2020 Fall EECS205002 Linear Algebra

2020/12/30 Quiz 4

1. (30%) True or False. For each item, if your answer is False, indicate which part of the statement is incorrect.

   (a) Two subspaces X and Y of $\mathbb{R}^n$ are said to be orthogonal complement if for all $x \in X$ and for all $y \in Y$ that $x^T y = 0$. Moreover, $\dim(X) + \dim(Y) = n$.

   False. add $X \oplus Y = \mathbb{R}^n$.

   (b) For a matrix A, the orthogonal complement of A's null space is A's row space.

   True.

   (c) The direction sum $W = U \oplus V$ means that every $w \in W$ can be expressed as $w = u + v$ for $u \in U$ and $v \in V$.

   False. add uniquely.

   (d) A vector norm $\|\cdot\|$ needs to satisfy the following conditions: For all x and y in a vector space

   I $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

   II $\|\alpha x\| = \alpha\|x\|$.

   III $\|x + y\| \leq \|x\| + \|y\|$

   False. II. $\|\alpha x\| = |\alpha|\|x\|$

   (e) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ are linearly independent if $< v_i, v_j >= 0$ for any $i \neq j$ and $< v_i, v_i >= 1$ for all i.

   True.

   (f) An $n \times n$ matrix $Q = [q_1, q_2, \ldots, q_n]$ is called an orthogonal matrix if $\{q_1, q_2, \ldots, q_n\}$ is an orthogonal set.

   False. should be an orthonormal set.

   (g) If x is an eigenvector of an $n \times n$ matrix A, αx is also an eigenvector of A for all $\alpha \in \mathbb{R}$.

   False. $\alpha \neq 0$.

   (h) If an $n \times n$ matrix A has a zero eigenvalue, A is singular.

   True.

(i) An $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.

False. If A has n distinct eigenvalues, A is diagonalizable. But the inverse is incorrect.

(j) The eigenvalues of a symmetric matrix are all real, and their corresponding eigenvectors must be orthogonal.

False. The eigenvector can be chosen to be orthogonal, but not necessary.

2. (10%) Find a basis for the orthogonal complement of the following subspace.

$$\text{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right)$$

Let

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}.$$

The null space of A is the orthogonal complement of the subspace. The reduced row-echelon form of A is

$$A \to \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}.$$

whose leading variable is x_1, x_2 and free variables are $x_3, x4$. Let $x_3 = \alpha, x_4 = \beta$. The null space has the form

$$N(A) = \begin{bmatrix}\alpha + 2\beta\\-2\alpha - 3\beta\\\alpha\\\beta\end{bmatrix} = \alpha\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \beta\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$$

3. (10%) For a matrix A, show that $N(A^TA) = N(A)$.

(a) If $x \in N(A)$, $Ax = 0$. Then $A^TAx = A^T0 = 0$, which shows $x \in N(A^TA)$. This means $N(A) \subseteq N(A^TA)$.

(b) If $x \in N(A^TA)$, $A^TAx = 0$. Then $x^TA^TAx = (Ax)^TAx = \|Ax\|^2 = 0$, which shows $Ax = 0$ and therefore $x \in N(A)$. This means $N(A^TA) \subseteq N(A)$.

(c) Based on (a) and (b), $N(A^TA) = N(A)$.

4. (10%) Given $A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Find all the eigenvalues of A and the eigenvector belonging to the largest eigenvalue. (No need to normalize it.)

The characteristic polynomial $p(\lambda) = \det(A - \lambda I) = \lambda(\lambda^2 - 3\lambda + 2)$, whose roots are $0, 1, 2$. To find the eigenvector of $\lambda = 2$, we need to compute the null space of $A - 2I = \begin{bmatrix} 2 & -5 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & -3 \end{bmatrix}$, which is $\text{span}([7\ 3\ 1]^T)$. So the eigenvector belonging to $\lambda = 2$ is

$$x = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

5. (10%) Let $p(x) = \alpha \det(A - xI) = 5x^3 - 6x^2 + 7x + 8$, and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. What are α, $\lambda_1\lambda_2\lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3$?

The leading term of $p(x)$ should be $(-1)^3 = -1$. So $\alpha = -5$.
The original $p(x) = -x^3 + (6/5)x^2 - (7/5)x - 8/5$.
So $\lambda_1\lambda_2\lambda_3 = -8/5$ and $\lambda_1 + \lambda_2 + \lambda_3 = 6/5$.

6. (10%) Let Q be an orthogonal matrix.

(a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.

(b) Show that $|\det(Q)| = 1$.

For (a), the eigenvalue/eigenvector of Q satisfies $Qx = \lambda x$. Since Q is orthogonal, $\|Qx\| = \|x\| = \|\lambda x\| = |\lambda|\|x\|$, and $\|x\| \neq 0$, we have $|\lambda| = 1$.

For (b), suppose Q is an $n \times n$ matrix, and it has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ Since $\det(Q) = \lambda_1\lambda_2\cdots\lambda_n$,

$$|\det(Q)| = |\lambda_1\lambda_2\cdots\lambda_n| = |\lambda_1| \times |\lambda_2| \times \cdots \times |\lambda_n| = 1 \times 1 \times \cdots \times 1 = 1.$$

7. (10%) Find ALL possible values of the scalar α that can make A defective

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{bmatrix}.$$

Solving the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = (\alpha - \lambda)(\lambda + 1)(\lambda - 3) = 0$, we have $\lambda = -1, 3, \alpha$. To make A defective, α must be -1 or 3. If $\alpha = -1$, A only has one linearly independent eigenvector belonging to -1, $[0\ 0\ 1]^T$, so A is defective. If $\alpha = 3$, A also has one linearly independent eigenvector belonging to -1, $[1\ 3\ 1]^T$, so A is defective. So both $\alpha = -1$ and $\alpha = 3$ can make A defective.

8. (10%) The unit circle is the set of points whose "distance" to $(0, 0)$ is 1. Different norms give different shapes of unit circles. For example, the unit circle of 2-norm, $\|[x, y]\|_2 = \sqrt{x^2 + y^2} = 1$, is shown in Figure 1.

   (a) Draw the unit circle of 1-norm, $\|[x, y]\|_1 = |x| + |y| = 1$.

   (b) Draw the unit circle of ∞-norm $\|[x, y]\|_\infty = \max(|x|, |y|) = 1$.

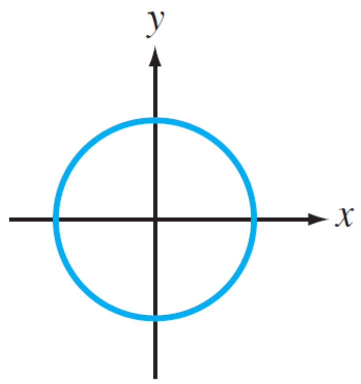
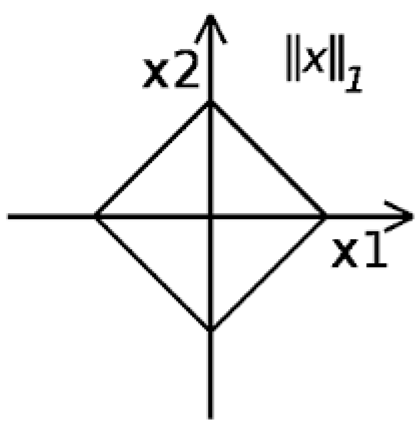
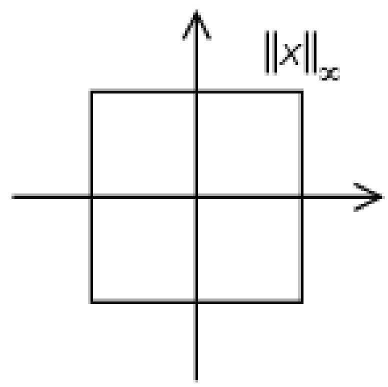


Figure 1: The unit circle of 2-norm





(a) The unit circle of 1-norm (b) The unit circle of ∞-norm.