2020 Fall EECS205002 Linear Algebra

2021/01/13 Quiz 5

1. (50%) Multiple choice questions. Each question may have 0, 1, or more correct choices. For each question, you need to choose all the correct items to get the credit.

   (a) For an $n \times n$ real matrix A is symmetric, which of the following statements are true?

      A. If (λ, x) is an eigenpair of A, $\lambda = x^T A x$.
      B. If λ is an eigenvalue of A, λ equals to its conjugate $\bar{\lambda}$.
      C. The eigenvectors of A belonging to distinct eigenvalues are linearly independent.
      D. There exists a set of n orthonormal eigenvectors of A.
      E. Matrix A is always diagonalizable.

      B, C, D, E

   (b) For an Householder reflector $H = I - 2uu^T$, where $\|u\| = 1$, which of the following statements are true?

      A. H is symmetric.
      B. $\|Hx\| = \|x\|$ for any vector x that can be pre-multiplied by H.
      C. $H^{-1} = H$.
      D. The eigenvalue of H are either 1 or -1.
      E. $(-1, u)$ is an eigenpair of H.

      A, B, C, D, E

   (c) For an $m \times n$ matrix A, $m \geq n$, which of the following statements are true?

      A. A can be factorized as $U\Sigma V^T$ uniquely, where U and V are orthogonal matrices, and Σ is a diagonal matrix.
      B. If $m = n$ and A has eigenvalue $\lambda_1, \ldots, \lambda_n$, the singular values of A are $|\lambda_1|, \ldots, |\lambda_n|$.
      C. The row vectors in V are the eigenvectors of $A^T A$.
      D. The column vectors in U are the eigenvectors of AA^T.
      E. For singular values $\sigma_1, \ldots, \sigma_n$, $Au_i = \sigma_i v_i$ for $i = 1, 2, \ldots, n$.

      D

(d) If an $m \times n$ matrix A has rank r, and the SVD of A is $A = U\Sigma V^T$, where $U = [u_1, u_2, \ldots, u_m]$ and $V = [v_1, v_2, \ldots, v_n]$, which of the following statements are true?

A. The number zero singular values is $m - r$.

B. $v_1, v_2, \ldots, v_r$ form an orthonormal basis for $R(A)$.

C. $v_{r+1}, v_{r+2}, \ldots, v_n$ form an orthonormal basis for $N(A)$.

D. $u_1, u_2, \ldots, u_r$ form an orthonormal basis for $R(A^T)$.

E. $u_{r+1}, u_{r+2}, \ldots, u_m$ form an orthonormal basis for $N(A^T)$.

C, E

(e) The Frobenius norm of an $m \times n$ matrix A is defined as $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$, which of the following statements are true?

A. If Q is an $m \times m$ orthogonal matrix, $\|Q\|_F = 1$.

B. If Q is an $m \times m$ orthogonal matrix, $\|QA\|_F = \|A\|_F$.

C. If A has singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$, $\|A\|_F = (\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2)^{1/2}$.

D. If A has rank r, for $0 < k < r$, the minimum of $\|A - X\|_F$ for all possible matrices X of rank k is $(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_k^2)^{1/2}$, where $\sigma_1, \ldots, \sigma_k$ are the k largest singular values of A.

E. If A has rank r, for $0 < k < r$, the matrix of rank k that makes $\|A - X\|_F$ minimum has Forbenius norm $(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_k^2)^{1/2}$, where $\sigma_1, \ldots, \sigma_k$ are the k largest singular values of A.

B, C, E

(f) Which operator L is a linear transformation?

A. $L([x_1, x_2]) = [x_2, 0, x_1 + x_2]$ where x_1, x_2 are scalars.

B. $L([x_1, x_2]) = \sqrt{x_1^2 + x_2^2}$ where x_1, x_2 are scalars.

C. $L([x_1, x_2]) = [x_1 + 1, x_2]$ where x_1, x_2 are scalars.

D. $L([x_1, x_2]) = [\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2]$ where x_1, x_2 are scalars, and $\theta \in [0, 2\pi)$.

E. $L(x) = Ax$ for an $m \times n$ matrix A and a vector $x \in \mathbb{R}^n$.

A, D, E

(g) If L is a linear transformation mapping a vector space V into a vector space W, which of the following statements are true?

A. The kernel of L is $\{v \in V | L(v) = 0_W\}$.

B. If S is a subspace of V, the image of S is also a subspace in V.

C. The range of L is the image of V.

D. If $L(x) = Ax$ for an $m \times n$ matrix A and a vector $x \in \mathbb{R}^n$, the kernel of L is the null space of A.

E. If $L(x) = Ax$ for an $m \times n$ matrix A and a vector $x \in \mathbb{R}^n$, the range of L is the row space of A.

A, C, D

(h) If L is a linear transformation from V into W, where V and W are vector spaces, which of the following statements are true?

A. L is **one-to-one** if $L(v_1) \neq L(v_2)$ implies $v_1 \neq v_2$.

B. L is **one-to-one** if and only if $\ker(L) = \{0_V\}$.

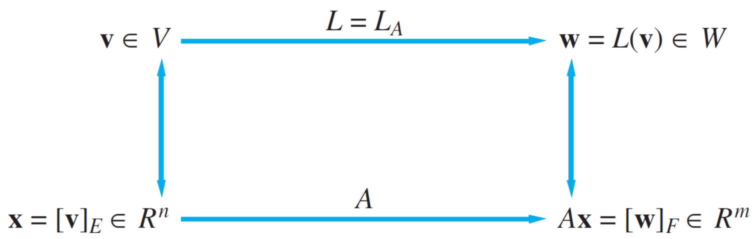
C. L maps V **onto** W if for each $w \in W$ there exists at most one $v \in V$ that $L(v) = w$.

D. L maps V **onto** W if the image of V is W.

E. If $L(x) = Ax$ is **one-to-one** and maps V **onto** W, A is nonsingular.

B, D, E

(i) The following figure illustrates the matrix representation theorem. Which explanations about the figure are correct?



A. L is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$.

B. $E = [e_1, e_2, \ldots, e_n]$ is an ordered basis of V;
$F = [f_1, f_2, \ldots, f_m]$ is an ordered basis of W.

C. $x = [v]_E$ means $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, $v = x_1 e_1 + \cdots + x_n e_n$;
$y = [w]_F$ means $y = [y_1, y_2, \ldots, y_m]^T \in \mathbb{R}^m$, $w = y_1 f_1 + \cdots + y_m f_m$.

D. A is the matrix representing L with respect to $\mathbb{R}^n$ and $\mathbb{R}^m$.

E. L maps v into w if and only if $y = Ax$.

B, C, E

(j) If L is a linear transformation mapping a vector space V into a vector space W, which of the following statements are true?

A. If $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$, L can be represented as $L(x) = Ax$ for $x \in \mathbb{R}^3$ and a 2×3 matrix A.

B. If $L([x_1, x_2, x_3]^T) = [x_1 + x_2, x_2 + x_3]^T$ for $x_1, x_2, x_3 \in \mathbb{R}$, then

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + x_2 L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + x_3 L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

C. If V has an ordered basis $E = [v_1, v_2]$, and W has an ordered basis F, for a vector v in V, we can represent L as $[L(v)]_F = A[v]_E$, where $A = [[L(v_1)]_E, [L(v_2)]_E]$.

D. If L is linear transformation that rotates an $\mathbb{R}^2$ vector $\vec{x}$ by degree θ, L can be represented as $L(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

E. If L is a differential operator for polynomial of degree 2, L can be represented as $L(ax^2 + bx + c) = A \begin{bmatrix} a & b & c \end{bmatrix}^T$, where A is a 2×3 matrix with column vector $L(x^2), L(x), L(1)$.

B

2. (10%) If an $n \times n$ matrix A has an eigenvector $e_1 = [1, 0, \ldots, 0]^T$, what kind of structure of A should be? Justify your answer.

A should be in the form of $\begin{bmatrix} \lambda & v^T \\ 0 & B \end{bmatrix}$. Since $Ae_1 = a_1 = \lambda e_1$, we know the first column vector $a_1 = \lambda e_1$.

3. (10%) If A is an $n \times n$ matrix which has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$, show that $|\lambda_1 \lambda_2 \cdots \lambda_n| = \sigma_1 \sigma_2 \cdots \sigma_n$.

We know that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. The SVD of A gives $A = U\Sigma V^T$, so $\det(A) = \det(U)\det(\Sigma)\det(V^T)$. Since U and V are orthogonal matrix, $|\det(U)| = |\det(V^T)| = 1$. Also, since Σ is a diagonal matrix, $\det(\Sigma) = \sigma_1 \sigma_2 \cdots \sigma_n$. In addition, all the singular values are nonnegative, so $|\det(\Sigma)| = |\sigma_1 \sigma_2 \cdots \sigma_n| = \sigma_1 \sigma_2 \cdots \sigma_n$. Putting them together, we have

$$|\det(A)| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\det(U)||\det(\Sigma)||\det(V^T)| = \sigma_1 \sigma_2 \cdots \sigma_n.$$

4. (10%) Let $L(x) = [x_1 + x_3, x_3 - x_2]$ for $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

(a) What is the kernel of L?

$$L(x) = Ax = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} x.$$

The kernel of L is equal to the null space of A, which is $\text{span}([-1, 1, 1])$.

(b) For $S = \text{span}([1, 1, 0]^T)$, what is the image of S?

$L([\alpha, \alpha, 0]^T) = [\alpha, -\alpha]$. So $L(S) = \text{span}([1, -1]^T)$.

5. (10%) Let $L([x_1,x_2,x_3]) = [x_1+x_2, x_3-x_1]^T$, $E=[u_1,u_2,u_3]$ be an ordered basis for $\mathbb{R}^3$ and $F=[b_1,b_2]$ be an ordered basis for $\mathbb{R}^2$, where

$$u_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, u_2 = \begin{bmatrix}1\\2\\1\end{bmatrix}, u_3 = \begin{bmatrix}-1\\1\\1\end{bmatrix}, b_1 = \begin{bmatrix}1\\-1\end{bmatrix}, b_2 = \begin{bmatrix}2\\-1\end{bmatrix}.$$

Find the matrix representation A of L with respect to E and F.

Using the formula of Theorem 4.2.2,

$$A = [[L(u_1)]_F, [L[u_2)]_F, [L(u_3)]_F],$$

where

$$L(u_i) = x_i b_1 + y_i b_2 = [b_1\ b_2]\begin{bmatrix}x_i\\y_i\end{bmatrix} = \begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}[L(u_i)]_F.$$

So

$$[L(u_i)]_F = \begin{bmatrix}1 & 2\\-1 & -1\end{bmatrix}^{-1} L(u_i) = \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix} L(u_i),$$

for $i=1,2,3$.

$$A = \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix}[L(u_1), L(u_2), L(u_3)] = \begin{bmatrix}-1 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 0\\-2 & 0 & 2\end{bmatrix} = \begin{bmatrix}3 & -3 & -4\\-1 & 3 & 2\end{bmatrix}$$

6. (10%) Matrix polynomial is a polynomial with square matrices as variables. For example, we can put an $n\times n$ matrix A into a polynomial $q(x) = 2x^2+3x+4$, and get a matrix polynomial $q(A) = 2A^2+3A+4I$. Let A be a symmetric matrix, and $p(x) = \det(A-xI)$ be A's characteristic polynomial. Show that $p(A)=O$, a zero matrix.

Since A is symmetric, A can be diagonalized by an orthogonal matrix U, $A = U\Lambda U^T$. The characteristic polynomial $p(x) = \det(A-xI)$ can be expressed as $p(x) = \sum_{i=0}^n a_i x^i$.

$$p(A) = \sum_{i=0}^n a_i A^i = \sum_{i=0}^n a_i (U\Lambda U^T)^i = \sum_{i=0}^n a_i U\Lambda^i U^T = U\left(\sum_{i=0}^n a_i\Lambda^i\right)U^T.$$

$$\sum_{i=0}^n a_i\Lambda^i = \sum_{i=0}^n a_i \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}^i = \sum_{i=0}^n \begin{bmatrix}a_i\lambda_1^i & & & \\ & a_i\lambda_2^i & & \\ & & \ddots & \\ & & & a_i\lambda_n^i\end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=0}^{n} a_i\lambda_1^i & & & \\ & \sum_{i=0}^{n} a_i\lambda_2^i & & \\ & & \ddots & \\ & & & \sum_{i=0}^{n} a_i\lambda_n^i \end{bmatrix} = \begin{bmatrix} p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_n) \end{bmatrix}$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $p(x)$, $p(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$, which means $\sum_{i=0}^{n} a_i\Lambda^i = O$. Therefore,

$$p(A) = U\left(\sum_{i=0}^{n} a_i\Lambda^i\right)U^T = UOU^T = O.$$

This result is called Cayley–Hamilton theorem, which holds for all kinds of square matrices, not only for symmetric matrices. The proof given here can also work for diagonalizable matrices.

PS: some of you use the following proof: since $p(x) = \det(A - xI)$, $p(A) = \det(A - AI) = 0$. This is not correct, because $p(A)$ is a matrix, but $\det(A - AI) = 0$ is a scalar. This is only true when A is an 1×1 matrix. So only partial credit (2pt) will be given.

7. (20%) There are four subspaces in an $m \times n$ matrix A, row space, column space, null space, and null space of A^T. Write down all the facts you have learned about those four subspaces.

   (a) The column space and row space have equal dimension r, which is the rank of the matrix.

   (b) The nullspace $N(A)$ has dimension $n - r$; the nullspace $N(A^T)$ has dimension $m - r$.

   (c) The orthogonal complement of the row space is $N(A)$; The orthogonal complement of the row space is $N(A^T)$.

   (d) Let $A = U\Sigma V^T$ be the SVD of A, where $U = [u_1, u_2, \ldots, u_m]$ is an $m \times m$ orthogonal matrix; $V = [v_1, v_2, \ldots, v_n]$ is an $n \times n$ orthogonal matrix; and Σ is an $m \times n$ diagonal matrix whose diagonal elements, $\sigma_1, \sigma_2, \ldots, \sigma_n$, are the singular values of A.

      A. The rank of A equals to the number of nonzero singular values.
      B. $v_1, v_2, \ldots, v_r$ form an orthonormal basis for the row space.
      C. $v_{r+1}, v_{r+2}, \ldots, v_n$ form an orthonormal basis for $N(A)$.
      D. $u_1, u_2, \ldots, u_r$ form an orthonormal basis for column space.
      E. $u_{r+1}, u_{r+2}, \ldots, u_m$ form an orthonormal basis for $N(A^T)$.

   You may reference the note, "The Four Fundamental Subspaces: 4 Lines" written by Gilbert Strang.
   `https://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf`