ECEN 611 Homework 4: Gap Function and Mutual Inductance for Salient Pole Rotor

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Table of Contents

Problem 3

Problem 3) Assume the steady state currents flowing in the conductors (consider sinusoidal winding distribution) of the device shown below are:

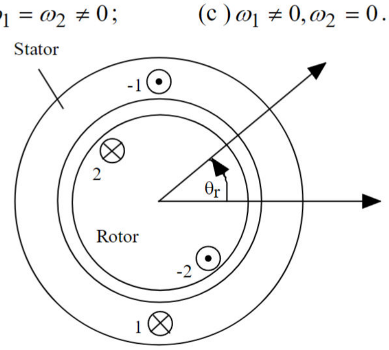
$$i_1 = I_{s1} \cos\omega_1 t \quad , \quad i_2 = I_{s2} \cos(\omega_2 t + \phi_2)$$

Assume also that during steady state operation the rotor speed is constant thus:

$$\theta_r = \omega_r t + \theta_r(0)$$

where $\theta_r(0)$ is the rotor displacement at time zero. Determine the rotor speeds at which the device produces a nonzero average torque during steady state operation if

(a) $\omega_1 = \omega_2 = 0$; (b) $\omega_1 = \omega_2 \neq 0$; (c) $\omega_1 \neq 0, \omega_2 = 0$.




```
syms I positive real
syms N_s

syms phi
syms theta_r
```

```
assume( 0 <= theta_r <= 2*pi )

NS = 1;

T = 2*pi;

digits(4)
```

Stator Counting & Winding Function

```
statorCountingFunction(phi) = piecewise( ...
    0 <= phi < pi/2, 0, ...
    pi/2 <= phi < 3/2*pi, -N_s, ...
    3/2*pi <= phi < T, 0 ...
);

disp("Original Stator Counting Function ϕ ∈ [0 T] :")
```

```
Original Stator Counting Function ϕ ∈ [0 T] :
```

```
disp(statorCountingFunction(phi))
```

$$\begin{cases} 0 & \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \\ -N_s & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\,\pi}{2}\right) \\ 0 & \text{if } \phi \in \left[\frac{3\,\pi}{2}, 2\,\pi\right) \end{cases}$$

```
statorCountingFunction_ext(phi) = piecewise( ...
    -2*T <= phi < -T, statorCountingFunction(phi+2*T), ...
    -T <= phi < 0, statorCountingFunction(phi+T), ...
    0 <= phi < T, statorCountingFunction(phi), ...
    T <= phi <= 2*T, statorCountingFunction(phi-T) ...
);

disp("Extended Stator Counting Function ϕ ∈ [-2T 2T] :")
```

```
Extended Stator Counting Function ϕ ∈ [-2T 2T] :
```

```
disp(statorCountingFunction_ext(phi))
```

$$\begin{cases} 0 & \text{if } \phi \in \left[-4\pi, -\frac{7\pi}{2}\right) \\ -N_s & \text{if } \phi \in \left[-\frac{7\pi}{2}, -\frac{5\pi}{2}\right) \\ 0 & \text{if } \phi \in \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \\ -N_s & \text{if } \phi \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \\ 0 & \text{if } \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ -N_s & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ 0 & \text{if } \phi \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right) \\ -N_s & \text{if } \phi \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right) \\ 0 & \text{if } \phi \in \left[\frac{7\pi}{2}, 4\pi\right) \end{cases}$$

```
statorCountingFunction_ext_avg = 1/T * int(statorCountingFunction_ext, phi, [0 T]);
statorWindingFunction_ext(phi) = statorCountingFunction_ext -
statorCountingFunction_ext_avg
```

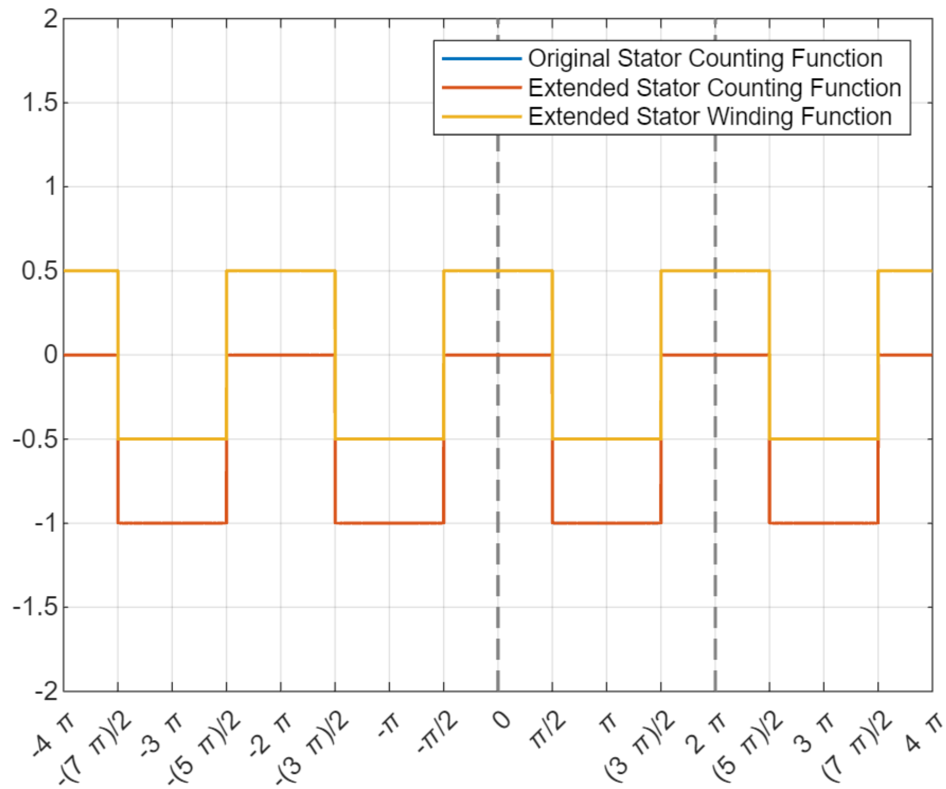
statorWindingFunction_ext(phi) =

$$\begin{cases} \frac{5734161139222659\,\pi\,N_s}{36028797018963968} & \text{if } \phi \in \left[-4\pi, -\frac{7\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} - N_s & \text{if } \phi \in \left[-\frac{7\pi}{2}, -\frac{5\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} & \text{if } \phi \in \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} - N_s & \text{if } \phi \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} & \text{if } \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} - N_s & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} & \text{if } \phi \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} - N_s & \text{if } \phi \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right) \\ \frac{5734161139222659\,\pi\,N_s}{36028797018963968} & \text{if } \phi \in \left[\frac{7\pi}{2}, 4\pi\right) \end{cases}$$

```
figure
fplot(subs(statorCountingFunction(phi), N_s, NS), [-2*T 2*T], ...
    "DisplayName", "Original Stator Counting Function", ...
```

```
    "LineWidth", 1.2)
hold on
fplot(subs(statorCountingFunction_ext(phi), N_s, NS), [-2*T 2*T], ...
    "DisplayName", "Extended Stator Counting Function", ...
    "LineWidth", 1.2)
fplot(subs(statorWindingFunction_ext(phi), N_s, NS), [-2*T 2*T], ...
    "DisplayName", "Extended Stator Winding Function", ...
    "LineWidth", 1.2)
hold off
ylim([-2 2])
grid on
legend

ax = gca;
S = sym(ax.XLim(1):pi/2:ax.XLim(2));
ax.XTick = double(S);
ax.XTickLabel = arrayfun(@texlabel,S,'UniformOutput',false);
```



Rotor Counting & Winding Function

```
syms N_r
NR = 1;

turnsRotorNum = [0      1     -1     0 ] * N_r;    % number of turns at each location:
replace Nt with 1 for simplicity
turnsRotorPhi = [0    T/4    T/2   T/4 ];  % angle of each location
```

```
turnsRotorNumLevel = cumsum(turnsRotorNum)
```

turnsRotorNumLevel = $\begin{pmatrix} 0 & N_r & 0 & 0 \end{pmatrix}$

```
rotor_winding_turning_phi = theta_r + cumsum(turnsRotorPhi)
```

rotor_winding_turning_phi =

$$\begin{pmatrix} \theta_r & \theta_r + \frac{\pi}{2} & \theta_r + \frac{3\,\pi}{2} & \theta_r + 2\,\pi \end{pmatrix}$$

```
rotorCountingFunction(phi) = turnsRotorNumLevel(1);

phiReference = 0;

for k = 1:length(rotor_winding_turning_phi)-1

    thisPhi = rotor_winding_turning_phi(k);
    nextPhi = rotor_winding_turning_phi(k+1);

    rotorCountingFunction(phi) = piecewise(thisPhi <= phi < nextPhi,
turnsRotorNumLevel(k), ...
                                           phiReference <= phi < thisPhi,
rotorCountingFunction);
end
```

Original Rotor Counting Function φ ∈ [θr θr+T]

```
disp("Original Rotor Counting Function: ");
```

Original Rotor Counting Function:

```
disp(rotorCountingFunction(phi));
```

$$\begin{cases} 0 & \text{if } 2\,\theta_r + 3\,\pi \leq 2\,\phi \wedge \phi < \theta_r + 2\,\pi \\ N_r & \text{if } 2\,\phi < 2\,\theta_r + 3\,\pi \wedge 2\,\theta_r + \pi \leq 2\,\phi \\ 0 & \text{if } 2\,\phi < 2\,\theta_r + \pi \wedge 0 \leq \phi \wedge (\phi < \theta_r \vee (\theta_r \leq \phi \wedge 2\,\phi < 2\,\theta_r + \pi)) \end{cases}$$

Extended Rotor Counting Function φ ∈ [θr-2T θr+2T]

```
rotorCountingFunction_ext(phi) = piecewise( ...
    theta_r - 2*T <= phi < theta_r - T, rotorCountingFunction(phi+2*T), ...
    theta_r - T <= phi < theta_r,       rotorCountingFunction(phi+T), ...
    theta_r <= phi < theta_r + T,       rotorCountingFunction(phi), ...
    theta_r + T <= phi < theta_r + 2*T, rotorCountingFunction(phi-T) ...
)
```

rotorCountingFunction_ext(phi) =

$$\begin{cases} 0 & \text{if } 2\,\theta_r \leq 2\,\phi + 5\,\pi \land \phi + 2\,\pi < \theta_r \\ N_r & \text{if } 2\,\phi + 5\,\pi < 2\,\theta_r \land 2\,\theta_r \leq 2\,\phi + 7\,\pi \land \phi \in \mathbb{R} \\ 0 & \text{if } (\phi < \theta_r \land 2\,\theta_r \leq 2\,\phi + \pi) \lor (2\,\phi + 7\,\pi < 2\,\theta_r \land \theta_r \leq \phi + 4\,\pi \land \phi \in \mathbb{R} \land (\phi + 4\,\pi < \theta_r \lor (2\,\phi + \\ N_r & \text{if } 2\,\theta_r \leq 2\,\phi + 3\,\pi \land \phi \in \mathbb{R} \land 2\,\phi + \pi < 2\,\theta_r \\ 0 & \text{if } (2\,\theta_r + 3\,\pi \leq 2\,\phi \land \phi < \theta_r + 2\,\pi) \lor (2\,\phi + 3\,\pi < 2\,\theta_r \land \theta_r \leq \phi + 2\,\pi \land \phi \in \mathbb{R} \land (\phi + 2\,\pi < \theta_r \lor \\ N_r & \text{if } 2\,\phi < 2\,\theta_r + 3\,\pi \land \phi \in \mathbb{R} \land 2\,\theta_r + \pi \leq 2\,\phi \\ 0 & \text{if } (2\,\theta_r + 7\,\pi \leq 2\,\phi \land \phi < \theta_r + 4\,\pi) \lor (\theta_r \leq \phi \land 2\,\phi < 2\,\theta_r + \pi \land (\phi < \theta_r \lor (\theta_r \leq \phi \land 2\,\phi < 2\,\theta_r + \\ N_r & \text{if } 2\,\phi < 2\,\theta_r + 7\,\pi \land 2\,\theta_r + 5\,\pi \leq 2\,\phi \land \phi \in \mathbb{R} \\ 0 & \text{if } 2\,\phi < 2\,\theta_r + 5\,\pi \land \theta_r + 2\,\pi \leq \phi \land \phi \in \mathbb{R} \land (\phi < \theta_r + 2\,\pi \lor (2\,\phi < 2\,\theta_r + 5\,\pi \land \theta_r + 2\,\pi \leq \phi)) \end{cases}$$

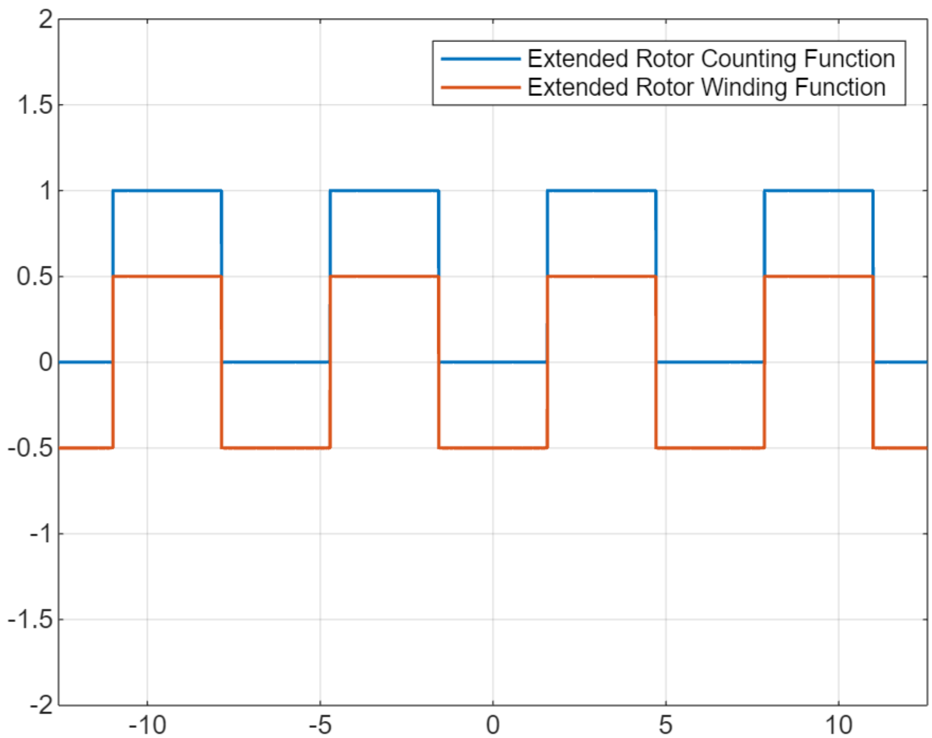
Extended Rotor Winding Function φ ∈ [θr-2T θr+2T]

```
% assume( theta < T )
rotorCountingFunction_ext_avg = 1/T * int(rotorCountingFunction_ext, phi, [0 T]);
rotorWindingFunction_ext = rotorCountingFunction_ext - rotorCountingFunction_ext_avg
```

```
rotorWindingFunction_ext(phi) =
```

$$
\begin{cases}
\frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\theta_r \leq 2\,\phi + 5\,\pi \land \theta_r \in \\
-\frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\theta_r \leq 2\,\phi + 5\,\pi \land (\theta_r \\
N_r - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} + \frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\phi + 5\,\pi < 2\,\theta_r \land 2\,\theta_r \\
N_r - \frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\phi + 5\,\pi < 2\,\theta_r \land 2\,\theta_r \\
\frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} & \text{if } \theta_r \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \land ((\phi < \\
-\frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } \left(\theta_r \in \left[\frac{3\pi}{2}, 2\,\pi\right] \lor \theta_r \right. \\
N_r - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} + \frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\theta_r \leq 2\,\phi + 3\,\pi \land \theta_r \in \\
N_r - \frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\theta_r \leq 2\,\phi + 3\,\pi \land (\theta_r \\
\frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} & \text{if } \theta_r \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \land ((2\,\theta_r \\
-\frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } \left(\theta_r \in \left[\frac{3\pi}{2}, 2\,\pi\right] \lor \theta_r \right. \\
N_r - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} + \frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 3\,\pi \land \theta_r \in \\
N_r - \frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 3\,\pi \land (\theta_r \\
\frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} & \text{if } \theta_r \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \land ((2\,\theta_r \\
-\frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } \left(\theta_r \in \left[\frac{3\pi}{2}, 2\,\pi\right] \lor \theta_r \right. \\
N_r - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} + \frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 7\,\pi \land 2\,\theta_r \\
N_r - \frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 7\,\pi \land 2\,\theta_r \\
\frac{5734161139222659\, N_r \left(\theta_r - \frac{3\pi}{2}\right)}{36028797018963968} - \frac{5734161139222659\, N_r \left(\theta_r - \frac{\pi}{2}\right)}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 5\,\pi \land \theta_r \in \\
-\frac{5734161139222659\, \pi\, N_r}{36028797018963968} & \text{if } 2\,\phi < 2\,\theta_r + 5\,\pi \land (\theta_r
\end{cases}
$$

```
figure
fplot(subs(rotorCountingFunction_ext(phi),[theta_r N_r], [0 NR]), [-2*T 2*T], ...
    "DisplayName", "Extended Rotor Counting Function", ...
    "LineWidth", 1.2)
hold on
fplot(subs(rotorWindingFunction_ext(phi),[theta_r N_r], [0 NR]), [-2*T 2*T], ...
    "DisplayName", "Extended Rotor Winding Function", ...
    "LineWidth", 1.2)
hold off
ylim([-2 2])
grid on
legend
```



Stator Winding Magnetizing Inductance

```
syms mu_o r l g
Lss = mu_o*r*l/g * int( statorWindingFunction_ext * statorWindingFunction_ext, ...
                    phi, [0 T]);

Lss_simplified = subs( simplify( vpa(Lss) ) )
```

Lss_simplified =

$$\frac{1.571\, N_s^2\, l\, \mu_o\, r}{g}$$

```
% Lss_simplified = subs( simplify( vpa(Lss) ), N_s^2*l*mu_o*r/g, 1 )
```

Rotor Winding Magnetizing Inductance

```
syms mu_o r l g

Lrr_integrand = subs(rotorWindingFunction_ext * rotorWindingFunction_ext, theta_r,
0:T/8:T);

tic
Lrr = vpa( simplify( mu_o*r*l/g * int( Lrr_integrand, phi, [0 T] ) ) )
```

Lrr =

$$\left(\frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g} \quad \frac{1.5}{}\right.$$

```
toc
```

Elapsed time is 1.354199 seconds.

```
Lrr = Lrr(1)
```

Lrr =

$$\frac{1.571\, N_r^{\,2}\, l\, \mu_o\, r}{g}$$

Rotor Winding Magnetizing Inductance is independent of rotor angle.

Stator-Rotor Winding Mutual Inductance

$$L_{BA} = \frac{\mu_0 r l}{g} \int_0^{2\pi} N_B(\phi) N_A(\phi) d\phi \qquad (1.75)$$

```
Lsr_integrand = simplify( vpa( rotorWindingFunction_ext *
statorWindingFunction_ext ) )
```

Lsr_integrand(phi) =

$$
\begin{cases}
-0.25\,N_r N_s & \text{if } \phi < -4.712 \wedge 1.571 < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 15.71 \\
0.25\,N_r N_s & \text{if } \phi + 6.283 < \theta_r \wedge \theta_r < 4.712 \wedge 2.0\,\theta_r \leq 2.0\,\phi + 15.71 \wedge \phi \in [-4.712, -1.571) \\
-0.25\,N_r N_s & \text{if } \phi + 6.283 < \theta_r \wedge \phi < -4.712 \wedge 2.0\,\theta_r \leq 2.0\,\phi + 15.71 \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571] \\
0.25\,N_r N_s & \text{if } \phi + 6.283 < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 15.71 \wedge \phi \in [-4.712, -1.571) \wedge (4.712 \leq \theta_r \vee \theta_r \in \\
-0.25\,N_r N_s & \text{if } \phi + 6.283 < \theta_r \wedge \phi \in [-1.571, 1.571) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \\
-0.25\,N_r N_s & \text{if } 1.571 < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 21.99 \wedge \phi < -7.854 \\
0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 21.99 \wedge 2.0\,\phi + 15.71 < 2.0\,\theta_r \wedge \theta_r \in (1.571, 4.712) \wedge \phi \in [-7.854, - \\
-0.25\,N_r N_s & \text{if } \theta_r < 4.712 \wedge 2.0\,\phi + 15.71 < 2.0\,\theta_r \wedge \phi \in [-4.712, -1.571) \\
-0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 21.99 \wedge \phi < -7.854 \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \\
0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 21.99 \wedge 2.0\,\phi + 15.71 < 2.0\,\theta_r \wedge \phi \in [-7.854, -4.712) \wedge (4.712 \leq \theta_r \\
-0.25\,N_r N_s & \text{if } 2.0\,\phi + 15.71 < 2.0\,\theta_r \wedge \phi \in [-4.712, -1.571) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [-12.57, -11.0) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \\
0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [-11.0, -7.854) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [-7.854, -4.712) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \\
0.25\,N_r N_s & \text{if } \phi \in [-4.712, -1.571) \wedge \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [-1.571, 1.571) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \\
0.25\,N_r N_s & \text{if } \phi \in [1.571, 4.712) \wedge \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [4.712, 7.854) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \\
0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [7.854, 11.0) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \phi \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge \phi \in [11.0, 12.57) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \vee (\theta_r \leq \phi \\
-0.25\,N_r N_s & \text{if } \phi \in [-12.57, -11.0) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.1 \\
0.25\,N_r N_s & \text{if } \phi \in [-11.0, -7.854) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.1 \\
-0.25\,N_r N_s & \text{if } \phi \in [-7.854, -4.712) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3. \\
0.25\,N_r N_s & \text{if } \phi \in [-4.712, -1.571) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3. \\
-0.25\,N_r N_s & \text{if } \phi \in [-1.571, 1.571) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.1 \\
0.25\,N_r N_s & \text{if } \phi \in [1.571, 4.712) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.14 \\
-0.25\,N_r N_s & \text{if } \phi \in [4.712, 7.854) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.14 \\
0.25\,N_r N_s & \text{if } \phi \in [7.854, 11.0) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \\
-0.25\,N_r N_s & \text{if } \phi \in [11.0, 12.57) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \wedge ((\phi < \theta_r \wedge 2.0\,\theta_r \leq 2.0\,\phi + 3.142) \\
-0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 9.425 \wedge \phi < -1.571 \wedge 1.571 < \theta_r \\
0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 9.425 \wedge 2.0\,\phi + 3.142 < 2.0\,\theta_r \wedge \theta_r \in (1.571, 4.712) \wedge \phi \in [-1.571, 1 \\
-0.25\,N_r N_s & \text{if } \theta_r < 4.712 \wedge 2.0\,\phi + 3.142 < 2.0\,\theta_r \wedge \phi \in [1.571, 4.712) \\
-0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 9.425 \wedge \phi < -1.571 \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \\
0.25\,N_r N_s & \text{if } 2.0\,\theta_r \leq 2.0\,\phi + 9.425 \wedge 2.0\,\phi + 3.142 < 2.0\,\theta_r \wedge \phi \in [-1.571, 1.571) \wedge (4.712 \leq \theta_r \vee \\
-0.25\,N_r N_s & \text{if } 2.0\,\phi + 3.142 < 2.0\,\theta_r \wedge \phi \in [1.571, 4.712) \wedge (4.712 \leq \theta_r \vee \theta_r \in [0, 1.571]) \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee (\theta_r \leq \phi + 6.283 \wedge 2 \\
0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee (\theta_r \leq \phi + 6.283 \wedge 2 \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee (\theta_r \leq \phi + 6.283 \wedge 2 \\
0.25\,N_r N_s & \text{if } \phi \in [-4.712, -1.571) \wedge \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0 \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee (\theta_r \leq \phi + 6.283 \wedge 2 \\
0.25\,N_r N_s & \text{if } \phi \in [1.571, 4.712) \wedge \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee \\
-0.25\,N_r N_s & \text{if } \theta_r \in (1.571, 4.712) \wedge ((\phi < \theta_r + 6.283 \wedge 2.0\,\theta_r + 9.425 \leq 2.0\,\phi) \vee (\theta_r \leq \phi + 6.283 \wedge 2
\end{cases}
$$

```
% Lsr_integrand = vpa( rotorWindingFunction_ext * statorWindingFunction_ext )
```

```
tic
Lsr = mu_o*r*l/g * int(Lsr_integrand, phi, [0 T]);
toc
```

```
Elapsed time is 22.924358 seconds.
```

```
Lsr_simplified = subs( simplify( vpa(Lsr) ), N_r*N_s*l*mu_o*r/g, 1 )
```

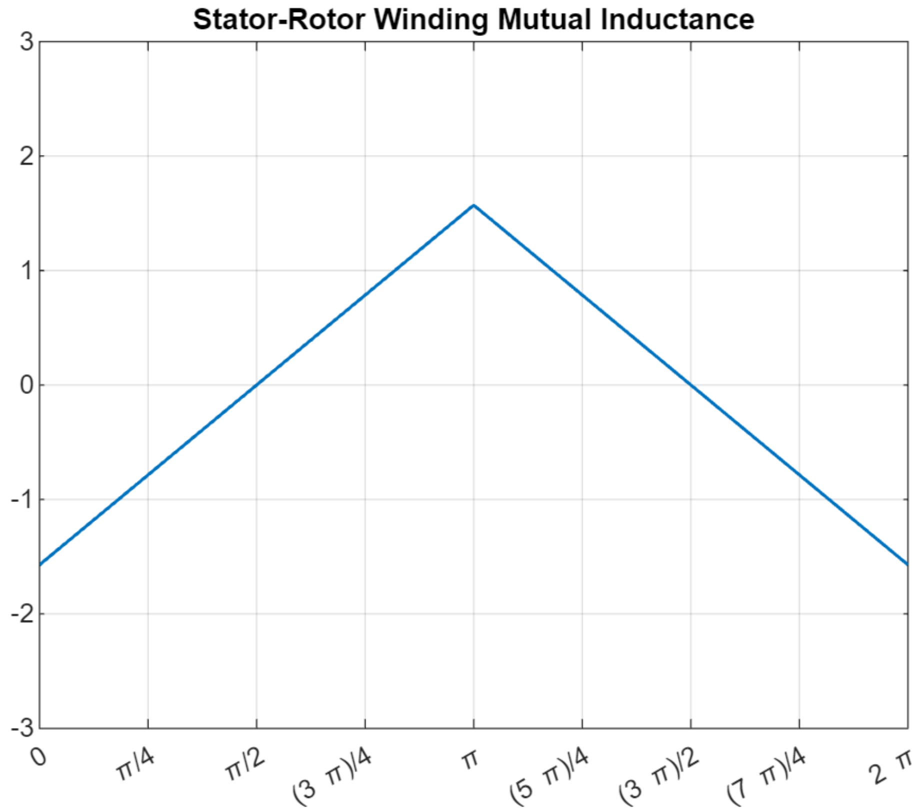
Lsr_simplified =

$$\begin{cases} -1.571 & \text{if } \theta_r = 0 \\ -9.183\text{e-}7 & \text{if } \theta_r = 4.712 \\ -3.673\text{e-}6 & \text{if } \theta_r = 4.712 \\ 1.571 & \text{if } \theta_r = 3.142 \\ 1.571 & \text{if } \theta_r = 3.142 \\ 2.356 - 0.5\,\theta_r & \text{if } \theta_r \in (4.712, 4.712] \\ 0.75\,\theta_r - 1.178 & \text{if } \theta_r \in [1.571, 1.571] \\ 1.571 & \text{if } \theta_r \in (3.142, 3.142) \\ 4.712 - 1.0\,\theta_r & \text{if } 4.712 < \theta_r \vee \theta_r \in (3.142, 4.712) \\ 1.0\,\theta_r - 1.571 & \text{if } \theta_r \in (0.0, 1.571) \vee \theta_r \in (1.571, 3.142) \end{cases}$$

```
figure

fplot(Lsr_simplified, [0 T], ...
    "LineWidth", 1.2)
ylim([-3 3])
grid on
title("Stator-Rotor Winding Mutual Inductance")

ax = gca;
S = sym(ax.XLim(1):pi/4:ax.XLim(2));
ax.XTick = double(S);
ax.XTickLabel = arrayfun(@texlabel,S,'UniformOutput',false);
```




```
% Fourier Series of Lsr
digits(4)

a0 = vpa( (1/T) * int( Lsr_simplified, theta_r, [0 T], ...
    'IgnoreSpecialCases', true) )
```

a0 = 6.047e-5

```
syms n

an = (2/T) * int( Lsr_simplified * cos(n * theta_r), theta_r, [0 T]);
bn = (2/T) * int( Lsr_simplified * sin(n * theta_r), theta_r, [0 T]);

orderOfHarmonics = 1:13;
an_13 = vpa( simplify( subs(an, n, orderOfHarmonics) ) );
bn_13 = vpa( simplify( subs(bn, n, orderOfHarmonics) ) );

tolerance = 1e-4;

an_13(abs(an_13) < tolerance) = 0
```

an_13 = $(-1.273 \quad 0 \quad -0.1415 \quad 0 \quad -0.05093 \quad 0 \quad -0.02599 \quad 0 \quad -0.01572 \quad 0 \quad -0.01052 \quad 0 \quad -0.007533)$

```
bn_13(abs(bn_13) < tolerance) = 0
```

bn_13 = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

```
Lsr_fourier_filtered = a0 + ...
    sum(an_13 .* cos(orderOfHarmonics .* theta_r) + ...
        bn_13 .* sin(orderOfHarmonics .* theta_r))
```

Lsr_fourier_filtered =

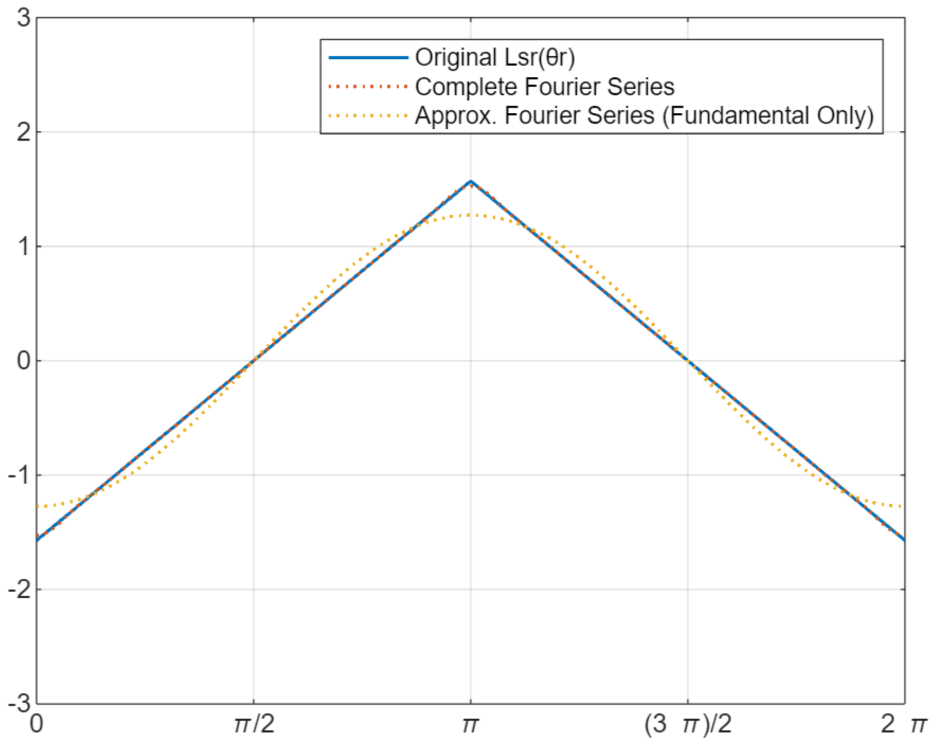
$6.047\mathrm{e}{-5} - 0.05093\cos(5\,\theta_r) - 0.02599\cos(7\,\theta_r) - 0.01572\cos(9\,\theta_r) - 0.01052\cos(11\,\theta_r) - 0.007533\cos$

```
Lsr_fourier_approx = an_13(1) * cos(orderOfHarmonics(1) * theta_r)
```

Lsr_fourier_approx = $-1.273\cos(\theta_r)$

```
figure
fplot(Lsr_simplified, [0 T], ...
      "DisplayName", "Original Lsr(θr)", ...
      "LineWidth", 1.2)
hold on
fplot(Lsr_fourier_filtered, [0 T], ...
      "DisplayName", "Complete Fourier Series", ...
      "LineStyle",":", ...
      "LineWidth", 1.2 ...
      )
fplot(Lsr_fourier_approx, [0 T], ...
      "DisplayName", "Approx. Fourier Series (Fundamental Only)", ...
      "LineStyle",":", ...
      "LineWidth", 1.2 ...
      )
hold off
ylim([-3 3])
grid on
legend

ax = gca;
S = sym(ax.XLim(1):pi/2:ax.XLim(2));
ax.XTick = double(S);
ax.XTickLabel = arrayfun(@texlabel,S,'UniformOutput',false);
```



Torque Production

```
syms I_s1 omega_1 I_s2 omega_2
syms t
syms phi_2 omega_r theta_r

magnitude = [I_s1 I_s2];
omega = [omega_1 omega_2];
angle = [ omega_1*t omega_2*t + phi_2 ];
current = magnitude .* cos(angle);
i1 = current(1)
```

i1 = $I_{s1}\cos(\omega_1 t)$

```
i2 = current(2)
```

i2 = $I_{s2}\cos(\phi_2 + \omega_2 t)$

Reluctance Torque

```
reluctanceTorque = 1/2 * (i1^2*diff(Lss,theta_r) + i2^2*diff(Lrr,theta_r))
```

reluctanceTorque = 0.0

Alignment Torque

```
disp(Lsr_fourier_approx)
```

$-1.273 \cos(\theta_r)$

```
alignmentTorque = i1 * i2 * diff(Lsr_fourier_approx, theta_r)
```

alignmentTorque = $1.273\, I_{s1}\, I_{s2} \cos(\phi_2 + \omega_2\, t) \cos(\omega_1\, t) \sin(\theta_r)$

Total Torque

```
totalTorque = simplify( reluctanceTorque + alignmentTorque )
```

totalTorque = $1.273\, I_{s1}\, I_{s2} \cos(\phi_2 + \omega_2\, t) \cos(\omega_1\, t) \sin(\theta_r)$

```
syms omega_r omega_r0 omega_m delta
totalTorque = subs(totalTorque, theta_r, omega_m*t + delta)
```

totalTorque = $1.273\, I_{s1}\, I_{s2} \cos(\phi_2 + \omega_2\, t) \sin(\delta + \omega_m\, t) \cos(\omega_1\, t)$

$$T = -I_{sm} I_{rm} M \cos\omega_s t\ \cos(\omega_r t + \alpha)\ \sin(\omega_m t + \delta)$$

$$T = -\frac{I_{\mathrm{sm}} I_{\mathrm{rm}} M}{4} \begin{bmatrix} \sin\left\{(\omega_{\mathrm{m}} + (\omega_{\mathrm{s}} + \omega_{\mathrm{r}}))t + \alpha + \delta\right\} + \\ \sin\left\{(\omega_{\mathrm{m}} - (\omega_{\mathrm{s}} + \omega_{\mathrm{r}}))t - \alpha + \delta\right\} + \\ \sin\left\{(\omega_{\mathrm{m}} + (\omega_{\mathrm{s}} - \omega_{\mathrm{r}}))t - \alpha + \delta\right\} + \\ \sin\left\{(\omega_{\mathrm{m}} - (\omega_{\mathrm{s}} - \omega_{\mathrm{r}}))t + \alpha + \delta\right\} \end{bmatrix}$$

Given the transformation above, it is clear that the average value of each term is zero unless the coefficient of t is zero.

Unfortunately, MATLAB seems incapable of performing such a transformation using Product-to-Sum Trigonometric Identities

cos(A)cos(B)=1/2�[cos(A+B)+cos(A−B)]

sin(A)cos(B)=1/2�[sin(A+B)+sin(A−B)]

as you can see if you uncomment the following code.

```
% expanded_totalTorque = expand(totalTorque)
% rewritten_totalTorque = rewrite(expanded_totalTorque, 'sin')
% simplified_totalTorque = simplify(rewritten_totalTorque)
% final_totalTorque = expand(simplified_totalTorque)
```

Although, MATLAB can do this the other way round:

```
syms A B
ccSum = 1/2*(cos(A+B)+cos(A-B))
```

ccSum =

$$\frac{\cos(A+B)}{2}+\frac{\cos(A-B)}{2}$$

```
ccProduct = simplify(ccSum)
```

ccProduct = $\cos(A)\cos(B)$

```
ssSum = 1/2*(sin(A+B)+sin(A-B))
```

ssSum =

$$\frac{\sin(A+B)}{2}+\frac{\sin(A-B)}{2}$$

```
ssProduct = simplify(ssSum)
```

ssProduct = $\cos(B)\sin(A)$

Well, let us realize it manually:

```
% Define the four sine terms
term1 = sin((omega_m + (omega_1 + omega_2))*t + phi_2 + delta);
term2 = sin((omega_m - (omega_1 + omega_2))*t - phi_2 + delta);
term3 = sin((omega_m + (omega_1 - omega_2))*t - phi_2 + delta);
term4 = sin((omega_m - (omega_1 - omega_2))*t + phi_2 + delta);

% Combine them with the coefficient
totalTorque_manual = (1.273 * I_s1 * I_s2 / 4) * (term1 + term2 + term3 + term4);

% Simplify and verify
simplify(vpa(totalTorque_manual))
```

ans = $1.273\, I_{s1}\, I_{s2} \cos(\phi_2+\omega_2\, t)\sin(\delta+\omega_m\, t)\cos(\omega_1\, t)$

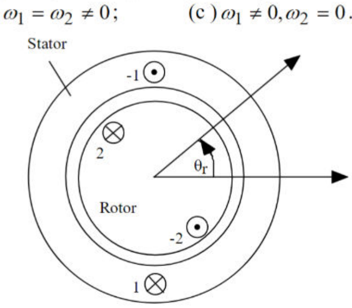
```
vpa(totalTorque)
```

ans = $1.273\, I_{s1}\, I_{s2} \cos(\phi_2+\omega_2\, t)\sin(\delta+\omega_m\, t)\cos(\omega_1\, t)$

Rotor Speed ωr

where $\theta_r(0)$ is the rotor displacement at time zero. Determine the rotor speeds at which the device produces a nonzero average torque during steady state operation if

(a) $\omega_1=\omega_2=0$; (b) $\omega_1=\omega_2\neq 0$; (c) $\omega_1\neq 0, \omega_2=0$.



Determine the rotor speeds at which the device produces a nonzero average torque during steady state operation if

(a) When ω1 = ω2 = 0

DC current supply. Single phase machine.

```
totalTorque_case_a = subs(totalTorque_manual, omega, [0 0])
```

totalTorque_case_a =

$$\frac{1273\, I_{s1}\, I_{s2}\, (2\sin(\delta - \phi_2 + \omega_m\, t) + 2\sin(\delta + \phi_2 + \omega_m\, t))}{4000}$$

```
totalTorque_case_a = subs(totalTorque_case_a, omega_m, 0)
```

totalTorque_case_a =

$$\frac{1273\, I_{s1}\, I_{s2}\, (2\sin(\delta - \phi_2) + 2\sin(\delta + \phi_2))}{4000}$$

```
% totalTorque_case_a_avg = 1/T * int( ...
%     subs(totalTorque_case_a, omega_r*t, theta_r), ...
%     theta_r, [0 T])
```

Only when $\omega_m = 0$ can the device produce a nonzero average torque.

(b) When ω1 = ω2 ≠ 0

Let's assume ω1 = ω2 = ω1

```
totalTorque_case_b = subs(totalTorque_manual, omega(2), omega_1)
```

totalTorque_case_b =

$$\frac{1273\, I_{s1}\, I_{s2}\, (\sin(\delta + \phi_2 + t\, (2\,\omega_1 + \omega_m)) + \sin(\delta - \phi_2 + \omega_m\, t) - \sin(\phi_2 - \delta + t\, (2\,\omega_1 - \omega_m)) + \sin(\delta + \phi_2 +}{4000}$$

```
% totalTorque_case_b_avg = simplify( ...
%     expand( ...
%         vpa( 1/T * int(totalTorque_case_b, t, [0 T/omega_m]) ) ...
%     ) ...
% );
```

Either

$$2\,\omega_1 + \omega_m = 0 \Rightarrow \omega_m = -2\omega_1$$

$$\text{and}$$

$$\delta - \phi_2 \neq 0 \Rightarrow \delta \neq \phi_2$$

```
totalTorque_case_b_nonZero = simplify(subs(totalTorque_case_b, omega_m, 2*omega_1))
```

totalTorque_case_b_nonZero =

$$\frac{1273\, I_{s1}\, I_{s2}\, \left(\sin(\delta - \phi_2) + \sin(\delta - \phi_2 + 2\,\omega_1\, t) + \sin(\delta + \phi_2 + 2\,\omega_1\, t) + \sin(\delta + \phi_2 + 4\,\omega_1\, t)\right)}{4000}$$

or

$$\omega_m = 0 \text{ and}$$
$$\delta \neq 0,\ \phi_2 \neq 0$$

```
totalTorque_case_b_nonZero = simplify(subs(totalTorque_case_b, omega_m, 0))
```

totalTorque_case_b_nonZero =

$$\frac{1273\, I_{s1}\, I_{s2}\, \sin(\delta)\, \left(\cos(\phi_2 + 2\,\omega_1\, t) + \cos(\phi_2)\right)}{2000}$$

will result in a non-zero torque.

(c) When ω1 ≠ 0, ω2 = 0

```
totalTorque_case_c = subs(totalTorque_manual, omega(2), 0)
```

totalTorque_case_c =

$$\frac{1273\, I_{s1}\, I_{s2}\, \left(\sin(\delta - \phi_2 + t\, (\omega_1 + \omega_m)) + \sin(\delta + \phi_2 - t\, (\omega_1 - \omega_m)) - \sin(\phi_2 - \delta + t\, (\omega_1 - \omega_m)) + \sin(\delta + \right.}{4000}$$

```
totalTorque_case_c_nonZero = subs(totalTorque_case_c, omega_m, [omega_1 -omega_1])
```

totalTorque_case_c_nonZero =

$$\left(\frac{1273\, I_{s1}\, I_{s2}\, \left(\sin(\delta - \phi_2) + \sin(\delta - \phi_2 + 2\,\omega_1\, t) + \sin(\delta + \phi_2) + \sin(\delta + \phi_2 + 2\,\omega_1\, t)\right)}{4000} \quad \frac{1273\, I_{s1}\, I_{s2}\, (\sin(\delta -}{}\right.$$

Only when $\omega_m = |\omega_1|$ the device can produce a non-zero torque.