Fall 2014

Soon Chee Loong

CSC418 c4soonch

<u>Assignment 2</u>

1. Viewing & Projection

a) The parameters of the 4 lines defined by the points are

$$l_{12} = p_1 \times p_2$$

$$= \begin{bmatrix} 0 \\ 0.2828 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0.1818 \\ 0.5143 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2315 \\ 0.1818 \\ -0.0514 \end{bmatrix}$$

$$l_{23} = p_2 \times p_3$$

$$= \begin{bmatrix} 0.1818 \\ 0.5143 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0.0952 \\ 0.6734 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1592 \\ -0.0866 \\ 0.0735 \end{bmatrix}$$

$$l_{34} = p_3 \times p_4$$

$$= \begin{bmatrix} 0.0952 \\ 0.6734 \\ 1 \end{bmatrix} \times \begin{bmatrix} -0.1053 \\ 0.4466 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2268 \\ -0.2005 \\ 0.1134 \end{bmatrix}$$

$$l_{41} = p_4 \times p_1$$

$$= \begin{bmatrix} -0.1053 \\ 0.4466 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0.2828 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1638 \\ 0.1053 \\ -0.0298 \end{bmatrix}$$

let $z = 1$,

$\therefore$ The 2 vanishing points are the point at which l_{12} and l_{34} intersect and point at which l_{23} and l_{41} intersect.

First vanishing point,

$$l_{12} \times l_{34}$$

$$= \begin{bmatrix} -0.2315 \\ 0.1818 \\ -0.0514 \end{bmatrix} \times \begin{bmatrix} 0.2268 \\ -0.2005 \\ 0.1134 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0103 \\ 0.0146 \\ 0.0052 \end{bmatrix} \cong \begin{bmatrix} 1.9895 \\ 2.8161 \\ 1 \end{bmatrix}$$

Second Vanishing point,

$$l_{23} \times l_{41}$$

$$= \begin{bmatrix} -0.1591 \\ -0.0866 \\ 0.0735 \end{bmatrix} \times \begin{bmatrix} 0.1638 \\ 0.1053 \\ -0.0298 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0052 \\ 0.0073 \\ -0.0026 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2.0080 \\ -2.8407 \\ 1 \end{bmatrix}$$

$\therefore$ 2 vanishing points in 2D are

$$\begin{bmatrix} 1.9895 \\ 2.8161 \end{bmatrix} \text{ and } \begin{bmatrix} 2.0080 \\ -2.8407 \end{bmatrix}$$

b) Show two arbitrary parallel lines in 3D that are infinitely long, that they project onto the same vanishing point as the points on the line move toward infinity.

let Line 1 = $\vec{l_1}(u_1) = \vec{P_1} + u_1\vec{d}$

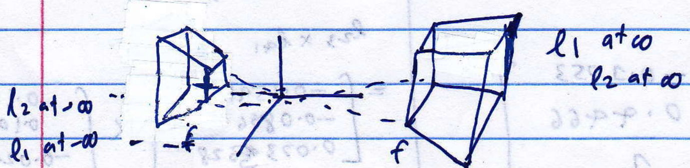
let Line 2 = $\vec{l_2}(u_2) = \vec{P_2} + u_2\vec{d}$

$\therefore$ The projected lines are

$$\vec{l_1}(u_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} P_1x + u_1d_x \\ P_1y + u_1d_y \\ P_1z + u_1d_z \\ 1 \end{bmatrix} = \begin{bmatrix} P_1x + u_1d_x \\ P_1y + u_1d_y \\ P_1z + u_1d_y \\ \frac{P_1z + u_1d_z}{f} \end{bmatrix} \cong \begin{bmatrix} \frac{f(P_1x + u_1d_x)}{P_1z + u_1d_z} \\ \frac{f(P_1y + u_1d_y)}{P_1z + u_1d_z} \\ f \\ 1 \end{bmatrix}$$

Similarly, $\vec{l_2}(u_2) = \begin{bmatrix} \frac{f(P_2x + u_2d_x)}{P_2z + u_2d_z} \\ \frac{f(P_2y + u_2d_y)}{P_2z + u_2d_z} \\ f \\ 1 \end{bmatrix}$

As the projection of these parallel lines intersect, it is independent of points P_1 and P_2. Since unprojected lines don't intersect, the projected intersection happens at $u_1 = \pm\infty$ and $u_2 = \pm\infty$. The sign depends on how the view frustrum is clipped.



$\therefore$ Projection intersection point is

$$\begin{bmatrix} \lim_{u_1 = \pm\infty} \frac{f(P_1x + u_1d_x)}{P_1z + u_1d_z} \\ \lim_{u_1 = \pm\infty} \frac{f(P_1y + u_1d_y)}{P_1z + u_1d_z} \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} \lim_{u_1 \to \pm\infty} \frac{fP_1x}{P_1z + u_1d_z} + \frac{fd_x}{\frac{P_1z}{u_1} + d_z} \\ \lim_{u_1 \to \pm\infty} \frac{fP_1y}{P_1z + u_1d_z} + \frac{fd_y}{\frac{P_1z}{u_1} + d_z} \\ f \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{fd_x}{d_z} \\ \frac{fd_y}{d_z} \\ f \\ 1 \end{bmatrix}$$

c) Let r_1 = ray from focal point of camera through one vanishing point on screen and r_2 = ray from focal point through the second vanishing point.

Are r_1 and r_2 orthogonal?

Let V_1 be the first vanishing point =

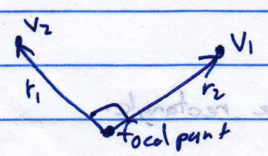
Let V_2 be the second vanishing point.

let f = focal point of camera

$$\Rightarrow r_1 = V_1 + (V_1 - f)t \quad \Rightarrow r_1 = \begin{bmatrix} 1.9895 \\ 2.8161 \end{bmatrix} + \begin{bmatrix} (1.9895 - f_x)t \\ (2.8161 - f_y)t \end{bmatrix}$$

$$r_2 = V_2 + (V_2 - f)t$$

$$\Rightarrow r_2 = \begin{bmatrix} 2.0080 \\ 2.8161 \end{bmatrix} + \begin{bmatrix} (2.0080 - f_x)t \\ (2.8161 - f_y)t \end{bmatrix}$$



$$r_1 = \begin{bmatrix} 1.9895 - f\frac{d_x}{d_z} \\ 2.8161 - f\frac{d_y}{d_z} \end{bmatrix} \cdot \begin{bmatrix} 2.0080 - f\frac{d_x}{d_z} \\ 2.8161 - f\frac{d_y}{d_z} \end{bmatrix}$$

Since r_1 and r_2 are formed by a quadrilateral,

$\therefore$ they must be orthogonal

Yes $r_1 \cdot r_2 = 0$ based on similar triangles

1. d) $\begin{bmatrix} x_1 \\ y_1 \\ -f \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ -f \end{bmatrix} = \begin{bmatrix} 1.9895 \\ 2.8161 \\ -f \end{bmatrix} \cdot \begin{bmatrix} 2.0080 \\ -2.8407 \\ -f \end{bmatrix} = 0$

$= 3.994916 - 7.99969527 + f^2 = 0$

$f^2 = 4.00477927$

$\Rightarrow f = \underline{\underline{2}}$

$\therefore$ Focal length is $\underline{\underline{2}}$

1. e) Find normal to the plane of the rectangle.

$(P_1) \times (P_2)$

$= \begin{bmatrix} 1.9895 \\ 2.8161 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2.0080 \\ -2.8407 \\ -2 \end{bmatrix}$

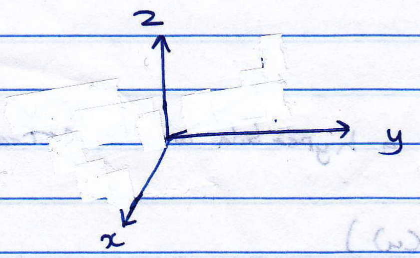
$= \underline{\underline{\begin{bmatrix} -11.3136 \\ -0.0370 \\ -11.3063 \end{bmatrix}}} = \vec{n}$

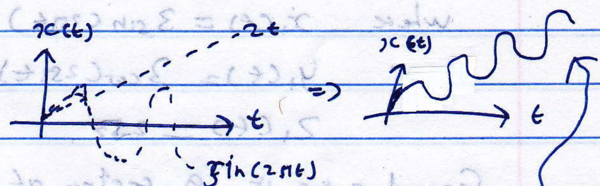
2. Surface of Revolution

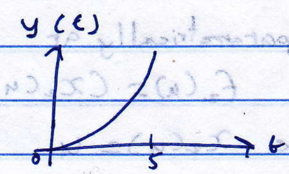
$x(t) = \sin(2\pi t) + 2t$

$y(t) = t^2 \quad , 0 \le t \le 5$

Form a surface of revolution by rotating curve about y-axis and give parametric formula to surface.







Each point of the curve traces out a circle parallel to the x-z plane.

The radius will be $|x(t)|$

$\bar{r}(\alpha, t) = (x(t) \qquad , \alpha \in [0, 2\pi)$

$$\text{Surface of revolution} = \left((\sin(2\pi t)+2t)\cos\theta,\ t^2,\ (\sin(2\pi t)+2t)\sin\theta \right)$$

$$0 \le t \le 5 \quad , 0 \le \theta \le 2\pi$$

Since radius corresponds to the x-value as seen in the diagram.

Rewriting,

surface of revolution $= \left((\sin(2\pi t)+2t)\cos\theta,\ t^2,\ (\sin(2\pi t)+2t)\sin\theta \right)$

where $0 \le t \le 5 \ , \ 0 \le \theta \le 2\pi$

3. Simple hyperboloid of one sheet is a 3D surface that can be define implicitly with $g(x,y,z)=0$

$$g(x,y,z)=x^2+y^2-z^2-1=0$$

$f_1(t) = (x_1(t), y_1(t), z_1(t))$

where $x_1(t) = 3\sin(2\pi t)$

$y_1(t) = 3\cos(2\pi t)$

$z_1(t) = 2\sqrt{2}$

Second curve is a section of part of a hyperbola on the surface, defined parametrically as

$$f_2(u) = (x_2(u), y_2(u), z_2(u))$$

Where $x_2(u) = u$

$y_2(u) = 0$

$z_2(u) = \sqrt{u^2-1}$

a) Compute values of t and u and the 3D position of the point where the curve intersect.

$g(x,y,z) = x^2+y^2-z^2-1=0$

Given 2 curves on surface, $f_1(t)$, $f_2(u)$

a) Find point of intersection:

$z_1(t) = z_2(u)$

$\Rightarrow 2\sqrt{2} = \sqrt{u^2-1} \Rightarrow u^2-1 = 4(2) = 8$

$\Rightarrow u^2 = 9$

$\Rightarrow u = \pm 3$ —— ①

$y_1(t) = y_2(u) \Rightarrow \cos(2\pi t) = 0 \Rightarrow 2\pi t = n\pi - \pi/2$, $n \in \mathbb{Z}$, n is an integer

$\Rightarrow t = \frac{n}{2} - \frac{1}{4}$

$x_1(t) = x_2(u) \Rightarrow 3\sin(2\pi t) = u \overset{\text{from ①}}{=} \pm 3$

$\Rightarrow \sin(2\pi t) = 1$ if $u=3 \Rightarrow t = n+\frac{1}{4}$, $n \in \mathbb{Z}$

$\sin(2\pi t) = -1$ if $u=-3 \Rightarrow t = n-\frac{1}{4}$, $n \in \mathbb{Z}$

$\therefore$ 2 point of intersection:

$P_1: u=3, t=n+\frac{1}{4}, (3, 0, 2\sqrt{2})$, $n \in \mathbb{Z}$

$P_2: u=-3, t=n-\frac{1}{4}, (-3, 0, 2\sqrt{2})$, $n \in \mathbb{Z}$

3. b) Compute the tangent of $f_1(t)$ and $f_2(u)$ of arbitrary t and u, and evaluate the tangents of each curve at the intersection point.

$f_1'(t) = (6\pi\cos(2\pi t), -6\pi\sin(2\pi t), 0)$

$f_2'(u) = \left(1, 0, \frac{u}{\sqrt{u^2-1}}\right)$

at P_1: $f_1' = (0, -6\pi, 0)$

$f_2' = \left(1, 0, \frac{3}{2\sqrt{2}}\right)$, $(3, 0, 2\sqrt{2}) = P_1$

at P_2: $f_1' = (0, 6\pi, 0)$

$f_2' = \left(1, 0, \frac{-3}{2\sqrt{2}}\right)$, $(-3, 0, 2\sqrt{2}) = P_2$

3. c) Take cross product of tangent computed in b and normalize it to get a unit normal at the intersection point. Choose normal that faces out from the surface.

Using P_1:

$$\vec{n} = \det\begin{bmatrix} i & j & k \\ 0 & -6\pi & 0 \\ 1 & 0 & 3/2\sqrt{2} \end{bmatrix} = \hat{i}\left(-\frac{9\pi}{\sqrt{2}}\right) + \hat{j}(0) + \hat{k}(6\pi)$$

$$= -\frac{9\pi}{\sqrt{2}}\hat{i} + 6\pi\hat{k}$$

$$\Rightarrow \|\vec{n}\| = \sqrt{\left(\frac{9\pi}{\sqrt{2}}\right)^2 + (6\pi)^2} = \sqrt{\frac{153\pi^2}{2}} = \frac{3\pi\sqrt{17}}{\sqrt{2}}$$

$$\Rightarrow \frac{\vec{n}}{\|\vec{n}\|} = \hat{n} = \frac{\sqrt{2}}{3\pi\sqrt{17}}\left(-\frac{9\pi}{\sqrt{2}}\hat{i} + 6\pi\hat{k}\right) = \frac{-3}{\sqrt{17}}\hat{i} + \frac{2\sqrt{2}}{\sqrt{17}}\hat{k}$$

Since it's at point $(3, 0, 2\sqrt{2})$, take the negative of this point out from the surface

$$\hat{n} = \frac{3}{\sqrt{17}}\hat{i} - \frac{2\sqrt{2}}{\sqrt{17}}\hat{k}$$

Similarly, using $P_2 \Rightarrow \vec{n}_2 = \hat{i}\left(-\frac{9\pi}{\sqrt{2}}\right) - 6\pi\hat{k}$ at point $(-3, 0, 2\sqrt{2})$, so pointing out of the surface. $\therefore$ $\hat{n}_2 = \frac{\vec{n}_2}{\|n_2\|} = \frac{-3}{\sqrt{17}}\hat{i} - \frac{2\sqrt{2}}{\sqrt{17}}\hat{k}$

3. d) Compute the normal at any point on the surface using implicit form of the surface.

$G(x,y,z) = x^2+y^2-\frac{z^2}{2}-1=0$

$\vec{n}$ is proportional to $\nabla G = (2x, 2y, -2z)$

$\Rightarrow \underline{\vec{n} = (2x, 2y, -2z)}$

3. e) Use the formula from d) to evaluate the normal at the point computed in a). Normalize the normal and compare with one computed in c).

At $P_1 = (x,y,z) = (3, 0, 2\sqrt{2})$

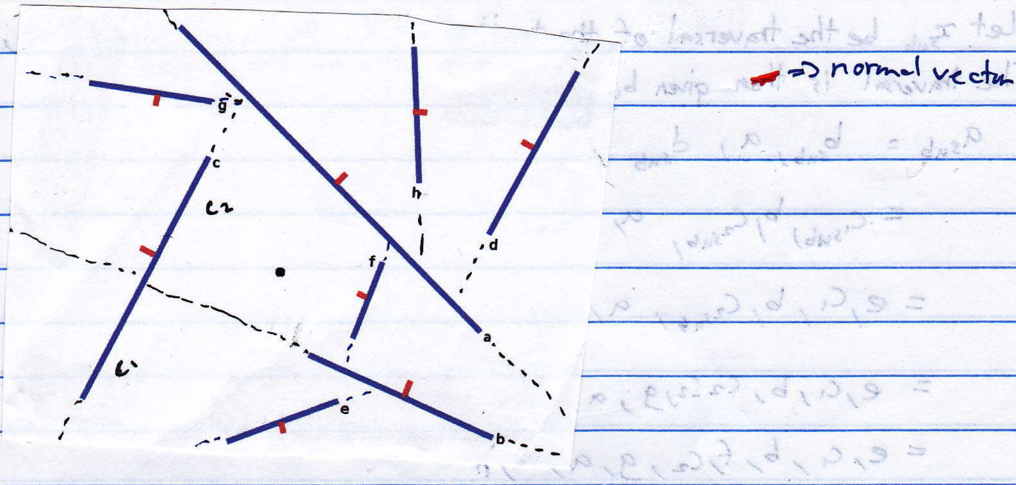
$\vec{n} = (6, 0, -4\sqrt{2})$, $\|\vec{n}\| = \sqrt{36+32} = \sqrt{68} = 2\sqrt{17}$

$\Rightarrow \hat{n}_1 = \frac{\vec{n}}{\|\vec{n}\|} = \left(\frac{3}{\sqrt{17}}, 0, \frac{-2\sqrt{2}}{\sqrt{17}} \right) = \frac{3}{\sqrt{17}}\hat{i} - \frac{2\sqrt{2}}{\sqrt{17}}\hat{k}$

At $P_2 = (x,y,z) = (-3, 0, 2\sqrt{2})$

$\Rightarrow \vec{n}_2 = \frac{-3}{\sqrt{17}}\hat{i} - \frac{2\sqrt{2}}{\sqrt{17}}\hat{k}$, which matches the previous result.

4.

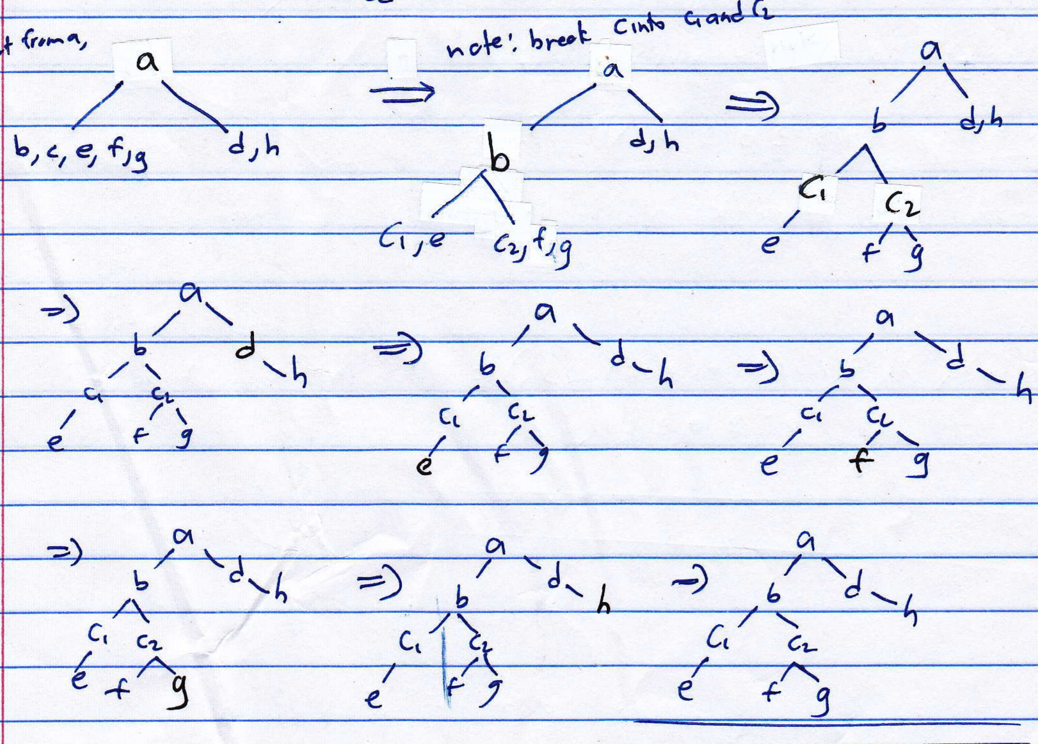


4. a) It is not possible to exclude any segments from rendering. This is because the height of the segments is not given. As the heights are not given the taller segments may still be visible although shorter segments may block them from the camera.

4. b) Let Left subtree = behind
Let Right subtree = in front

Start from a,

note: break c into c_1 and c_2



4. c) Let x_{sub} be the traversal of the subtree at x. The traversal draws back, current, then front in a inorder tree traversal.

The traversal is then given by

$$a_{sub} = b_{sub}, a, d_{sub}$$

$$= c_{1_{sub}}, b, c_{2_{sub}}, a, d_{sub}$$

$$= e, c_1, b, c_{2_{sub}}, a, d_{sub}$$

$$= e, c_1, b, f, c_2, g, a, d_{sub}$$

$$= e, c_1, b, f, c_2, g, a, d, h$$

$\therefore$ The traversal is given by

$$= \underline{e, c_1, b, f, c_2, g, a, d, h} \quad \text{(from left to right)}$$