Proof. We know that $\binom{n}{k}$ is the number of binary words of length n with exactly k digits 1. There are two kinds of words like that: those that start with 1 and those that start with 0. How many words of each kind are there?
When we delete the first digit, we are left with a word of length $n-1$. For the words of the first kind, this word of length $n-1$ must contain $k-1$ digits 1. Thus there are $\binom{n-1}{k-1}$ words of the first kind.
Similarly, for a word of second kind we are left with a word of length $n-1$ that contains k digits 1. Thus there are $\binom{n-1}{k}$ words of the second kind.
Since every word is either of the first kind or of the second kind but not both, identity (**??**) holds. □

Proof of Theorem **??**. Let us write the numbers $\binom{n}{k}$ in a triangle similar to the Pascal triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

The top-left neighbor of the number $\binom{n}{k}$ is $\binom{n-1}{k-1}$, the top-right neighbor is $\binom{n-1}{k}$. By Lemma **??**, the numbers in the $\binom{n}{k}$-triangle satisfy the same rule that the numbers in the Pascal triangle: each number is the sum of its top-left and top-right neighbors. The outermost numbers $\binom{n}{0}$ and $\binom{n}{n}$ are also the same as in the Pascal triangle:

$$\binom{n}{0} = \binom{n}{n} = 1.$$

It follows that the $\binom{n}{k}$-triangle coincides with the Pascal triangle. (The formal argument here is proof by induction: if the n-th line of the $\binom{n}{k}$-triangle coincides with the n-th line of the Pascal triangle, then their $(n+1)$-st lines also coincide.)

□