Applied Optimization
Exercise 1 - Convex Sets

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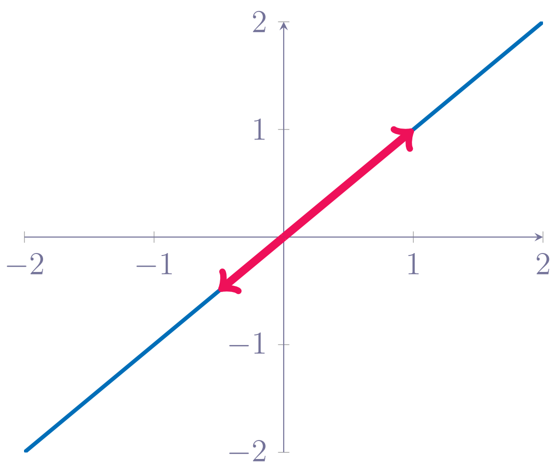
1 Convex Sets

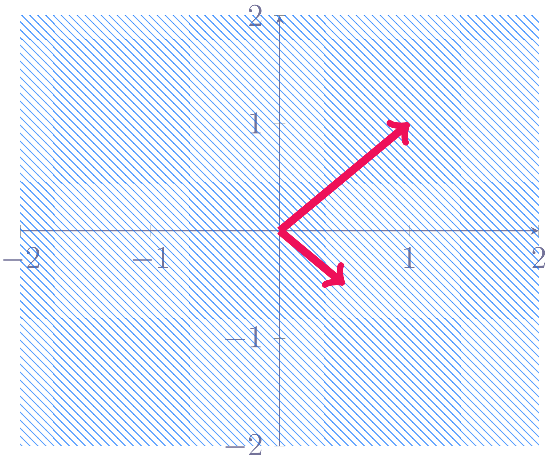
1. Example sets

Sketch the following sets in R^2

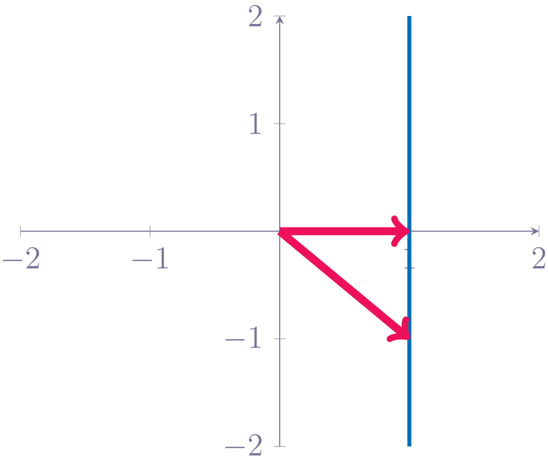
1. $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-0.5\\-0.5\end{pmatrix}\right\}$



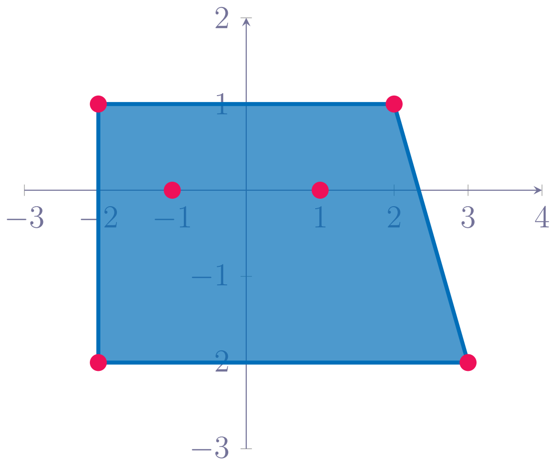
2. $\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0.5\\-0.5\end{pmatrix}\right\}$



3. $\operatorname{aff}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$



4. $\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix}\right\}$



2. Convexity

Let $C \in R^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in R$ satisfy $\theta_i \geq 0, \theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

3. Linear Equations

Show that the solution set of linear equations $\{x|Ax = b\}$ with $x \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$ is an affine set.

4. Linear Inequations

1. Show that the solution set of linear inequations $\{x|Ax \preceq b, Cx = d\}$ with $x \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$, $C \in R^{k\times n}$ and $d \in R^k$ is a convex set. Here $\preceq$ means componentwise less or equal.
2. Is it an affine set?

5. Voronoi description of halfspace

Let a and b be distinct points in R^n. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x|\, \|x-a\|^2 \leq \|x-b\|^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

2 Convex Illumination Problem

Show that the solution $p^* = (p_1^*, p_2^*, ..., p_*^n)^T \in R^n$ of the non-convex illumination problem from the lecture

$$\begin{array}{ll} \text{minimize} & \max\limits_{k=1...m} |\log I_k - \log I_{des}| \\ \text{subject to} & 0 \leq p_j \leq p_{max}, \quad j = 1...n \end{array}$$

with $I_k = \sum_{j=1}^{n} a_{kj}p_j$ for geometric constants $a_{jk} \in R$, a constant desired illumination $I_{des} \in R$ and an upper bound $p_{max} \in R$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{array}{ll} \text{minimize} & \max_{k=1...m} h(I_k/I_{des}) \\ \text{subject to} & 0 \leq p_j \leq p_{max}, \quad j = 1...n \end{array}$$

with $h(u) = max\{u, \frac{1}{u}\}$.