2020 Fall EECS205002 Linear Algebra

2020/12/30 Quiz 4

1. (30%) True or False. For each item, if your answer is False, indicate which part of the statement is incorrect.
   (a) Two subspaces X and Y of $\mathbb{R}^n$ are said to be orthogonal complement if for all $x \in X$ and for all $y \in Y$ that $x^T y = 0$. Moreover, $\dim(X) + \dim(Y) = n$.
   (b) For a matrix A, the orthogonal complement of A's null space is A's row space.
   (c) The direction sum $W = U \oplus V$ means that every $w \in W$ can be expressed as $w = u + v$ for $u \in U$ and $v \in V$.
   (d) A vector norm $\|\cdot\|$ needs to satisfy the following conditions: For all x and y in a vector space
      I $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
      II $\|\alpha x\| = \alpha \|x\|$.
      III $\|x + y\| \leq \|x\| + \|y\|$
   (e) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ are linearly independent if $< v_i, v_j >= 0$ for any $i \neq j$ and $< v_i, v_i >= 1$ for all i.
   (f) An $n \times n$ matrix $Q = [q_1, q_2, \ldots, q_n]$ is called an orthogonal matrix if $\{q_1, q_2, \ldots, q_n\}$ is an orthogonal set.
   (g) If x is an eigenvector of an $n \times n$ matrix A, αx is also an eigenvector of A for all $\alpha \in \mathbb{R}$.
   (h) If an $n \times n$ matrix A has a zero eigenvalue, A is singular.
   (i) An $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.
   (j) The eigenvalues of a symmetric matrix are all real, and their corresponding eigenvectors must be orthogonal.

2. (10%) Find a basis for the orthogonal complement of the following subspace.

$$\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right)$$

3. (10%) For a matrix A, show that $N(A^T A) = N(A)$.

4. (10%) Given $A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Find all the eigenvalues of A and the eigenvector belonging to the largest eigenvalue. (No need to normalize it.)

5. (10%) Let $p(x) = \alpha \det(A - xI) = 5x^3 - 6x^2 + 7x + 8$, and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. What are α, $\lambda_1\lambda_2\lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3$?

6. (10%) Let Q be an orthogonal matrix.

   (a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.

   (b) Show that $|\det(Q)| = 1$.

7. (10%) Find ALL possible values of the scalar α that can make A defective

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{bmatrix}.$$

8. (10%) The unit circle is the set of points whose "distance" to $(0, 0)$ is 1. Different norms give different shapes of unit circles. For example, the unit circle of 2-norm, $\|[x, y]\|_2 = \sqrt{x^2 + y^2} = 1$, is shown in Figure 1.

   (a) Draw the unit circle of 1-norm, $\|[x, y]\|_1 = |x| + |y| = 1$.

   (b) Draw the unit circle of ∞-norm $\|[x, y]\|_\infty = \max(|x|, |y|) = 1$.

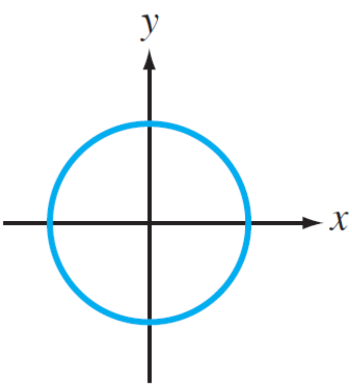

Figure 1: The unit circle of 2-norm