Notes on Semi-Isomorphism

Let G and H be simple graphs with adjacency matrices A and B, respectively. We say that these graphs are *semi-isomorphic* provided there are permutation matrices P and Q such that $AP = QB$. Notation $G \cong_s H$.

We also recall that G and H are *fractionally isomorphic* if there is a doubly stochastic matrix S such that $AS = SB$. Notation: $G \cong_f H$.

Proposition 1. *Semi-isomorphism is an equivalence relation.*

Proof. It is easy to see that $\cong_s$ is reflexive: $AI = IA$.

Suppose $G \cong_s H$. Let A and B be the adjacency matrices, and let P and Q be permutation matrices with $AP = QB$. Taking the transpose of both sides gives

$$(AP)' = (QB)' \Rightarrow P'A = BQ' \Rightarrow BQ' = P'A$$

and so $H \cong_s G$. Therefore $\cong_s$ is symmetric.

Finally, suppose G, H, and K are graphs with $G \cong_s H \cong_s K$. Let A, B, and C be their adjacency matrices. We therefore have permutation matrices P, Q, R, and S such that

$$AP = QB \quad \text{and} \quad BR = SC.$$

The first gives $Q'AP = B$ and we substitute into the second to give $[Q'AP]R = SC$. Multiplying through by Q gives $A[PR] = [QS]C$, and so $G \cong_s K$. Therefore $\cong_s$ is transitive. $\square$

Proposition 2. *Let $n \geq 3$ be an integer. Then $C_n + C_n \cong_s C_{2n}$ if and only if n is odd.*

Proof. Let A_0 be the adjacency matrix of C_n. For example, for $n = 7$ we have

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Indexing vertices from 0, vertex k is adjacent to $k \pm 1 \pmod 2$.

and note that

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix}$$

is the adjacency matrix of $C_n + C_n$.

Now let

$$B = \begin{bmatrix} 0 & A_0 \\ A_0 & 0 \end{bmatrix}$$

and let H be the graph whose adjacency matrix is B.

Computational evidence only so far. This should be doable. $\square$

Proposition 3. *Let G and H be graphs. Then*

$$G \cong H \Rightarrow G \cong_s H \Rightarrow G \cong_f H$$

but neither reverse implication holds.

Proof. The first implication is trivial.

Suppose $G \cong_s H$. Let A and B be their adjacency matrices and let P and Q be permutation matrices with $AP = QB$. Transpose both sides to give $(AP)' = (QB)' \Rightarrow P'A = BQ' \Rightarrow P(P'A)Q = P(BQ')Q \Rightarrow AQ = PB$.

Add the equations $AP = QB$ and $AQ = PB$ to give $A(P + Q) = (P + Q)B$. Note that $S = \frac{1}{2}(P + Q)$ is doubly stochastic and that $AS = SB$. Therefore $G \cong_f H$.

Calculations show

$$\begin{array}{ccc} C_{10} \cong_s C_5 + C_5 & \text{but} & C_{10} \ncong C_5 + C_5 \\ C_{10} \cong_f C_4 + C_6 & \text{but} & C_{10} \ncong_s C_4 + C_6 \end{array}$$

and so neither reverse implication holds. $\square$

Conjecture 4. *Let G and H be graphs with adjacency matrices A and B, respectively. Suppose that $G \cong_f H$ with $AS = SB$ where S is a doubly stochastic matrix whose entries consist only of the values 0, $\frac{1}{2}$, and 1. Then $G \cong_s H$.*

This is false! Counterexample: $2C_4$ and C_8 are not semi-isomorphic but they are fractionally isomorphic. Let A and B be the adjacency matrices of these two graphs:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and let

$$S = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We have $AS = SB$.

Let G be a graph with adjacency matrix A. A *semi-automorphism* of G is a pair of permutation matrices (P, Q) such that $AP = QA$. The set of all semi-automorphisms of G is $\mathrm{Aut}_s(G)$.

Proposition 5. *If* $(P, Q) \in \mathrm{Aut}_s(G)$*, then* $(Q', P') \in \mathrm{Aut}_s(G)$.

Proof. $(P, Q) \in \mathrm{Aut}_s(G) \Rightarrow AP = QA \Rightarrow (AP)' = (QA)' \Rightarrow P'A = AQ' \Rightarrow AQ' = P'A \Rightarrow (Q', P') \in \mathrm{Aut}_s(G)$. □

Proposition 6. *If* $(P, Q) \in \mathrm{Aut}_s(G)$*, then* $(P', Q') \in \mathrm{Aut}_s(G)$.

Proof. $(P, Q) \in \mathrm{Aut}_s(G) \Rightarrow AP = QA \Rightarrow Q'AP = A \Rightarrow Q'A = AP' \Rightarrow AP' = Q'A \Rightarrow (P', Q') \in \mathrm{Aut}_s(G)$. □

Combining these two results gives this:

Proposition 7. *If* $(P, Q) \in \mathrm{Aut}_s(G)$*, then* $(Q, P) \in \mathrm{Aut}_s(G)$. □

Proposition 8. *If* $(P, Q), (X, Y) \in \mathrm{Aut}_s(G)$*, then* $(PX, QY) \in \mathrm{Aut}_s(G)$.

Proof. Since $(X, Y) \in \mathrm{Aut}_s(G)$, we also have $(X', Y') \in \mathrm{Aut}_s(G)$. This gives

$$AP = QA \quad \text{and} \quad AX' = Y'A$$

and so

$$Q'AP = A = YAX'.$$

Left multiply by Q and right multiply by X to give

$$Q[Q'AP]X = Q[YAX']X \Rightarrow A(PX) = (QY)A$$

and so $(PX, QY) \in \mathrm{Aut}_s(G)$. □

We now define an operation for $\mathrm{Aut}_s(G)$. For $(P, Q), (X, Y) \in \mathrm{Aut}_s(G)$, let

$$(P, Q) * (X, Y) = (PQ, XY).$$

Note that (I, I) is an identity element for this operation and the inverse of (P, Q) is (P', Q'). Therefore $\mathrm{Aut}_s(G)$, together with the operation $*$, is a group that we call the *semi-automorphism group* of the graph G.

Note that P is an automorphism of G iff $(P, P) \in \mathrm{Aut}_s(G)$. This gives the following:

Proposition 9. *The automorphism group* $\mathrm{Aut}(G)$ *of a graph G is (isomorphic to) a subgroup of its semi-automorphism group* $\mathrm{Aut}_s(G)$. □

Proposition 10. *Let G, H_1, H_2 be graphs with $G \cong_s H_1$ and $H_1 \cong H_2$. Then $G \cong_s H_2$.*

Proof. Let A, B_1, B_2 be the adjacency matrices of G, H_1, H_2, respectively. Then we have

$$AP = QB_1 \qquad \text{and} \qquad R'B_1R = B_2$$

where P, Q, R are permutation matrices. Rewrite the second equation as $B_1 = RB_2R'$ and substitute in the first to give

$$AP = Q[RB_2R'] \Rightarrow A(PR) = (QR)B_2$$

and so $G \cong_s H_2$. $\square$

Given a graph G with adjacency matrix A, we can find all graphs semi-isomorphic to G by considering all permutation matrices P and Q for which $Q'AP$ is a legitimate adjacency matrix. If G has n vertices, there are $(n!)^2$ possibilities to consider. However, it is sufficient to just consider matrices of the form AP (there are "only" $n!$ to consider) because if $B = AP$ is an adjacency matrix of a graph H, then $AP = IB$ and so $G \cong_s H$. Conversely, if $G \cong_s H$ with $AP = QB$, then $A(PQ') = QBQ'$. Note that QBQ' is the adjacency matrix of a graph isomorphic to H. Thus listing all adjacency matrices of the form AP will identify all graphs (up to isomorphism) that are semi-isomorphic to G.