Evan's Conjecture

Let v be a b-bit word. Treat v as a binary number to get its value. For example, $\mathrm{val}(1, 1, 0, 0) = 1100_{\mathrm{TWO}} = 12$. We have $v \prec w$ exactly when $\mathrm{val}\, v < \mathrm{val}\, w$ and $\mathrm{val}\, \rho v < \mathrm{val}\, \rho w$.

- A *type I* transformation replaces a 0 in v with a 1.

  Example: $0010\underline{0}101 \rightarrow 0011\underline{1}101$.

- A *type H* transformation replaces a substring in v of the form[1] $011\ldots10$ with its complement, i.e., $100\ldots01$.

  Example: $00\underline{11001} \rightarrow 0\underline{100101}$.

Proposition 1. *Suppose $v, w \in W_b$ and w is derived from v by an I transformation, than $v \prec w$.*

Proof. Let $v = (x, 0, y)$ and $w = (x, 1, y)$. Then $\mathrm{val}\, w = \mathrm{val}\, v + 2^k > \mathrm{val}\, v$ for some k. Likewise, $\mathrm{val}\, \rho w = \mathrm{val}\, \rho v + 2^j > \mathrm{val}\, \rho v$ for some j. Therefore $v \prec w$. $\square$

Proposition 2. *Suppose $v, w \in W_b$ and w is derived from v by an H transformation, than $v \prec w$.*

Proof. Let $v = (x, 01\ldots10, y)$ and $w = (x, 10\ldots01, y)$. Note that $10\ldots01_{\mathrm{TWO}} > 01\ldots10_{\mathrm{TWO}}$ and therefore $\mathrm{val}\, w > \mathrm{val}\, v$. Likewise, $\mathrm{val}\, \rho w > \mathrm{val}\, \rho v$. Therefore $v \prec w$. $\square$

Corrolary 3. *Let $v, w \in W_b$. If w is obtained from v by a series of I and H transformations, then $v \preceq w$.* $\square$

The interesting part is the converse.

Theorem 4. *Let $v, w \in W_b$ with $v \prec w$. Then w is obtained from v by a sequence of I and H transformations.*

Proof. The proof is by strong induction on b. It is easy to verify for W_1, W_2, and W_3. Suppose the result has been shown for all W_i with $i < b$.

Let $v, w \in W_b$. We consider the first and last bits of both words, and break into cases. All told there are 16 possibilities: Four possibilities for v: $(0, \bar{v}, 0)$, $(0, \bar{v}, 1)$, $(1, \bar{v}, 0)$, and $(1, \bar{v}, 1)$. And likewise for w.

Cases as follows:

---

[1] The substring begins and ends with a 0 and has one or more consecutive 1s in between.

1. $v = (0, \bar{v}, 0)$ and $w = (0, \bar{w}, 0)$: Induction on $\bar{v} \prec \bar{w}$.

2. $v = (0, \bar{v}, 0)$ and $w = (0, \bar{w}, 1)$: Cannot happen because $\rho v = (0, \rho\bar{v}, 0) \not\prec (1, \rho\bar{w}, 0) = \rho w$.

3. $v = (0, \bar{v}, 0)$ and $w = (1, \bar{w}, 0)$:

4. $v = (0, \bar{v}, 0)$ and $w = (1, \bar{w}, 1)$:

---

5. $v = (0, \bar{v}, 1)$ and $w = (0, \bar{w}, 0)$:

6. $v = (0, \bar{v}, 1)$ and $w = (0, \bar{w}, 1)$:

7. $v = (0, \bar{v}, 1)$ and $w = (1, \bar{w}, 0)$:

8. $v = (0, \bar{v}, 1)$ and $w = (1, \bar{w}, 1)$:

---

9. $v = (1, \bar{v}, 0)$ and $w = (0, \bar{w}, 0)$:

10. $v = (1, \bar{v}, 0)$ and $w = (0, \bar{w}, 1)$:

11. $v = (1, \bar{v}, 0)$ and $w = (1, \bar{w}, 0)$:

12. $v = (1, \bar{v}, 0)$ and $w = (1, \bar{w}, 1)$:

---

13. $v = (1, \bar{v}, 1)$ and $w = (0, \bar{w}, 0)$:

14. $v = (1, \bar{v}, 1)$ and $w = (0, \bar{w}, 1)$:

15. $v = (1, \bar{v}, 1)$ and $w = (1, \bar{w}, 0)$:

16. $v = (1, \bar{v}, 1)$ and $w = (1, \bar{w}, 1)$:

$\square$

Recall that, in a poset, v is *covered by* w provided $v \prec w$ and there is no element x with $v \prec x \prec w$.

Corrolary 5. *For $v, w \in W_b$, v is covered by w if and only if w is obtained from v by a type I or type H transformation.* $\square$

In particular, if $w_1 \prec w_2 \prec \cdots \prec w_t$ is a maximum chain in W_b, then each w_{i+1} is obtained from w_i by a type I or a type H transformation.