5. The Sampling Theorem

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You have probably already been exposed to the Sampling Theorem, which says that if a waveform is bandwidth-limited to W Hz, then it can be represented exactly by its samples if they are taken at a rate of $2W$ samples/second or more. Here we will review the Sampling Theorem. We will interpret the Sampling Theorem as an orthonormal expansion of bandlimited waveforms. The main reason for discussing the Sampling Theorem is to make concrete the notion that a bandlimited signal of bandwidth W and duration T has roughly $2WT$ degrees of freedom for T large.

1 Review of Fourier transforms

The Fourier transform of a waveform $u(t)$ is defined as

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-j2\pi ft}\, dt. \tag{1}$$

This is a mapping from one function to another function. The first function is the waveform $u(t)$ mapping $\mathbb{R} \to \mathbb{R}$. As a waveform, it maps time (in units of seconds) into real valued amplitudes. The same defininition can be used if $u(t)$ is a complex valued function of time, *i.e.*, a mapping $\mathbb{R} \to \mathbb{C}$. Later, when we study modulation, we often consider the Fourier transform for complex valued functions. For now we can restrict our attention to real valued functions, but will often write equations in such a way as to make the generalization to complex valued functions easy. For example, we use the notation $|u(t)|^2$ to denote $u^2(t)$ for real functions, but more generally to denote $u(t)u^*(t)$.

For the most part, we restrict our attention to finite-energy functions, *i.e.*, functions for which the integral $\int_{-\infty}^{\infty} |u(t)|^2\, dt$ exists with a finite value. There are some very common functions that are not finite-energy, including any non-zero constant function and any sinusoid. Impulse functions are also not included as finite-energy functions (in fact, impulses are not really functions at all since, for example, the property that a unit impulse has an integral equal to 1 does not follow from the value of the function at all t). Even if one ignores the fact that impulses are not really functions, the energy of an impulse in any reasonable sense is infinite.

In what follows, we explicitly define a *waveform* as a finite-energy function. Obviously sinusoids and other infinite energy functions are very important for many types of applications, and generalized functions such as impulses are similarly important. However, the properties of waveforms that we will rely on most strongly here are not shared by infinite-energy functions. It should be noticed that physical waveforms do not persist forever, and do not take on infinite values, so we are not restricting the set of physical waveforms of interest, but only the set of models for those waveforms.

There is a very important theorem, due to Plancherel, that states that if $u(t)$ is finite-energy, then the integral $U(f)$ in (1) exists for each f, and $U(f)$ satisfies the inverse Fourier transform,

$$u(t) = \int_{-\infty}^{\infty} U(f)e^{j2\pi ft}\, df. \tag{2}$$

To be more precise, define $u_B(t)$ as

$$u_B(t) = \int_{-B}^{B} U(f)e^{j2\pi ft}\, df. \tag{3}$$

In the context of Plancherel's Theorem, (2) is shorthand for

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} |u(t) - u_B(t)|^2\, dt = 0 \tag{4}$$

This says that the energy difference between $u(t)$ and the finite bandwidth transforms in (3) goes to zero with increasing B. In essence this allows $u(t)$ to differ from the inverse transform of its transform at isolated points. For example, if $u(t)$ is discontinuous at a number of points, we would not expect the inverse of its transform to converge to $u(t)$ at those points. In fact, changing $u(t)$ at a single point cannot change its Fourier transform.

The Fourier transform itself in (1) must be interpreted in the same way. Define

$$U_A(f) = \int_{-A}^{A} u(t)e^{-j2\pi ft}\, dt \tag{5}$$

Then, according to Plancherel's theorem, there is a function $U(f)$ such that

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |U(f) - U_A(f)|^2\, df = 0 \tag{6}$$

A final part of this result is that the energy in $u(t)$ is the same as the energy in $U(f)$, *i.e.*,

$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = \int_{-\infty}^{\infty} |U(f)|^2\, df. \tag{7}$$

Thus, every finite-energy function has a finite-energy transform with the same energy, and these transforms satisfy the convergence properties in (4) and (6).

These complicated sounding conditions say that for finite-energy functions, we can approximate $u(t)$ by truncating the waveform within very large limits, $(-A, A)$. The Fourier transform will not be substantially changed for A sufficiently large. Similarly, we can truncate $U(f)$ within very large frequency limits and the time function $u(t)$ will not be substantially changed. In other words, finite-energy functions are essentially constrained within sufficiently large time and frequency limits. This is important for engineering, since when we model a function, we don't much care what it does before the device in question was built or after it is discarded. We also don't much care what happens in frequency ranges beyond those of interest. Thus we prefer models in which these extraneous effects can be ignored. One problem with Fourier transforms is that parts of the function outside the time range of interest can have an important effect inside the frequency range of interest, and vice-versa. Restricting ourselves to finite-energy models avoids the worst of these problems.

We will give more interpretations of what these conditions mean later. One should not view sinusoids, impulses, and other infinite-energy functions as second class citizens in the function world. They are very useful in studying linear filters, for example. Unfortunately, they are usually poor models for the output of analog sources.

The above theorem is a far more mathematical statement than engineers are used to, and, in fact, we have not even stated it with perfect precision. However, we hope to subsequently convince the reader that waveforms, *i.e.*, finite-energy functions, are the appropriate kinds of functions to study for both analog sources and for channel inputs and outputs. We also hope to convince the reader that the above theorem is meaningful in both an intuitive and an engineering sense. You will often see finite-energy functions referred to as $\mathcal{L}_2$ functions.

We next review a few properties of Fourier transforms. In the following short table, $u(t)$ and $U(f)$ are a Fourier transform pair, written $u(t) \leftrightarrow U(f)$.

$$\begin{aligned}
\text{waveform} \quad &\leftrightarrow \quad \text{Fourier transform} \\
u^*(-t) \quad &\leftrightarrow \quad U^*(f) && (8) \\
u(t-\tau) \quad &\leftrightarrow \quad e^{-j2\pi f\tau}U(f) && (9) \\
u(t/T) \quad &\leftrightarrow \quad T\,U(fT) && (10) \\
\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad &\leftrightarrow \quad \mathrm{rect}(f) = \begin{cases} 1 & \text{for} \quad |f| \le 1/2 \\ 0 & \text{for} \quad |f| > 1/2 \end{cases} && (11) \\
\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau \quad &\leftrightarrow \quad U(f)V(f) && (12) \\
\int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau \quad &\leftrightarrow \quad U(f)V^*(f) && (13)
\end{aligned}$$

Equations (8-10) follow directly from (1) by substitution and changing the variable of integration. Equation (11) follows from the inverse transform (2). The convolution theorem (12) follows by Fourier transforming the left side to get $\iint u(\tau)v(t-\tau)e^{-j2\pi ft}d\tau dt$. Multiplying and dividing by $e^{-j2\pi f\tau}$ yields the right side. Equation (13) gives the transform of the cross-correlation of $u(t)$ and $v(t)$; it follows by using (8) in (12).

Two useful special cases of any Fourier transform pair are:

$$u(0) = \int_{-\infty}^{\infty} U(f)\,df \tag{14}$$

$$U(0) = \int_{-\infty}^{\infty} u(t)\, dt \tag{15}$$

These are useful sanity checks on any Fourier transform pair, and in particular are often the best way to compute or check multiplicative constants. They are also useful for deriving various properties of Fourier transforms. The most useful of these is *Parseval's theorem*, which follows from applying (14) to (13):

$$\int_{-\infty}^{\infty} u(t)v^*(t)\, dt = \int_{-\infty}^{\infty} U(f)V^*(f)\, df. \tag{16}$$

As a corollary, replacing $v(t)$ by $u(t)$ in (16), we derive (7), *i.e.*

$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = \int_{-\infty}^{\infty} |U(f)|^2\, df.$$

For another corollary, define two waveforms $u(t)$ and $v(t)$ to be *orthogonal* if $\int u(t)v^*(t)\, dt = 0$. From (16), we see that $u(t)$ and $v(t)$ are orthogonal if and only if $U(f)$ and $V(f)$ are orthogonal; *i.e.*,

$$\int_{-\infty}^{\infty} u(t)v^*(t)\, dt = 0 \quad \text{if and only if} \quad \int_{-\infty}^{\infty} U(f)V^*(f)\, df = 0. \tag{17}$$

Finally, we note that there is almost complete symmetry between time-domain and frequency-domain functions in Fourier transform theory, apart from a difference in sign in (1) and (2). Indeed, if $u(t) \leftrightarrow U(f)$ are a Fourier transform pair, then so are $U(t) \leftrightarrow u(-f)$. Because of this time-frequency symmetry, we may translate any time-domain result to the frequency domain with the appropriate sign changes, and *vice versa*.

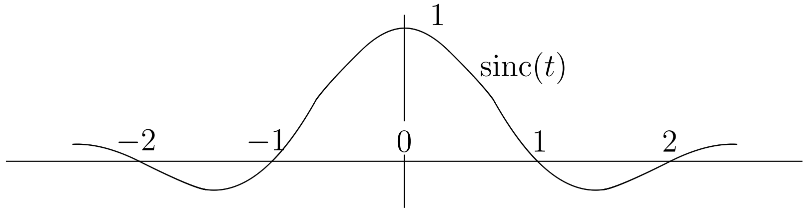
2 The Sampling Theorem

The Sampling Theorem shows that a continuous-time band-limited signal may be represented perfectly by its samples at uniform intervals of T seconds if T is small enough. In other words, the continuous-time signal may be reconstructed perfectly from its samples; sampling at a high enough rate is information-lossless.

An $\mathcal{L}_2$ waveform $u(t)$ will be said to be *band-limited* to a frequency W Hz if its Fourier transform satisfies $U(f) = 0$ for $|f| > W$. The Sampling Theorem is then as follows:

Theorem 2.1 (Sampling Theorem) *Let $u(t)$ be a waveform (*i.e.*, a finite energy function) that is band-limited to W Hz. Then $u(t)$ at all times t can be recovered from its samples $\{u(nT), n \in \mathbb{Z}\}$ at integer multiples of $T = 1/(2W)$ sec, as follows:*

$$u(t) = \sum_{n=-\infty}^{\infty} u(nT)\ \mathrm{sinc}\left(\frac{t - nT}{T}\right). \tag{18}$$



The sinc function, $\text{sinc}(t) = \sin(\pi t)/(\pi t)$ is illustrated in the following figure:

We see that $\text{sinc}(t/T)$ has the value 1 at $t = 0$ and 0 at $t = nT$ for all integers $n \neq 0$. It is non-zero at all times other than these integer times. Also, from (10) and (11), note that the Fourier transform of $\text{sinc}(t/T)$ is equal to T for $|f| < W$ and 0 for $f > W$. More generally, applying the shifting rule of (9), we have

$$\text{sinc}\left(\frac{t - nT}{T}\right) \quad \leftrightarrow \quad \begin{cases} Te^{-j2\pi fnT} & \text{for} \quad |f| < 1/(2T) \\ 0 & \text{for} \quad |f| > 1/(2T) \end{cases} \tag{19}$$

From this, we see that the right side of (18) is in fact band-limited to W.

We can interpret the right side of (18) as follows. Suppose we start with a real sequence $\ldots, u(-T), u(0), u(T), u(2T), \ldots$. If we multiply $u(nT)$ by $\text{sinc}((t-nT)/T)$, then the resulting waveform is band-limited to W and has the value $u(nT)$ at $t = nT$ and 0 at $t = jT$ for each integer $j \neq n$. Thus, when we add the waveforms $u(nT)\text{sinc}((t - nT)/T)$ for all n, the sum $u(t)$ is band-limited to W and has value $u(nT)$ at time $t = nT$ for each integer $n \in \mathbb{Z}$.

Thus we have shown that the right side of (18) is a waveform that is band-limited to W and has the values $\ldots, u(-T), u(0), u(T), u(2T), \ldots$ at times $t = nT$. However, this is not the Sampling Theorem; rather, it is a sort of modulation theorem, showing that from a sequence $\{u(nT)\}$ we can create a band-limited waveform having those values at times $t = nT$. We have not shown that we can go the other way— *i.e.*, start with an arbitrary finite-energy band-limited waveform and represent it by its samples.

We postpone this proof until reviewing the the Fourier series expansion of a time-limited waveform. We assume the following result is familiar to you.

Proposition 1 (Time-domain Fourier series expansion) *If $u(t)$ is a waveform that is time-limited to some interval $[\tau, \tau + T]$ of length T, then $u(t)$ may be expanded as*

$$u(t) = \sum_{n=-\infty}^{\infty} u_n\, e^{j2\pi nt/T}, \quad \tau \leq t \leq \tau + T, \tag{20}$$

where the Fourier series coefficients u_n are given by

$$u_n = \frac{1}{T}\int_{\tau}^{\tau+T} u(t)e^{-j2\pi nt/T}\, dt. \tag{21}$$

Note that the Fourier series bears a close resemblance to the Fourier transform. The Fourier transform expresses an arbitrary (non-time-limited) waveform by a function $\mathbb{R} \to \mathbb{C}$. In contrast, the Fourier series represents a time-limited waveform by a sequence of values, $\mathbb{Z} \to \mathbb{C}$. Comparing (21) with (1), we see that $U(n/T) = Tu_n$.

By time-frequency symmetry, we have equally a Fourier series expansion of a band-limited Fourier transform, as follows:

Proposition 2 (Frequency-domain Fourier series expansion) *If $U(f)$ is a Fourier transform of a waveform that is band-limited to $-W \le f \le W$, then $U(f)$ may be written as*

$$U(f) = \sum_{n=-\infty}^{\infty} \hat{u}_n e^{-j2\pi nfT}, \qquad -W \le f \le W, \tag{22}$$

where $T = 1/(2W)$, for some set of Fourier series coefficients $\hat{u}_n$.

Now, taking the inverse Fourier transform of both sides in (22) and using (19), we have

$$u(t) = \sum_{n=-\infty}^{\infty} \frac{\hat{u}_n}{T} \operatorname{sinc}\left(\frac{t-nT}{T}\right). \tag{23}$$

By evaluating both sides at $t = nT$, we see that $u(nT) = \hat{u}_n/T$. This completes the proof of the Sampling Theorem, and shows that the sampling theorem is in fact just a form of the Fourier series applied in the frequency domain.

3 Sampling a waveform

We have seen that if a waveform is bandlimited to W, then its samples at intervals $T = 1/(2W)$ specify the function. We will shortly also show that

$$\int |u(t)|^2 \, dt = \sum T|u(nT)|^2 \tag{24}$$

In other words, the energy in $u(t)$ is equal to the weighted sum of the squares of the samples.

Suppose we then quantize these samples to values $v(nT)$. Suppose these samples are encoded and then decoded at the receiver. Suppose the samples $v(nT)$ are then used to approximate the waveform $u(t)$ by

$$v(t) = \sum_{n=-\infty}^{\infty} v(nT) \operatorname{sinc}\left(\frac{t-nT}{T}\right). \tag{25}$$

It then follows that

$$\int_{-\infty}^{\infty} |v(t) - u(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} T|v(nT) - u(nT)|^2. \tag{26}$$

In other words, the mean square error in the quantization directly gives the mean square error in the waveform representation. This is one of the major reasons why mean square error is such a convenient measure of distortion – it carries over directly from samples to waveforms.

The Sampling Theorem is very convenient for sampling band-limited waveforms. However, most source waveforms are not quite band-limited. This then leads to first filtering the source waveform to achieve band-limiting and then sampling. This adds another source of mean squared error, since the part of the waveform that is out of band gets represented by 0.

There is another rather peculiar mathematical issue with the Sampling Theorem. The sinc function is non-zero over all non-integer times. Thus recreating the waveform at the receiver from a set of samples requires infinite delay. Practically, of course, these sinc functions can be truncated, but then the resulting waveform is no longer band-limited. There is a similar problem at the source. A truly band-limited waveform cannot be selected in real time. It must be created at time $-\infty$. Again, this is not a practical problem, but it says that band-limited functions are only approximations, so that the clean result of the Sampling Theorem is not quite as nice as it seems at first.

Another approach to a more general form of sampling is to use the time-domain Fourier series over intervals of some arbitrary duration T. Then an arbitrary waveform can be represented by the coefficients of its Fourier series, recalculated for each interval T. The Fourier series within each interval must be truncated in order to represent each interval by a finite number of samples, but this is often preferable to the truncation inherent to using the Sampling Theorem because of the nature of the source. For example, the structure of voice is such that Fourier coefficients, repeated each time interval, often provide a more convenient discrete time set of samples than sampling at a much higher rate.

4 Orthogonal waveforms and orthogonal expansions

Recall that two functions, $v(t)$ and $w(t)$ are *orthogonal* if $\int_{-\infty}^{\infty} v(t)w^*(t)\,dt = 0$. In this section, we interpret the Fourier series of a time-limited function $u(t)$ as a linear combination of orthogonal functions. We then illustrate some of the consequences of this orthogonal expansion. We next show that the Sampling Theorem is also an expansion in terms of orthogonal functions.

4.1 The Fourier series as an orthogonal expansion

Assume that $u(t)$ is a finite-energy complex-valued function, non-zero only for $\tau \leq t \leq T + \tau$. For $-\infty < n < \infty$, let

$$\theta_n(t) = \begin{cases} e^{j2\pi nt/T} & \text{for} \quad \tau \leq t \leq \tau + T \\ 0 & \text{otherwise} \end{cases}$$

The Fourier series expansion of $u(t)$ is

$$u(t) = \sum_{n=-\infty}^{\infty} u_n\theta_n(t) \qquad \text{where} \tag{27}$$

$$u_n = \frac{1}{T}\int_{-\infty}^{\infty} u(t)\theta_n^*(t)\,dt \tag{28}$$

We now show that the functions $\{\theta_n(t)\}$ are orthogonal:

$$\int_{-\infty}^{\infty} \theta_n(t)\theta_j^*(t)\,dt = \int_{t=\tau}^{\tau+T} e^{j2\pi(n-j)t/T}\,dt = \begin{cases} 0 & \text{for} \quad n \neq j \\ T & \text{for} \quad n = j \end{cases} \tag{29}$$

where we have used the fact that the integral is over an integer number of cycles. We next show that (28) is a consequence of (29). In particular, assume that (27) is valid for some set of coefficients $\{u_n\}$. Then the integral $\int u(t)\theta_j^*(t)$ can be calculated as follows:

$$\int_{-\infty}^{\infty} u(t)\theta_j^*(t)\,dt = \sum_{n=-\infty}^{\infty} u_n \int_{-\infty}^{\infty} \theta_n(t)\theta_j^*(t)\,dt = u_jT \tag{30}$$

where we have used (29). This is equivalent to (28).

Note that the argument in (30) does not use the particular form of the orthogonal functions in the Fourier series; it only uses the orthogonality (plus the fact that each orthogonal function has energy T). We will use this argument many times for many different expansions in terms of orthogonal functions. On the other hand, the argument in (30) does not help to justify that any time limited waveform can be expanded as a linear combination of the sinusoids in (27). We come back to this issue later.

We can now use the orthogonality again to find the energy in the time-limited waveform $u(t)$ in terms of the Fourier coefficients $\{u_n\}$. Using first (27) and then (30),

$$\begin{aligned} \int_{-\infty}^{\infty} |u(t)|^2\,dt &= \int_{-\infty}^{\infty} u(t)u^*(t)\,dt = \int_{-\infty}^{\infty} u(t) \sum_{n=-\infty}^{\infty} u_n^*\theta_n^*(t)\,dt \\ &= \sum_{n=-\infty}^{\infty} u_n^* \int_{-\infty}^{\infty} u(t)\theta_n^*(t)\,dt \\ &= \sum_{n=-\infty}^{\infty} u_n^*Tu_n = T\sum_{n=-\infty}^{\infty} |u_n|^2 \end{aligned} \tag{31}$$

Assume that the 'samples' $\{u_n\}$ are quantized into points $\{v_n\}$, which are then encoded, transmitted, decoded, and converted into the waveform $v(t) = \sum_n v_n\theta_n(t)$. Then, from (31), the squared error is

$$\int_{-\infty}^{\infty} |u(t) - v(t)|^2\,dt = T\sum_{n=-\infty}^{\infty} |u_n - v_n|^2. \tag{32}$$

In summary, we have shown that the Fourier series is an orthogonal expansion. If $u(t) = \sum_n u_n\theta_n(t)$, where $\{\theta_n(t)\}$ are orthogonal functions each with energy T, then

- $u_n = (1/T) \int u(t)\theta_n^*(t)\, dt$;
- $u(t)$ has energy equal to $T \sum |u_n|^2$.
- If $v(t) = \sum_n v_n\theta_n(t)$, then the energy in $u(t) - v(t)$ is $T \sum_n |u_n - v_n|^2$.

4.2 The Sampling Theorem as an orthogonal expansion

Assume that $u(t)$ is a finite-energy function band-limited to $|f| \le W$ Hertz. Let $\psi_n(t) = \text{sinc}\{(t - nt)/T\}$ where $T = 1/(2W)$. The Sampling Theorem then says that

$$u(t) = \sum_{n=-\infty}^{\infty} u_n \psi_n(t) \tag{33}$$

where $u_n = u(nt)$. We now show that $\{\psi_n(t)\}$ is an orthogonal set of functions. Recall that two functions are orthogonal if and only if their Fourier transforms are orthogonal. We have the Fourier transform pair:

$$\psi_n(t) = \text{sinc}\{(t - nt)/T\} \quad \leftrightarrow \quad \Psi_n(f) = \begin{cases} Te^{-j2\pi nfT} & \text{for} \quad |f| \le W \\ 0 & \text{for} \quad |f| > W \end{cases}$$

To show that the Fourier transforms are orthogonal,

$$\int_{-\infty}^{\infty} \Psi_n(f)\Psi_m^*(f)\ df = \int_{-W}^{W} T^2 e^{j2\pi(m-n)f}\ df \tag{34}$$

For $m \ne n$, this is the integral of a sinusoid over an integer number of cycles, which is therefore 0. Thus $\{\Psi_n(f)\}$ is a set of orthogonal functions. For $m = n$, the integral in (34) is $2WT^2 = T$. It follows that the functions $\{\psi_n(t)\}$ are also orthogonal. From Parseval, each has energy T, *i.e.*,

$$\int_{-\infty}^{\infty} \psi_n(t)\psi_m^*(t)\, dt = \begin{cases} 0 & \text{for} \quad m \ne n \\ T & \text{for} \quad m = n \end{cases} \tag{35}$$

As with the Fourier series, we next use (35) to evaluate the coefficients u_n in (33).

$$\int_{-\infty}^{\infty} u(t)\psi_m^*(t)\, dt = \sum_{n=-\infty}^{\infty} u_n \int_{-\infty}^{\infty} \psi_n(t)\psi_m^*(t)\, dt = u_m T \tag{36}$$

This gives us an alternative expression for $u_n = u(nt) = (1/T) \int u(t)\psi_n^*(t)\, dt$. We can also use (36) to derive (24). As with the Fourier coefficients, we use (34) and then (36),

$$\begin{aligned} \int_{-\infty}^{\infty} |u(t)|^2\, dt &= \int_{-\infty}^{\infty} u(t)u^*(t)\, dt = \int_{-\infty}^{\infty} u(t) \sum_{n=-\infty}^{\infty} u_n^*\psi_n^*(t)\, dt \\ &= \sum_{n=-\infty}^{\infty} u_n^* \int_{-\infty}^{\infty} u(t)\psi_n^*(t)\, dt \\ &= \sum_{n=-\infty}^{\infty} u_n^* T u_n = T \sum_{n=-\infty}^{\infty} |u_n|^2 \end{aligned} \tag{37}$$

In summary, we have shown that the Sampling Theorem is an orthogonal expansion. If $u(t) = \sum_n u_n \psi_n(t)$, where $\{\psi_n(t)\}$ are orthogonal functions each with energy T, then

- $u_n = (1/T) \int u(t) \psi_n^*(t)\, dt$;
- $u(t)$ has energy equal to $T \sum |u_n|^2$.