Quantum Mechanics: Detailed Solution for Assignement 1
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1 Early Experimental Discoveries Unexplained by Classical Physics

Several key experiments revealed limitations in classical physics, which eventually led to the development of quantum mechanics. These discoveries include:

1.1 Blackbody Radiation

Classical physics, through the Rayleigh-Jeans law, predicted that the intensity of radiation emitted by a blackbody at thermal equilibrium should increase infinitely as the wavelength of the radiation decreases. This led to the so-called **ultraviolet catastrophe**, where classical theory suggested that a blackbody should emit an infinite amount of energy at short wavelengths, which was clearly not observed experimentally.

Quantum Explanation: In 1900, Max Planck introduced the concept that energy is quantized, meaning that it can only be emitted or absorbed in discrete amounts called "quanta" (or photons). He proposed that the energy E of these quanta is proportional to the frequency ν of radiation:

$$E = h\nu$$

where h is Plancks constant. This successfully explained the observed blackbody spectrum and avoided the ultraviolet catastrophe.

1.2 Photoelectric Effect

Classical wave theory predicted that the energy of light should depend on its intensity, not its frequency. However, experiments showed that light below a certain frequency, regardless of intensity, could not eject electrons from a metal surface. This was incompatible with classical physics.

Quantum Explanation: In 1905, Albert Einstein extended Plancks quantum theory by suggesting that light consists of particles, or **photons**, and that the energy of each photon is related to its frequency by $E = h\nu$. He proposed that for electrons to be ejected, the energy of an individual photon must be greater than the work function of the material. This explained why no electrons were ejected below a certain threshold frequency, regardless of light intensity.

1.3 Atomic Spectra

Classical physics could not explain why atoms emit light only at certain discrete wavelengths, producing **line spectra**. According to classical electromagnetism, an electron orbiting a nucleus should continuously radiate energy and spiral into the nucleus, but this does not occur.

Quantum Explanation: Niels Bohr proposed in 1913 that electrons exist in discrete energy levels or orbits around the nucleus, and that they can only transition between these levels by absorbing or emitting specific amounts of energy. The energy difference between two levels corresponds to the energy of the emitted or absorbed photon:

$$\Delta E = h\nu$$

This model explained the discrete lines in atomic spectra, especially the hydrogen spectrum, by quantizing the energy levels of electrons.

2 Postulates of Quantum Mechanics

Quantum mechanics is based on a set of fundamental principles or postulates, which describe the behavior of physical systems at the microscopic level:

2.1 State of a System Wavefunction (Ψ)

The state of any quantum mechanical system is fully described by a **wavefunction** $\Psi(x,t)$, which contains all the information about the system's properties and evolves over time according to the Schrdinger equation.

2.2 Probability Interpretation

The square of the absolute value of the wavefunction $|\Psi(x,t)|^2$ represents the **probability density** of finding a particle at a position x at time t. The total probability of finding the particle somewhere in space must be 1, meaning the wavefunction must be **normalized**:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

2.3 Operators and Observables

In quantum mechanics, physical quantities (such as momentum, energy, position) are represented by mathematical operators. An **observable** is any physical quantity that can be measured, and each observable is associated with a corresponding operator. For example, the momentum operator is:

$$\hat{p} = -i\hbar \frac{d}{dx}$$

where $\hbar$ is the reduced Plancks constant. The possible outcomes of measuring an observable are the **eigenvalues** of the corresponding operator. When an observable is measured, the system collapses into an **eigenstate** of the operator associated with that observable.

2.4 Superposition Principle

Quantum states can exist in **superpositions**, meaning a system can be in a combination of multiple states at once. This principle explains phenomena like interference and entanglement. Mathematically, if Ψ_1 and Ψ_2 are two possible states of a system, any linear combination $c_1\Psi_1 + c_2\Psi_2$ is also a valid state.

2.5 Evolution of a System Schrdinger Equation

The time evolution of the wavefunction is governed by the **Schrdinger equation**. For a particle moving in a potential $V(x)$, the time-dependent Schrdinger equation is:

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t)$$

This equation describes how the wavefunction changes with time and allows us to predict the systems future behavior.

3 Wavefunction in Quantum Mechanics

The **wavefunction** $\Psi(x,t)$ is a central concept in quantum mechanics, describing the quantum state of a system. It is a complex-valued function that contains all the information about a particle's position, momentum, and other observable quantities. The physical interpretation of the wavefunction is probabilistic: the square of the wavefunctions magnitude $|\Psi(x,t)|^2$ gives the probability density of finding the particle at a particular location at a given time.

3.1 Characteristics of Acceptable Wavefunctions

1. Normalizability: An acceptable wavefunction must be **normalizable**, meaning the total probability of finding the particle somewhere in space is 1.

Mathematically, this requires that:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

This ensures that the wavefunction corresponds to a physically meaningful state.

2. Continuity: The wavefunction must be **continuous** and smooth, except at points where there are infinite potential barriers. Both the wavefunction $\Psi(x,t)$ and its first derivative $\frac{d\Psi}{dx}$ must be continuous.

3. Finite Value: The wavefunction must remain **finite** at all points in space, as an infinite value would imply an infinite probability density, which is unphysical.

4. Single-Valued: The wavefunction must be **single-valued**, meaning that for each point in space and time, the wavefunction can only take on one value.

These characteristics ensure that the wavefunction describes a physically valid and meaningful quantum state. If any of these conditions are violated, the wavefunction does not correspond to a possible state of the system.