2020 Fall EECS205002 Linear Algebra

2020/11/4 Quiz 2

1. (25%) What are the definitions of the following terms?

   (a) For a 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the minor and the cofactor of a_{23}.

   Let $M_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$.

   The minor of a_{23} is $\det(M_{23})$ and the cofactor of a_{23} is $(-1)^{2+3}\det(M_{23})$.

   (b) Skew symmetric matrix

   A matrix A that satisfies the property $A = -A^T$.

   (c) Spanning set for a vector space V

   A set of vectors $v_1, v_2, \ldots, v_n$ in V is called a spanning set for V if for any vector $x \in V$, x can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$.

   (d) Null space of a matrix A

   N(A) = {x l Ax = 0}.

   (e) Linear independence

   A set of vectors $v_1, v_2, \ldots, v_n$ in a vector space is called linear independence if $c_1v_1 + c_2v_2 + \cdots c_nv_n = 0$ implies $c_1 = c_2 = \cdots = c_n = 0$.

2. (9%) What is $\det(EA)$ for an $n \times n$ matrix A with an elementary matrix E of type I, type II, and type III? Express your answer in terms of $\det(A)$.

$$\det(EA) = \begin{cases} -\det(A) & \text{type I} \\ \alpha\det(A) & \text{type II} \\ \det(A) & \text{type III} \end{cases}$$

3. (10%) Use mathematical induction to show that for an $n \times n$ matrix A and $\alpha \in \mathbb{R}$, $\det(\alpha A) = \alpha^n \det(A)$.

   For $k = 1$, $\det(\alpha A) = \det(\alpha[a_{11}]) = \alpha a_{11} = \alpha\det(A)$.
   Assume for $k = n$, $\det(\alpha A) = \alpha^n \det(A)$.

For $k = n + 1$, using cofactor expansion of the first row,

$$\begin{aligned}
\det(\alpha A) &= \sum_{i=1}^{n+1} \alpha a_{1i}(-1)^{1+i} \det(\alpha M_{1i}) \\
&= \sum_{i=1}^{n+1} \alpha a_{1i}(-1)^{1+i} \alpha^n \det(M_{1i}) \\
&= \alpha^{n+1} \sum_{i=1}^{n+1} a_{1i}(-1)^{1+i} \det(M_{1i}) = \alpha^{n+1} \det(A).
\end{aligned}$$

4. (10%) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$. What is the product of A and the adjoint matrix of A?

   The product of A and its adjoint matrix equals to $\det(A)I$. So you only need to compute $\det(A)$. Let's use the first column for cofactor expansion. $\det(A) = 1(3 - 8) - 2(2 - 6) = 3$. So the answer is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

5. (16%) Let $A = \begin{bmatrix} -2 & 4 & 4 \\ 2 & -8 & 0 \\ 8 & -20 & -12 \end{bmatrix}$.

   (a) What is the null space of A?

   Use Gaussian elimination to convert A into a reduced row echelon form

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   The solution for $Ax = 0$ has the form $x = \alpha[4, 1, 1]^T$ for $\alpha \in \mathbb{R}$. So

$$N(A) = \alpha \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\right).$$

   (b) Is the set of vectors $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a spanning set of the null space of A. Justify your answer.

This is asking whether $\alpha[4,1,1]^T$ can be expressed as the linear combination of $[0,1,2]^T$ and $[1,0,1]^T$ for any α.

$$y_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

This gives us three equations

$$\begin{array}{rcl} & y_2 & = 4\alpha \\ y_1 & & = \alpha \\ 2y_1 + & y_2 & = \alpha \end{array}$$

The linear system is inconsistent. So it is NOT a spanning set of $N(A)$.

6. (10%) Are vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent? Justify your answer.

Let's verify this using the definition.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives us a homogeneous linear system

$$\begin{array}{cccccl} c_1 & + & & & c_3 & = 0 \\ 2c_1 & + & c_2 & & & = 0 \\ 3c_1 & + & 5c_2 & + & c_3 & = 0 \end{array},$$

which has a unique answer $c_1 = c_2 = c_3 = 0$. So they are linearly independent.

7. (10%) Let A and B be $k \times k$ matrices and let

$$M = \begin{bmatrix} O & A \\ B & O \end{bmatrix}.$$

Show that $\det(M) = (-1)^k \det(A)\det(B)$.

(Hint: You can use the result $\det\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right) = \det(A)\det(B)$ directly.)

There are many different ways to prove this. Here gives two proofs.

(a) By exchanging the rows of M, row 1 and row k+1, row 2 and row k+2, ... row k and row 2k, using the type I elementary row operations, one can convert M into another form.

$$M = \begin{bmatrix} O & A \\ B & O \end{bmatrix} = E_1 E_2 \cdots E_k \begin{bmatrix} B & O \\ O & A \end{bmatrix}.$$

So

$$\det(M) = \det\left(E_1 E_2 \cdots E_k \begin{bmatrix} B & O \\ O & A \end{bmatrix}\right) = \det(E_1) \cdots \det(E_k) \det\left(\begin{bmatrix} B & O \\ O & A \end{bmatrix}\right).$$

Because $\det(E_i) = -1$ (type I), and $\det\left(\begin{bmatrix} B & O \\ O & A \end{bmatrix}\right) = \det(A)\det(B)$,

$$\det(M) = (-1)^k \det(A)\det(B).$$

(b) You can also show that

$$\begin{bmatrix} O & A \\ B & O \end{bmatrix} = \begin{bmatrix} O & A \\ I & O \end{bmatrix} \begin{bmatrix} B & O \\ O & I \end{bmatrix}$$

and use induction to show that

$$\det\left(\begin{bmatrix} O & A \\ I & O \end{bmatrix}\right) = (-1)^k \det(A).$$

So

$$\det\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) = \det\left(\begin{bmatrix} O & A \\ I & O \end{bmatrix}\right) \det\left(\begin{bmatrix} B & O \\ O & I \end{bmatrix}\right) = (-1)^k \det(A)\det(B).$$

8. (10%) Let A be an $n \times n$ matrix and let $x_1, x_2, \ldots x_k$ be vectors in $\mathbb{R}^n$, where $k < n$. If the vectors $y_i = Ax_i$ for $i = 1, 2, \ldots k$ are linearly independent,

(a) Show that the vectors $x_1, x_2, \ldots, x_k$ are linearly independent.

If $x_1, x_2, \ldots, x_k$ are linearly dependent, the equation $c_1x_1 + c_2x_2 + \cdots c_kx_k = 0$ has a nonzero solution (some of the c_i are nonzero). We use this nonzero solution in the linear combination of y_i.

$$\begin{aligned} c_1y_1 + c_2y_2 + \cdots c_ky_k &= c_1Ax_1 + c_2Ax_2 + \cdots c_kAx_k \\ &= A(c_1x_1 + c_2x_2 + \cdots c_kx_k) \\ &= A0 = 0 \end{aligned}$$

Thus, we find a set of coefficients c_i, not all zeros, that makes the linear combination of y_i equal to 0. This implies y_i are linearly dependent, which contradicts the given condition. So $x_1, x_2, \ldots, x_k$ are linearly independent.

(b) Given the conditions that $\{y_1, y_2, \ldots y_k\}$ are linearly independent, and $\{x_1, x_2, \ldots x_k\}$ are linearly independent. Under what kind of conditions that matrix A can be singular?

Claim: If $\text{span}(x_1, x_2, \ldots, x_k) \cap N(A) = \{0\}$, A can be singular.
Since A is singular, there exist nonzero vectors v so that $Av = 0$, $v \in N(A)$. If $\text{span}(x_1, x_2, \ldots, x_k) \cap N(A) \neq \{0\}$, there exists a nonzero vector $v \in N(A)$ that can be expressed as the linear combination of $x_1, x_2, \ldots, x_k$,

$$v = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$$

Since $v \in N(A)$,

$$Av = A(c_1 x_1 + c_2 x_2 + \cdots + c_k x_k) = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k = 0.$$

Which means we can find a set of $c_1, c_2, \ldots, c_k$, not all zeros, that makes the linear combination of y_i zero. This implies y_i are linearly dependent, which contradicts the given condition. Thus, A can be singular if $\text{span}(x_1, x_2, \ldots, x_k) \cap N(A) = \{0\}$.