Assignment 1
2021 Fall EECS205002 Linear Algebra

(1)

Myles is in a fitness program that requires him to eat a certain amount of macros in his diet. As an international student in Taiwan, he has to manage his money properly in order not to starve at the end of the month.

His diet requires 5 main nutrients, protein, carbohydrates, fats, fiber, and minerals from his daily meals. Due to his packed schedule, Myles could only acquire 5 dishes for consumption: roasted chicken breast, greek salad, brown rice, eggs, and baked oats. Each chicken breast costs 30 NT, a bowl of greek salad costs 40 NT, a bowl of brown rice costs 10 NT, an egg costs 5 NT, and each cup of baked oats costs 20 NT. Each day, Myles requires at least 1800 calories, 180g of protein, 140 g of carbs, 60 g of fats, 30 g of fibers and 10gr of minerals. The nutritional content per unit of each portion of food is shown below:

Types of dishes	Calories	Nutrition facts (g)				
		Protein	Carbs	Fats	Fibers	Sodium
Chicken Breast	210	50	2	6	1	2
Greek Salad	180	8	12	6	10	1
Brown Rice	220	5	40	2	4	1
Eggs	60	6	1	5	1	1
Baked Oats	400	18	30	6	15	3

Suppose Myles plans to eat *a* pieces of roasted chicken breast , *b* bowl of greek salad, *c* bowl of brown rice, *d* number of eggs, and *e* cup of baked oats daily. The minimum cost of his meal per day is expressed as follows :

$$30a + 40b + 10c + 5d + 20e$$

where the coefficients in front of the variables are the price of each dish. But, he has the following constraints:

$200a + 180b + 220c + 60d + 400e \geq 1800$ // the minimum calorie intake

$50a + 8b + 5c + 6d + 18e \geq 180$ // minimum grams of protein

$2a + 12b + 40c + 1d + 30e \geq 140$ // minimum grams of carbs

$6a + 6b + 2c + 5d + 6e \geq 60$ //minimum grams of fats

$1a + 10b + 4c + 1d + 15e \geq 30$ //minimum grams of fibers

$2a + 1b + 1c + 1d + 3e \geq 10$ // minimum grams of sodium

$a, b, c, d, e \geq 0$ // number of dishes should be nonnegative

The problem to minimize the cost of his meal is called *Linear Programming*, which is expressed as follows:

$$\min_{a,b,c,d,e} Cost\,(a, b, c, d, e) = 30a + 40b + 10c + 5d + 20e$$

Subject to

$$200a + 180b + 220c + 60d + 400e \geq 1800$$

$$50a + 8b + 5c + 6d + 18e \geq 180$$

$$2a + 12b + 40c + 1d + 30e \geq 140$$

$$6a + 6b + 2c + 5d + 6e \geq 60$$

$$1a + 10b + 4c + 1d + 15e \geq 30$$

$$2a + 1b + 1c + 1d + 3e \geq 10$$

$$a, b, c, d, e \geq 0$$

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$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & & & \\ \vdots & & & \\ a_{N1} & \cdots & & a_{NN} \end{bmatrix}}_{N} \Bigg\} N \qquad A' = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & & & \\ \vdots & & & \\ (a_{N1}+x_{N1}) & \cdots & & (a_{NN}+x_{NN}) \end{bmatrix}}_{N} \Bigg\} N$$

- Show that $A' = A + uv^T$, where u and v are two vectors.

$$A' - A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & & & 0 \\ x_{N1} & \cdots & & x_{NN} \end{bmatrix}$$

$uv^T \rightarrow u = N \times 1$
$v^T = 1 \times N$

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_N \\ u_2v_2 & & & \\ \vdots & & & \\ u_Nv_1 & \cdots & & u_Nv_N \end{bmatrix}$$

thus if u and v^T are:

$$uv^T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} = A' - A$$

uv^T is equal to $A' - A$, and the matrix from vector multiplication of uv^T is equal to $A' - A$. therefore, it is proven that $A' = A + uv^T$

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- Show that the inverse of A' can be computed by

$$A'^{-1} = (A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$$

* property of identity matrix
$IA = AI = A$.

$A.A^{-1} = I$

$$I = A' . A'^{-1}$$

$$I = (A + uv^T) . \left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \right)$$

$$I = AA^{-1} + uv^TA^{-1} - \frac{AA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} - \frac{uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$$

$$I = I + uv^TA^{-1} - \left(\frac{I\,uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \right)$$

$$I = I + uv^TA^{-1} - \left(\frac{u\,(v^TA^{-1} + v^TA^{-1}uv^TA^{-1})}{1+v^TA^{-1}u} \right)$$

$$I = I + uv^TA^{-1} - \left(\frac{u\,(v^TA^{-1})(\cancel{1 + uv^TA^{-1}})}{(\cancel{1+v^TA^{-1}u})} \right)$$

$$I = I + uv^TA^{-1} - uv^TA^{-1}$$

$$I = I + 0$$

$$I = I \quad \text{proven}_{//}$$

A'^{-1} when multiplied by A' results in identity matrix.
Therefore, it is proven that A'^{-1} can be computed by $A^{-1} - \dfrac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

(5) Approach for optimal solution

The simplex method would reduce time in finding the optimal solution as it does not need to enumerate all the intersections in the solution set.

It is a search procedure that goes through the set of feasible solutions one at a time until the optimal solution is identified. It generates and tests the solution and at every iteration, chooses the variable that can make the biggest modification toward the minimum solution. In other words, it ensures that at every point, the objective function increases, or is unaffected.

To solve using the simplex method, a linear program must be converted into the standard form, which has three requirements: it must be a maximization problem, all constraints must be less than or equal to inequality, and all variables must be non negative.

Second, we have to determine slack variables, which are additional variables introduced to linear constraints to transform them from inequality to equality constraints. After that, set up the initial tableau, the augmented matrix of the system of equations. Separate the last row using a line (the objective function).

Third, Select the pivot column by choosing the most negative number. If all the numbers in the bottom row are zero or positive, then the basic solution is the optimal solution. Then, select the pivot in the pivot column. It must always be a positive number, for each positive entry *b* in the pivot column, compute the ratio *a/b* where a is the number of the rightmost column in the row (test ratio). And choose the smallest one. *b* is the pivot variable.

Then use the pivot to clear the pivot column, and this will give the next tableau. Repeat step 3-5 until there are no more negative numbers in the bottom row.

This method usually takes 2m to 3m iterations at most (m = number of constraints). Which drastically limits the possibilities of the solutions, and therefore shifting it to a certain part of the solution set, and lastly toward the final solution. This method can actually evaluate whether no solution actually exists. But all variables must be non-negative, constraints are inequalities, and a minimizing problem must be changed into a maximizing problem by multiplying the function by -1.

In conclusion, it produces an optimal solution that satisfies the constraints and produces the optimal value from the objective function without enumerating all intersections.

References

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