IV. Two stage rocket

Problem Statement

Imagine you are preparing the new Falcon launch vehicle for its first mission from Kennedy Space Center. The vehicle must deliver a total ΔV (ΔV_{design}) of 10,000 m/s. The total mass of the second stage, including structure and propellant, is 12,000 kg, 9000 kg of which is propellant. The payload mass is 2000 kg. The I_{sp} of the first stage is 350 seconds and of the second stage is 400 seconds. The structural mass of the first stage is 8000 kg. What mass of propellant must be loaded on the first stage to achieve the required ΔV_{design}? What is the vehicle's total mass at lift-off?

Problem Summary

Given: 2 stages

$m_{payload} = 2000 \text{ kg}$

$m_{structure\text{-}2} + m_{propellant\text{-}2} = 12{,}000 \text{ kg}$

$m_{propellant\text{-}2} = 9000 \text{ kg}$

$m_{structure\text{-}1} = 8000 \text{ kg}$

$I_{sp\text{-}1} = 350 \text{ s}$

$I_{sp\text{-}2} = 400 \text{ s}$

$\Delta V_{design} = 10{,}000 \text{ m/s}$

Find: $m_{propellant\text{-}1}$

$m_{initial}$

Conceptual Solution

1) Determine the $\Delta V_{stage\,2}$

$$\Delta V_{stage\,2} = I_{sp\,2} g_o \times \ln\left(\frac{m_{structure\,2} + m_{propellant\,2} + m_{payload}}{m_{structure\,2} + m_{payload}}\right)$$

2) Determine the required ΔV of stage 1

$$\Delta V_{stage\,1} = \Delta V_{design} - \Delta V_{stage\,2}$$

3) Determine the initial mass of stage 1

$$\Delta V_{stage\,1} = I_{sp\,1} g_o \times \ln\left(\frac{m_{initial}}{m_{structure\,2} + m_{propellant\,2} + m_{payload} + m_{structure\,1}}\right)$$

4) Determine the mass propellant in stage 1

$$m_{propellant\,1} = m_{initial} - (m_{stucture\,1} + m_{structure\,2} + m_{propellant\,2} + m_{payload})$$

Analytical Solution

1) Determine

$$\Delta V_{stage\,2} = I_{sp\,2} g_o \times \ln\left(\frac{m_{structure\,2} + m_{propellant\,2} + m_{payload}}{m_{stucture\,2} + m_{payload}}\right)$$

$$= (400 \text{ s})(9.81 \text{ m/s}^2) \ln\left(\frac{12{,}000\text{kg} + 2000\text{kg}}{3000\text{kg} + 2000\text{kg}}\right)$$

$$\Delta V_{stage\,2} = 4040 \text{ m/s}$$

2) Determine the required ΔV of the first stage

$$\Delta V_{stage\,1} = \Delta V_{design} - \Delta V_{stage\,2}$$

$$= 10{,}000 \text{ m/s} - 4040 \text{ m/s}$$

$$\Delta V_{stage\,1} = 5960 \text{ m/s}$$

3) Determine the initial mass of stage 1

$$\Delta V_{stage\,1} = I_{sp\,1} g_o \times \ln\left(\frac{m_{initial}}{m_{structure\,1} + m_{structure\,2} + m_{propellant\,2} + m_{payload}}\right)$$

$$m_{initial} = (8000 \text{ kg} + 3000 \text{ kg} + 9000 \text{ kg} + 2000 \text{ kg})\, e^{\left[\frac{5960\text{m/s}}{(350\text{s})(9.81\text{m/s}^2)}\right]}$$

$$m_{initial} = 124{,}821 \text{ kg}$$

4) Determine mass of propellant in stage 1

$$m_{propellant\,1} = m_{initial} - (m_{structure\,1} + m_{structure\,2} + m_{propellant\,2} + m_{payload})$$

$$= 124{,}821 - (8000 \text{ kg} + 3000 \text{ kg} + 9000 \text{ kg} + 2000\text{kg})$$

$$m_{propellant\text{-}1} = 102{,}821 \text{ kg}$$

Interpreting the Results

The total mass of this launch vehicle at lift-off is 124,821 kg. About 82% of this mass is propellant in the first stage alone (102,821 kg/124,821 kg). Less than 2% of the total lift-off mass is payload (2000 kg/124,821 kg).

I. Orbiting satellites (Komp. exerc. 1.3)

$$\left.\begin{aligned} M_1 &= \frac{4\pi R_1^3}{3}\cdot \rho_1 \\ M_2 &= \frac{4\pi R_2^3}{3}\cdot \rho_2 \end{aligned}\right\}\text{ spheres}$$

Gravitational force $= G\,\dfrac{M_1 m}{R_1^2} \equiv m\cdot\dfrac{v_1^2}{R_1}$ = centripetal force; analogously for M_2

We get $\quad v_1^2 = G\cdot\dfrac{M_1}{R_1}\quad$ and $\quad v_2^2 = G\cdot\dfrac{M_2}{R_2}$

Orbital period T: $\quad v = \dfrac{2\pi R}{T}\quad$ in general

The ratio T_1/T_2:

$$\frac{2\pi R_1}{v_1}\Big/\frac{2\pi R_2}{v_2} = \frac{R_1}{R_2}\cdot\frac{v_2}{v_1} = \frac{R_1}{R_2}\cdot\frac{\sqrt{G\cdot\frac{M_2}{R_2}}}{\sqrt{G\,\frac{M_1}{R_1}}}$$

$$= \frac{R_1}{R_2}\sqrt{\frac{R_2^3\rho_2}{R_2}\Big/\frac{R_1^3\rho_1}{R_1}}$$

$$= \frac{R_1}{R_2}\cdot\frac{R_2}{R_1}\cdot\sqrt{\frac{\rho_2}{\rho_1}}$$

$$\Rightarrow T_1/T_2 = \sqrt{\rho_2/\rho_1}$$

So, knowing T_1 and ρ_1 for the Earth we can calculate ρ_2 by measuring T_2! If $\rho_1 = \rho_2$ then T is independent of R!
(Cf. exercise 1.4 in compendium!)

VI Transfer orbit

(First a solution based on lectured formula, then a more penetrating solution based on direct use of Kepler's laws)

Look at point A.

$$v_{in}^{A} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{r_1}\right)} \qquad \text{circular orbit in} \quad a = r_1$$

$$v_{out}^{A} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)}$$

here a is the half major axis of the elliptic orbit $2a = r_1 + r_2 = 7r_1$

$$= \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{3.5 \cdot r_1}\right)} = \sqrt{\frac{\mu}{r_1}} \cdot 1.31$$

speed increase: $v_{out}^{A} - v_{in}^{A} = 1.31\sqrt{\frac{\mu}{r_1}} - 1.00\sqrt{\frac{\mu}{r_1}} = 0.31\sqrt{\frac{\mu}{r_1}}$

Look at point B

$$v_{in}^{B} = \sqrt{\mu\left(\frac{2}{7r_1 - 1r_1} - \frac{1}{3.5\,r_1}\right)} = \sqrt{\frac{\mu}{r_1}\left(\frac{r_1}{6 \cdot 3.5\,r_1}\right)} = \frac{1}{\sqrt{21}}\sqrt{\frac{\mu}{r_1}}$$

$$v_{out}^{B} = \sqrt{\frac{\mu}{r_2}} = \frac{1}{\sqrt{6}}\sqrt{\frac{\mu}{r_1}}$$

speed increase: $v_{out}^{B} - v_{in}^{B} = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{21}}\right)\sqrt{\frac{\mu}{r_1}} \approx 0.19\sqrt{\frac{\mu}{r_1}}$

(obs! $v_{in}^{A} = \sqrt{\frac{\mu}{r_1}} = \frac{2\pi r_1}{T_1}$)

Since we have now calculated the necessary speed increases we can – basically from the rocket equation – deduce the propellant use, thrusts, etc for the rockets chosen

VI. Transfer orbit and Kepler's laws

$r_2 = 6r_1 \Rightarrow$ the axis of the elliptic orbit will be

$$2a = r_1 + r_2 = 7r_1$$

I. Now we use Keplers third law (T^2 prop a^3)

Orbit time is proportional to $a^{3/2}$

$\Rightarrow$ for inner circular orbit $T_1 = \text{const} \cdot r_1^{3/2}$

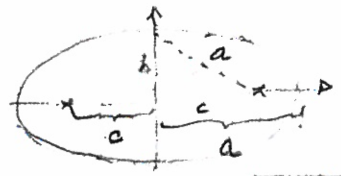
for outer circular orbit $T_2 = \text{const} \cdot r_2^{3/2} = \text{const} \cdot (6r_1)^{3/2}$

for elliptic orbit $T_e = \text{const} \cdot (3.5 \cdot r_1)^{3/2}$

II. We now look for the area of the three orbits in order to be able to use Kepler's second law ('The area speed is constant in the elliptical orbit')

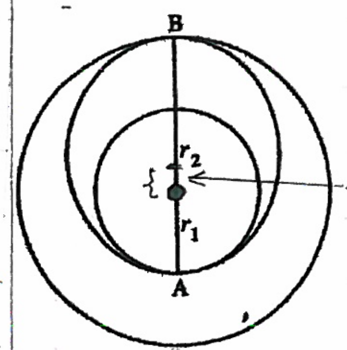
Study an ellipse (the area of a circle is trivial)

For the ellipse



$a^2 = b^2 + c^2$

Area $= \pi \cdot a \cdot b = \pi \cdot a \cdot \sqrt{a^2 - c^2}$ How big is c?



In our case c is the distance between the focus Earth and the midpoint of the ellipse, i.e.

this distance

which must be given by

$$c = a - r_1 = \frac{r_1 + r_2}{2} - r_1 = \frac{r_2 - r_1}{2} = \frac{6r_1 - r_1}{2}$$

$$\Rightarrow c = \frac{6r_1 - r_1}{2} = 2{,}5\, r_1$$

$$\Rightarrow a = 3{,}5\, r_1 \quad \text{(from above)}$$

$$\Rightarrow b = \sqrt{a^2 - c^2} = 2{,}45\, r_1$$

Area $= \pi \cdot 3.5 \cdot 2.45\, r_1^2 = \pi \cdot 8.57 \cdot r_1^2$ for the ellipse

Area inner orbit $= \pi r_1^2$

Area outer orbit $= 36\pi r_1^2$

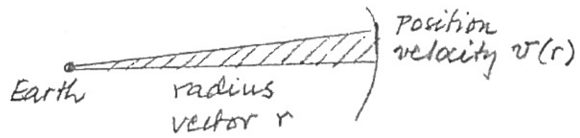
Area speed constant! (Kepler's second law). How big is it?

Transfer orbit $\bar{v}_A^e = \pi \cdot 8.57 \cdot r_1^2 / \text{const}\,(3.5 r_1)^{3/2}$

Inner orbit $\bar{v}_A^1 = \pi r_1^2 / \text{const}\, r_1^{3/2} = \pi r_1^2 / T_1$

Outer orbit $\bar{v}_A^2 = 36\pi r_1^2 / \text{const}\,(6 r_1)^{3/2}$

III When we know the area speed we can deduce the velocity at a given orbit position:



We know the area speed v_A and look for $v(r)$. The triangle (hatched) has an area which can be written

$$\frac{r \cdot v(r) \cdot \Delta t}{2}$$

where Δt is the time to travel the base of the triangle

Therefore the area speed will be

$$v_A = \frac{r \cdot v(r) \cdot \Delta t}{2} \Big/ \Delta t = \frac{r \cdot v(r)}{2}$$

Thus, $v(r) = 2\, v_A / r$

IV We can now get the velocities at points A and B of the orbits

Look first at point A

Velocity in inner orbit at point A: $v = 2 v_A^1 / r_1 = 2 \frac{\pi r_1^2}{r_1} / T_1$

$= 2\pi r_1 / T_1$ (which is obviously correct!)

Velocity of transfer orbit in point A: $2 \cdot \pi \cdot 8.57 \cdot r_1^2 / \text{const} \, (3.5 r_1)^{3/2} / r_1 =$

$$= 2 \cdot \pi \cdot 8.57 r_1 / \text{const} \, (3.5 r_1)^{3/2}$$

$$= 2\pi \cdot 8.57 r_1 / 3.5^{3/2} \cdot T_1 \quad = \frac{2\pi \cdot r_1}{T_1} (4.66)$$

Speed increase in point A is thus

$$= (1.31 - 1.00) \frac{2\pi r_1}{T_1} = 0.31 \cdot \frac{2\pi r_1}{T_1}$$

Look then at point B

Velocity in transfer orbit in B: $2 \cdot \pi \cdot 8.57 \cdot r_1^2 / \text{const} (3.5 r_1)^{3/2} / 6 r_1$

$$= \frac{2\pi r_1}{T_1} \cdot \left(\frac{1.43}{3.5^{3/2}} \right) = \frac{2\pi r_1}{T_1} \cdot 0.22$$

Outer orbit in B: $2 \cdot 36 \pi r_1^2 / \text{const} \, (6 r_1)^{3/2} / 6 r_1$

$$= 2 \cdot 6 \pi r_1 \; / \; 6^{3/2} \cdot T_1$$

$$= \frac{2\pi r_1}{T_1} \left(\frac{6}{6^{3/2}} \right) = \frac{2 \cdot \pi r_1}{T_1} \cdot 0.41$$

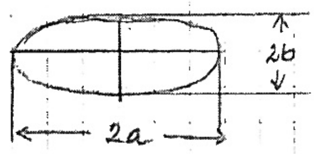
Speed increase in point B is thus

$$= (0.41 - 0.22) \cdot \frac{2\pi r_1}{T_1} = 0.19 \cdot \frac{2\pi r_1}{T_1}$$

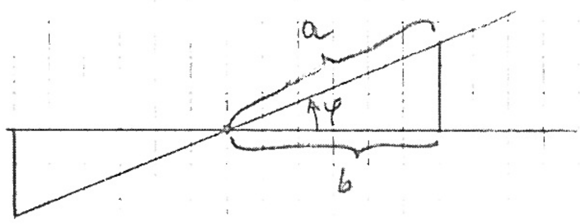
(Since we have now calculated the necessary speed increases we can deduce the thrusts and propellant us for the rocket used)

Area of ellipse

The ellipse has a major axis - $2a$ - and a minoraxis $2b$.



Look at it sidewise and turn it around the minor axis!



Turn it so that the projection of the semimajor axis a is exactly b.

Then the ellipse is projected down onto a circle with radius b!

The area A_{ellips} is then projected onto an area $A_{circle} = \pi b^2$

But then $A_{ellipse} \cdot \cos\varphi = A_{circle}$

We get $\cos\varphi$ from the figure: $\cos\varphi = \frac{b}{a}$

$$\Rightarrow \quad A_{ellipse} = \frac{A_{circle}}{b} \cdot a = \frac{\pi b^2 a}{b} = \pi \cdot a \cdot b$$

VII Satellite entry into atmosphere

Energy in orbit = kinetic + potential energy

a. Kinetic energy: $\frac{mv^2}{2}$, with $v = \sqrt{r \cdot g}$

$$= m \cdot (r \cdot g)/2 \qquad r = R_0 + \Delta r$$

b. Potential energy at altitude r

$$= \int_{R_0}^{r} F \cdot dr = G \cdot M \cdot m \int_{R_0}^{r} \frac{1}{r^2} dr$$

$$= G M \cdot m \left[\frac{1}{R_0} - \frac{1}{r}\right] \quad \left\{ \begin{array}{l} \text{with } g = \frac{GM}{R_0^2} \\ \text{and } r = R_0 + \Delta r \end{array} \right.$$

$$= g R_0^2 \cdot m \left(\frac{r - R_0}{r R_0}\right) = g R_0\, m \frac{\Delta r}{r}$$

Let us assume that $r = 6400 + 600$ [km] $= R_0 + \Delta r$

Kinetic energy: $m \cdot 7000 \cdot 10^3 \cdot 10/2$ [J]

Potential energy: $10 \cdot m \cdot 6400 \cdot 10^3 \left(\frac{600}{7000}\right) = m \cdot 6400 \cdot 10^3 \left(\frac{60}{70}\right)$ [J]

(The ratio is about 16 % between potential and kinetic energy)

Now, assume that half the total energy goes into heat stored in the satellite. Temperature increase $\Delta T \Rightarrow$

$$m \cdot 35000 \cdot 10^3 + m \cdot 5500 \cdot 10^3 = c \cdot m \cdot \Delta T \cdot 2 \quad \text{with } c = 448 \left[\frac{J}{kg \cdot K}\right]$$

$$\Rightarrow \Delta T \simeq 40 \cdot 10^3 \text{ degrees}$$

Higher than melting and boiling point of iron (1535 °C and 2750 °C respectively). But the iron will evaporate, disintegrate and radiate heat underways! To prevent heating use shielding (cf. Space Shuttle!)