

## EVAN'S THEOREM

Let  $v$  be a  $b$ -bit word. Treat  $v$  as a binary number to get its value. For example,  $\text{val}(1, 1, 0, 0) = 1100_{\text{two}} = 12$ . We have  $v \prec w$  exactly when  $\text{val } v < \text{val } w$  and  $\text{val } \rho v < \text{val } \rho w$ .

- A *type I* transformation replaces a 0 in  $v$  with a 1.

Example:  $0010\underline{1}01 \rightarrow 001101$ .

- A *type H* transformation replaces a substring in  $v$  of the form<sup>1</sup> 011...10 with its complement, i.e., 100...01.

Example:  $0011001 \rightarrow 0100101$ .

**Proposition 1.** Suppose  $v, w \in W_b$  and  $w$  is derived from  $v$  by an *I* transformation, then  $v \prec w$ .

*Proof.* Let  $v = (x, 0, y)$  and  $w = (x, 1, y)$ . Then  $\text{val } w = \text{val } v + 2^k > \text{val } v$  for some  $k$ . Likewise,  $\text{val } \rho w = \text{val } \rho v + 2^j > \text{val } \rho v$  for some  $j$ . Therefore  $v \prec w$ .  $\square$

**Proposition 2.** Suppose  $v, w \in W_b$  and  $w$  is derived from  $v$  by an *H* transformation, then  $v \prec w$ .

*Proof.* Let  $v = (x, 01\dots10, y)$  and  $w = (x, 10\dots01, y)$ . Note that  $10\dots01_{\text{two}} > 01\dots10_{\text{two}}$  and therefore  $\text{val } w > \text{val } v$ . Likewise,  $\text{val } \rho w > \text{val } \rho v$ . Therefore  $v \prec w$ .  $\square$

**Corollary 3.** Let  $v, w \in W_b$ . If  $w$  is obtained from  $v$  by a series of *I* and *H* transformations, then  $v \preceq w$ .  $\square$

The interesting part is the converse.

**Theorem 4.** Let  $v, w \in W_b$  with  $v \prec w$ . Then  $w$  is obtained from  $v$  by a sequence of *I* and *H* transformations.

*Proof.* The proof is by strong induction on  $b$ . It is easy to verify for  $W_1$ ,  $W_2$ , and  $W_3$ . Suppose the result has been shown for all  $W_i$  with  $i < b$ .

Let  $v, w \in W_b$ . We consider the first and last bits of both words, and break into cases. All told there are 16 possibilities: Four possibilities for  $v$ :  $(0, \bar{v}, 0)$ ,  $(0, \bar{v}, 1)$ ,  $(1, \bar{v}, 0)$ , and  $(1, \bar{v}, 1)$ , and likewise for  $w$ .

Cases as follows:

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<sup>1</sup>The substring begins and ends with a 0 and has one or more consecutive 1s in between.

1.  $v = (0, \bar{v}, 0)$  and  $w = (0, \bar{w}, 0)$ : Induction on  $\bar{v} \prec \bar{w}$ .
  2.  $v = (0, \bar{v}, 0)$  and  $w = (0, \bar{w}, 1)$ : Induction on  $(\bar{v}, 0) \prec (\bar{w}, 1)$ .
  3.  $v = (0, \bar{v}, 0)$  and  $w = (1, \bar{w}, 0)$ : Induction on  $(0, \bar{v}) \prec (1, \bar{w})$ .
  4.  $v = (0, \bar{v}, 0)$  and  $w = (1, \bar{w}, 1)$ : Use I transformations to go from  $v = (0, \bar{v}, 0)$  to  $(0, 11\dots 1, 0)$ , then one H transformation to  $(1, 00\dots 0, 1)$ , and then I transformations to  $w = (1, \bar{w}, 1)$ .
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5.  $v = (0, \bar{v}, 1)$  and  $w = (0, \bar{w}, 0)$ : Contradiction:  $\text{val } \rho v > \text{val } \rho w$ .
  6.  $v = (0, \bar{v}, 1)$  and  $w = (0, \bar{w}, 1)$ : Induction on  $(\bar{v}, 1) \prec (\bar{w}, 1)$ .
  7.  $v = (0, \bar{v}, 1)$  and  $w = (1, \bar{w}, 0)$ : Contradiction:  $\text{val } \rho v > \text{val } \rho w$ .
  8.  $v = (0, \bar{v}, 1)$  and  $w = (1, \bar{w}, 1)$ : Induction on  $(0, \bar{v}) \prec (1, \bar{w})$ .
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9.  $v = (1, \bar{v}, 0)$  and  $w = (0, \bar{w}, 0)$ : Contradiction:  $\text{val } v > \text{val } w$ .
  10.  $v = (1, \bar{v}, 0)$  and  $w = (0, \bar{w}, 1)$ : Contradiction:  $\text{val } v > \text{val } w$ .
  11.  $v = (1, \bar{v}, 0)$  and  $w = (1, \bar{w}, 0)$ : Induction on  $(1, \bar{v}) \prec (1, \bar{w})$ .
  12.  $v = (1, \bar{v}, 0)$  and  $w = (1, \bar{w}, 1)$ : Induction on  $(1, \bar{v}) \prec (1, \bar{w})$ .
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13.  $v = (1, \bar{v}, 1)$  and  $w = (0, \bar{w}, 0)$ : Contradiction:  $\text{val } v > \text{val } w$ .
14.  $v = (1, \bar{v}, 1)$  and  $w = (0, \bar{w}, 1)$ : Contradiction:  $\text{val } v > \text{val } w$ .
15.  $v = (1, \bar{v}, 1)$  and  $w = (1, \bar{w}, 0)$ : Contradiction:  $\text{val } \rho v > \text{val } \rho w$ .
16.  $v = (1, \bar{v}, 1)$  and  $w = (1, \bar{w}, 1)$ : Induction on  $\bar{v} \prec \bar{w}$ .  $\square$

Recall that, in a poset,  $v$  is *covered by*  $w$  provided  $v \prec w$  and there is no element  $x$  with  $v \prec x \prec w$ .

**Corollary 5.** *For  $v, w \in W_b$ ,  $v$  is covered by  $w$  if and only if  $w$  is obtained from  $v$  by a type I or type H transformation.*  $\square$

In particular, if  $w_1 \prec w_2 \prec \dots \prec w_t$  is a maximum chain in  $W_b$ , then each  $w_{i+1}$  is obtained from  $w_i$  by a type I or a type H transformation.