

Applied Optimization

Exercise 1 - Convex Sets

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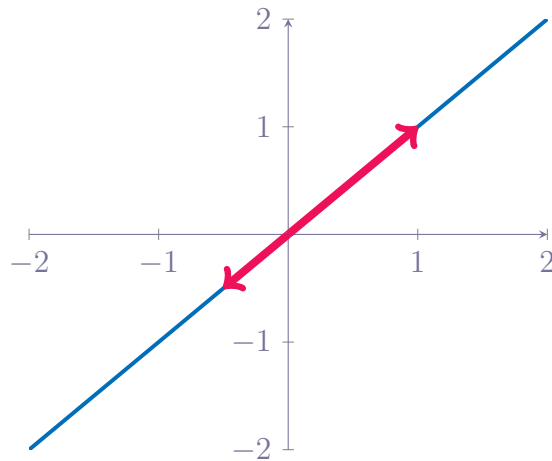
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1 Convex Sets

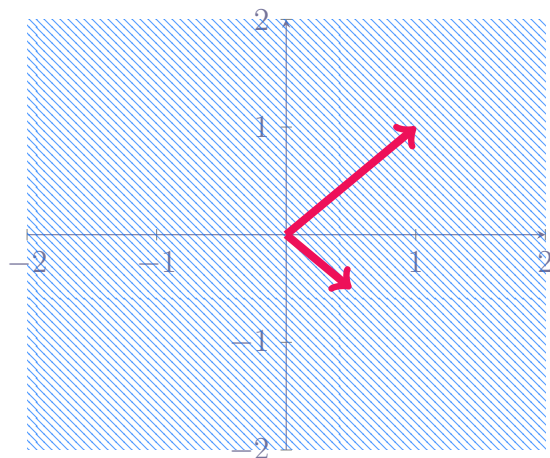
1. Example sets

Sketch the following sets in R^2

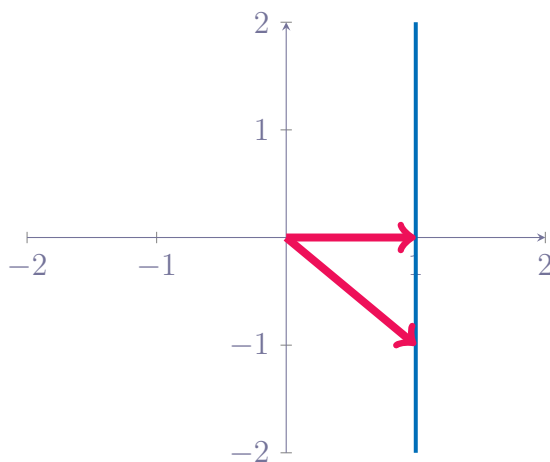
1. $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \right\}$



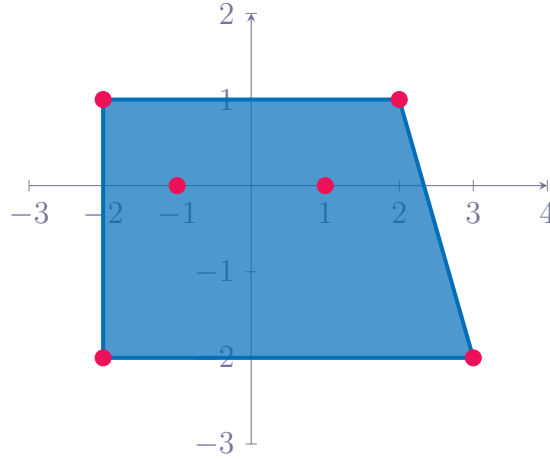
2. $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$



3. $\text{aff} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$



4. $\text{conv} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}$



2. Convexity

Let $C \in R^n$ be a convex set, with $x_1, \dots, x_k \in C$, and let $\theta_1, \dots, \theta_k \in R$ satisfy $\theta_i \geq 0, \theta_1 + \dots + \theta_k = 1$. Show that $\theta_1 x_1 + \dots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k .) Hint. Use induction on k .

Show. that $\theta_1 + \dots + \theta_k \in C$

Proof. Base Case $k = 2$ By definition, it holds for $k = 2$.

Inductive Step Assume it is correct for $k = n$, then for $k = n + 1$, we have:

$$x_1, \dots, x_{n+1} \in C$$

$$\theta_1, \dots, \theta_{n+1} \in C$$

with $\theta_i \geq 0$ and $\sum_{i=1}^{n+1} \theta_i = 1 \Rightarrow \theta_{n+1} = 1 - \sum_{i=1}^n \theta_i$

$$\Rightarrow \sum_{i=1}^{n+1} \theta_i x_i = \sum_{i=1}^n \theta_i x_i + \theta_{n+1} x_{n+1} = \sum_{i=1}^n \theta_i x_i + x_{n+1} = \sum_{i=1}^n \theta_i x_{n+1}$$

$$= \underbrace{x_{n+1}}_{\in C} \underbrace{\sum_{i=1}^n \theta_i (x_i - x_{n+1})}_{\substack{\in C \text{ by hypothesis on } k = n \\ \text{because } \sum_{i=1}^n \theta_i \in [0, 1]}} \} \Rightarrow \in C$$

□

3. Linear Equations

Show that the solution set of linear equations $\{x | Ax = b\}$ with $x \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ is an affine set.

Proof. Let x_0, x_1 be solutions of linear equations, i.e. $Ax_0 = b$ and $Ax_1 = b$

Then

$$\begin{aligned} A((1 - \beta)x_0 + \beta x_1) &= (1 - \beta)Ax_0 + \beta Ax_1 \\ &= (1 - \beta)b + \beta b = b \end{aligned}$$

\Rightarrow The set is affine. □

4. Linear Inequations

1. Show that the solution set of linear inequations $\{x | Ax \preceq b, Cx = d\}$ with $x \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$, $C \in R^{k \times n}$ and $d \in R^k$ is a convex set. Here \preceq means componentwise less or equal.

Proof. Let x_0, x_1 be solutions of linear inequations, i.e. $Ax_0 \preceq b, Cx_0 = d$ and $Ax_1 \preceq b, Cx_1 = d$, for $\beta \in [0, 1]$

Then

$$A((1 - \beta)x_0 + \beta x_1) = \underbrace{(1 - \beta)}_{\geq 0} \underbrace{Ax_0}_{\preceq b} + \underbrace{\beta}_{\geq 0} \underbrace{Ax_1}_{\preceq b} \preceq (1 - \beta)b + \beta b = b \quad (1)$$

and

$$C((1 - \beta)x_0 + \beta x_1) = (1 - \beta)Cx_0 + \beta Cx_1 = (1 - \beta)d + \beta d = d$$

\Rightarrow convex set. □

2. Is it an affine set?

No, cf [Equation 1](#), if we don't have the $a_i \geq 0$ for $\forall i$ constraint, it removes the constraint on the $\beta \in [0, 1]$, which then means we could have $(1 - \beta) < 0$. Therefore the step $(1 - \beta)Ax_0 \preceq b$ is not doable, we might have $(1 - \beta)Ax_0 \succeq b$. ■

5. Voronoi description of halfspace

Let a and b be distinct points in R^n . Show that the set of all points that are closer (in Euclidean norm) to a than b , i.e., $\{x \mid \|x - a\|^2 \leq \|x - b\|^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Proof. Be $c \in R^n$ a vector, $c = b - a$ be $d \in R$, $d = c^T x_0 = c^T \frac{a+b}{2}$

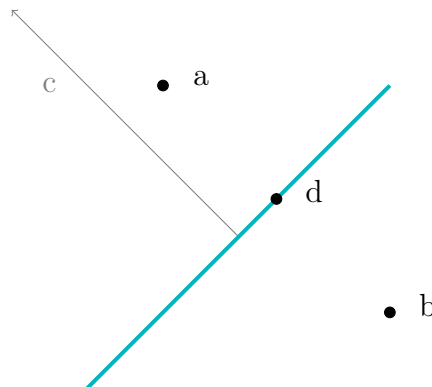
then we have the halfspace

$$\left\{ x \mid (b - a)^T x \leq \frac{(b - a)^T (b - a)}{2} \right\}$$

□

Check.

$$\begin{aligned} & \|x - a\|^2 \leq \|x - b\|^2 \\ \Leftrightarrow (x - a)^T (x - a) & \leq (x - b)^T (x - b) \\ \Leftrightarrow \cancel{x^T x} - 2a^T x + a^T a & \leq \cancel{x^T x} - 2b^T x + b^T b \\ \Leftrightarrow (b - a)^T x & \leq (b^T b - a^T a) \frac{1}{2} \end{aligned}$$



2 Convex Illumination Problem

Show that the solution $p^* = (p_1^*, p_2^*, \dots, p_n^*)^T \in R^n$ of the non-convex illumination problem from the lecture

$$\begin{aligned} & \text{minimize} && \max_{k=1\dots m} |\log I_k - \log I_{des}| \\ & \text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1\dots n \end{aligned}$$

with $I_k = \sum_{j=1}^n a_{kj} p_j$ for geometric constants $a_{jk} \in R$, a constant desired illumination $I_{des} \in R$ and an upper bound $p_{max} \in R$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{aligned} & \text{minimize} && \max_{k=1\dots m} h(I_k/I_{des}) \\ & \text{subject to} && 0 \leq p_j \leq p_{max}, \quad j = 1\dots n \end{aligned}$$

with $h(u) = \max\{u, \frac{1}{u}\}$.

Proof.

$$\begin{aligned} & \max_{k=1,\dots,m} |\log(I_k) - \log(I_{des})| && \left| \begin{array}{l} \text{log rules} \end{array} \right. \\ & = \max_{k=1,\dots,m} \left| \log\left(\frac{I_k}{I_{des}}\right) \right| && \left| \begin{array}{l} \text{remove } | \cdot | \end{array} \right. \\ & = \max_{k=1,\dots,m} \max \left\{ \log\left(\frac{I_k}{I_{des}}\right), \log\left(\frac{I_{des}}{I_k}\right) \right\} && \left| \begin{array}{l} \text{max log} = \text{log max} \end{array} \right. \\ & = \log \max_{k=1,\dots,m} \max \left\{ \frac{I_k}{I_{des}}, \frac{I_{des}}{I_k} \right\} && \left| \begin{array}{l} \text{with } h(u) = \max\left\{u, \frac{1}{u}\right\} \end{array} \right. \\ & = \log \max_{k=1,\dots,m} h\left(\frac{I_k}{I_{des}}\right) \end{aligned}$$

The log is a monotonically increasing function, i.e. $\forall x, y$ with $x \leq y$ we have $\log(x) \leq \log(y)$. So when minimizing, we can remove it without affecting the solution.

$$\Rightarrow \max_{k=1,\dots,m} h\left(\frac{I_k}{I_{des}}\right) \text{ with } h(u) = \max\left\{u, \frac{1}{u}\right\}.$$

□