# 2020 Fall EECS205002 Linear Algebra

2020/12/2 Quiz 3

1. (50%) Multiple choice questions. Each question may have 0, 1, or more correct choices.

   (If you answer an option incorrectly, 3 points will be deducted, if you answer two or more options incorrectly, 5 points will be deducted)

   (a) If the vectors $v_1, v_2, \ldots, v_n$ form a basis for a vector space $V$, which statements are true?

      A. $v_1, v_2, \ldots, v_n$ are linearly independent.
      B. Any vector in $V$ can be expressed as a unique linear combination of $v_1, v_2, \ldots, v_n$.
      C. The dimension of $V$ is $n$.
      D. Any $m$ vectors in $V$, $m > n$ cannot be linearly independent.
      E. Any $m$ vectors in $V$, $m > n$ can span $V$.

      A, B, C, D

   (b) If $V$ is a vector space of dimension $n > 0$, which statements are true?

      A. Any set of $n$ linearly independent vectors spans $V$.
      B. Any $n$ vectors that span $V$ forms a basis for $V$.
      C. No set of fewer than $n$ vectors can span $V$.
      D. Any subset of fewer than $n$ linearly independent vectors cannot be a basis for $V$.
      E. Any spanning set containing more than $n$ vectors can form a basis for $V$.

      A, B, C, D

   (c) Which statements are true?

      A. The rank of a matrix $A$ is the dimension of $A$'s row space.
      B. The dimension of $A$'s row space equals to the dimension of $A$'s column space.
      C. The row space of a matrix $A$ is the space spanned by $A$'s row vectors; the column space of a matrix $A$ is the space spanned by $A$'s column vectors;
      D. For an $m \times n$ matrix, the rank of $A$ plus the nullity of $A^T$ equals to $m$.
      E. The nullity of $A$ is the dimension of $A$'s null space.

      A, B, C, D, E

   (d) Let $U$ be a reduced row echelon form of a matrix $A$, which of the following statements are true?

      A. $U$ has the same row space as $A$.
      B. The rank of $U$ is the same as $A$'s rank.
      C. The column space of $U$ is the same as the column space of $A$.
      D. The column vectors of $U$ that contains leading variables are linearly independent.
      E. The column vectors of $A$ that corresponds to the column vectors of $U$ containing leading variables form a basis of $A$'s column space.

      A, B, D, E

   (e) For an $m \times n$ matrix $A$, which of the statements are true?

      A. A linear system $Ax = b$ is consistent if and only if $b$ is in the column space of $A$.
      B. The linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ if and only if the column vectors of $A$ are linearly independent.
      C. The system $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$ if and only if the column vectors of $A$ span $\mathbb{R}^m$.
      D. For $m = n$, matrix $A$ is nonsingular if and only if the row vectors of $A$ form a basis for $\mathbb{R}^n$.
      E. The linear system $Ax = b$ has a unique solution for every $b \in \mathbb{R}^m$ if and only if $A$ is nonsingular.

A, D, E

(f) An $m \times n$ matrix $A$ has a right inverse if there exists an $n \times m$ matrix $C$ such that $AC = I_m$. The matrix $A$ is said to have a left inverse if there exists an $n \times m$ matrix $D$ such that $DA = I_n$. Based on the above definitions, which of the statements are true?

A. If $A$ has a right inverse, then the column vectors of A span $\mathbb{R}^m$.

B. If $A$ has a right inverse, then $n \geq m$.

C. If $A^T$ has a left inverse, then $A$ has a right inverse.

D. If $A$ has a left inverse then the columns of $A$ are linearly independent.

E. If $A$ has both left inverse and right inverse, $A$ must be nonsingular.

A, B, C, D, E

Here give a short explanation for each item.

A. If $A$ has a right inverse, $AC = I_m$. Any vector $x$ in $\mathbb{R}^m$ can be expressed as

$$x = I_m x = ACx = Ay.$$

where $y = Cx$. Thus, any vector $x$ in $\mathbb{R}^m$ can be expressed as a linear combination of $A$'s column vectors. So the column vectors of A span $\mathbb{R}^m$.

B. If $A$ has a right inverse, from A, we know that $A$'s column vectors span $\mathbb{R}^m$. Therefore, the number of column vectors, which is $n$, must be larger than or equal to $m$. (Theorem 3.4.4 (i)).

C. If $A^T$ has a left inverse, $A^T C = I_n$. Since $I_n = I_n^T = (A^T C)^T = C^T (A^T)^T = C^T A$, $A$ has a right inverse.

D. Suppose $A$'s column vectors are linearly dependent. The homogeneous system, $Ax = 0$, has a nontrivial solution, $x \neq 0$. Now multiplying both sides by $A$'s left inverse, say $D$.

$$D(Ax) = (DA)x = I_n x = x \neq 0.$$

but $D(Ax)$ also equals to $D(Ax) = D0 = 0$, which is a contradiction of above equation. So $A$'s column vectors must be linearly independent.

E. If $A$ has both left inverse and right inverse, $XA = I$ and $AY = I$, which implies

$$X = X(AY) = (XA)Y = Y.$$

The left inverse must equal to the right inverse. $A$ is nonsingular.
You can also prove it using the properties from (A) and (D). If $A$ has a left inverse, the column vectors of A span $\mathbb{R}^m$. If $A$ has a right inverse, the column vectors of A are linear independent. So if $A$ has both inverses, $A$'s column vectors form a basis of $\mathbb{R}^m$. By Corollary 3.6.4, $A$ must be nonsingular.

(g) Which of the statements are true?

A. $\vec{x}^T \vec{y} = \|\vec{y}\| \|\vec{x}\| \cos(\theta)$, where $\theta$ is the angle between $\vec{x}$ and $\vec{y}$.

B. Two vectors are orthogonal if their scalar product equals to 0.

C. The projection matrix of $\vec{x}$ is $\dfrac{\vec{x}\vec{x}^T}{\vec{x}^T\vec{x}}$.

D. The normal vector of a plane $ax + by + cz + d = 0$ is $[a - d, b - d, c - d]^T$.

E. Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$. $\|\vec{x}\| + \|\vec{y}\| \leq \|\vec{x} + \vec{y}\|$.

A, B, C

(h) For a linear least square problem $\min_{\vec{x}} \|A\vec{x} - \vec{b}\|$, which of the statements are true?

A. The problem is equivalent to find a vector in $A$'s row space whose distance to $\vec{b}$ is minimum.

B. For an $m \times n$ matrix $A$, $A^T A$ is nonsingular if and only if $A$ has rank $n$.

C. The optimal solution must be orthogonal to $A$'s column vectors.

D. The solution of this problem must satisfy the normal equation: $A^T A\vec{x} = A^T \vec{b}$.

E. The linear system $A^T A\vec{x} = A^T \vec{b}$ has no solution if $A^T A$ is singular.

B, D

(i) For any pair of vectors $x$, $y$ in a vector space $V$, and $< x, y >$ is an inner product operation, which of the statements are true?

A. $< x, y >$ is a real number.

B. $< x, x >$ is a positive number.

C. $< x, x >= 0$ if and only if $x = 0$.

D. $< x, y >=< y, x >$.

E. $< (\alpha + \beta)x, y >= \alpha < x, y > +\beta < x, y >$ for $\alpha, \beta \in \mathbb{R}$.

A, C, D, E or A, C, D

(j) If $v \neq 0$ and $p$ is the vector projection of $u$ onto $v$.

A. $p = \dfrac{< u, v >}{< v, v >} v$.

B. $v - p$ and $p$ are orthogonal.

C. $p$ is a scalar multiple of $v$.

D. $u$ is a scalar multiple of $v$.

E. The length of $p$ is $\dfrac{< u, v >}{\|v\|}$.

A, C

2. (10%) Find a subset from the following vectors to form a basis for $\mathbb{R}^3$.

$$\vec{x}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \vec{x}_2 = \begin{bmatrix}2\\5\\4\end{bmatrix}, \vec{x}_3 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \vec{x}_4 = \begin{bmatrix}2\\7\\4\end{bmatrix}, \vec{x}_5 = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

This is equivalent to find a basis of $A$'s column vectors. We can use its reduced row echelon form to find the basis.

$$A = \begin{bmatrix}1 & 2 & 1 & 2 & 1\\2 & 5 & 3 & 7 & 1\\2 & 4 & 2 & 4 & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1 & 2 & 1\\0 & 1 & 1 & 3 & -1\\0 & 0 & 0 & 0 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -4 & 0\\0 & 1 & 1 & 3 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

The leading variables are $x_1, x_2$, and $x_5$. So the $\vec{x}_1, \vec{x}_2, \vec{x}_5$ form a basis for $\mathbb{R}^3$.

3. (10%) Find the point on the line $y = 2x + 4$ that is closest to the point $(5, 2)$.

There are many ways to do that. Here shows two.

Method I:

Let $(0, 4)$ be the new origin. The vector $u = (1, 2)$ is the vector in the direction of $y - 4 = 2x$. Let $v = (5, 2) - (0, 4) = (5, -2)$. The vector project of $v$ onto $u$ is

$$p = \frac{v^T u}{u^T u} u = (1/5, 2/5)$$

Now transform this point back to the original coordinate with origin $(0, 0)$, which is $(0.2, 4.4)$.

Method II:

The point on the line has the form $p = (t, 2t+4)$. Let $u = (5, 2) - p = (5-t, 2-2t-4) = (5-t, -2-2t)$. If $p$ is the point with the shortest distance to $(5, 2)$, $u$ must be orthogonal to $v = (1, 2)$.

$$v^T u = (1, 2)(5 - t, -2 - 2t)^T = 5 - t - 4 - 4t = 0$$

The solution is $t = 1/5$, which means $p = (0.2, 4.4)$.

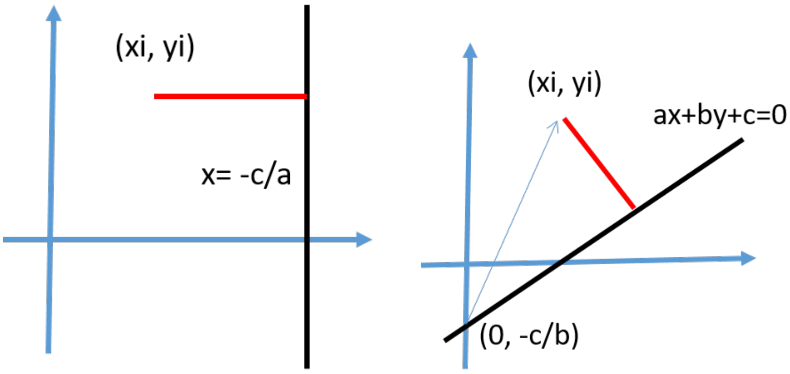
4. (10%) There are four points $(0, 3), (1, 2), (2, 4), (3, 4)$. Find a line $y = ax + b$ so that the summation of their squared distance to the line is minimum, as requested below.

$$\min_{a,b} \sum_{i=1}^{4} (y_i - (ax_i + b))^2$$

From the description, we need to solve the normal equation $A^T Ax = A^T y$, where

$$A = \begin{bmatrix}0 & 1\\1 & 1\\2 & 1\\3 & 1\end{bmatrix}, x = \begin{bmatrix}a\\b\end{bmatrix}, y = \begin{bmatrix}3\\2\\4\\4\end{bmatrix}$$

The solution is $a = 0.5, b = 2.5$, so the line is $y = 0.5x + 2.5$.




Case I: $b = 0$ Case II: $b \neq 0$

Figure 1: The cases of question 6.

5. (10%) Show that if $U$ and $V$ are subspaces of $\mathbb{R}^n$, then $\dim(U + V) \leq \dim(U) + \dim(V)$ where

$$U + V = \{\vec{z} | \vec{z} = \vec{u} + \vec{v} \text{ where } \vec{u} \in U \text{ and } \vec{v} \in V\}.$$

Let $u_1, u_2, \dots, u_k$ be a basis of $U$ and $v_1, v_2, \dots, v_l$ be a basis of $V$. We have $\dim(U) = k$ and $\dim(V) = l$. For any vector $z \in U + V$, $z$ can be expressed as

$$z = u + v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_l v_l.$$

Thus, the vectors $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_l$ span $U + V$. So $k + l = \dim(U) + \dim(V)$ must be larger than or equal to $\dim(U + V)$. (see Theorem 3.4.4 (i)).

6. (10%) Show the distance of a point $(x_i, y_i)$ to the line $ax + by + c = 0$ is $|ax_i + by_i + c|$ where $a^2 + b^2 = 1$.

There are several ways to prove it. Here we use the method in linear algebra to prove it. There are two cases of the equations, as shown in Figure 1.

Case I: If $b = 0$, in that case, $a = \pm 1$. The distance from $(x_i, y_i)$ to $x = \pm c$ is $|ax_i + c| = |ax_i + by_i + c|$.

Case II: If $b \neq 0$, the y-intercept is $-c/b$. There are two vectors, one is the normal vector of $ax+by+c = 0$, which is $\vec{u} = (a, b)$, another is from $(0, -c/b)$ to $(x_i, y_i)$, which is $\vec{v} = (x_i, y_i + c/b)$. The distance is length of the projected vector from $\vec{v}$ to the normal vector $\vec{u}$,

$$\left\| \frac{\vec{v}^T \vec{u}}{\|\vec{u}\|^2} \vec{u} \right\| = \left\| (ax_i + by_i + c) \begin{bmatrix} a \\ b \end{bmatrix} \right\| = |ax_i + by_i + c| \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = |ax_i + by_i + c|.$$