Applied Mathematics Letters 96 (2019) 216–222

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Decay of the Kolmogorov N-width for wave problems

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ARTICLE INFO

Article history:
Received 21 March 2019
Received in revised form 7 May 2019
Accepted 7 May 2019
Available online 14 May 2019

Keywords:
Kolmogorov N-width
Wave equation

ABSTRACT

The Kolmogorov N-width $d_N(\mathcal{M})$ describes the rate of the worst-case error (w.r.t. a subset $\mathcal{M} \subset H$ of a normed space H) arising from a projection onto the best-possible linear subspace of H of dimension $N \in \mathbb{N}$. Thus, $d_N(\mathcal{M})$ sets a limit to any projection-based approximation such as determined by the reduced basis method. While it is known that $d_N(\mathcal{M})$ decays exponentially fast for many linear coercive parameterized partial differential equations, i.e., $d_N(\mathcal{M}) = \mathcal{O}(e^{-\beta N})$, we show in this note, that only $d_N(\mathcal{M}) = \mathcal{O}(N^{-1/2})$ for initial–boundary-value problems of the hyperbolic wave equation with discontinuous initial conditions. This is aligned with the known slow decay of $d_N(\mathcal{M})$ for the linear transport problem.

1. Introduction

The Kolmogorov N-width is a classical concept of (nonlinear) approximation theory as it describes the error arising from a projection onto the best-possible space of a given dimension $N \in \mathbb{N}$, [1]. This error is measured for a class $\mathcal{M}$ of objects in the sense that the *worst* error over $\mathcal{M}$ is considered. Here, we focus on subsets $\mathcal{M} \subset H$, where H is some Banach or Hilbert space with norm $\|\cdot\|_H$. Then, the Kolmogorov N-width is defined as

$$d_N(\mathcal{M}) := \inf_{V_N \subset H;\ \dim V_N = N} \sup_{u \in \mathcal{M}} \inf_{v_N \in V_N} \|u - v_N\|_H, \tag{1.1}$$

where V_N are linear subspaces. The corresponding approximation scheme is nonlinear as one is looking for the best possible linear space of dimension N. Due to the infimum, the decay of $d_N(\mathcal{M})$ as $N \to \infty$ sets a lower bound for the best possible approximation of all elements in $\mathcal{M}$ by a linear approximation in V_N.

Particular interest arises if the set $\mathcal{M}$ is chosen as a set of solutions of certain equations such as partial differential equations (PDEs), which is the reason why sometimes (even though slightly misleading) $\mathcal{M}$ is termed as 'solution manifold'. In that setting, one considers a *parameterized* PDE (PPDE) with a suitable solution u_μ and μ ranges over some parameter set $\mathcal{D}$, i.e., $\mathcal{M} \equiv \mathcal{M}(\mathcal{D}) := \{u_\mu : \mu \in \mathcal{D}\}$, where we will skip the dependence on $\mathcal{D}$ for notational convenience. As a consequence, the decay of the Kolmogorov N-width

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https://doi.org/10.1016/j.aml.2019.05.013
0893-9659/

is of particular interest for projection-based model reduction. There, given a PPDE and a parameter set $\mathcal{D}$, one wishes to construct a possibly optimal linear subspace V_N in an offline phase in order to highly efficiently compute a reduced approximation with N degrees of freedom (in V_N) in an online phase. For more details on projection-based model reduction and the reduced basis method, we refer the reader e.g. to the recent surveys [2–4].

It has been proven that for certain linear, coercive parameterized problems, the Kolmogorov N-width decays exponentially fast, i.e.,

$$d_N(\mathcal{M}) \le Ce^{-\beta N}$$

with some constants $C < \infty$ and $\beta > 0$, see e.g. [5,6]. This extremely fast decay is at the heart of any model reduction strategy (based upon a projection to V_N) since it allows us to choose a very moderate N to achieve small approximation errors. It is worth mentioning that this rate can in fact be achieved numerically by determining V_N by a greedy-type algorithm.

However, the situation dramatically changes when leaving the elliptic and parabolic realm. In fact, it has been proven in [6] that d_N decays for certain first-order linear transport problems at most with the rate $N^{-1/2}$. This in turn implies that projection-based approximation schemes for transport problems severely lack efficiency, [7,8]. In this note, we consider hyperbolic problems and show in a similar way as in [6] that

$$d_N(\mathcal{M}) \ge \tfrac{1}{4} N^{-1/2},$$

(Theorem 4.5) for an example of the second-order wave equation. In Section 2, we describe the Cauchy problem of a second-order wave equation with discontinuous initial conditions and review the distributional solution concept. Section 3 is devoted to the investigation of a corresponding initial–boundary value problem and Section 4 contains the proof of Theorem 4.5.

2. Distributional solution of the wave equation on $\mathbb{R}$

We start by considering the univariate wave equation on the spatial domain $\Omega := \mathbb{R}$ and on the time interval $I := \mathbb{R}^+$ (i.e., a Cauchy problem) for a real-valued parameter $\mu \ge 0$ with discontinuous initial values, i.e.,

$$\partial_{tt} u_\mu(t,x) - \mu^2 \cdot \partial_{xx} u_\mu(t,x) = 0 \quad \text{for} \quad (t,x) \in \Omega_I := I \times \Omega, \tag{2.1a}$$

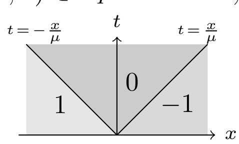
$$u_\mu(0,x) = u_0(x) := \begin{cases} 1, & \text{if } x < 0, \\ -1, & \text{if } x \ge 0, \end{cases} \quad x \in \Omega, \tag{2.1b}$$

$$\partial_t u_\mu(0,x) = 0, \quad x \in \Omega. \tag{2.1c}$$

This initial value problem has no classical solution, so that we consider a weak solution concept, namely we look for solutions in the distributional sense, which is known to be appropriate for hyperbolic problems.

Lemma 2.1. *Let $\mu \ge 0$. A distributional solution of (2.1) is given, for $(t,x) \in \Omega_I = \mathbb{R}^+ \times \mathbb{R}$, by*

$$u_\mu(t,x) = \begin{cases} 1, & \text{if } x < -\mu t, \\ -1, & \text{if } x \ge \mu t, \\ 0, & \text{else.} \end{cases}$$



Proof. We start by considering the following initial value problem

$$\begin{aligned} \partial_{tt} G_\mu(t,x) - \mu^2 \cdot \partial_{xx} G_\mu(t,x) = 0 \quad &\text{for} \quad (t,x) \in \Omega_I, \\ G_\mu(0,x) = 0, \quad \partial_t G_\mu(0,x) = \delta(x), \quad &x \in \Omega, \end{aligned} \tag{2.2}$$

where $\delta(\cdot)$ denotes Dirac's δ-distribution at 0. A solution G_μ of (2.2) is called *fundamental solution* (see e.g. [9, Ch. 5]) and can easily be seen to read $G_\mu(t,x) = \frac{1}{2\mu}\big(H(x+\mu t) - H(x-\mu t)\big)$, where $H(x) := \int_{-\infty}^{x} \delta(y)dy$ denotes the Heaviside step function with distributional derivative $H' = \delta$. Hence, the distributional derivative of G_μ w.r.t. t reads

$$\partial_t G_\mu(t,x) = \frac{1}{2}\big(\delta(x+\mu t) + \delta(x-\mu t)\big) \tag{2.3}$$

and it is obvious that $G_\mu(0,x) = 0$ as well as $\partial_t G_\mu(0,x) = \delta(x)$ for $x \in \mathbb{R}$. By using the properties of the Dirac's δ-distribution (see e.g. [10]) we observe that $\partial_{tt} G_\mu(t,x) = \frac{\mu}{2}\big(\delta(x+\mu t) - \delta(x-\mu t)\big)$ and $\partial_{xx} G_\mu(t,x) = \frac{1}{2\mu}\big(\delta(x+\mu t) - \delta(x-\mu t)\big)$ in the distributional sense. Hence, G_μ satisfies (2.2).

Now, we consider the original problem (2.1). To this end, the following relation of the fundamental solution G_μ of (2.2) and the solution u_μ of (2.1) is well-known [9],

$$u_\mu(t,x) = \int_{\mathbb{R}} \partial_t G_\mu(t,x-y) u_\mu(0,y)dy + \int_{\mathbb{R}} G_\mu(t,x-y)\partial_t u_\mu(0,y)dy.$$

Finally, inserting $\partial_t G_\mu$ from (2.3), the initial condition $u_\mu(0,\cdot) = u_0(\cdot)$ in $\mathbb{R}$, and the Neumann initial condition $\partial_t u_\mu(0,\cdot) = 0$ in $\mathbb{R}$, yields

$$\begin{aligned} u_\mu(t,x) &= \tfrac{1}{2} \int_{\mathbb{R}} \big(\delta(x-y+\mu t) + \delta(x-y-\mu t)\big) u_0(y)\, dy \\ &= \tfrac{1}{2}\Big[u_0(x+\mu t) + u_0(x-\mu t)\Big] = \begin{cases} 1, & \text{if } x < -\mu t, \\ -1, & \text{if } x \geq \mu t, \\ 0, & \text{else}, \end{cases} \end{aligned}$$

which proves the claim. □

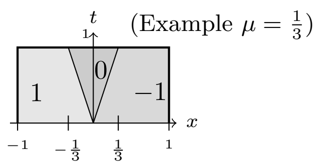
3. The wave equation on the interval

Let us consider the wave equation (2.1a), but now on the bounded space–time domain $\Omega_I := (0,1)\times(-1,1)$ with Dirichlet boundary conditions

$$u_\mu(t,-1) = 1, \quad u_\mu(t,1) = -1, \qquad \text{for} \quad t \in I := (0,1), \tag{2.1d}$$

and the initial conditions (2.1b),(2.1c). It is readily seen that the functions φ_μ defined by

$$\varphi_\mu(t,x) := \begin{cases} 1, & \text{if } x < -\mu t, \\ -1, & \text{if } x \geq \mu t, \\ 0, & \text{else}, \end{cases} \tag{3.1}$$



for $(t,x) \in \overline{\Omega}_I = [0,1]\times[-1,1]$ are contained in the solution manifold of ((2.1)a–d), i.e.,

$$\{\varphi_\mu : \mu \in \mathcal{D}\} \subset \mathcal{M} \equiv \mathcal{M}(\mathcal{D}) := \{u_\mu : \mu \in \mathcal{D} := [0,1]\} \subset L_2(\Omega_I). \tag{3.2}$$

In fact, by Lemma 2.1, φ_μ solves ((2.1)a–c) on $\mathbb{R}^+ \times \mathbb{R}$ and they also satisfy the boundary conditions (2.1d). The next step is the consideration of a specific family of functions to be defined now. For some $M \in \mathbb{N}$ and $1 \leq m \leq M$, let

$$\psi_{M,m}(t,x) := \begin{cases} 1, & \text{if } x \in \big[-\frac{m}{M}t, -\frac{m-1}{M}t\big), \\ -1, & \text{if } x \in \big[\frac{m-1}{M}t, \frac{m}{M}t\big), \\ 0, & \text{else}, \end{cases} \qquad \text{for} \quad (t,x) \in \bar{\Omega}_I, \tag{3.3}$$

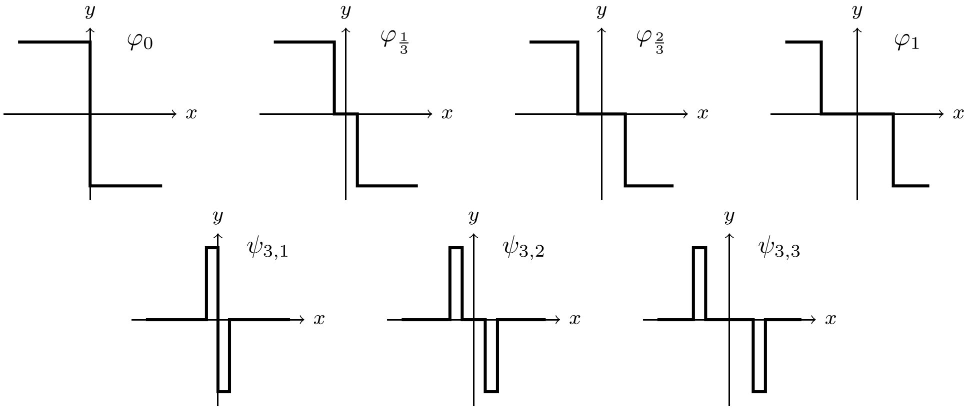


Fig. 1. Top: functions φ_μ for $\mu = 0, \frac{1}{3}, \frac{2}{3}, 1$. Bottom: functions $\psi_{M,m}$ for $M = 3$ and $m = 1, 2, 3$. All for $t = \frac{1}{2}$ fixed on $[-1, 1]$.

and we collect all $\psi_{M,m}$, $m = 1, \dots, M$ in

$$\Psi_M := \{\psi_{M,m} : 1 \le m \le M\}. \tag{3.4}$$

Note, that Ψ_M can be generated by

$$\Phi_M := \{\varphi_{\frac{m}{M}} : 0 \le m \le M\} \subset \{\varphi_\mu : \mu \in \mathcal{D}\}, \tag{3.5}$$

as follows $\psi_{M,m} = \varphi_{\frac{m-1}{M}} - \varphi_{\frac{m}{M}}$, $1 \le m \le M$, which in fact can be easily seen; see also Fig. 1. We will see later that $d_N(\Phi_M) \ge \frac{1}{2} d_N(\Psi_M)$. Moreover $\|\psi_{M,m}\|_{L_2(\Omega_I)} = \sqrt{1/M}$ and these functions are pairwise orthogonal, i.e.

$$\big(\psi_{M,m_1}, \psi_{M,m_2}\big)_{L_2(\Omega_I)} = \int_0^1 \int_{-1}^1 \psi_{M,m_1}(t,x)\, \psi_{M,m_2}(t,x)\, dx\, dt = \tfrac{1}{M}\, \delta_{m_1,m_2},$$

where δ_{m_1,m_2} denotes the Kronecker-δ for $m_1, m_2 \in \{1, \dots, M\}$. Thus,

$$\tilde{\Psi}_M := \{\tilde{\psi}_{M,m} : 1 \le m \le M\}, \qquad \tilde{\psi}_{M,m} := \sqrt{M}\, \psi_{M,m}, 1 \le m \le M, \tag{3.6}$$

is a set of orthonormal functions.

4. Kolmogorov N-width of sets of orthonormal elements

Let us start by introducing the notation $\mathcal{V}_N := \{V_N \subset H : \text{linear space with } \dim(V_N) = N\}$, so that the Kolmogorov N-width in (1.1) can be rephrased as

$$d_N(\mathcal{M}) := \inf_{V_N \in \mathcal{V}_N} \sup_{u \in \mathcal{M}} \inf_{v_N \in V_N} \|u - v_N\|_H.$$

We are going to determine either the exact value or lower bounds of $d_N(\mathcal{M})$ for certain sets of functions.

Lemma 4.1. *The canonical orthonormal basis $\{e_1, \dots, e_{2N}\}$ of $H := (\mathbb{R}^{2N}, \|\cdot\|_2)$ has the Kolmogorov N-width $d_N(\{e_1, \dots, e_{2N}\}) = \frac{1}{\sqrt{2}}$.*

Proof. Let $V_N = \{v = \sum_{j=1}^N a_j d_j \mid a_1, \dots, a_N \in \mathbb{R}\} \in \mathcal{V}_N$, with $\{d_1, \dots, d_N\}$ being an arbitrary set of orthonormal vectors in H. Thus, V_N is an arbitrary linear subspace of H of dimension N. Then, for any

$k \in \{1, \ldots, 2N\}$ and the canonical basis vector $e_k \in \mathbb{R}^{2N}$, we get

$$\sigma_{V_N}(k)^2 := \inf_{v \in V_N} \|e_k - v\|_2^2 = \|e_k - P_{V_N}(e_k)\|_2^2 = \left\| e_k - \sum_{j=1}^N (d_j)_k d_j \right\|_2^2,$$

where $P_{V_N}(e_k) = \sum_{j=1}^N \langle e_k, d_j \rangle d_j = \sum_{j=1}^N (d_j)_k d_j$ is the orthogonal projection of e_k onto V_N. Then,

$$\|P_{V_N}(e_k)\|_2^2 = \left\langle \sum_{j=1}^N (d_j)_k d_j, \sum_{l=1}^N (d_l)_k d_l \right\rangle = \sum_{j=1}^N (d_j)_k \left\langle d_j, \sum_{l=1}^N (d_l)_k d_l \right\rangle = \sum_{j=1}^N (d_j)_k^2.$$

Next, for $k \in \{1, \ldots, 2N\}$, we get,[1]

$$\begin{aligned} \sigma_{V_N}(k)^2 &= \|e_k - P_{V_N}(e_k)\|_2^2 = \|P_{V_N}(e_k)\|_2^2 - (P_{V_N}(e_k))_k^2 + \big(1 - (P_{V_N}(e_k))_k\big)^2 \\ &= \sum_{j=1}^N (d_j)_k^2 - \Big(\sum_{j=1}^N (d_j)_k^2\Big)^2 + 1 - 2\sum_{j=1}^N (d_j)_k^2 + \Big(\sum_{j=1}^N (d_j)_k^2\Big)^2 = 1 - \sum_{j=1}^N (d_j)_k^2. \end{aligned} \tag{4.1}$$

Let us now assume that

$$\sum_{j=1}^N (d_j)_k^2 > \frac{1}{2} \quad \text{for all} \quad k \in \{1, \ldots, 2N\}. \tag{4.2}$$

Then, we would have that

$$N = \sum_{j=1}^N \|d_j\|_2^2 = \sum_{j=1}^N \sum_{k=1}^{2N} (d_j)_k^2 = \sum_{k=1}^{2N} \sum_{j=1}^N (d_j)_k^2 > 2N \cdot \tfrac{1}{2} = N,$$

which is a contradiction, so that (4.2) must be wrong and we conclude that there exists a $k^* \in \{1, \ldots, 2N\}$ such that $\sum_{j=1}^N (d_j)_{k^*}^2 \le \frac{1}{2}$. This yields by (4.1) that $\sigma_{V_N}(k^*)^2 = 1 - \sum_{j=1}^N (d_j)_{k^*}^2 \ge \frac{1}{2}$. By using this k^*, this leads us to

$$d_N(\{e_1, \ldots, e_{2N}\}) = \inf_{V_N \in \mathcal{V}_N} \sup_{k \in \{1,\ldots,2N\}} \inf_{v \in V_N} \|e_k - v\|_2 \ge \inf_{V_N \in \mathcal{V}_N} \sigma_{V_N}(k^*) \ge \tfrac{1}{\sqrt{2}}.$$

To show equality, we consider $V_N := \operatorname{span}\{d_j : j = 1, \ldots, N\}$ generated by orthonormal vectors $d_j := \frac{1}{\sqrt{2}}(e_{2j-1} + e_{2j})$. Then, for any even $k \in \{2, 4, \ldots, 2N\}$ (and analogous for odd k) we get by (4.1) that

$$\sigma_{V_N}(k)^2 = 1 - \sum_{j=1}^N (d_j)_k^2 = 1 - \Big(\frac{1}{\sqrt{2}}(e_{k-1} + e_k)\Big)_k^2 = 1 - \Big(\frac{1}{\sqrt{2}}\Big)^2 = \frac{1}{2},$$

which proves the claim. $\square$

Remark 4.2. We note that, more general, for $k \in \mathbb{N}$, it holds that $d_N(\{e_1, \ldots, e_{kN}\}) = \sqrt{\frac{k-1}{k}}$, which can easily be proven following the above lines.

Having these preparations at hand, we can now estimate the Kolmogorov N-width for arbitrary orthonormal sets in Hilbert spaces.

Lemma 4.3. *Let H be an infinite-dimensional Hilbert space and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_{2N}\} \subset H$ any orthonormal set of size $2N$. Then, $d_N(\{\tilde{\psi}_1, \ldots, \tilde{\psi}_{2N}\}) = \frac{1}{\sqrt{2}}$.*

[1] We also refer to [11,12] where it was proven that $\|P\| = \|I - P\|$ for any idempotent operator $P \ne 0$, i.e., (4.1).

Proof. Since $V_N := \arg\inf_{V_N \in \mathcal{V}_N} \sup_{w \in \{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\}} \inf_{v \in V_N} \|w-v\|_H \subset \operatorname{span}\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\}$, we can consider the subspace $\operatorname{span}\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\} \subset H$ instead of the whole H. The space $\operatorname{span}\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\}$ with norm $\|\cdot\|_H$ can be isometrically mapped to $\mathbb{R}^{2N}$ with canonical orthonormal basis $\{e_1,\dots,e_{2N}\}$ and Euclidean norm $\|\cdot\|_2$. In fact, by defining the map $f : \operatorname{span}\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\} \to \mathbb{R}^{2N}$ with $f(v) := \sum_{i=1}^{2N} (v,\tilde{\psi}_i)_H\, e_i$ for $v, w \in \operatorname{span}\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\}$ we get

$$\begin{aligned}
\|f(w)-f(v)\|_2^2 &= \left\| \sum_{i=1}^{2N} (w-v,\tilde{\psi}_i)_H\, e_i \right\|_2^2 = \sum_{i=1}^{2N} (w-v,\tilde{\psi}_i)_H^2 \|e_i\|_2^2 = \sum_{i=1}^{2N} (w-v,\tilde{\psi}_i)_H^2 \\
&= \sum_{i=1}^{2N} (w-v,\tilde{\psi}_i)_H^2 \|\tilde{\psi}_i\|_H^2 = \left\| \sum_{i=1}^{2N} (w-v,\tilde{\psi}_i)_H\, \tilde{\psi}_i \right\|_H^2 = \|w-v\|_H^2.
\end{aligned}$$

Choosing $w = \tilde{\psi}_k, k \in \{1,\dots,2N\}$, we have $f(w) = \sum_{i=1}^{2N} (\tilde{\psi}_k,\tilde{\psi}_i)_H e_i = e_k$. Thus, Lemma 4.1, yields $d_N(\{\tilde{\psi}_1,\dots,\tilde{\psi}_{2N}\}) = d_N(\{e_1,\dots,e_{2N}\}) = \frac{1}{\sqrt{2}}$, which proves the claim. □

Proposition 4.4. *Let $\mathcal{M}$ be the solution manifold of ((2.1)a–d) in (3.2) and Φ_M, Ψ_M defined in (3.4), (3.5), $M \in \mathbb{N}$. Then, $d_N(\mathcal{M}) \geq d_N(\Phi_M) \geq \frac{1}{2} d_N(\Psi_M)$ for $N \in \mathbb{N}$ and $H = L_2(\Omega_I)$.*

Proof. By (3.2), we have $\Phi_M = \{\varphi_{\frac{m}{M}} : 0 \leq m \leq M\} \subset \{\varphi_\mu \mid \mu \in \mathcal{D}\} \subset \mathcal{M}$, so that the first inequality is immediate. For the proof of the second inequality, we use the abbreviation $\|\cdot\| = \|\cdot\|_{L_2(\Omega_I)}$. First, we denote some optimizing spaces and functions, $m \in \{m^*-1, m^*\}$

$$\begin{aligned}
V_N^{\Psi_M} &:= \arg \inf_{V_N \in \mathcal{V}_N} \sup_{\psi \in \Psi_M} \inf_{v \in V_N} \|\psi - v\|, & \psi_{M,m^*} &:= \arg \sup_{\psi \in \Psi_M} \inf_{v \in V_N^{\psi}} \|\psi - v\|, \\
V_N^{m} &:= \arg \inf_{V_N \in \mathcal{V}_N} \inf_{v \in V_N} \|\varphi_{\frac{m}{M}} - v\|, & v^{m} &:= \arg \inf_{v \in V_N^{m}} \|\varphi_{\frac{m}{M}} - v\|.
\end{aligned}$$

With those notations, we get

$$\begin{aligned}
d_N(\Psi_M) &= \inf_{V_N \in \mathcal{V}_N} \sup_{\psi \in \Psi_M} \inf_{v \in V_N} \|\psi - v\| = \inf_{v \in V_N^{\Psi_M}} \|\psi_{M,m^*} - v\| \\
&\leq \|\psi_{M,m^*} - (v^{m^*} - v^{m^*-1})\| = \|(\varphi_{\frac{m^*-1}{M}} - \varphi_{\frac{m^*}{M}}) - (v^{m^*} - v^{m^*-1})\| \\
&\leq \|\varphi_{\frac{m^*-1}{M}} - v^{m^*-1}\| + \|\varphi_{\frac{m^*}{M}} - v^{m^*}\| = \inf_{v \in V_N^{m^*-1}} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{v \in V_N^{m^*}} \|\varphi_{\frac{m^*}{M}} - v\| \\
&= \inf_{V_N \in \mathcal{V}_N} \inf_{v \in V_N} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{V_N \in \mathcal{V}_N} \inf_{v \in V_N} \|\varphi_{\frac{m^*}{M}} - v\| \leq \inf_{v \in W_N} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{v \in W_N} \|\varphi_{\frac{m^*}{M}} - v\|,
\end{aligned}$$

where $W_N := \arg\inf_{V_N \in \mathcal{V}_N} \big(\inf_{v \in V_N} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{v \in V_N} \|\varphi_{\frac{m^*}{M}} - v\| \big)$. This gives

$$\begin{aligned}
\inf_{v \in W_N} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{v \in W_N} \|\varphi_{\frac{m^*}{M}} - v\| &= \inf_{V_N \in \mathcal{V}_N} \Big(\inf_{v \in V_N} \|\varphi_{\frac{m^*-1}{M}} - v\| + \inf_{v \in V_N} \|\varphi_{\frac{m^*}{M}} - v\| \Big) \\
&\leq \inf_{V_N \in \mathcal{V}_N} \Big(2 \sup_{\varphi \in \Phi_M} \inf_{v \in V_N} \|\varphi - v\| \Big) = 2 \cdot d_N(\Phi_M),
\end{aligned}$$

which proves the second inequality. □

We can now prove the main result of this note.

Theorem 4.5. *For $\mathcal{M}$ being defined as in (3.2), we have that $d_N(\mathcal{M}) \geq \frac{1}{4} N^{-1/2}$ for $H = L_2(\Omega_I)$.*

Proof. Using Proposition 4.4 with $M = 2N$ (which in fact maximizes $d_N(\Psi_M)$) yields $d_N(\mathcal{M}) \geq d_N(\Phi_{2N}) \geq \frac{1}{2} \cdot d_N(\Psi_{2N})$. Since V_N is a linear space, we have

$$d_N(\Psi_{2N}) = d_N(\{\psi_{2N,n} : 1 \leq n \leq 2N\}) = \frac{1}{\sqrt{2N}}\, d_N(\{\sqrt{2N}\psi_{2N,n} : 1 \leq n \leq 2N\}) = \frac{1}{\sqrt{2N}}\, d_N(\tilde{\Psi}_{2N}).$$

Applying now Lemma 4.3 for the orthonormal functions previously defined in (3.6) gives $\frac{1}{2}\, d_N(\Psi_{2N}) = \frac{1}{2}\, \frac{1}{\sqrt{2N}} d_N(\tilde{\Psi}_{2N}) = \frac{1}{2} \frac{1}{\sqrt{2N}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{4} N^{-1/2}$, which completes the proof. $\square$

Theorem 4.5 shows the same decay of $d_N(\mathcal{M})$ as for linear advection problems, [6]. Thus, transport and hyperbolic parameterized problems are expected to admit a significantly slower decay as for certain elliptic and parabolic problems as mentioned in the introduction. We note, that this result is *not* limited to the specific discontinuous initial conditions (2.1b). In fact, also for continuous initial conditions with a smooth 'jump', one can construct similar orthogonal functions like (3.3) yielding the slow decay result.

Remark 4.6. (a) Replacing (2.1b), (2.1c) by $u_\mu(0, x) \equiv 0$ and $\partial_t u_\mu(0, x) = \delta(x)$, $x \in \Omega$, results in traveling Dirac deltas. Considering (2.2) on $\Omega_I := (0, 1) \times (-1, 1)$ with homogeneous Dirichlet boundary conditions and slightly adapting the above reasoning yields $d_N(\mathcal{M}) \geq \frac{1}{8}\, N^{-1/2}$ for $H = L_2(\Omega_I)$ in this case. (b) The above Hilbert space techniques cannot directly be used to estimate the N-width of $\{\partial_t u_\mu : \mu \in \mathcal{D}\}$ if Dirac deltas are present.

Remark 4.7. By considering $V_N := \mathrm{span}(\tilde{\Psi}_{N-1} \cup \{\varphi_1\})$, see (3.6) and (3.1), we can show that $\frac{1}{4} N^{-1/2} \leq d_N(\mathcal{M}) \leq \frac{1}{2}(N-1)^{-1/2}$. Our conjecture is that the latter estimate is in fact sharp, which will be subject to future research.

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