

EVAN'S THEOREM

Let v be a b -bit word. Treat v as a binary number to get its value. For example, $\text{val}(1, 1, 0, 0) = 1100_{\text{TWO}} = 12$. We have $v \prec w$ exactly when $\text{val } v < \text{val } w$ and $\text{val } \rho v < \text{val } \rho w$.

- A *type I* transformation replaces a 0 in v with a 1.

Example: $0010101 \rightarrow 0011101$.

- A *type H* transformation replaces a substring in v of the form¹ $011 \dots 10$ with its complement, i.e., $100 \dots 01$.

Example: $0011001 \rightarrow 0100101$.

Proposition 1. Suppose $v, w \in W_b$ and w is derived from v by an *I* transformation, then $v \prec w$.

Proof. Let $v = (x, 0, y)$ and $w = (x, 1, y)$. Then $\text{val } w = \text{val } v + 2^k > \text{val } v$ for some k . Likewise, $\text{val } \rho w = \text{val } \rho v + 2^j > \text{val } \rho v$ for some j . Therefore $v \prec w$. \square

Proposition 2. Suppose $v, w \in W_b$ and w is derived from v by an *H* transformation, then $v \prec w$.

Proof. Let $v = (x, 01 \dots 10, y)$ and $w = (x, 10 \dots 01, y)$. Note that $10 \dots 01_{\text{TWO}} > 01 \dots 10_{\text{TWO}}$ and therefore $\text{val } w > \text{val } v$. Likewise, $\text{val } \rho w > \text{val } \rho v$. Therefore $v \prec w$. \square

Corollary 3. Let $v, w \in W_b$. If w is obtained from v by a series of *I* and *H* transformations, then $v \preceq w$. \square

The interesting part is the converse.

Theorem 4. Let $v, w \in W_b$ with $v \prec w$. Then w is obtained from v by a sequence of *I* and *H* transformations.

Proof. The proof is by strong induction on b . It is easy to verify for W_1 , W_2 , and W_3 . Suppose the result has been shown for all W_i with $i < b$.

Let $v, w \in W_b$. We consider the first and last bits of both words, and break into cases. All told there are 16 possibilities: Four possibilities for v : $(0, \bar{v}, 0)$, $(0, \bar{v}, 1)$, $(1, \bar{v}, 0)$, and $(1, \bar{v}, 1)$, and likewise for w .

Cases as follows:

¹The substring begins and ends with a 0 and has one or more consecutive 1s in between.

1. $v = (0, \bar{v}, 0)$ and $w = (0, \bar{w}, 0)$: Induction on $\bar{v} \prec \bar{w}$.
 2. $v = (0, \bar{v}, 0)$ and $w = (0, \bar{w}, 1)$: Induction on $(\bar{v}, 0) \prec (\bar{w}, 1)$.
 3. $v = (0, \bar{v}, 0)$ and $w = (1, \bar{w}, 0)$: Induction on $(0, \bar{v}) \prec (1, \bar{w})$.
 4. $v = (0, \bar{v}, 0)$ and $w = (1, \bar{w}, 1)$: Use I transformations to go from $v = (0, \bar{v}, 0)$ to $(0, 11 \dots 1, 0)$, then one H transformation to $(1, 00 \dots 0, 1)$, and then I transformations to $w = (1, \bar{w}, 1)$.
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5. $v = (0, \bar{v}, 1)$ and $w = (0, \bar{w}, 0)$: Contradiction: $\text{val } \rho v > \text{val } \rho w$.
 6. $v = (0, \bar{v}, 1)$ and $w = (0, \bar{w}, 1)$: Induction on $(\bar{v}, 1) \prec (\bar{w}, 1)$.
 7. $v = (0, \bar{v}, 1)$ and $w = (1, \bar{w}, 0)$: Contradiction: $\text{val } \rho v > \text{val } \rho w$.
 8. $v = (0, \bar{v}, 1)$ and $w = (1, \bar{w}, 1)$: Induction on $(0, \bar{v}) \prec (1, \bar{w})$.
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9. $v = (1, \bar{v}, 0)$ and $w = (0, \bar{w}, 0)$: Contradiction: $\text{val } v > \text{val } w$.
 10. $v = (1, \bar{v}, 0)$ and $w = (0, \bar{w}, 1)$: Contradiction: $\text{val } v > \text{val } w$.
 11. $v = (1, \bar{v}, 0)$ and $w = (1, \bar{w}, 0)$: Induction on $(1, \bar{v}) \prec (1, \bar{w})$.
 12. $v = (1, \bar{v}, 0)$ and $w = (1, \bar{w}, 1)$: Induction on $(1, \bar{v}) \prec (1, \bar{w})$.
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13. $v = (1, \bar{v}, 1)$ and $w = (0, \bar{w}, 0)$: Contradiction: $\text{val } v > \text{val } w$.
14. $v = (1, \bar{v}, 1)$ and $w = (0, \bar{w}, 1)$: Contradiction: $\text{val } v > \text{val } w$.
15. $v = (1, \bar{v}, 1)$ and $w = (1, \bar{w}, 0)$: Contradiction: $\text{val } \rho v > \text{val } \rho w$.
16. $v = (1, \bar{v}, 1)$ and $w = (1, \bar{w}, 1)$: Induction on $\bar{v} \prec \bar{w}$. □

Recall that, in a poset, v is *covered* by w provided $v \prec w$ and there is no element x with $v \prec x \prec w$.

Corollary 5. *For $v, w \in W_b$, v is covered by w if and only if w is obtained from v by a type I or type H transformation.* □

In particular, if $w_1 \prec w_2 \prec \dots \prec w_t$ is a maximum chain in W_b , then each w_{i+1} is obtained from w_i by a type I or a type H transformation.