2020 Fall EECS205002 Linear Algebra

Name: ID:

2020/10/7 Quiz 1

1. (40%) Multiple choice questions. Each question may have 0, 1, or more correct choices. For each question, you need to choose all the correct items to get the credit.

   (a) Which matrices are in the row echelon form?

   $$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$$

   A, B

   (b) Which statement(s) indicates that an $n \times n$ matrix A is nonsingular.

   A. There exists a matrix B such that BA equals to an $n \times n$ identity matrix.
   B. $Ax = b$ is consistent.
   C. A is row equivalent to I.
   D. $Ax = 0$ has a nontrivial solution.
   E. $A^T = A$.

   A, C

   (c) Which matrices are elementary matrices?

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   A, D, E

   (d) For a linear system $Ax = b$, which condition guarantees it is consistent.

   A. A is nonsingular.
   B. $Ax = b$ is an underdetermined system.
   C. $Ax = b$ is an overdetermined system.
   D. $b = 0$.
   E. $Ax = 0$ has a solution.

   A, D

   (e) Which statements are true?

   A. All types of elementary matrices are invertible, and the inverse matrix is also an elementary matrix of the same type.
   B. A symmetric upper triangular matrix must be a diagonal matrix.
   C. A matrix in the row echelon form must be an upper triangular matrix.
   D. An overdetermined system must be inconsistent.
   E. If $E_k \dots E_2 E_1 A = I$, then $A^{-1} = E_1^{-1} E_2^{-1} \dots E_k^{-1}$.

   A, B, C

   (f) In the following coefficient matrix of a linear system, which variables are free variables?

   $$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}$$

   C

(g) Let $F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Which of the following matrix X makes $XF = FX$?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, E = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

A, D, E

(h) For matrix A, B, C, and scalar α, β, which statements are true?

A. $A(B+C) = AB + AC$.
B. If A is nonsingular, $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$.
C. If A and B are nonsingular, $(BA)^{-1} = B^{-1}A^{-1}$.
D. $(ABC)^T = C^T(AB)^T = (CB)^T A^T$.
E. $(A+B+C)^T = A^T + C^T + B^T$.

A, B, E

2. (20%) For the following linear system

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 6 \end{aligned}$$

(a) Write its augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 6 \end{array}\right]$$

(b) Convert the augmented matrix to the row echelon form.

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 6 \end{array}\right] \xrightarrow{E_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 1 & 2 \\ 3 & 4 & 2 & 6 \end{array}\right] \xrightarrow{E_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -4 \\ 3 & 4 & 2 & 6 \end{array}\right] \xrightarrow{E_3}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & -1 & -3 \end{array}\right] \xrightarrow{E_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{E_5} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

(c) Indicate which elementary matrices are used for (b).

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) Find all the solutions of the linear system.

$$x_1 = -2,\ x_2 = 1,\ x_3 = 4$$

3. (20%) Prove the following statements

(a) If A is nonsingular, then A^T is nonsingular.

To show A^T is nonsingular, we only need to find its inverse. The guess is $(A^{-1})^T$. So what we need to prove is their products, from either side, are the identity matrix.

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$
$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

(b) If A is a symmetric nonsingular matrix, then A^{-1} is also symmetric. (You can use the result in (a))
What we need to prove is $(A^{-1})^T = A^{-1}$. But from (a), we have known that $(A^{-1})^T = (A^T)^{-1}$. Also, since A is symmetric, $A^T = A$. So

$$(A^{-1})^T = (A^T)^{-1} = (A)^{-1}$$

4. (10%) Let U be an $n \times n$ matrix in the row echelon form. Show that if any diagonal element of U is 0, the last row of U must be entirely 0.

First, the matrix A in the row echelon form is an upper triangular matrix, which mean $a_{i,j} = 0$ for $i > j$. Suppose $a_{i,i} = 0$. The number of zeros in row i must be larger than or equal to i. Since the row echelon form requires the number of zeros increases row by row. So the number of zeros in row k will be larger than or equal to k for $i \le k \le n$. But U is an $n \times n$ matrix, so the row n will have n zeros, which means the entire row (the last row) is a zero vector.

- If you only list the definition of row echelon form, you will get 2 points.
- If you only give an example, you will get 1 point. If you also give some explanation about the example, you will have 2 points.

5. (10%) In the movie Tenet, there is a sator matrix, as shown in the figure below. Show it is a symmetric matrix.

Since $a_{i,j} = a_{j,i}$, it is a symmetric matrix.