Paul's Poset Poser

- For a positive integer b let W_b be the set of all length-b binary words.
- For $w \in W_b$, let $\operatorname{val} w$ be the integer represented in binary by w; that is

  $$\operatorname{val} w = \sum_{i=1}^{b} w_i 2^{b-i}.$$

  For example, $\operatorname{val}(1,1,0,1) = 1101_{\text{TWO}} = 13$.
- For $w \in W_b$, let $\rho(w)$ be the *reversal* of w. That is

  $$\rho(d_1, d_2, \ldots, d_b) = (d_b, d_{b-1}, \ldots, d_1).$$
- For words $v, w \in W_b$, define $v \prec w$ provided $\operatorname{val} v < \operatorname{val} w$ and $\operatorname{val} \rho(v) < \operatorname{val} \rho(w)$. Note that $(W_b, \prec)$ is a poset.
- A *chain* is a subset of W_b containing pairwise comparable words and an *antichain* is a subset of W_b containing pairwise incomparable words.
- Let $c(b)$ be the size of a maximum chain in W_b and $a(b)$ be the size of a maximum antichain in W_b.
- If v and w are binary words, vw is their concatenation.

Conjecture 1. *For a positive integer b,*

$$c(b) = \begin{cases} 3 \cdot 2^{b/2-1} & \text{if } b \text{ is even and} \\ 2^{(b+1)/2} & \text{if } b \text{ is odd.} \end{cases}$$

Conjecture 2. *For a positive integer b, $a(b) = c(b) - 1$.*

Proposition 3 (Chain lower bound)**.** *For a positive integer b,*

$$c(b) \geq \begin{cases} 3 \cdot 2^{b/2-1} & \text{if } b \text{ is even and} \\ 2^{(b+1)/2} & \text{if } b \text{ is odd.} \end{cases}$$

Proof sketch. Observe that if $w_1 \prec w_2 \prec \cdots \prec w_n$ is a chain in W_b, then

$$0w_10 \prec 0w_20 \prec \cdots \prec 0w_n0 \prec 1w_11 \prec 1w_21 \prec \cdots \prec 1w_n1$$

is a chain in W_{b+2} that is twice as long.

We see that $c(1) = 2$ because $0 \prec 1$ is a chain in W_1 and that $c(2) = 3$ because $00 \prec 01 \prec 11$ is a chain in W_2 (and easy to check we cannot make a longer one).

The result now follows by induction on b. □

Alternative proof sketch for odd b. The palindromes of W_b form a chain. When b is odd, the number of palindromes is $2^{(b+1)/2}$. This argument doesn't work for even b. □