Paul's Poset Poser

- For a positive integer b let W_b be the set of all length-b binary words.
- For $w \in W_b$, let $\operatorname{val} w$ be the integer represented in binary by w; that is

$$\operatorname{val} w = \sum_{i=1}^{b} w_i 2^{b-i}.$$

  For example, $\operatorname{val}(1,1,0,1) = 1101_{\text{TWO}} = 13$.

- For $v, w \in W_b$, we have $v < w$ exactly when $\operatorname{val} v < \operatorname{val} w$.
- For $w \in W_b$, let $\rho(w)$ be the *reversal* of w. That is

$$\rho(d_1, d_2, \ldots, d_b) = (d_b, d_{b-1}, \ldots, d_1).$$

- A *chain* is a sequence of words $w_1, w_2, \ldots, w_n \in W_b$ such that $w_1 < w_2 < \cdots < w_n$ and $\rho(w_1) < \rho(w_2) < \cdots < \rho(w_n)$.
- An *antichain* is a sequence of words $w_1, w_2, \ldots, w_n \in W_b$ such that $w_1 < w_2 < \cdots < w_n$ and $\rho(w_1) > \rho(w_2) > \cdots > \rho(w_n)$.
- Let $c(b)$ be the size of a maximum chain in W_b and $a(b)$ be the size of a maximum antichain in W_b.
- If v and w are binary words, vw is their concatenation.

Conjecture 1. *For a positive integer b,*

$$c(b) = \begin{cases} 3 \cdot 2^{b/2-1} & \text{if } b \text{ is even and} \\ 2^{(b+1)/2} & \text{if } b \text{ is odd.} \end{cases}$$

Conjecture 2. *For a positive integer b,* $a(b) = c(b) - 1$.

Proposition 3 (Chain lower bound)**.** *For a positive integer b,*

$$c(b) \geq \begin{cases} 3 \cdot 2^{b/2-1} & \textit{if } b \textit{ is even and} \\ 2^{(b+1)/2} & \textit{if } b \textit{ is odd.} \end{cases}$$

Proof sketch. Observe that if $w_1, w_2, \ldots, w_n$ is a chain in W_b, then

$$0w_10, 0w_20, \ldots, 0w_n0, 1w_11, 1w_21, \ldots 1w_n1$$

is a chain in W_{b+2} that is twice as long.

We see that $c(1) = 2$ because $0, 1$ is a chain and that $c(2) = 3$ because $00, 01, 11$ is a chain (and easy to check we cannot make a longer one).

The result now follows by induction on b. □

Alternative proof sketch for odd b. The palindromes of W_b form a chain. When b is odd, the number of palindromes is $2^{(b+1)/2}$. □