Past, Present and Future of Time-Dependent Scheduling

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Scheduling Seminar
https://schedulingseminar.com/
May 25th, 2022

Lecture outline

Introduction: Classical scheduling theory

- There are two domains in scheduling theory today:
  - Classical scheduling theory
  - Non-classical scheduling theory
- Main assumptions of classical scheduling theory:

(M1) Each machine is continuously available
(M2) Each machine can handle at most one job at a time
(M3) Machine speeds are fixed and known in advance
(J1) Each job may be performed only by one of machines
(J2) Job processing times do not overlap
(J3) Job parameters are numbers known in advance
(F) The quality of a schedule is measured by a single-valued criterion function

Introduction: Classical scheduling theory

- There are many monographs on classical scheduling theory:

Introduction: Non-classical scheduling theory

- If at least one of assumptions (M1)-(M3), (J1)-(J3) or (F) is not satisfied, we deal with non-classical scheduling theory
- The most of research in non-classical scheduling theory concerns scheduling problems with a modification of the (J3) assumption
- In these problems, jobs have variable processing times
- In the lecture, we will consider scheduling problems with variable job processing times

Introduction: Variable processing times

- Three main models of variable job processing times exist:
  - resource-dependent
  - position-dependent
  - time-dependent
- Resource-dependent job processing times are functions of the amount of allocated resource (Vickson, 1980; Nowicki & Zdrzałka, 1990; Shabtay & Steiner, 2007; Shioura, Shakhlevich & Strusevich, 2018; Błażewicz et al, 2019)
- Position-dependent job processing times are functions of the position of job in schedule (Gawiejnowicz, 1996; Bachman & Janiak, 2004; Biskup, 2008; Agnetis et al, 2014; Strusevich & Rustogi, 2017; Azzouz, Ennigrou & Ben Said, 2018)
- Time-dependent job processing times are functions of the starting time of job (Melnikov & Shafransky, 1980; Gupta & Gupta, 1988; Gawiejnowicz, 1996; Alidaee & Womer, 1999; Cheng, Ding & Lin, 2004; Gawiejnowicz, 2008; Błażewicz et al, 2019; Sedding, 2020; Gawiejnowicz, 2020)

Introduction: Variable processing times

- We will consider scheduling problems with time-dependent job processing times
- Scheduling problems with time-dependent job processing times are called time-dependent scheduling problems
- We will focus on time-dependent scheduling problems with
  - deteriorating jobs, when the job processing times are non-decreasing functions of the job starting times, and
  - shortening jobs, when the job processing times are non-increasing functions of the job starting times
- Remaining assumptions of the problems will be the same as in classical scheduling theory

Time-dependent scheduling: Origins and main dates

1974–1978 – variable job processing times as realizations of random variables (Holloway & Nelson, 1974; Picard & Queyranne, 1978)

1979-1980 – variable processing times of deteriorating jobs as functions of the job starting times (Melnikov & Shafransky, 1979, 1980) – **beginning of time-dependent scheduling**

1984-1995 – linearly deteriorating jobs (Tanaev, Gordon & Shafransky, 1984, 1994; Wajs, 1986; Gupta & Gupta, 1988; Browne & Yechiali, 1990; Gawiejnowicz & Pankowska, 1995)

1990 – non-linearly deterioratong jobs (Kunnathur & Gupta, 1990; Alidaee, 1990)

1993 – linearly shortening jobs (Ho, Leung & Wei, 1993)

Time-dependent scheduling: Origins and main dates

1994 – proportionally deteriorating jobs (Mosheiov, 1994)

1996–2004 – the first reviews of time-dependent scheduling (Gawiejnowicz, 1996; Alidaee & Womer, 1999; Cheng, Ding & Lin, 2004)

2001 – the first paper on time-dependent scheduling on dedicated machines (Kononov & Gawiejnowicz, 2001)

2003 – the first paper on time-dependent scheduling on a machine with limited availability (Wu & Lee, 2003)

2006 – the first paper on bi-criterion time-dependent scheduling (Gawiejnowicz, Kurc & Pankowska, 2006)

Time-dependent scheduling: Origins and main dates

2008 – the first monograph on time-dependent scheduling (Gawiejnowicz, 2008)

2008 – the first paper on two-agent time-dependent scheduling (Liu & Tang, 2008)

2009-2014 – the first papers on equivalent, conjugate and isomorphic time-dependent scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009; Gawiejnowicz & Kononov, 2014)

2009 – the first paper on time-dependent scheduling with job rejection (Cheng & Sun, 2009)

Time-dependent scheduling: Origins and main dates

2010–2014 – the first papers on time-dependent scheduling with mixed job processing times (Gawiejnowicz & Lin, 2010; Dębczyński & Gawiejnowicz, 2013; Dębczyński, 2014)

2016–2020 – the first papers on time-dependent scheduling with alterable job processing times (Jaehn & Sedding, 2016; Sedding, 2020)

2016 – the first international conference devoted to scheduling problems with variable job processing times (IWDSP 2016)

2020 – a new review of time-dependent scheduling (Gawiejnowicz, 2020)

2020 – the second monograph on time-dependent scheduling (Gawiejnowicz, 2020)

Time-dependent scheduling: Applications

- Simultaneous repayment of multiple loans (Gupta, Kunnathur & Dandapani, 1987)
- Recognizing of aerial threats (Ho, Leung & Wei, 1993)
- Scheduling maintenance activities (Mosheiov, 1994)
- Planning derusting procedures (Gawiejnowicz, Kurc & Pankowska, 2006)
- Modeling fire-fighting problems (Rachaniotis & Pappis, 2006)
- Modeling health care problems (Wu, Dong & Cheng, 2014; Zhang, Wang & Wang, 2015)
- Transport problems in car production industry (Jaehn & Sedding, 2016; Sedding, 2020)

The most recent list of known applications of time-dependent scheduling is given in monograph Gawiejnowicz, 2020

Time-dependent scheduling: Theoretical tools

- Proof techniques of classical scheduling theory: adjacent job interchange technique, mathematical induction, direct proof, proof by a contradiction
- Priority-generating functions (Tanaev, Gordon & Shafransky, 1994; Strusevich & Rustogi, 2017)
- Methods of minimizing a function on a set of permutations (Strusevich & Rustogi, 2017)
- Signatures (Gawiejnowicz, Kurc & Pankowska, 2002, 2006)
- Matrix methods (Gawiejnowicz 2008, 2020)
- Methods of solving multiplicative problems (Ng, Barketau, Cheng & Kovalyov, 2010)
- Properties of pairs of mutually related scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009; Gawiejnowicz & Kononov, 2014)
- New methods of NP-completeness proving (Cheng, Shafransky & Ng, 2016)
- Properties of function composition operator (Kawase, Makino & Seimi, 2018)

Time-dependent scheduling: Notation

- Scheduling problems are denoted with the use of three-field notation (Graham, Lawler, Lenstra & Rinnooy Kan, 1979)
- To cover various forms of variable job processing times, a few extensions of the three-field notation were proposed (Agnetis, Billaut, Gawiejnowicz, Pacciarelli & Soukhal, 2014; Gawiejnowicz, 2008; Strusevich & Rustogi, 2017; Błażewicz et al, 2019; Gawiejnowicz, 2020)

Examples of the use of extended three-field notation

- $1|p_j = b_j(a + bt)|f_{\max}$ – a single machine problem with proportional-linear processing times and criterion $f_{\max}$
- $P2|p_j = b_j t|\sum C_j$ – two parallel-identical machine problem with proportional processing times and criterion $\sum C_j$
- $O2|p_{ij} = a_{ij} + b_{ij} t|C_{\max}$ – two open shop problem with linear processing times and criterion $C_{\max}$

Main results: Single machine: Proportional job processing times

Theorem (Mosheiov, 1994)

(a) Problem $1|p_j = b_j t|C_{\max}$ is solvable in $O(n)$ time,

$$C_{\max}(\sigma) = t_0 \prod_{j=1}^{n} \left(1 + b_{[j]}\right)$$

and it does not depend on schedule σ.

(b) Problem $1|p_j = b_j t|L_{\max}$ is solvable in $O(n \log n)$ time by scheduling job in non-decreasing order of job due dates (EDD order).

(c) Problem $1|p_j = b_j t|f_{\max}$ is solvable in $O(n^2)$ time by scheduling jobs using modified Lawler's algorithm.

Main results: Single machine: Proportional job processing times

Theorem (Mosheiov, 1994)

(a) Problem $1|p_j = b_j t|\sum C_j$ is solvable in $O(n \log n)$ time by scheduling job in non-decreasing order of job deterioration rates (SDR order) and

$$\sum C_j(\sigma) = t_0 \sum_{j=1}^{n} \prod_{k=1}^{j} \left(1 + b_{[k]}\right).$$

(b) Problem $1|p_j = b_j t|\sum w_j C_j$ is solvable in $O(n \log n)$ time by scheduling jobs in non-decreasing order of ratios $\frac{b_j}{w_j(1+b_j)}$.

(c) Problem $1|p_j = b_j t|\sum U_j$ is solvable in $O(n \log n)$ time by scheduling job using modified Moore's algorithm.

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Main results: Single machine: Proportional-linear job processing times

Theorem (Kononov, 1998)

(a) Problem $1|p_j = b_j(a + bt)|C_{\max}$ is solvable in $O(n)$ time,

$$C_{\max}(\sigma) = \left(t_0 + \frac{a}{b}\right) \prod_{j=1}^{n} \left(1 + b_{[j]} b\right) - \frac{a}{b}$$

does not depend on schedule σ.

(b) Problem $1|p_j = b_j(a + bt)|L_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs in the EDD order.

(c) Problem $1|p_j = b_j(a + bt)|f_{\max}$ is solvable in $O(n^2)$ time by scheduling jobs using modified Lawler's algorithm.

Main results: Single machine: Proportional-linear job processing times

Theorem (Strusevich & Rustogi, 2017)

(a) Problem $1|p_j = b_j(a+bt)|\sum C_j$ is solvable in $O(n\log n)$ time by scheduling jobs in the SDR order and

$$\sum C_j(\sigma) = \left(t_0 + \frac{a}{b}\right)\sum_{j=1}^{n}\prod_{k=1}^{j}\left(1 + b_{[k]}b\right) - \frac{na}{b}.$$

(b) If $a = 1$, then problem $1|p_j = b_j(a+bt)|\sum w_j C_j$ is solvable in $O(n\log n)$ time by scheduling job in non-increasing order of ratios $\frac{w_j(1+b_j b)}{b_j b}$.

(c) If $a = 1$ and $b = 0$, then problem $1|p_j = b_j(a+bt)|\sum T_j$ is weakly NP-hard.

(d) Problem $1|p_j = b_j(a+bt)|\sum U_j$ is solvable in $O(n\log n)$ time by scheduling job using modified Moore's algorithm.

Main results: Single machine: Linear job processing times

Theorem (Wajs, 1986; Gupta & Gupta, 1988; Tanaev, Gordon & Shafransky, 1994; Gawiejnowicz & Pankowska, 1995)

Problem $1|p_j = a_j + b_j t|C_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs in non-increasing order of ratios $\frac{b_j}{a_j}$ and

$$C_{\max}(\sigma) = \sum_{j=1}^{n} a_{[j]} \prod_{k=j+1}^{n} (1 + b_{[k]}) + t_0 \prod_{j=1}^{n} (1 + b_{[j]}).$$

Theorem (Tanaev, Gordon & Shafransky, 1994; Gawiejnowicz, 2008; Gordon, Potts, Strusevich & Whitehead, 2008)

Problem $1|p_j = a_j + b_j t, \delta|C_{\max}$ is solvable in at most $O(n^2)$ time, provided that precedence constraints δ are in the form of chains, a tree or a series-parallel digraph.

Main results: Single machine: Linear job processing times

- If in problem $1|p_j = a_j + b_j t|C_{\max}$ we replace $C_{\max}$ with $\sum C_j$, the time complexity of the new problem, $1|p_j = a_j + b_j t|\sum C_j$, is unknown even if $a_j = 1$ for all j
- For a given b, job completion times for problem $1|p_j = 1 + b_j t|\sum C_j$ are as follows:

$$\begin{aligned} C_{[0]} &= 1, \\ C_{[j]} &= C_{[j-1]} + p_j(C_{[j-1]}) = 1 + \beta_{[j]} C_{[j-1]}, \end{aligned} \tag{1}$$

where $\beta_{[j]} = 1 + b_{[j]}$ for $1 \leqslant j \leqslant n$

- Recurrence formulae (1) can be rewritten in matrix form:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -\beta_1 & 1 & \dots & 0 & 0 \\ 0 & -\beta_2 & \dots & 0 & 0 \\ \vdots & & \dots & & \vdots \\ 0 & 0 & \dots & -\beta_n & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{2}$$

- Matrix Eq. (2) can be rewritten as $A(b)C(b) = d(1)$, where $d(1) = [1, \ldots, 1]^\top \in \mathbb{R}^{n+1}$, $C(b) = [C_0, \ldots, C_n]^\top \in \mathbb{R}^{n+1}$
- The determinant $\det(A(b)) = 1$ and hence the inverse $A^{-1}(b)$ to the matrix $A(b)$ exists,

$$A^{-1}(a) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ \beta_1 & 1 & \ldots & 0 & 0 \\ \beta_1\beta_2 & \beta_2 & \ldots & 0 & 0 \\ \beta_1\beta_2\beta_3 & \beta_2\beta_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \beta_1\beta_2\ldots\beta_n & \beta_2\beta_3\ldots\beta_n & \ldots & \beta_n & 1 \end{pmatrix}$$

- Knowing $A^{-1}(b)$, we can find the components $C_i(b)$ of the vector $C(b) = A^{-1}(b)d(1)$
- Expressing a time-dependent scheduling problem in a matrix form is called matrix approach and it was introduced by Gawiejnowicz, Kurc & Pankowska, 2002

- $\sum C_j$ and $C_{\max}$ criteria are two limit cases of norm $\|C(b)\|_p$
- The norm is very-well known in optimization theory, but seems to be unexplored in scheduling theory

Definition

Given any $p \geqslant 1$, the l_p-norm of vector $x \in \mathbb{R}^n$ is as follows:

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, & 1 \leqslant p < +\infty, \\ \max_{1 \leqslant i \leqslant n}\{|x_i|\}, & p = +\infty \end{cases}$$

- It is easy to note that $\sum C_j \equiv l_1$ and $C_{\max} \equiv l_\infty$
- An interesting question is how the l_p norm behaves for $1 < p < +\infty$

Main results: Single machine: Linear job processing times

- Let $A(b)$ denote the matrix composed of coefficients of recurrence equations, which specify job completion times C_j for a given schedule for problem $1|p_j = 1 + b_j t|\sum C_j$, defined by vector $b = (b_0, b_1, \ldots, b_n)$, $j = 0, 1, \ldots, n$
- Then, if we replace criteria $C_{\max}$ and $\sum C_j$ by appropriate norm l_p, $1 \leqslant p \leqslant +\infty$, there holds the following result

Theorem (Gawiejnowicz & Kurc, 2015)

If $A(b)C(b) = d$ is a matrix equation defining schedule b for an instance of problem $1|p_j = 1 + b_j t|\|C(b)\|_p$, then

$$\log \|C(b)\|_p \leqslant \frac{1}{p} \log \|C(b)\|_1 + \left(1 - \frac{1}{p}\right) \log \|C(b)\|_\infty .$$

- Other properties of problem $1|p_j = 1 + b_j t|\|C(b)\|_p$ are discussed by Gawiejnowicz & Kurc, 2015

Main results: Single machine: Linear job processing times

Theorem (Kononov, 1997; Bachman & Janiak, 2000)

(a1) Problem $1|p_j = a_j + b_j t|L_{\max}$ is weakly NP-hard, even if only one coefficient $a_k \neq 0$ for some $1 \leqslant k \leqslant n$, and due dates of all jobs with $a_j = 0$, $j \neq k$, are equal.

(a2) Problem $1|p_j = a_j + b_j t|L_{\max}$ is weakly NP-hard, even if only two distinct due dates exist.

(b) Problem $1|p_j = a_j + b_j t|f_{\max}$ is weakly NP-hard.

Theorem (Bachman, Janiak & Kovalyov, 2002)

(a) Problem $1|p_j = a_j + b_j t|\sum w_j C_j$ is weakly NP-hard.

(b) Problem $1|p_j = a_j + b_j t|\sum U_j$ is weakly NP-hard.

(c) Problem $1|p_j = a_j + b_j t|\sum T_j$ is weakly NP-hard.

Main results: Single machine: Non-linear job processing times

Theorem (Gawiejnowicz, 1997; Melnikov & Shafransky, 1980; Strusevich & Rustogi, 2017)

(a) If $f(t) \geqslant 0$ for $t \geqslant t_0$ and $f(t)$ is non-decreasing, then problem $1|p_j = a_j + f(t)|C_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs in non-decreasing order of basic job processing times a_j (SPT order).

(b) If $f(t) \geqslant 0$ for $t \geqslant t_0$ and $f(t)$ is non-decreasing, then problem $1|p_j = a_j + f(t)|\sum C_j$ is solvable in $O(n \log n)$ time by scheduling jobs in the SPT order.

Main results: Single machine: Non-linear job processing times

Theorem (Kononov, 1998)

If $f(t)$ is a convex (concave) function for $t \geqslant 0$, $f(t_0) > 0$, and if $t_1 + b_j f(t_1) \leqslant t_2 + b_j f(t_2)$ for all $t_2 > t_1 \geqslant t_0$ and all jobs, then

(a) problem $1|p_j = b_j f(t)|C_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs in non-decreasing (non-increasing) order of coefficients b_j;

(b) problem $1|p_j = b_j f(t)|L_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs in non-decreasing (non-increasing) order of sums $b_j + d_j$.

Main results: Parallel machines: Proportional job processing times

Theorem (Kononov, 1996, 1997; Mosheiov, 1998)

(a) Problem $P2|p_j = b_j t|C_{\max}$ is weakly NP-hard.

(b) Problem $P|p_j = b_j t|C_{\max}$ is strongly NP-hard.

Theorem (Cheng & Sun, 2007)

(a) If $t_0 = 1$ and $b_j \in (0, 1]$, then for the LS algorithm applied to problem $P2|p_j = b_j t|C_{\max}$ there holds inequality

$$\frac{C_{\max}(LS)}{C_{\max}(OPT)} \leqslant \sqrt{2}.$$

(b) If $t_0 = 1$ and $b_j \in (0, \alpha]$, where $0 < \alpha \leqslant 1$, then for the LS algorithm applied to problem $Pm|p_j = b_j t|C_{\max}$ there holds inequality

$$\frac{C_{\max}(LS)}{C_{\max}(OPT)} \leqslant 2^{\frac{m-1}{m}}.$$

Main results: Parallel machines: Proportional job processing times

Theorem (Cheng, Wang & He, 2009)

If $t_0 = 1$, then for the LS and LDR algorithms applied to problem $Pm|p_j = b_j t|C_{\max}$ there hold inequalities:

$$\frac{\log C_{\max}(LS)}{\log C_{\max}(OPT)} \leqslant 2 - \frac{1}{m}$$

and

$$\frac{\log C_{\max}(LDR)}{\log C_{\max}(OPT)} \leqslant \frac{4}{3} - \frac{1}{3m}.$$

Main results: Parallel machines: Proportional job processing times

- The latter results show significant similarity to well-known results of classical scheduling theory
- Some authors (Cheng & Ding, 2000; Cheng, Ding & Lin, 2004; Gawiejnowicz, Kurc & Pankowska, 2006) observed that there exist pairs of time-dependent scheduling problems which have similar properties
- One group of these similarities, for proportional case, may be explained with the use of the notion of isomorphic scheduling problems (Gawiejnowicz & Kononov, 2014)
- Before we introduce this notion, we need a few definitions

Main results: Parallel machines: Proportional job processing times

Definition (Gawiejnowicz & Kononov, 2014)

Let I_Π, $\sigma = (s_1, \ldots, s_k, C_1, \ldots, C_k, \mu_1, \ldots, \mu_k)$ and $f_\Pi(C_1, \ldots, C_k)$ denote an instance of an optimization problem Π, a feasible solution to the instance and the value of its criterion function, respectively.

Problem Π_1 is said to be *(γ, θ)–reducible* to problem Π_2 if there exist two strictly increasing continuous functions, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, such that the following two conditions hold:

1) for any instance I_{Π_1} of problem Π_1 there exists an instance I_{Π_2} of problem Π_2 such that function γ transforms any feasible solution σ of instance I_{Π_1} into feasible solution $\sigma_d = (\gamma(s_1), \ldots, \gamma(s_k), \gamma(C_1), \ldots, \gamma(C_k), \mu_1, \ldots, \mu_k)$ of instance I_{Π_2}, and for any feasible solution $\tau_d = (s'_1, \ldots, s'_k, C'_1, \ldots, C'_k, \mu'_1, \mu'_2, \ldots, \mu'_k)$ of instance I_{Π_2} solution $\tau = (\gamma^{-1}(s'_1), \ldots, \gamma^{-1}(s'_k), \gamma^{-1}(C'_1), \ldots, \gamma^{-1}(C'_k), \mu'_1, \ldots, \mu'_k)$ is a feasible solution of instance I_{Π_1};

2) for any feasible solution σ of instance I_{Π_1} criterion functions f_{Π_1} and f_{Π_2} satisfy equality $f_{\Pi_2}(\gamma(C_1), \ldots, \gamma(C_k)) = \theta(f_{\Pi_1}(C_1, \ldots, C_k))$.

Main results: Parallel machines: Proportional job processing times

Definition (Gawiejnowicz & Kononov, 2014)

Let I_Π and σ denote an instance of a decision problem Π and a feasible solution to I_Π, respectively. Problem Π_1 is said to be *γ–reducible* to problem Π_2 if there exists a strictly increasing continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any instance I_{Π_1} of problem Π_1 there exists an instance I_{Π_2} of problem Π_2 such that function γ transforms any feasible solution σ of instance I_{Π_1} into feasible solution σ_d of instance I_{Π_2}, and for any feasible solution τ_d of instance I_{Π_2} solution τ is a feasible solution of instance I_{Π_1}.

Property (Gawiejnowicz & Kononov, 2014)

If problem Π_1 is (γ, θ)–reducible (γ–reducible) to problem Π_2, then problem Π_2 is $(\gamma^{-1}, \theta^{-1})$–reducible ($\gamma^{-1}$–reducible) to problem Π_1.

Main results: Parallel machines: Proportional job processing times

Definition (Gawiejnowicz & Kononov, 2014)

(γ, θ)–*reducible* or γ–*reducible* scheduling problems are called *isomorphic problems*.

Lemma (Gawiejnowicz & Kononov, 2014)

Let problem Π_2 be (γ, θ)–reducible to problem Π_1. Then if schedule $\sigma^\star = (s_1^\star, \ldots, s_k^\star,\ C_1^\star, \ldots, C_k^\star, \mu_1^\star, \ldots, \mu_k^\star)$ is optimal for instance I_{Π_1} of problem Π_1, then schedule $\sigma_d^\star = (\gamma(s_1^\star), \ldots, \gamma(s_k^\star), \gamma(C_1^\star), \ldots, \gamma(C_k^\star), \mu_1^\star, \ldots, \mu_k^\star)$ is optimal for instance I_{Π_2} of problem Π_2 and vice versa.

Main results: Parallel machines: Proportional job processing times

Theorem (Chen, 1996; Kononov, 1997; Ji & Cheng, 2009)

(a) Problem $P2|p_j = b_j t|\sum C_j$ is weakly NP-hard.

(b) Problem $P|p_j = b_j t|\sum C_j$ is strongly NP-hard.

Theorem (Chen, 1996)

For the SDR algorithm applied to problem $P2|p_j = b_j t|\sum C_j$ there holds inequality

$$\frac{\sum C_j(SDR)}{\sum C_j(OPT)} \leqslant \max\left\{\frac{1+b_n}{1+b_1}, \frac{2}{n-1} + \frac{(1+b_1)(1+b_n)}{1+b_2}\right\}.$$

Main results: Parallel machines: Proportional-linear job processing times

- Let $GP||C_{\max}$ denote a generic scheduling problem with fixed job processing times

Theorem (Gawiejnowicz & Kononov, 2014)

Problem $GP||C_{\max}$ is (γ, θ)–reducible to problem $GP|p_j = b_j(a + bt)|C_{\max}$ with $\gamma = \theta = 2^x - \frac{a}{b}$.

Theorem (Gawiejnowicz & Kononov, 2014)

Let A be an approximation algorithm for problem $GP||C_{\max}$ such that

$$\frac{C_{\max}(A)}{C_{\max}(OPT)} \leqslant r_A < +\infty.$$

Then for approximation algorithm $\bar{A}$ for problem $GP|p_j = b_j(a + bt)|C_{\max}$ there holds inequality

$$\frac{\log(C_{\max}(A) + \frac{a}{b})}{\log(C_{\max}(OPT) + \frac{a}{b})} = \frac{C_{\max}(A)}{C_{\max}(OPT)}.$$

Main results: Dedicated machines: Proportional job processing times

Theorem (Kononov, 1996; Mosheiov, 2002)

Problem $F2|p_{ij} = b_{ij}t|C_{\max}$ is solvable in $O(n \log n)$ using modified Johnson's algorithm.

Theorem (Kononov, 1996; Mosheiov, 2002; Thörnblad & Patriksson, 2011)

(a) Problem $F3|p_{ij} = b_{ij}t|C_{\max}$ is strongly NP-hard.

(b) For problem $F3|p_{ij} = b_{ij}t, b_{i1} = b_{i3} = b|C_{\max}$ does not exist a polynomial-time approximation algorithm with a constant worst-case ratio, unless P=NP.

Main results: Dedicated machines: Proportional job processing times

Theorem (Kononov, 1996; Mosheiov, 2002)

Problem $O2|p_{ij} = b_{ij}t|C_{\max}$ is solvable in $O(n)$ time by scheduling jobs using modified Gonzalez-Sahni's algorithm.

- Recently, applying the notion of isomorphic problems, there has been shown the following result

Theorem (Gawiejnowicz & Kolińska, 2020)

Problem $O2|p_{ij} = b_{ij}t|C_{\max}$ is solvable in $O(n)$ time by scheduling jobs using LADR rule.

Main results: Dedicated machines: Proportional job processing times

Theorem (Kononov, 1996; Kononov & Gawiejnowicz, 2001)

(a) Problem $O3|p_{ij} = b_{ij}t|C_{\max}$ is weakly NP-hard.

(b) Problem $O3|p_{ij} = b_{ij}t, b_{3j} = b|C_{\max}$ is weakly NP-hard.

Theorem (Mosheiov, 2002)

Problem $J2|p_{ij} = b_{ij}t|C_{\max}$ is weakly NP-hard.

Main results: Dedicated machines: Linear job processing times

Theorem (Kononov & Gawiejnowicz, 2001)

(a) Problem $F2|p_{ij} = a_{ij} + b_{ij}t|C_{\max}$ is strongly NP-hard.

(b) Problem $O2|p_{ij} = a_{ij} + b_{ij}t|C_{\max}$ is weakly NP-hard.

Theorem (Kononov & Gawiejnowicz, 2001)

(a) Problem $F2|p_{ij} = a_{ij} + b_{ij}t|\sum C_j$ is strongly NP-hard.

(b) Problem $O2|p_{ij} = a_{ij} + b_{ij}t|\sum C_j$ is weakly NP-hard.

Main results: Summary

- Exact algorithms (Kunnathur & Gupta, 1990; Kovalyov & van de Velde, 1998; Wu & Lee, 2006; Lee, Wu & Chung, 2008; Ouazene & Yalaoui, 2018)
- Approximation algorithms and approximation schemes (Hsieh & Bricker, 1997; Kovalyov & Kubiak, 1998; Woeginger, 2000; Ji & Cheng, 2009; Halman, 2020)
- Heuristic algorithms (Alidaee, 1990; Mosheiov, 1991, 1996; Hsu & Lin, 2003; Gawiejnowicz, Kurc & Pankowska, 2006)
- Meta-heuristic algorithms (Hindi & Mhlanga, 2001; Wu, Lee & Shiau, 2007; Gawiejnowicz & Suwalski, 2014; Lu, Liu, Pei, Thai & Pardalos, 2018)

Main results: Summary

- Time-dependent scheduling on machines with limited availability (Wu & Lee, 2003; Ji, He & Cheng, 2006; Gawiejnowicz, 2007; Gawiejnowicz & Kononov, 2010; Ji & Cheng, 2010)
- Two-criteria time-dependent scheduling (Gawiejnowicz, Kurc & Pankowska, 2006; Cheng, Tadikamalla, Shang & Zhang, 2014, 2015)
- Two-agent time-dependent scheduling (Liu & Tang, 2008; Liu, Yi & Zhou, 2011; Gawiejnowicz & Suwalski, 2014)
- Time-dependent scheduling with job rejection (Cheng & Sun, 2009; Li & Zhao, 2015)
- Time-dependent scheduling games (Li, Liu & Li, 2014; Chen, Lin, Tan & Yan, 2017)

Main results: Summary

- Equivalent time-dependent scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009)
- Conjugate time-dependent scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009)
- Mixed problems of time-dependent scheduling (Gawiejnowicz & Lin, 2010; Dębczyński & Gawiejnowicz, 2013; Dębczyński, 2014)
- Isomorphic scheduling problems (Gawiejnowicz & Kononov, 2014; Gawiejnowicz & Kolińska)

Open problems: Single machine

- Problem $1|p_j = 1 + b_j t|\sum C_j$ is the most important open single machine time-dependent scheduling problem
- In the problem, a set of $n+1$ jobs $J_0, J_1, \ldots, J_n$ has to be executed on a single machine
- The processing time of job J_j at time t equals $p_j(t) = 1 + b_j t$, where t denotes the starting time of job J_j and $b_j > 0$ is deterioration rate of the job
- The criterion of schedule optimality is $\sum_{j=0}^{n} C_{[j]}$, where

$$\begin{array}{rcl} C_{[0]} & = & 1 \\ C_{[1]} & = & \beta_{[1]} C_{[0]} + 1 \\ \vdots & & \vdots \\ C_{[n]} & = & \beta_{[n]} C_{[n-1]} + 1 \end{array}$$

and $\beta_{[i]} := 1 + b_{[i]}$ for $1 \leqslant i \leqslant n$

Open problems: Single machine

- Research on problem $1|p_j = 1 + b_j t|\sum C_j$ was initiated by Mosheiov (1991) who formulated it and proved the main its properties
- A special case of the problem, $1|p_j = b_j t|\sum C_j$, is solvable in $O(n \log n)$ time (Mosheiov, 1994)
- Gawiejnowicz et al (2006) proposed for problem $1|p_j = 1 + b_j t|\sum C_j$ two greedy algorithms, based on signatures
- Ocetkiewicz (2010) proposed for a special case of problem $1|p_j = 1 + b_j t|\sum C_j$ approximation scheme (FPTAS)
- Gawiejnowicz & Kurc (2015) generalized results by Mosheiov (1991) to case of l_p norm
- Gawiejnowicz & Kurc (2020) gave a new upper bound on the power of the set of all possible optimal schedules for problem $1|p_j = 1 + b_j t|\sum C_j$

Open problems: Single machine

- An instance of $1|p_j = 1 + b_j t|\sum C_j$ may be identified with sequence $b = (b_0, b_1, \ldots, b_n)$ and any rearrangement of b may be identified with a schedule for the problem
- The total completion time for problem $1|p_j = 1 + b_j t|\sum C_j$ and schedule σ can be computed using the formula

$$\sum C_j(\sigma) = \sum_{i=1}^{n} \sum_{j=1}^{i} \prod_{k=j+1}^{n} \beta_{[k]} + t_0 \sum_{i=1}^{n} \prod_{j=1}^{n} \beta_{[j]},$$

- The above formula is a special case of the formula for $\sum w_j C_j(\sigma)$ for problem $1|p_j = a_j + b_j t|\sum w_j C_j$,

$$\sum w_j C_j(\sigma) = \sum_{i=1}^{n} w_{[i]} \sum_{j=1}^{i} a_{[j]} \prod_{k=j+1}^{n} \beta_{[k]} + t_0 \sum_{i=1}^{n} w_{[i]} \prod_{j=1}^{n} \beta_{[j]}.$$

- Problem $1|p_j = a_j + b_j t|\sum w_j C_j$ is weakly NP-hard, while problem $1|p_j = 1 + b_j t|\sum C_j$ is open

Open problems: Single machine

The largest job property (Mosheiov, 1991)

In an optimal schedule for problem $1|p_j = 1 + b_j t|\sum C_j$ first is scheduled job with the largest deterioration rate.

The symmetry property (Mosheiov, 1991)

If $b = (b_0, b_1, \ldots, b_n)$ is an optimal schedule for problem $1|p_j = 1 + b_j t|\sum C_j$, then also $\bar{b} = (b_0, b_n, \ldots, b_1)$ is optimal.

Definition

A schedule is V-shaped, if jobs are ordered non-increasingly (non-decreasingly), when the jobs are before the job (after the job) with the smallest deterioration rate.

Theorem (necessary condition no. 1, Mosheiov, 1991)

An optimal schedule for problem $1|p_j = 1 + b_j t|\sum C_j$ is V-shaped.

Open problems: Single machine

- For problem $1|p_j = 1 + b_j t|\sum C_j$ was proposed (Gawiejnowicz, Kurc & Pankowska, 2002, 2006) a greedy algorithm, based on properties of functions $S^{-}(\beta)$ and $S^{+}(\beta)$ of sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of coefficients $\beta_j := 1 + b_j$
- Let $\bar{\beta} = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ denote the sequence of job deterioration rates ordered reversely compared to β and

$$F(\beta) = \sum_{j=1}^{n} \sum_{i=1}^{j} \prod_{k=j}^{j} \beta_k,$$

$$M(\beta) = 1 + \sum_{i=1}^{n} \prod_{k=i}^{n} \beta_k.$$

Open problems: Single machine

- Then the functions, signatures, are defined as follows:

$$S^{-}(\beta) = M(\bar{\beta}) - M(\beta) = \sum_{i=1}^{n} \prod_{j=1}^{i} \beta_j - \sum_{i=1}^{n} \prod_{j=i}^{n} \beta_j,$$

$$S^{+}(\beta) = M(\bar{\beta}) + M(\beta),$$

- Let $(\beta^1|\beta^2|\beta^3)$ and B denote the sequence consisting of sequences β^1, β^2 and β^3 (in that order) and the product of all β_j, respectively

Open problems: Single machine

- The main properties of signatures are described by the following lemma

Lemat (Gawiejnowicz, Kurc & Pankowska, 2006)

For a given sequence β and numbers $a > 1, b > 1$, there hold equalities:
(a) $F(a|\beta|b) = F(\beta) + aM(\bar{\beta}) + bM(\beta) + aBb$,
(b) $F(b|\beta|a) = F(\beta) + bM(\bar{\beta}) + aM(\beta) + aBb$,
(c) $F(a|\beta|b) - F(b|\beta|a) = (a-b)S^{-}(\beta)$,
(d) $F(a|\beta|b) + F(b|\beta|a) = (a+b)S^{+}(\beta) + 2(F(\beta) + aBb)$.

Open problems: Single machine

Theorem (Gawiejnowicz, Kurc & Pankowska, 2006)

(a) For a given sequence β and numbers $a > 1$, $b > 1$, there holds equivalence

$$F(a|\beta|b) \leqslant F(b|\beta|a) \text{ iff } (a-b)S^{-}(\beta) \leqslant 0.$$

Moreover, a similar equivalence holds if symbol ' $\leqslant$ ' will be replaced with symbol ' $\geqslant$ '.
(b) Let sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be ordered non-decreasingly, let $u = (u_1, u_2, \ldots, u_{k-1})$ be a V-shaped schedule consisting of the first $k \geqslant 1$ elements β, let $a = \beta_k > 1$, $b = \beta_{k+1} > 1$, where $1 < k < n$, and let $a \leqslant b$. Then

$$\text{if } S^{-}(u) \geqslant 0, \text{ then } F(a|u|b) \leqslant F(b|u|a).$$

Moreover, a similar implication holds, if in the above inequality symbol ' $\geqslant$ ' will be replaced with symbol ' $\leqslant$ ', and symbol ' $\leqslant$ ' will be replaced with symbol ' $\geqslant$ '.

- Based on this theorem, the following greedy algorithm for problem $1|p_j = 1 + b_j t|\sum C_j$ can be formulated

Open problems: Single machine

Step 1. { `Initialization` }

Create $b^{\nearrow} = (b_{[1]}, b_{[2]}, \ldots, b_{[n]}, b_{[0]})$ by sorting b in non-decreasing order

Step 2. { `Main loop` }

If n is odd **then**

begin

$u := (b_{[1]})$

for $i := 2$ **to** $n - 1$ **step** 2 **do**

if $S^{-}(u) \leqslant 0$ **then** $u := (b_{[i+1]}|u|b_{[i]})$

else $u := (b_{[i]}|u|b_{[i+1]})$

end

else { n `is even` }

begin

$u := (b_{[1]}, b_{[2]})$

for $i := 3$ **to** $n - 1$ **step** 2 **do**

if $S^{-}(u) \leqslant 0$ **then** $u := (b_{[i+1]}|u|b_{[i]})$

else $u := (b_{[i]}|u|b_{[i+1]})$

end

Step 3. { `Final sequence` }

return $(b_{[0]}|u)$

Open problems: Single machine

Effectiveness of the greedy algorithm for $1|p_j = 1 + (1 + j)t| \sum C_j$

n	OPT	GA	Mosheiov's H_1	Mosheiov's H_2
3	14	$\star$	$\star$	$\star$
4	51	$\star$	$\star$	0.078431372549
5	221	$\star$	0.004524886878	0.072398190045
6	1,162	$\star$	0.002581755594	0.048192771084
7	7,386	$\star$	0.002301651774	0.034118602762
8	55,207	$\star$	0.001104932346	0.027605194993
9	473,945	$\star$	0.000730042515	0.019799765796
10	4,580,090	$\star$	0.000360691602	0.014409323834
11	49,097,362	$\star$	0.000212516510	0.011143164881
12	577,329,127	$\star$	0.000120697184	0.008978497286
13	7,382,689,709	$\star$	0.000077018407	0.007387560110
14	101,952,444,582	$\star$	0.000050799302	0.006206066040
15	1,511,666,077,882	$\star$	0.000035594077	0.005301824889
16	23,947,081,624,255	$\star$	0.000025806657	0.004588409164
17	403,593,295,119,129	$\star$	0.000019304148	0.004012957606
18	7,209,715,929,612,834	$\star$	0.000014771383	0.003541247187
19	136,066,769,455,072,000	$\star$	0.000011521732	0.003149219157
20	2,705,070,072,148,870,000	$\star$	0.000009131012	0.002819572095

Open problems: Single machine

Let

$$\Delta_k(r,q) = \sum_{i=1}^{q-k-1} \prod_{j=i}^{q-k-1} \beta_j - \sum_{i=q-k+1}^{q-1} \prod_{j=q-k+1}^{i} \beta_j - \frac{1}{a_q} \sum_{i=q+1}^{n} \prod_{j=q-k+1}^{i} \beta_j$$

and

$$\nabla_k(r,q) = \frac{1}{a_r} \sum_{i=1}^{r-1} \prod_{j=i}^{r+k-1} \beta_j + \sum_{i=r+1}^{r+k-1} \prod_{j=i}^{r+k-1} \beta_j - \sum_{i=r+k+1}^{n} \prod_{j=r+k+1}^{i} \beta_j,$$

where $1 \leqslant r < q \leqslant n$ i $k = 1, 2, \ldots, q-r$.

Open problems: Single machine

Theorem (necessary condition no. 2, Gawiejnowicz & Kurc, 2020)

Let $b = (b_1, b_2, \ldots, b_n)$ be an optimal schedule for problem $1|p_j = 1 + b_j t|\sum C_j$. Then (i) b is V-shaped, the smallest element in b is b_m, where $1 < m < n$, and hold the following inequalities (ii)

$$\Delta_1(m-1, m+1) = \sum_{j=1}^{m-1} \prod_{k=j}^{m-1} \beta_k - \sum_{i=m+2}^{n} \prod_{k=m+2}^{i} \beta_k \geqslant 0,$$

$$\nabla_1(m-1, m+1) = \sum_{j=1}^{m-2} \prod_{k=j}^{m-2} \beta_k - \sum_{i=m+1}^{n} \prod_{k=m+1}^{i} \beta_k \leqslant 0.$$

Open problems: Single machine

- Let $V_I(b)$ and $V_{II}(b)$ denote the sets of schedules which satisfy the necessary condition no. 1 and no. 2, respectively
- Let $1 < u < v$, where $u = \min\{b_i : i = 1, 2, \ldots, n\}$ and $v = \max\{b_i : i = 1, 2, \ldots, n\}$
- Let $d_n = n \times \frac{\log u}{\log u + \log v}$, $g_n = 1 + n \times \frac{\log v}{\log u + \log v}$
- Let $D := \{k \in \mathbb{N} : d_n < k < g_n, 1 < k < n\}$ and let $V_D(b)$ denote the set of all V-shaped schedules which can be generated from sequence b, provided that $b_m \in D$

Theorem (Gawiejnowicz & Kurc, 2020)

Let $c(n) = \sqrt{\frac{2}{\pi n}}\, 2^n \left(1 + O\left(\frac{1}{n}\right)\right)$. Then

$$|V_{II}(b)| \leqslant |V_D(b)| \leqslant \left(1 + \frac{\log v - \log u}{\log v + \log u} n\right) \times c(n)$$

and, if v is sufficiently close to u,

$$|V_D(b)| \geqslant c(n).$$

Open problems: Single machine

- The time complexity of single machine problems of scheduling proportional and linear jobs with arbitrary precedence constraints is unknown
- The following two problems are the main candidates to study:
  - $1|p_j = b_j t, prec|\sum w_j C_j$,
  - $1|p_j = a_j + b_j t, prec|C_{\max}$
- Establishing the status of time complexity of the second of these problems will allow to establish the status of time complexity of problem $1|p_j = a_j - b_j t, prec|\sum w_j C_j$ using the notion of conjugated time-dependent scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009)

Open problems: Single machine

Definition

Problem $1|p_j = h_j + \gamma_j t|\sum v_j C_j$ is conjugated to problem $1|p_j = b_j + \alpha_j t|\sum w_j C_j$, if for any schedule σ for problem $1|p_j = b_j + \alpha_j t|\sum w_j C_j$ there exists schedule $\varrho \equiv ((h_i, g_i, v_i))_{i=1}^n$ for problem $1|p_j = h_j + \gamma_j t|\sum v_j C_j$ such that there holds the equality

$$C_{\max}(\sigma) C_{\max}(\varrho) + \sum_{j=0}^{n} w_j h_j = \sum_{j=0}^{n} w_j C_j(\sigma) + \sum_{j=0}^{n} v_j C_j(\varrho),$$

where $C_0(\varrho) := h_0$, $g_j = 1 + \gamma_j$, $h_j \geqslant 0$, $v_j \geqslant 0$, $C_j(\varrho) = g_j C_{j-1}(\varrho) + h_j$ for $1 \leqslant j \leqslant n$.

Theorem (Gawiejnowicz, Kurc & Pankowska, 2009)

Let $B_j = \frac{b_j}{1+b_j}$ and $a_j > -1$ for all j. Then problems $1|p_j = a_j - B_j t|\sum w_j C_j$ and $1|p_j = w_j + b_j t|\sum a_j C_j$ are conjugated.

Open problems: Single machine

- A separate group of pairs of problems constitute those related to the notion of equivalent time-dependent scheduling problems (Gawiejnowicz, Kurc & Pankowska, 2009)
- The notion uses a general transformation of an arbitrary instance of an *initial problem* into an instance of a *transformed problem*
- The initial problem is a time-dependent scheduling problem with the total weighted starting time criterion
- The transformed problem is a time-dependent scheduling problem with a similar criterion but with other job processing times and job weights

Theorem (Gawiejnowicz, Kurc & Pankowska, 2009)

Let $\beta_j = 1 + b_j$ for all j. Then the following pairs of problems are equivalent:
(a) $1|p_j = b_j t|\sum \beta_j C_j$ i $1|p_j = 1 + b_j(1+t)|C_{\max}$
(b) $1|p_j = a_j + bt|\sum C_j$ i $1|p_j = 1 + bt|\sum a_j C_j$
(c) $1|p_j = bt|\sum a_j C_j$ i $1|p_j = a_j + bt|C_{\max}$

Open problems: Parallel machines

- Interesting candidates to study are the following two parallel-machine time-dependent scheduling problems with non-linear job processing times:
  - $Pm|p_j = a_j + f(t)|C_{\max}$
  - $Pm|p_j = b_j f(t)|\sum C_j$
- We know that if $f(t) \geqslant 0$ for $t \geqslant t_0$ and $f(t)$ is non-decreasing, then there is an optimal schedule for the first problem, where jobs are scheduled in the SPT order (Gawiejnowicz, 2020)
- We also know that the first problem is polynomially solvable for $m = 1$ (Gawiejnowicz, 1997)
- The second problem is polynomially solvable for $m = 1$ and convex or concave functions (Kononov, 1998)
- It seems that appropriate choice of conditions on $f(t)$ may lead to polynomial solvability of these problems

Open problems: Dedicated machines

- Good candidates for study are the following time-dependent scheduling problems on two dedicated machines:
  - $F2|p_{ij} = b_{ij}t|\sum C_j$
  - $O2|p_{ij} = b_{ij}t|\sum C_j$
- Counterparts of the problems with fixed processing times are NP-hard (Garey, Johnson & Sethi, 1976; Achugbue & Chin, 1982)
- Establishing of time complexity of these problems is a challenge in view of their multiplicative nature
- It seems that useful would be the consideration of well-chosen special cases, e.g. similar ones to those for problem $1|p_j = a_j + b_j t|\sum w_j C_j + \theta L_{\max}$ (Gawiejnowicz & Suwalski, 2014)

Conclusions

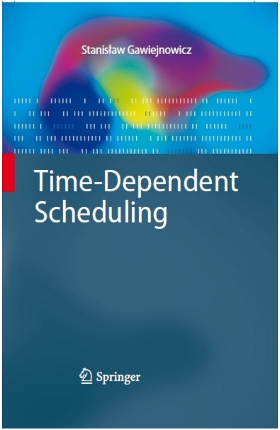
- We presented the subject, main ideas and the place of time-dependent scheduling in non-classical scheduling theory
- We gave a brief description of main research directions in time-dependent scheduling
- We described the most important results of time-dependent scheduling
- Finally, we sketched the present status of research on several open time-dependent scheduling problems

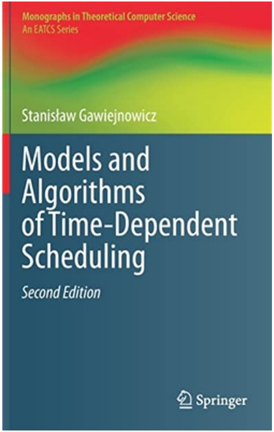
References

- As it is today, the literature on time-dependent scheduling counts ca. 350 positions
- Majority of these positions, ca. 90%, were published in JCR journals
- Ca. 60% of references concerns time-dependent scheduling on a single machine
- Ca. 25% of references concerns time-dependent scheduling on parallel machines
- Finally, ca. 15% of references concerns time-dependent scheduling on dedicated machines

References: Monographs

- S. Gawiejnowicz, Time-Dependent Scheduling, Springer, 2008, 377pp.
- S. Gawiejnowicz, Models and Algorithms of Time-Dependent Scheduling, Springer, 2020, 538pp.





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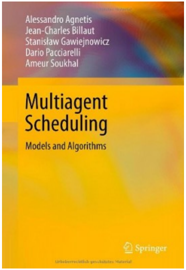
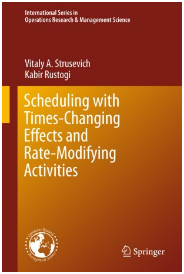
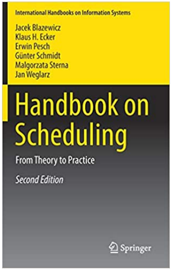
- A. Agnetis, J-C. Billaut, S. Gawiejnowicz, D. Pacciarelli, A. Soukhal, Multi-Agent Scheduling, Springer, 2014, 271pp.
- V.A. Strusevich, K. Rustogi, Scheduling with Times-Changing Effects and Rate-Modifying Activities, Springer, 2017, 455pp.
- J. Błażewicz, K. Ecker, E. Pesch, G. Schmidt, M. Sterna, J. Węglarz, Handbook on Scheduling, Springer, 2019, 833pp.







- Some other monographs also include bibliographic remarks on time-dependent scheduling literature (see, e.g., Pinedo, 2016)

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- S. Gawiejnowicz, Brief survey of continuous models of scheduling, Foundations of Computing and Decision Sciences, 21 (1996), 81–100.
- B. Alidaee, N.K. Womer, Scheduling with time-dependent processing times: review and extensions, Journal of the Operational Research Society, 50 (1999), 711-720.
- T.C-E. Cheng, Q. Ding, B.M-T. Lin, A concise survey of scheduling with time-dependent processing times, European Journal of Operational Research, 152 (2004), 1–13.
- S. Gawiejnowicz, A review of four decades of time-dependent scheduling: main results, new topics, and open problems, Journal of Scheduling, 23 (2020), 3–47.

References: IWDSP

International Workshop on Dynamic Scheduling Problems is a series of workshops focused on non-classical scheduling problems

- IWDSP 2016 – https://iwdsp2016.wmi.amu.edu.pl
- IWDSP 2018 – https://iwdsp2018.wmi.amu.edu.pl
- IWDSP 2021 – https://iwdsp2021.wmi.amu.edu.pl

- **The next IWDSP is planned in June 2023 in Switzerland**

Though it is hard to predict the future, it seems that due to the attractiveness of research topics, interesting open problems and numerous applications, time-dependent scheduling will remain one of main domains in non-classical scheduling theory

Thank you for your attention!