Channel Coding Theory

Exam: 28/06/2007

Advices: If you consider necessary to make some assumption, state it and develop the solution consistently. Begin with the easiest question first.

Good Luck!

A. *Exercise 1*

Given the cyclic code with generator polynomial $(6,5)_8$ (octal notation) determine:

1) The weight enumerating function (WEF).
2) The minimum distance.
3) The probability of undetected errors assuming a binary symmetric channel (BSC) with $\varepsilon = 10^{-2}$.
4) Find the percentage of error patterns with 3 and 4 errors that cannot be detected by the code (assume an error detection strategy).
5) Find the fraction of undetectable bursts of length 12 and motivate the result.

Note: the code parameters (n, k) are not given but you can determine them.

Hints: In order to determine the WEF work on the dual code. Take into account the fundament properties of linear cyclic codes. Take into account that if the codeword length n is odd and $X + 1$ is not a factor of $g(X)$ the code contains a codeword consisting of all 1's.

Recall: $(a+b)^n = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} a^{n-j} b^j$.

Solution to 1: The string 110101 corresponds to the octal notation $(6,5)_8$. The corresponding polynomial is

$$g(z) = z^5 + z^4 + z^2 + 1. \tag{1}$$

Let us determine first the codeword length. It can be obtained by dividing the polynomial $z^n + 1$ by $g(z)$. The division end when we obtain a rest with the following structure

$$z^{n-m} + z^{n-m-1} + z^{n-m-3} + z^{n-m-5}. \tag{2}$$

Then, n is obtain by solving the equation $n - m = 5$. In this case, $n = 15$. The cyclic code with generator function $g(z)$ is then an $(n, n-p) = (15, 10)$ cyclic code, where p i the degree of $g(z)$. .

We can determine the dual code. From the division made in the previous step we can readily write the parity check polynomial $h(z)$ such that $h(z)g(z) = z^{15} + 1$. The division yields

$$h(z) = z^{10} + z^9 + z^8 + z^6 + z^5 + z^2 + 1. \tag{3}$$

The generator polynomial of the $(15, 5)$ dual code is

$$g_d(z) = z^{10}h(z^{-1}) = z^{10} + z^8 + z^5 + z^4 + z^2 + z + 1. \tag{4}$$

We can now determine the WEF of the dual code. It has 32 codewords. Because of the properties of cyclic codes we can easily get all of them:

- The all zero codeword belongs to the codebook because of the linearity of cyclic codes.
- The all 1 codewords belongs to the codes. In fact, it is easy to check that $z + 1$ is not a factor of $g_d(z)$.
- Given a codeword we can obtain other codewords by shifting of the given one. If the codeword does not have any special structure from the shift (in contrast the previous cases of all zero and all one codewords) we can get 15 different codewords. It it easy to check that the data word (00001) is coded in the codeword (000010100110111) with weight $w(000010100110111) = 7$. Then, the codebook of the dual code $\mathcal{C}_d$ has 15 codewords of weight 7. It is also easy to find a second codeword that is not a cyclic shift of the previous ones. The data word (00011) is encoded in the codeword (000111101011001) with weight $w(000111101011001) = 8$. In the dual code there are 15 codewords with weight 8.

From the previous considerations we have 1 codeword with weight 0, 1 codeword with weight 15, 15 codewords with weight 7 and 15 codewords with weight 8 (in total 32 codewords). We can write down the WEF of the dual code

$$B(z) = 1 + 15z^7 + 15z^8 + z^{15}. \tag{5}$$

By applying the Mac Williams identity we can find the WEF of $\mathcal{C}$

$$\begin{aligned} A(z) &= 2^{-(n-k)}(1+z)^n B\left(\frac{(1-z)}{1+z}\right) & (6)\\ &= 2^{-5}\left(1 + 15\left(\frac{(1-z)}{1+z}\right)^7 + 15\left(\frac{(1-z)}{1+z}\right)^8 + \left(\frac{(1-z)}{1+z}\right)^{15}\right) & (7)\\ &= \frac{1}{32}\left(1 + \sum_{k=0}^{7}\left[30\begin{pmatrix}7\\k\end{pmatrix}(-1)^k + \begin{pmatrix}15\\2k\end{pmatrix}\right]z^{2k}\right). & (8) \end{aligned}$$

Solution to 5: From the Theorems in slides 62 and 63 we know as follows:

- All error bursts of length lower or equal that $n - k = 15 - 10$ are detectable.
- The fraction of undetectable bursts of length $n - k + 1 = 11$ is $2^{n-k-1} = 2^{-4}$.
- The fraction of undetectable bursts of length higher than $n - k + 1 = 11$ is $2^{n-k} = 2^{-5}$.

B. *Exercise 2*

Given the cyclic code in Exercise 1 draw the corresponding Meggitt decoder. Describe in detail how to determine the error pattern detection circuit.

C. *Exercise 3*

A. Given the LDPC ensemble $(\Lambda(x) = 9x^4, P(x) = 6x^6)$ determine the **design rate**.

B. Consider the realization of the previous LDPC ensemble corresponding to the permutation $\sigma = (9, 13, 25, 36, 7, 8, 10, 14, 1, 15, 16, 17, 2, 3, 4, 11, 5, 19, 20, 21, 12, 18, 26, 32, 6, 27, 28, 29, 22, 23, 24, 33, 30, 34, 35, 31)$ and draw the Tanner graph.

C. Determine the generator matrix $\boldsymbol{G}$ and determine the **rate** of the corresponding code.

D. Consider the Tanner graph associated to the generator matrix $\boldsymbol{G}$ (Tanner graph without parallel edges) and determine the variable and node distribution from an edge perspective $\lambda(x)$ and $\rho(x)$.

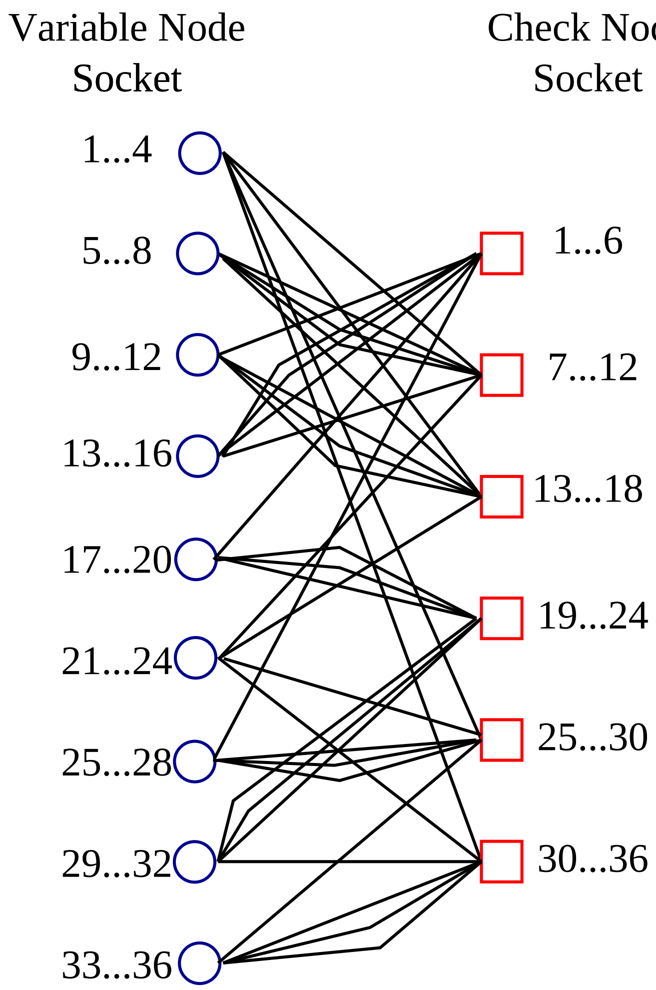
D. Determine a stopping set with 2 or 3 variable nodes if there is one.

E. Given the LDPC ensemble $(\lambda(x), \rho(x), n)$ determine the **design rate** and the **rate** as $n \to +\infty$ (assume that the conditions on the asymptotic relation between rate and design rate are satisfied).

F. Determine the density evolution function.

G. Apply the message passing algorithm for the received signal $\boldsymbol{r} = (0, 0, 0, \Delta, 0, 0, 0, \Delta, \Delta)$.

Solution to item B: In order to determine the Tanner graph realization of the LDPC ensemble $(\Lambda(x) = 9x^4, P(x) = 6x^6)$ corresponding to the permutation σ we apply the definition of LDPC ensemble at page 41 of the LDPC handouts. We draw 9 variable nodes each of them with 4 sockets and 6 check nodes, each of them with 6 sockets. We enumerate the sockets on both sides from 1 to 32. Then, we connect the i-th socket of the variable node to the socket of the check node indicated in the ith position of the permutation σ, e.g. socket $i = 3$ in the variable node side is connected to socket 25 i the check node side. The following figure shows the Tanner graph.



D. *Exercise 4*

Given the convolutional code $\mathcal{C}_1$ having generator polynomials $g_1^{(1)} = 1 + D + D^3$ and $g_2^{(1)} = 1 + D + D^2 + D^4$ draw the corresponding feedforward encoder and the feedback systematic encoder. Discuss briefly differences and similarities of the two encoders in terms of WER and BER.

Given the convolutional code $\mathcal{C}_2$ defined by the polynomials $g_1^{(2)} = 1 + D^3 + D^4$ and $g_2^{(2)} = 1 + D + D^2 + D^3 + D^4$ draw the parallel concatenated encoder with interleaving based on $\mathcal{C}_1$ and $\mathcal{C}_2$ and rate $R = 1/3$. How to get from the previous code a code of rate $R = 1/2$?

Describe the impact of the implementation of the convolutional encoders (feedforward encoders versus systematic feedback encoders) and the interleaver on the performance of turbo codes. Provide an intuitive explanation.

Provide also an explanation for the error floor of turbo codes and how to modify it.

E. *Exercise 5*

Consider a theoretical multiple access system in figure -E with 3 active users impaired by additive white noise with equiprobable random values $w[k] \in \{\pm 1\}$ (obtained by binary quantization of an additive white

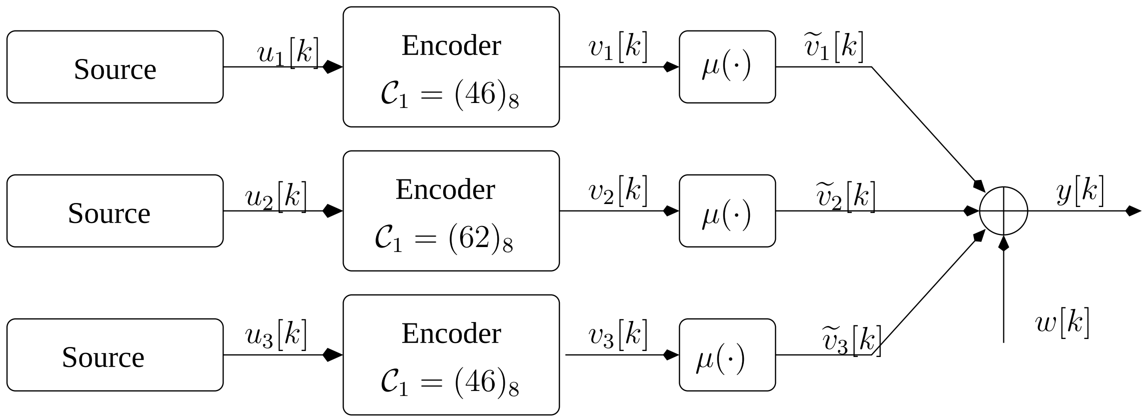


Fig. 1. Multiple access channel

Gaussian channel). The datawords $\boldsymbol{u}_1,\ \boldsymbol{u}_2,\ \boldsymbol{u}_3$ are encoded by cyclic codes $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3$. $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3$ are defined in octal notation by the factors of $(z+1)^{15}$ $(46)_8,\ (62)_8$ and (46), respectively. The binary symbols of the codewords $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3$ are mapped by the function $\mu:\ 0,1 \rightarrow \{\pm 1\}$ with $\mu(0) = -1$ and $\mu(1) = +1$ and then transmitted synchronously through the multiple access channel.

The k-th received symbol $y[k]$ is given by

$$y[k] = \mu\{v_1[k]\} + \mu\{v_2[k]\} + \mu\{v_3[k]\} + w[k]$$

with $\boldsymbol{v}_1 = \boldsymbol{G}_1\boldsymbol{u}_1,\ \boldsymbol{v}_2 = \boldsymbol{G}_2\boldsymbol{u}_2,\ \boldsymbol{v}_3 = \boldsymbol{G}_3\boldsymbol{u}_3$ and $\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3$ the generating matrices.

Draw the joint code/channel factor graph . Write the expressions for the functions at the factor nodes and label the variable nodes. What is the size of the messages?