VI Transfer orbit

(First a solution based on lectured formula, then a more penetrating solution based on direct use of Kepler's laws)

Look at point A.

$$v_{in}^{A} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{r_1}\right)}$$ circular orbit in $a = r_1$

$$v_{out}^{A} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)}$$ here a is the half major axis of the elliptic orbit $2a = r_1 + r_2 = 7r_1$

$$= \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{3.5\cdot r_1}\right)} = \sqrt{\frac{\mu}{r_1}} \cdot 1.31$$

speed increase: $v_{out}^{A} - v_{in}^{A} = 1.31\sqrt{\frac{\mu}{r_1}} - 1.00\sqrt{\frac{\mu}{r_1}} = 0.31\sqrt{\frac{\mu}{r_1}}$

Look at point B

$$v_{in}^{B} = \sqrt{\mu\left(\frac{2}{7r_1 - 1r_1} - \frac{1}{3.5\,r_1}\right)} = \sqrt{\frac{\mu}{r_1}\left(\frac{r_1}{6\cdot 3.5\,r_1}\right)} = \frac{1}{\sqrt{21}}\sqrt{\frac{\mu}{r_1}}$$

$$v_{out}^{B} = \sqrt{\frac{\mu}{r_2}} = \frac{1}{\sqrt{6}}\sqrt{\frac{\mu}{r_1}}$$

speed increase: $v_{out}^{B} - v_{in}^{B} = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{21}}\right)\sqrt{\frac{\mu}{r_1}} \approx 0.19\sqrt{\frac{\mu}{r_1}}$

(obs! $v_{in}^{A} = \sqrt{\frac{\mu}{r_1}} = \frac{2\pi r_1}{T_1}$)

Since we have now calculated the necessary speed increases we can – basically from the rocket equation – deduce the propellant use, thrusts, etc for the rockets chosen

VI. Transfer orbit and Kepler's laws

$r_2 = 6 r_1 \quad \Rightarrow$ the axis of the elliptic orbit will be

$$2a = r_1 + r_2 = 7 r_1$$

I. Now we use Keplers third law $\left(T^2 \text{ prop } a^3 \right)$

Orbit time is proportional to $a^{3/2}$

$\Rightarrow$ for inner circular orbit $T_1 = \text{const} \cdot r_1^{3/2}$

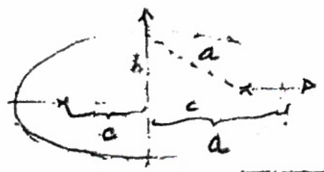
for outer circular orbit $T_2 = \text{const} \cdot r_2^{3/2} = \text{const} \cdot (6 r_1)^{3/2}$

for elliptic orbit $T_e = \text{const} \cdot (3.5 \cdot r_1)^{3/2}$

II. We now look for the area of the three orbits in order to be able to use Kepler's second law ('The area speed is constant in the elliptical orbit')

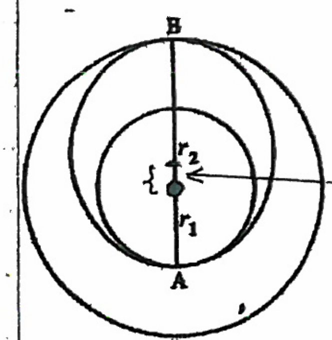
Study an ellipse (the area of a circle is trivial)

For the ellipse



$a^2 = b^2 + c^2$

Area $= \pi \cdot a \cdot b \quad = \pi \cdot a \cdot \sqrt{a^2 - c^2}$ How big is c?



In our case c is the distance between the focus Earth and the ~~midpoint of~~ the ellipse, i.e., this distance

which must be given by

$$c = a - r_1 = \frac{r_1 + r_2}{2} - r_1 = \frac{r_2 - r_1}{2} = \frac{6 r_1 - r_1}{2}$$

$\Rightarrow c = \frac{6 r_1 - r_1}{2} = 2.5 r_1$

$\Rightarrow a = 3.5 r_1$ (from above)

$\Rightarrow b = \sqrt{a^2 - c^2} = 2.45 r_1$

Area $= \pi \cdot 3.5 \cdot 2.45 \, r_1^2 = \pi \cdot 8.57 \cdot r_1^2$ for the ellipse

Area inner orbit $= \pi r_1^2$

Area outer orbit $= 36\pi r_1^2$

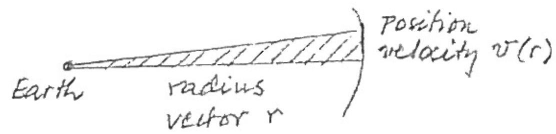
Area speed constant! (Kepler's second law). How big is it?

Transfer orbit $\bar{v}_A^e = \pi \cdot 8.57 \cdot r_1^2 / \text{const}\,(3.5 r_1)^{3/2}$

Inner orbit $\bar{v}_A^1 = \pi r_1^2 / \text{const}\, r_1^{3/2} = \pi r_1^2 / T_1$

Outer orbit $\bar{v}_A^2 = 36\pi r_1^2 / \text{const}\,(6 r_1)^{3/2}$

III When we know the area speed we can deduce the velocity at a given orbit position:



We know the area speed v_A and look for $v(r)$. The triangle (hatched) has an area which can be written

$$\frac{r \cdot v(r) \cdot \Delta t}{2}$$ where Δt is the time to travel the base of the triangle

Therefore the area speed will be

$$v_A = \frac{r \cdot v(r) \cdot \Delta t}{2} \Big/ \Delta t = \frac{r \cdot v(r)}{2}$$

Thus, $v(r) = 2\, v_A / r$

IV We can now get the velocities at points A and B of the orbits

Look first at point A

Velocity in inner orbit at point A: $v = 2 v_A^1 / r_1 = 2 \frac{\pi r_1^2}{r_1} / T_1$

$= 2\pi r_1 / T_1$ (which is obviously correct!)

Velocity of transfer orbit in point A: $2 \cdot \pi \cdot 8.57 \cdot r_1^2 / const\,(3.5 r_1)^{3/2} / r_1 =$

$= 2\pi \cdot 8.57 r_1 / const\,(3.5 r_1)^{3/2}$

$= 2\pi \cdot 8.57 r_1 / 3{,}5^{3/2} \cdot T_1 \quad = \frac{2\pi \cdot r_1}{T_1}(4.66)$

Speed increase in point A is thus

$= (1.31 - 1{,}00)\frac{2\pi r_1}{T_1} = 0.31 \cdot \frac{2\pi r_1}{T_1}$

Look then at point B

Velocity in transfer orbit in B: $2 \cdot \pi \cdot 8.57 \cdot r_1^2 / const\,(3.5 r_1)^{3/2} / 6 r_1$

$= \frac{2\pi r_1}{T_1} \cdot \left(\frac{1{,}43}{3.5^{3/2}}\right) = \frac{2\pi r_1}{T_1} \cdot 0.22$

Outer orbit in B: $2 \cdot 36 \pi r_1^2 / const\,(6 r_1)^{3/2} / 6 r_1$

$= 2 \cdot 6 \pi r_1 / 6^{3/2} \cdot T_1$

$= \frac{2\pi r_1}{T_1}\left(\frac{6}{6^{3/2}}\right) = \frac{2\pi r_1}{T_1} \cdot 0{,}41$

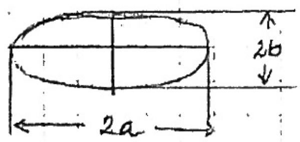
Speed increase in point B is thus

$= (0.41 - 0.22) \cdot \frac{2\pi r_1}{T_1} = 0.19 \cdot \frac{2\pi r_1}{T_1}$

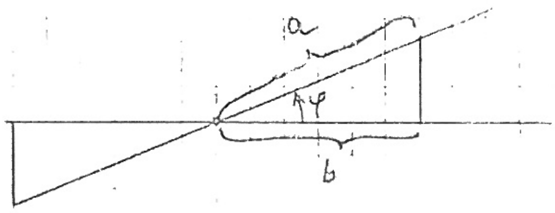
(Since we have now calculated the necessary speed increases we can deduce the thrusts and propellant us for the rocket used)

Area of ellipse Comment

The ellipse has a major axis 2a and a minor axis 2b.



Look at it sidewise and turn it around the minor axis!



Turn it so that the projection of the semimajor axis a is exactly b.

Then the ellipse is projected down onto a circle with radius b!

The area A_{ellips} is then projected onto an area $A_{circle} = \pi b^2$

But then $A_{ellipse} \cdot \cos\varphi = A_{circle}$

We get $\cos\varphi$ from the figure: $\cos\varphi = \frac{b}{a}$

$$\Rightarrow \quad A_{ellipse} = \frac{A_{circle}}{b} \cdot a = \frac{\pi b^2 a}{b} = \pi \cdot a \cdot b$$