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1. $x(t) = t\sin(t)$
$y(t) = t\cos(t)$ $\Big\}$ = spiral

a) ⇒ Tangent is given by $[x'(t), y'(t)]$

$x'(t) = \sin(t) + t\cos(t)$
$y'(t) = \cos(t) - t\sin(t)$

⇒ Tangent Vector = $[\sin(t) + t\cos(t), \cos(t) - t\sin(t)]$

⇒ A Normal Vector = $[\cos(t) - t\sin(t), -\sin(t) - t\cos(t)]$

b) Intersection of spiral & line $ax + by + c = 0$ is,

⇒ Both have sam x & y at intersection ⇒ both have same points at intersection.

⇒ $\underline{a(t\sin(t)) + b(t\cos(t)) + c = 0}$

if $c = 0$

⇒ $a\,t\sin(t) = -b(t\cos(t))$

⇒ $a\sin(t) = -b\cos(t)$

⇒ $\frac{b}{a} = -\tan(t)$

⇒ $\tan(t) = -\frac{b}{a}$

⇒ $t = \tan^{-1}(-\frac{b}{a})$

⇒ $x(\tan^{-1}(-\frac{b}{a})) = \tan^{-1}(-\frac{b}{a})\sin(\tan^{-1}(-\frac{b}{a}))$

$y(\tan^{-1}(-\frac{b}{a})) = \tan^{-1}(-\frac{b}{a})\cos(\tan^{-1}(-\frac{b}{a}))$

∴ intersection points ~~is~~

$\underline{\left(\tan^{-1}(-\frac{b}{a})\sin(\tan^{-1}(-\frac{b}{a})), \tan^{-1}(-\frac{b}{a})\cos(\tan^{-1}(-\frac{b}{a}))\right)}$

$\underline{\text{when } c = 0}$

2. To transform 2D shape from left into right,

need :

i) Scale up by $\sqrt{2}$ (from graph)

ii) Flip/Reflect across y-axis

iii) Rotate counterclockwise by $\pi/4$ from origin (0,0)

iv) Translate by (4, 4)

Scale up by $\sqrt{2}$ = $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_1$, $\sqrt{2}$ since $\frac{6\sqrt{2}}{6} = \sqrt{2}$ (from graph)

Flip across y-axis = $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_2$

Rotate counterclockwise by $\pi/4$ from (0, 0) = $\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_3$

$\pi/4$ since $\theta = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$ (from graph)

Translate by (4, 4) = $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = M_4$

$\therefore$ Transformation Matrix = $M_4 M_3 M_2 M_1 = M$

$M_2 M_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$M_3 M_2 M_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow M = M_4 M_3 M_2 M_1 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 4 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = M$

$\therefore$ Single transformation Matrix = $\begin{bmatrix} -1 & -1 & 4 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

3. Derive homography s.t. $(0,0), (0,1), (1,0), (1,1) = (x_i, y_i)$

maps to $(4,2), (3.5,1), (3,1.5), (3,1) = (x_i'', y_i'')$

a)

$\Rightarrow$ Need solve

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x_i' \\ y_i' \\ w_i' \end{bmatrix}$$

, (x_i, y_i) is point before transformation, whereas (x_i'', y_i'') is the corresponding point after transformation

$$\cong \begin{bmatrix} x_i'/w_i' \\ y_i'/w_i' \\ 1 \end{bmatrix} = \begin{bmatrix} x_i'' \\ y_i'' \\ 1 \end{bmatrix}$$

$\Rightarrow \quad a x_i + b y_i + c = x_i'$

$d x_i + e y_i + f = y_i'$

$g x_i + h y_i + i = w_i'$

To simplify problem, set $i = 1$

$$\Rightarrow \frac{a x_i + b y_i + c}{g x_i + h y_i + 1} = x_i''$$

$\Rightarrow a x_i + b y_i + c - x_i''(g x_i + h y_i + 1) = 0$ — (A)

Similarly $d x_i + e y_i + f - y_i''(g x_i + h y_i + 1) = 0$ — (B)

From (A) & (B), can obtain 8 equations,

From $(0,0) \to (4,2)$

$\Rightarrow a(0) + b(0) + c - 4(g(0) + h(0) + 1) = 0$

$\Rightarrow c - 4 = 0 \Rightarrow c = 4$ — (7)

$\Rightarrow d(0) + e(0) + f - 2(g(0) + h(0) + 1) = 0$

$\Rightarrow f - 2 = 0 \Rightarrow f = 2$ — (8)

From $(0,1) \to (3.5,1)$

$\Rightarrow a(0) + b(1) + c - 3.5(g(0) + h(1) + 1) = 0$

$\Rightarrow b + c - 3.5h - 3.5 = 0 \Rightarrow b - 3.5h + 0.5 = 0$ — (1)

$\Rightarrow d(0) + e(1) + f - 1(g(0) + h(1) + 1) = 0$

$\Rightarrow e + 2 - h - 1 = 0 \Rightarrow e - h + 1 = 0$ — (2)

From $(1,0) \to (3,1.5)$

$a(1) + b(0) + c - 3(g(1) + h(0) + 1) = 0$

$\Rightarrow a + 4 - 3g - 3 = 0 \Rightarrow a - 3g + 1 = 0$ — (3)

$d(1) + e(0) + f - 1.5(g(1) + h(0) + 1) = 0$

$\Rightarrow d + 2 - 1.5g - 1.5 = 0 \Rightarrow d - 1.5g + 0.5 = 0$ — (4)

From $(1,1) \to (3,1)$

$\Rightarrow a(1) + b(1) + c - 3(g(1) + h(1) + 1) = 0$

$\Rightarrow a + b + 4 - 3g - 3h - 3 = 0 \Rightarrow a + b - 3g - 3h + 1 = 0$ — (5)

$\Rightarrow d(1) + e(1) + f - 1(g(1) + h(1) + 1) = 0$

$\Rightarrow d + e + 2 - g - h - 1 = 0 \Rightarrow d + e - g - h + 1 = 0$ — (6)

From ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧

$$
\begin{array}{c}
 \\ ⑦ \\ ⑧ \\ ① \\ ② \\ \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -3.5 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1.5 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & -3 & -3 \\
0 & 0 & 0 & 1 & 1 & 0 & -1 & -1
\end{bmatrix}
\end{array}
\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}
=
\begin{bmatrix} 4 \\ 2 \\ -0.5 \\ -1 \\ -1 \\ -0.5 \\ -1 \\ -1 \end{bmatrix}
$$

From MatLab

$\Rightarrow a = 2$

$b = 3$

$c = 4$

$d = 1$

$e = 0$

$f = 2$

$g = 1$

$h = 1$

a)

$$\Rightarrow \text{Homography} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

3. b) $(2,1) \to (x'', y'')$?

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(2) + 3(1) + 4(1) \\ 1(2) + 0(1) + 2(1) \\ 1(2) + 1(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 4+3+4 \\ 2+0+2 \\ 2+1+1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 4 \end{bmatrix}$$

$$\cong \begin{bmatrix} 11/4 \\ 4/4 \\ 4/4 \end{bmatrix} = \begin{bmatrix} 11/4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}$$

$\therefore$ $(2,1)$ gets mapped to $\underline{(11/4, 1)}$

3. c) The homography does not represent an affine transformation as the last row is not $[0, 0, 1]$ $\Rightarrow$ does not have form $\begin{bmatrix} [A] & [t] \\ 0 \quad 0 & 1 \end{bmatrix}$

$\Rightarrow$ Can't show $y = Ax + t$ $\forall (x,y)$ above. $\therefore$ <u>NOT Affine</u>

4. a) Show any 2D rotation can be achieved as a series of 3 consecutive shears.

$$\text{Shear for } x\text{-axis} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Shear for } y_{axis} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any 2D rotation can be achieved using a shear along 1-axis followed by shear along other axis, followed by shear along first axis

$$\Rightarrow 2D \text{ rotation} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ \beta & \gamma\beta+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c|ccc} & C_1 & C_2 & C_3 \\ \hline R_1 & & & \\ R_2 & & & \\ R_3 & & & \end{array} \begin{bmatrix} 1+\alpha\beta & \gamma+\alpha\beta\gamma+\alpha & 0 \\ \beta & \gamma\beta+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_1$$

Rows; C_1, C_2, C_3 = Columns;

$$2D \text{ rotation} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = M_2$$

$\Rightarrow$ Set $M_1 = M_2$ & equate corresponding elements

$\Rightarrow$ $1+\alpha\beta = \cos(\theta)$ — ① From (R_1, C_1)

$\beta = \sin(\theta)$ — ② From (R_2, C_1)

Substitute ② into ①

$\Rightarrow$ $1+\alpha\sin(\theta) = \cos(\theta)$

$$\Rightarrow \alpha = \frac{\cos(\theta)-1}{\sin(\theta)} = -\tan(\theta/2) \Rightarrow \alpha = -\tan(\theta/2) \text{ — ③}$$

From (R_2, C_2)

$\Rightarrow$ $\cos\theta = \gamma\beta + 1$

From ① $\Rightarrow$ $1+\alpha\beta = \gamma\beta+1$

$\Rightarrow \gamma = \alpha = -\tan(\theta/2) \Rightarrow \gamma = -\tan(\theta/2)$ — ④

$\therefore$ From ②, ③ & ④

$$\Rightarrow \text{Using } \left.\begin{aligned} \alpha &= -\tan(\theta/2) \\ \beta &= \sin(\theta) \\ \gamma &= -\tan(\theta/2) \end{aligned}\right\}$$

$\Rightarrow$ Any 2D rotation can be achieved using 3 consecutive shears with order

$$\begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ where } \alpha, \beta, \gamma \text{ have values}$$

4. b) Rotations that cannot be achieved by approach in 4 a) are rotations that do not occur at the origin (0, 0)