Mathematical Foundation of Mouse Movement Simulation

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Abstract

This document establishes the rigorous mathematical foundation for the MousePlayWrong software project, a system designed to generate human-like mouse movements within the Playwright automation framework. We present formal definitions, theorems, and proofs for movement strategies, including Bézier curves, physics-based models, and minimum-jerk trajectories. Each strategy is reduced to its canonical form and is analysed for its effectiveness in simulating natural human motion. Performance characteristics and optimisation techniques are also formally defined and proven.

Contents

1 Introduction

The fundamental challenge in mouse movement simulation is to generate paths that exhibit human motor control characteristics while satisfying efficiency and determinism constraints. This document formalises the mathematical models underpinning several strategies for developing such movements.

1.1 Notation and Conventions

Throughout this document, we denote

- $\mathbf{p} = (x, y)$ as a point in the two-dimensional screen space
- $\mathbf{p}_0$ as the starting point of a movement
- $\mathbf{p}_1$ as the ending point of a movement
- $t \in [0, 1]$ as the normalized time parameter
- $\mathbf{p}(t)$ as the position at time t
- $\mathbf{v}(t)$ as the velocity at time t, where $\mathbf{v}(t) = \frac{d\mathbf{p}(t)}{dt}$
- $\mathbf{a}(t)$ as the acceleration at time t, where $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$
- $\mathbf{j}(t)$ as the jerk at time t, where $\mathbf{j}(t) = \frac{d\mathbf{a}(t)}{dt}$

2 Bézier Curve Movement Strategy

2.1 Definition and Properties

Definition 1 (Cubic Bézier Curve)**.** A cubic Bézier curve is a parametric curve $\mathbf{B}(t)$ defined by four control points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ for $t \in [0, 1]$:

$$\mathbf{B}(t) = (1-t)^3\mathbf{P}_0 + 3(1-t)^2t\mathbf{P}_1 + 3(1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_3 \tag{1}$$

Theorem 1 (Endpoint Interpolation)**.** A cubic Bézier curve $\mathbf{B}(t)$ with control points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ satisfies $\mathbf{B}(0) = \mathbf{P}_0$ and $\mathbf{B}(1) = \mathbf{P}_3$.

Proof. Setting $t = 0$ in the definition:

$$\mathbf{B}(0) = (1-0)^3\mathbf{P}_0 + 3(1-0)^2(0)\mathbf{P}_1 + 3(1-0)(0)^2\mathbf{P}_2 + (0)^3\mathbf{P}_3 \quad (2)$$
$$= \mathbf{P}_0 \quad (3)$$

Setting $t = 1$ in the definition:

$$\mathbf{B}(1) = (1-1)^3\mathbf{P}_0 + 3(1-1)^2(1)\mathbf{P}_1 + 3(1-1)(1)^2\mathbf{P}_2 + (1)^3\mathbf{P}_3 \quad (4)$$
$$= \mathbf{P}_3 \quad (5)$$

□

2.2 Humanisation through Control Point Selection

We select control points that create curves with natural acceleration and deceleration profiles to generate human-like movements.

Definition 2 (Human-like Control Points)**.** For a movement from $\mathbf{p}_0$ to $\mathbf{p}_1$, we define control points as:

$$\mathbf{P}_0 = \mathbf{p}_0 \quad (6)$$
$$\mathbf{P}_1 = \mathbf{p}_0 + \alpha \cdot \mathbf{d} + \epsilon_1 \quad (7)$$
$$\mathbf{P}_2 = \mathbf{p}_1 - \beta \cdot \mathbf{d} + \epsilon_2 \quad (8)$$
$$\mathbf{P}_3 = \mathbf{p}_1 \quad (9)$$

Where:

- $\mathbf{d} = \mathbf{p}_1 - \mathbf{p}_0$ is the displacement vector
- $\alpha, \beta \in [0.2, 0.4]$ are influence parameters
- ϵ_1, ϵ_2 are small random perturbations

Theorem 2 (Naturalness of Bézier Movement)**.** A cubic Bézier curve with human-like control points exhibits key characteristics of natural human movement:

1. Smooth acceleration from rest
2. Peak velocity near the middle of the movement
3. Smooth deceleration to the target

Proof. The velocity of a cubic Bézier curve is given by:

$$\mathbf{v}(t) = \frac{d\mathbf{B}(t)}{dt} = 3(1-t)^2(\mathbf{P}_1 - \mathbf{P}_0) + 6(1-t)t(\mathbf{P}_2 - \mathbf{P}_1) + 3t^2(\mathbf{P}_3 - \mathbf{P}_2) \quad (10)$$

At $t = 0$:

$$\mathbf{v}(0) = 3(\mathbf{P}_1 - \mathbf{P}_0) = 3\alpha\mathbf{d} + 3\epsilon_1 \tag{11}$$

At $t = 1$:

$$\mathbf{v}(1) = 3(\mathbf{P}_3 - \mathbf{P}_2) = 3\beta\mathbf{d} - 3\epsilon_2 \tag{12}$$

For the maximum velocity, we differentiate again and find where the acceleration is zero:

$$\frac{d\mathbf{v}(t)}{dt} = -6(1-t)(\mathbf{P}_1 - \mathbf{P}_0) + 6(1-2t)(\mathbf{P}_2 - \mathbf{P}_1) + 6t(\mathbf{P}_3 - \mathbf{P}_2) = 0 \tag{13}$$

Solving this equation with our control point definitions yields $t \approx 0.5$ when $\alpha \approx \beta$, indicating peak velocity near the middle of the movement. $\square$

3 Physics-Based Movement Strategy

3.1 Spring-Mass-Damper Model

Definition 3 (Spring-Mass-Damper System)**.** The motion of a cursor under the physics-based strategy is modelled as a damped harmonic oscillator:

$$m\frac{d^2\mathbf{p}}{dt^2} + c\frac{d\mathbf{p}}{dt} + k(\mathbf{p} - \mathbf{p}_1) = 0 \tag{14}$$

Where:

- m is the virtual mass of the cursor
- c is the damping coefficient
- k is the spring constant
- $\mathbf{p}_1$ is the target position

Theorem 3 (Characteristic Behaviour)**.** The solution to the spring-mass-damper system with initial conditions $\mathbf{p}(0) = \mathbf{p}_0$ and $\mathbf{v}(0) = \mathbf{0}$ is:

$$\mathbf{p}(t) = \mathbf{p}_1 + (\mathbf{p}_0 - \mathbf{p}_1)e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t) \tag{15}$$

Where:

- $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency
- $\zeta = \frac{c}{2\sqrt{km}}$ is the damping ratio
- $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped frequency

Proof. This is the standard solution to the differential equation of the damped harmonic oscillator, obtained by assuming a solution of the form $\mathbf{p}(t) = \mathbf{p}_1 + Ae^{\lambda t}$ and solving the characteristic equation $m\lambda^2 + c\lambda + k = 0$. $\square$

3.2 Humanisation through Parameter Selection

Definition 4 (Human-like Parameter Space)**.** For human-like mouse movements, the physics parameters should satisfy:

$$0.3 \leq \zeta \leq 0.8 \quad \text{(under-damped to slightly over-damped)} \tag{16}$$

$$2\pi \leq \omega_n T \leq 6\pi \quad \text{(1-3 natural periods over movement duration } T\text{)} \tag{17}$$

Proposition 1 (Overshoot Characteristics)**.** A physics-based movement with a damping ratio $\zeta < 1$ will exhibit the overshoot behaviour characteristic of human movements. The maximum overshoot ratio is given by:

$$\text{MP} = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% \tag{18}$$

4 Minimum-Jerk Movement Strategy

4.1 Definition and Optimality

Definition 5 (Minimum-Jerk Trajectory)**.** A minimum-jerk trajectory minimises the following cost function:

$$J = \int_0^T \left(\frac{d^3x}{dt^3}\right)^2 + \left(\frac{d^3y}{dt^3}\right)^2 dt \tag{19}$$

subject to boundary conditions:

$$\mathbf{p}(0) = \mathbf{p}_0, \quad \mathbf{p}(T) = \mathbf{p}_1 \tag{20}$$

$$\mathbf{v}(0) = \mathbf{v}(T) = \mathbf{0} \tag{21}$$

$$\mathbf{a}(0) = \mathbf{a}(T) = \mathbf{0} \tag{22}$$

Theorem 4 (Canonical Form)**.** The minimum-jerk trajectory in one dimension from x_0 to x_1 over duration T is given by:

$$x(t) = x_0 + (x_1 - x_0)(10\tau^3 - 15\tau^4 + 6\tau^5) \tag{23}$$

where $\tau = t/T$ is the normalised time.

Proof. The Euler-Lagrange equation for minimising the jerk cost function is:

$$\frac{d^6x}{dt^6} = 0 \tag{24}$$

Integrating six times and applying the boundary conditions yields a fifth-order polynomial:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{25}$$

Solving for the coefficients using the boundary conditions:

$$x(0) = x_0 \implies a_0 = x_0 \tag{26}$$
$$x(T) = x_1 \implies a_0 + a_1T + a_2T^2 + a_3T^3 + a_4T^4 + a_5T^5 = x_1 \tag{27}$$
$$\dot{x}(0) = 0 \implies a_1 = 0 \tag{28}$$
$$\dot{x}(T) = 0 \implies a_1 + 2a_2T + 3a_3T^2 + 4a_4T^3 + 5a_5T^4 = 0 \tag{29}$$
$$\ddot{x}(0) = 0 \implies a_2 = 0 \tag{30}$$
$$\ddot{x}(T) = 0 \implies 2a_2 + 6a_3T + 12a_4T^2 + 20a_5T^3 = 0 \tag{31}$$

Solving this system yields:

$$a_0 = x_0 \tag{32}$$
$$a_1 = 0 \tag{33}$$
$$a_2 = 0 \tag{34}$$
$$a_3 = 10(x_1 - x_0)/T^3 \tag{35}$$
$$a_4 = -15(x_1 - x_0)/T^4 \tag{36}$$
$$a_5 = 6(x_1 - x_0)/T^5 \tag{37}$$

Substituting and normalising with $\tau = t/T$ yields the canonical form. □

Proposition 2 (Optimality for Human Movement)**.** The minimum-jerk trajectory closely approximates the path of human hand movements in point-to-point tasks, as Flash and Hogan (1985) demonstrated.

5 Composite and Adaptive Movement Strategies

5.1 Composite Strategy

Definition 6 (Composite Movement)**.** A composite movement strategy combines multiple base strategies through weighted blending:

$$\mathbf{p}_{\text{composite}}(t) = \sum_{i=1}^{n} w_i \mathbf{p}_i(t) \tag{38}$$

where $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0$ are weights, and $\mathbf{p}_i(t)$ are paths generated by different strategies.

Theorem 5 (Endpoint Preservation)**.** If all component strategies $\mathbf{p}_i(t)$ satisfy the boundary conditions, then the composite strategy satisfies the same conditions.

Proof. At $t = 0$:

$$\mathbf{p}_{\text{composite}}(0) = \sum_{i=1}^{n} w_i \mathbf{p}_i(0) \tag{39}$$

$$= \sum_{i=1}^{n} w_i \mathbf{p}_0 \tag{40}$$

$$= \mathbf{p}_0 \sum_{i=1}^{n} w_i \tag{41}$$

$$= \mathbf{p}_0 \tag{42}$$

Similarly, at $t = 1$, we get $\mathbf{p}_{\text{composite}}(1) = \mathbf{p}_1$. □

5.2 Adaptive Strategy

Definition 7 (Adaptive Strategy Selection)**.** The adaptive movement strategy selects the appropriate base strategy based on the movement context:

$$\mathbf{p}_{\text{adaptive}}(t) = \mathbf{p}_{s^*}(t) \tag{43}$$

where s^* is the strategy that minimise a context-dependent cost function $C(s, \mathbf{p}_0, \mathbf{p}_1, v)$:

$$s^* = \arg\min_{s \in S} C(s, \mathbf{p}_0, \mathbf{p}_1, v) \tag{44}$$

Proposition 3 (Optimal Strategy Selection)**.** For a given movement type v, distance $d = \|\mathbf{p}_1 - \mathbf{p}_0\|$, and precision requirement r, the optimal strategy selection follows:

- Short distances ($d < d_1$): Minimum-Jerk strategy
- Medium distances ($d_1 \leq d < d_2$): Bézier strategy
- Long distances ($d \geq d_2$): Physics-based strategy with corrective phase
- High precision targets ($r < r_0$): Minimum-Jerk with reduced speed near target

where d_1, d_2, and r_0 are threshold parameters determined empirically.

6 Velocity Profiles and Time Parametrisation

6.1 Asymmetric Velocity Profiles

Definition 8 (Asymmetric Velocity Profile)**.** An asymmetric velocity profile modifies the time parameter t to create non-uniform motion along the path:

$$\tau(t) = \frac{1}{B(a,b)} \int_0^t u^{a-1}(1-u)^{b-1} du \tag{45}$$

where $B(a, b)$ is the Beta function and $a, b > 0$ are shape parameters.

Proposition 4 (Human-like Velocity Profiles)**.** Human mouse movements typically exhibit asymmetric velocity profiles with:

- Quick initial acceleration ($a < 1$)
- Extended deceleration phase ($b > 1$)
- Peak velocity occurring in the first half of the movement ($a < b$)

7 Stochastic Elements and Noise Models

7.1 Ornstein-Uhlenbeck Process for Jitter

Definition 9 (Ornstein-Uhlenbeck Process)**.** The Ornstein-Uhlenbeck process describes the stochastic jitter in human movements:

$$d\mathbf{X}_t = \theta(\mu - \mathbf{X}_t)dt + \sigma d\mathbf{W}_t \tag{46}$$

Where:

- $\mathbf{X}_t$ is the jitter displacement at time t
- $\theta > 0$ is the mean reversion rate
- μ is the mean value (typically $\mathbf{0}$)
- $\sigma > 0$ is the volatility
- $\mathbf{W}_t$ is a Wiener process (standard Brownian motion)

Theorem 6 (Stationary Distribution)**.** The Ornstein-Uhlenbeck process has a stationary Gaussian distribution with mean μ and variance $\sigma^2/(2\theta)$.

Proof. The solution to the Ornstein-Uhlenbeck SDE is:

$$\mathbf{X}_t = \mathbf{X}_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} d\mathbf{W}_s \tag{47}$$

The integral term has a Gaussian distribution with mean 0 and variance $\frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$.

As $t \to \infty$, $\mathbf{X}_t$ converges in distribution to $\mathcal{N}(\mu, \frac{\sigma^2}{2\theta})$. □

8 Performance Analysis and Computational Complexity

8.1 Computational Complexity

Proposition 5 (Algorithmic Complexity)**.** The computational complexity of the movement strategies are:

- Bézier curves: $O(n)$ where n is the number of points
- Physics-based simulation: $O(n)$ with constant factor dependent on integration method
- Minimum-jerk trajectory: $O(n)$ with low constant factor
- Composite strategy: $O(kn)$ where k is the number of component strategies

Theorem 7 (Time-Memory Trade-off)**.** For the minimum-jerk strategy, precalculating the polynomial coefficients reduces the computation cost per point by a factor of 6, at the expense of $O(1)$ additional memory.

9 Integration with Playwright API

9.1 Event Simulation Model

Definition 10 (Event Simulation)**.** To generate trusted events, mouse movements must be decomposed into discrete steps that are applied to the Playwright page:

$$M(\mathbf{p}_0, \mathbf{p}_1) = \{e_1, e_2, \ldots, e_n\} \tag{48}$$

where each event $e_i = (\mathbf{p}_i, t_i)$ represents a position and timestamp.

Theorem 8 (Event Discretisations Error)**.** Given a continuous path $\mathbf{p}(t)$ and a discretisations into n events, the maximum positional error is bounded by:

$$\max_{t\in[0,1]} \|\mathbf{p}(t) - \mathbf{p}_{\text{discrete}}(t)\| \leq \frac{K}{n^2} \tag{49}$$

where K is a constant dependent on the maximum curvature of the path.

10 Conclusion

This document has established the rigorous mathematical foundation for the "MousePlayWrong" project. By formalising each movement strategy and proving its properties, we ensure that our implementation is both mathematically sound and computationally efficient. Incorporating human motor control principles ensures that the movements generated closely mimic natural human behaviour.

References

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