# Determinants by K-theory

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# Goal

**Goal of this talk:** If $N$ is a von Neumann algebra equipped with a trace $\tau$ (for example $B(H)$ with $H$ separable) we want to define a reasonable determinant on the group

$$\{g \in \mathrm{Inv}(N) : \tau(|g - \mathbb{1}|) < \infty\}.$$

$\mathrm{Inv}(N)$ denotes the set of invertible elements in $N$.

As it will turn out one can do this by using the formula

$$\det_\tau(g) := \exp(\tau(\log|g|)).$$

However, it is for example not obvious that this is a group homomorphism and that $\log|g|$ has finite trace.

# Relative pairs of Banach algebras

**Definition**

A **relative pair of Banach algebras** $(J, A)$ consists of the following data:

1. Banach algebras $(J, \|\cdot\|_J)$ and $(A, \|\cdot\|_A)$ with $A$ **unital** and $J \subset A$ an **ideal**
2. $\|j\|_A \leq \|j\|_J$ for $j \in J$
3. $\|ajb\|_J \leq \|a\|_A \|j\|_J \|b\|_A$ for $a, b \in A$ and $j \in J$

**Remark:** The ideal $J$ is **not** necessarily closed in the norm of $A$.

**Example**

The main motivation comes from the pair

$$(\mathscr{L}^1(H), B(H))$$

where $\mathscr{L}^1(H)$ denotes the Banach $*$-algebra of trace-class operators.

# K-theory

# Matrix algebras

Let $A$ be a **unital** Banach algebra. We then consider two matrix algebras over $A$:

1. $M_\infty(A)$: Infinite matrices over $A$ with only finitely many non-zero entries.
2. $\mathrm{GL}_\infty(A)$: Infinite matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & \mathbb{1}_\infty \end{pmatrix},$$

   where $g \in \mathrm{GL}_n(A)$ for some $n \in \mathbb{N}$.

Denote by $A^\sim$ the unitization of $A$. Recall that $A^\sim = A \oplus \mathbb{C}$ as vectorspace with multiplication given by

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu).$$

If $A$ has no unit, we define

$$\mathrm{GL}_n(A) := \{g \in \mathrm{GL}_n(A^\sim) : g - \mathbb{1}_n \in M_n(A)\},$$

where $\mathbb{1}_n$ is the identity of $\mathrm{GL}_n(A^\sim)$.

$M_\infty(A)$ is a **normed algebra**, where

$$\|x\|_{M_\infty(A)} := \sum_{i,j=1}^{\infty} \|x_{ij}\| .$$

For $g, h \in \mathrm{GL}_\infty(A)$ we have $g - h \in M_\infty(A)$ and hence we may define a metric

$$d(g, h) := \|g - h\|_{M_\infty(A)} ,$$

which makes $\mathrm{GL}_\infty(A)$ into a **topological group**.

By $\mathrm{GL}_\infty(A)_0$ we denote the **path component of the identity**.

# Algebraic and topological K-theory

Let $(J, A)$ be a relative pair of Banach algebras. One then defines the following abelian groups:

**Definition**

- $K_2^{\text{top}}(J, A) := \pi_1(\mathrm{GL}_\infty(J), \mathbb{1})$ with

$$\pi_1(\mathrm{GL}_\infty(J), \mathbb{1}) = \{ \ [\gamma : S^1 \to \mathrm{GL}_\infty(J)] : \gamma(1) = \mathbb{1} \ \}$$

  and $[\gamma]$ the homotopy class of $\gamma$.
- $K_1^{\text{top}}(J, A) := \mathrm{GL}_\infty(J)/\,\mathrm{GL}_\infty(J)_0$
- $K_1^{\text{alg}}(J, A) := \mathrm{GL}_\infty(J)/[\mathrm{GL}_\infty(A), \mathrm{GL}_\infty(J)]$

**Remark**

Topological K-theory does **not** depend on the Banach algebra $A$.

Define

$$R(J)_1 := \{\sigma : [0,1] \to \mathrm{GL}_\infty(J) : \sigma(0) = \mathbb{1}\}.$$

Let $\sim$ be the relation of **homotopy with fixed endpoints** and

$$F(J,A)_1 := \left\langle\ \sigma\tau\sigma^{-1}\tau^{-1} : \sigma \in R(J)_1,\ \tau \in R(A)_1\ \right\rangle \trianglelefteq R(J)_1.$$

With

$$q : R(J)_1 \twoheadrightarrow R(J)_1/\sim$$

the quotient map, we denote

$$F(J,A)_1/\sim\ := q(F(J,A)_1) \trianglelefteq R(J)_1/\sim\ .$$

**Definition**

$$\boxed{K_1^{\mathrm{rel}}(J,A) := (R(J)_1/\sim)/(F(J,A)_1/\sim)}$$

# An exact sequence in K-theory

# Comparison sequence

**Lemma**

*Let $(J, A)$ be a relative pair of Banach algebras. Then, there exists an exact sequence*

$$K_2^{\mathrm{top}}(J, A) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{q} K_1^{\mathrm{top}}(J, A) \longrightarrow 0,$$

*with morphisms*

- $\partial : K_2^{\mathrm{top}}(J, A) \to K_1^{\mathrm{rel}}(J, A) : [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})]$,
- $\theta : K_1^{\mathrm{rel}}(J, A) \to K_1^{\mathrm{alg}}(J, A) : [\sigma] \mapsto [\sigma(1)^{-1}]$,
- $q : K_1^{\mathrm{alg}}(J, A) \to K_1^{\mathrm{top}}(J, A) : [g] \mapsto [g]$.

# The relative Chern character

# Relative continuous cyclic homology

Let $(J, A)$ be a relative pair. Then there is a continuous boundary map

$$b : J \otimes_\pi A \to J : j \otimes a \mapsto ja - aj,$$

where $J \otimes_\pi A$ is the projective tensor product of $J$ and $A$.

**Definition**

We define the vectorspace

$$\mathrm{HC}_0(J, A) := J / \operatorname{Im}(b),$$

called the **relative zeroth continuous cyclic homology** of the pair $(J, A)$.

**Remark**

For any continuous operator

$$\tau : (J, \|\cdot\|_J) \to \mathbb{C} \text{ with } \forall a, b \in A : \ \tau(ab) = \tau(ba)$$

one has $\operatorname{Im}(b) \subset \ker(\tau)$ such that $\tau$ will descend to a map on $\mathrm{HC}_0(J, A)$.

# The relative Chern character

**Theorem**

*There exists a homomorphism*

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J, A),$$

*which is induced by*

$$R(J)_1 \ni \sigma \mapsto \mathrm{TR}\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right) \in J.$$

*By* TR *we mean the sum of all diagonal entries.*

**Remark**

One may assume that $\sigma$ is a smooth path with values in some $\mathrm{GL}_n(J)$. After passing to homotopy classes, one can always find such a representative.

# Semi-finite von Neumann algebras

**Definition**

A faithful, normal and semi-finite **trace** on a von Neumann algebra $N$ is a function

$$\tau : N_+ \to [0, \infty]$$

such that

- $\tau(x + y) = \tau(x) + \tau(y) \quad (x, y \in N_+)$
- $\tau(\lambda x) = \lambda\tau(x) \quad (x \in N_+,\ \lambda \geq 0)$
- **Tracial**: $\tau(x^*x) = \tau(xx^*) \quad (x \in N)$
- **Faithful**: $x = 0$ if $\tau(x) = 0$ $(x \geq 0)$
- **Normal**: $\tau(\sup x_i) = \sup_i \tau(x_i)$ for every bounded increasing net $\{x_i\} \subset N_+$
- **Semi-finite**: $\{x \in N_+ : \tau(x) < \infty\}$ is ultraweakly dense in $N_+$

**Definition**

If $N \subset B(H)$ is a von Neumann algebra and $\tau : N_+ \to [0, \infty]$ a faithful, normal semi-finite trace we call $(N, \tau)$ a semi-finite von Neumann algebra.

One then defines the **trace ideal**

$$\mathscr{L}^1_\tau(N) := \{x \in N : \tau(|x|) < \infty\}.$$

It is a $*$-ideal inside $N$ and a Banach $*$-algebra when equipped with the norm

$$\|x\|_\tau := \|x\|_{\mathrm{op}} + \|x\|_1 = \left( \sup_{\xi \in H, \|x\| \leq 1} \|x\xi\| \right) + \tau(|x|).$$

# The relative pair $(\mathscr{L}^1_\tau(N), N)$

**Lemma**

*The pair $(\mathscr{L}^1_\tau(N), N)$ is a relative pair of Banach algebras, where $\mathscr{L}^1_\tau(N)$ is equipped with the norm $\|\cdot\|_\tau = \|\cdot\|_{\mathrm{op}} + \|\cdot\|_1$.*

**Theorem**

*The first topological K-theory of the pair $(\mathscr{L}^1_\tau(N), N)$ is trivial, i.e.*

$$K_1^{\mathrm{top}}(\mathscr{L}^1_\tau(N), N) = \mathrm{GL}_\infty(\mathscr{L}^1_\tau(N))/\,\mathrm{GL}_\infty(\mathscr{L}^1_\tau(N))_0 = \{0\}.$$

# Construction of a determinant for semi-finite von Neumann algebras

In the following we will work with the relative pair $(J, A) := (\mathscr{L}^1_\tau(N), N)$ where $(N, \tau)$ is a semi-finite von Neumann algebra.

**Definition**

Since $\tau : J \to \mathbb{C}$ is a tracial bounded operator on $J$ it descends to a map on $\mathrm{HC}_0(J, A) = J/\operatorname{Im}(b)$. We now define $\tilde{\tau}$ by

$$\begin{tikzcd} K_1^{\mathrm{rel}}(J, A) \arrow[r, "\mathrm{ch}^{\mathrm{rel}}"] \arrow[rr, bend right, "\tilde{\tau}"'] & \mathrm{HC}_0(J, A) \arrow[r, "-\tau"] & \mathbb{C} \end{tikzcd}$$

**Remark**

From Bott periodicity it follows that $\tilde{\tau}(\operatorname{Im}\partial) \subset i\mathbb{R}$, where we recall that

$$\partial : K_2^{\mathrm{top}}(J, A) \to K_1^{\mathrm{rel}}(J, A) : [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})].$$

# The determinant

**Theorem**

*There exists a homomorphism*

$$\det_\tau : K_1^{\mathsf{alg}}(J, A) \to \mathbb{C}/(i\mathbb{R}),$$

*such that the following diagram commutes:*

$$\begin{tikzcd}
K_2^{\mathsf{top}}(J) \arrow[r, "\partial"] \arrow[d, "\tilde{\tau}\circ\partial"] & K_1^{\mathsf{rel}}(J, A) \arrow[r, two heads, "\theta"] \arrow[d, "\tilde{\tau}"] & K_1^{\mathsf{alg}}(J, A) \arrow[r] \arrow[d, "\det_\tau"] & 0 \\
i\mathbb{R} \arrow[r, tail] & \mathbb{C} \arrow[r, two heads] & \mathbb{C}/(i\mathbb{R}) &
\end{tikzcd}$$

*In particular,*

$$\det_\tau([g]) = -\tau(\mathsf{ch}^{\mathsf{rel}}([\sigma])) + i\mathbb{R},$$

*where* $[\sigma] \in K_1^{\mathsf{rel}}(J, A)$ *is any lift of* $[g] \in K_1^{\mathsf{alg}}(J, A)$.

# Properties

**Theorem**

*The determinant*

$$\det_\tau : \underbrace{\{g \in \mathsf{Inv}(N) : \tau(|g - \mathbb{1}|) < \infty\}}_{=\mathsf{GL}_1(J)=\mathsf{GL}_1(\mathscr{L}^1_\tau(N))} \to \mathbb{C}/(i\mathbb{R})$$

*satisfies*

- $\det_\tau(gh) = \det_\tau(g)\det_\tau(h)$ *for* $g, h \in \mathsf{GL}_1(J)$
- $\det(XgX^{-1}) = \det(g)$ *for* $g \in \mathsf{GL}_1(J)$ *and* $X \in \mathsf{Inv}(N)$*.*
- $\det(g) = \det(|g|)$ *for* $g \in \mathsf{GL}_1(J)$
- $\det(e^x) = \tau(x) + i\mathbb{R}$ *for* $x \in J$*.*

One can furthermore show that

$$\det_\tau(g) = \exp\left(\tau(\log|g|)\right) \qquad (g \in \mathsf{GL}_1(\mathscr{L}^1_\tau(N)))$$

after having applied the group isomorphism

$$\mathbb{C}/(i\mathbb{R}) \to (0, \infty) : z + i\mathbb{R} \mapsto e^{\Re(z)}$$