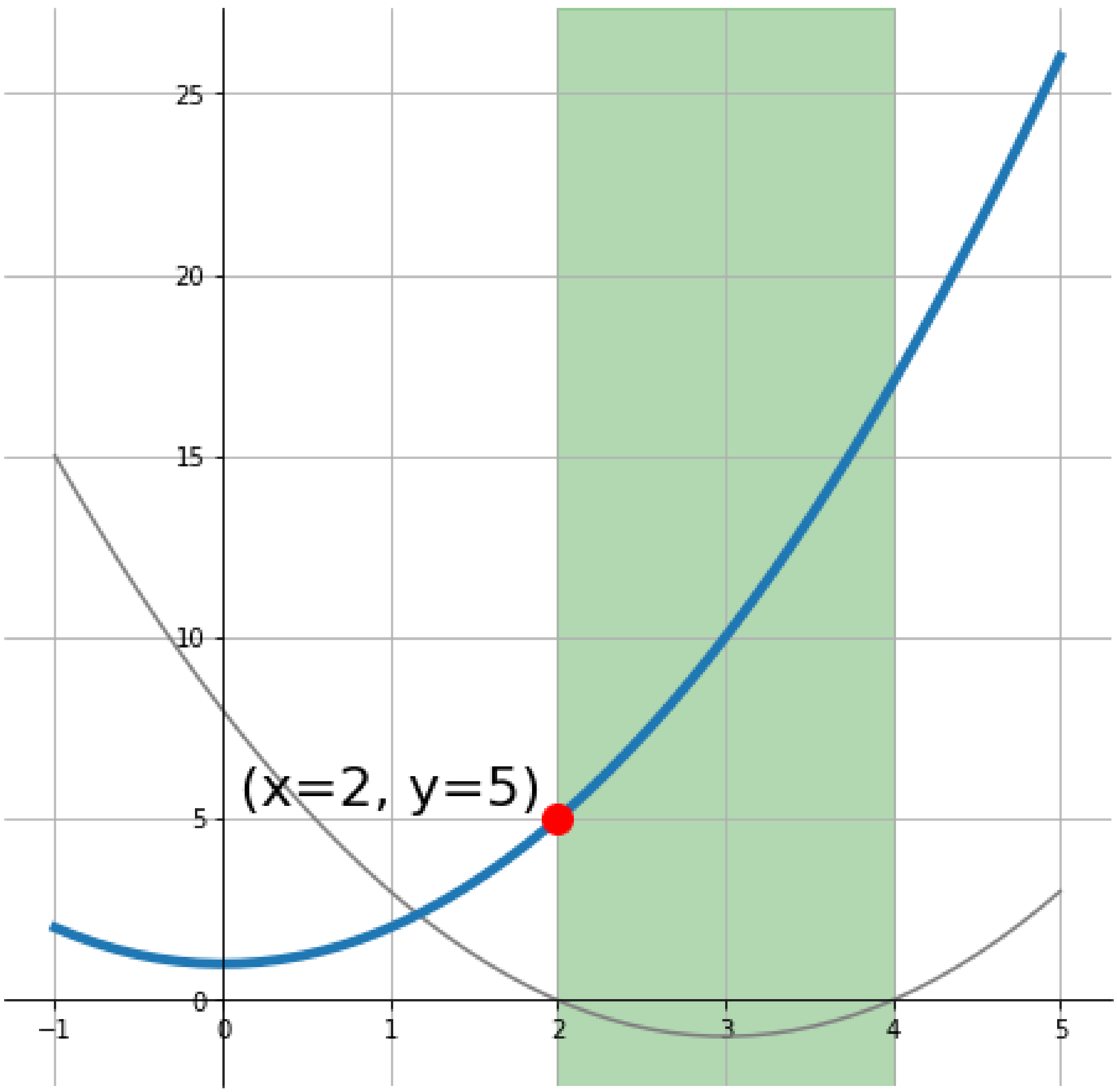
```python
def objective_function(X):
    return X ** 2 + 1
```

```
X = np.linspace(-1, 5)
ineq_Y = ineq(X)

Y = objective_function(X)

x_opti = 2
y_opti = objective_function(x_opti)

fig, ax = plt.subplots()
ax.axvspan(root_low, root_high, color="green", alpha=0.3)
ax.plot(X, ineq_Y, 'gray')
ax.plot(X, Y,
        x_opti, y_opti, 'ro', markersize=12, linewidth=4)
ax.text(x_opti - 0.1, y_opti + 0.35, f"(x={x_opti}, y={y_opti})", size=22, ha="right")
center_axis(ax)
```



Plot the Lagrangian function with a few positive values of λ

Verify the lower bound property

Compute and sketch the Lagrange dual function g

(c)

State the dual problem

Verify concave maximization problem

Find the dual optimal value and optimal solution λ

Does strong duality hold?

KKT Condition (2 pts)

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 - 2x_2^2 && (3) \\ \text{subject to} \quad & (x_1+4)^2 - 2 \leq x_2 && (4) \\ & x_1 - x_2 + 4 = 0 && (5) \\ & x_1 \geq -10 && (6) \end{aligned}$$

(1) sketch the problem and graphically determine the primal solution x^*.

(2) verify your x^* by determining suitable λ^* and v^* such that the KKT conditions are satisfied for (x^*, λ^*, v^*).

(1) Sketch the problem and determine x^*

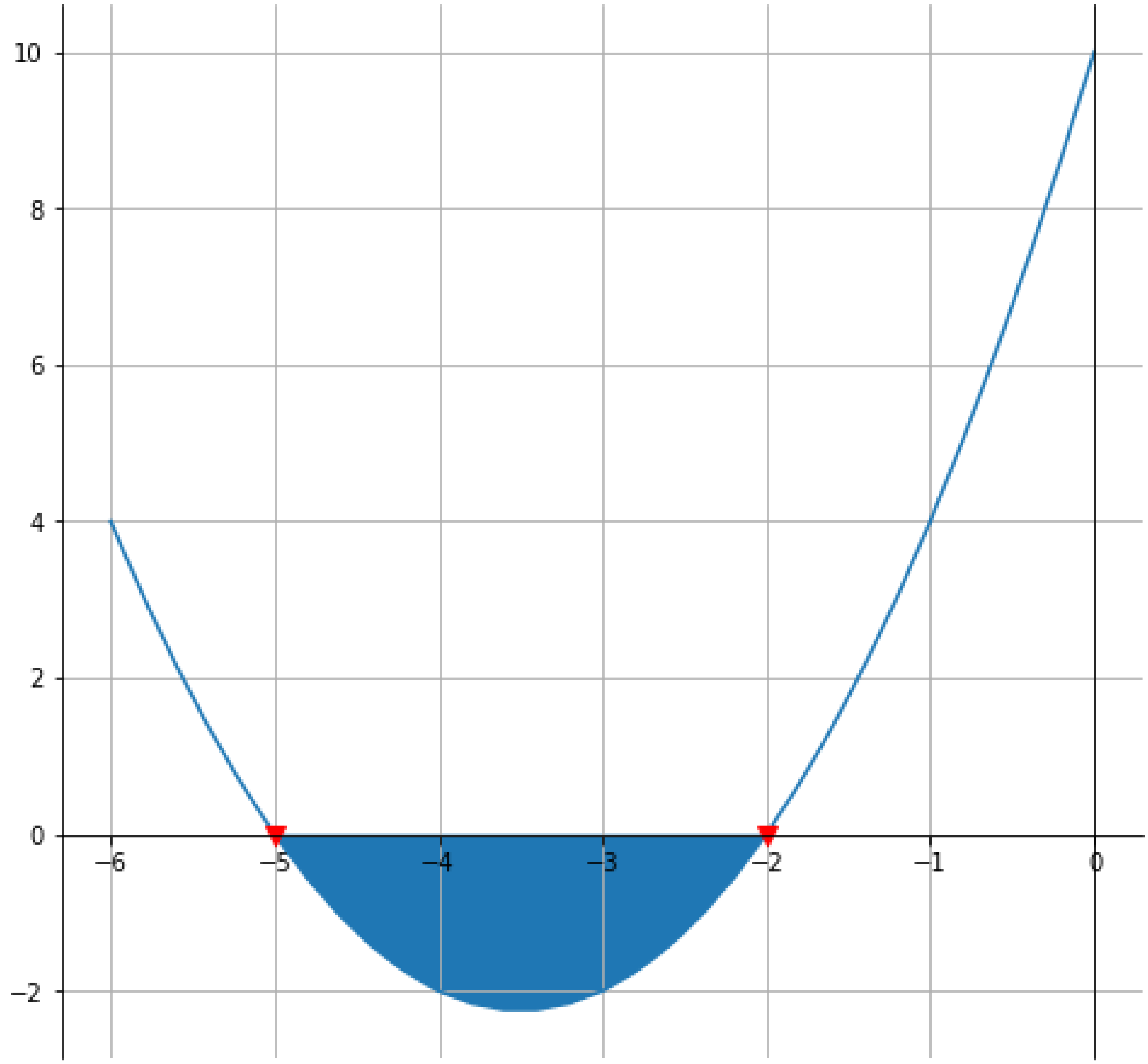
1. Expand 1st inequality: $\Rightarrow (x_1+4)^2 - x_2 \leq 2$
2. Rewrite 2nd constraint: $\Rightarrow x_1 + 4 = x_2$
3. Plug 2nd constraint in 1st inequality: $\Rightarrow (x_1+4)^2 - (x_1+4) \leq 2$
4. Expand/simplify $\Rightarrow x_1^2 + 8x_1 + 16 - x_1 - 4 \leq 2 \Leftrightarrow x_1^2 + 7x_1 + 10 \leq 0$

```
roots = solve(x ** 2 + 7 * x + 10)
root_low, root_high = roots

def ineq1(X):
    return X ** 2 + 7 * X + 10

ineq1_X = np.linspace(int(root_low) - 1, max(int(root_high) + 1, 0), 31)
ineq1_Y = ineq1(ineq1_X)


fig, ax = plt.subplots()
ax.plot(ineq1_X, ineq1_Y,
        roots, [0, 0], 'rv', markersize=8)
ax.fill_between(ineq1_X, 0, ineq1_Y, where=(ineq1_Y <= 0))
center_right_axis(ax)
```



⇒ with inequality 1, x_1 must be equal to $[-5, -2]$, which respect and is more strict than the last inequality $x_1 \geq -10$.

⇒ using the second constraint, x_2 must be equal to $[-1, 2]$.

```
def objective_function(X, Y):
    return X**2 - 2 * Y ** 2

X = np.linspace(-5, -2, 31)
Y = np.linspace(-1, 2, 31)

X, Y = np.meshgrid(X, Y)
Z = objective_function(X, Y)

ax = plt.axes(projection='3d', xlabel="x$_1$ = [-5, -2]", ylabel="$x_2$ = [-1, 2]")
ax.plot_surface(X, Y, Z, rstride=1, cstride=1,
                cmap='viridis', edgecolor='none')
```

```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x13510a5e0>
```

```
min_x1 = -2
```

```python
min_x2 = np.array([-1, 2])
z = objective_function(min_x1, min_x2)

min_index = int(z.argmin())
# print(f"(x={min_x1}, y={min_x2[min_index]}, z={min(z)})")

from IPython.display import Math
Math("$\\boldsymbol{{x^* = \\binom{{{}}}{{{}}}}} \\quad z = {}$".format(min_x1, min_x2[min_index], min(z))
```

(2) Verify x^* by determining suitable λ^* and v^* *st.* KKT conditions are satisfied