EURECOM 14/06/2010
SSP Statistical Signal Processing

TD2: Optimal and Adaptive Filtering, Equalization Partial Solutions

1 Wiener Filtering

Problem 1. A Wiener filtering problem

In this problem we want to estimate the signal x_k from a measurement y_k using a filter $H(z)$ such that the estimate $\hat{x}_k$ minimizes $\mathrm{E}(x_k - \hat{x}_k)^2$. This is clearly a Wiener filter problem, but it can also be solved directly by realizing that $X(z) = (G(z) + z^{-d})Y(z)$. This means that by choosing $H(z) = G(z) + z^{-d}$ the estimation error is zero. In terms of a Wiener filter, the optimal $H(z)$ is given by $S_{yy}^{-1}(z)S_{xy}(z) = S_{yy}^{-1}(z)(G(z) + z^{-d})S_{yy}(z) = G(z) + z^{-d}$.

Problem 2. Constrained and Unconstrained Two-Channel Wiener Filtering

(a) Using the two measurements, y_{1k} and y_{2k} of the signal x_k, we want to choose two filters $H_1(z)$ and $H_2(z)$ satisfying

$$H_1(z) + H_2(z) = 1$$

such that the sum of their outputs is a MMSE estimate of x_k. The noise components are assumed independent, to have zero-mean and have power spectra $S_{v_1v_1}(z)$ and $S_{v_2v_2}(z)$. Before finding the exact solution, we can find the form of the solution by inspection. Assume for a particular frequency $S_{v_1v_1}(z) \gg S_{v_2v_2}(z)$, so that the measurement y_{2k} is more reliable at that frequency. In this case $|H_1(z)| \approx 1$ and $|H_2(z)| \approx 0$. The same must hold for the opposite case.
Writing the output in the time–domain

$$\begin{aligned}\hat{x}_k &= x_k * (h_{1k} + h_{2k}) + v_{1k} * h_1(k) + v_{2k} * h_2(k) \\ &= x_k + (v_{1k} - v_{2k}) * h_{1k} + v_{2k} \qquad (1)\end{aligned}$$

we have the simple Wiener filter problem: find the filter $H_1(z)$ which when given an input $v_{1k} - v_{2k}$ yields a MMSE estimate of $-v_{2k}$. The optimal $H_1(z)$ is then given by

$$H_1(z) = S^{-1}_{v_1-v_2,v_1-v_2}(z)\; S_{-v_2,v_1-v_2}(z) = \frac{S_{v_2,v_2}(z)}{S_{v_1,v_1}(z) + S_{v_2,v_2}(z)}$$

From the constraint equation, the other filter is given by

$$H_2(z) = \frac{S_{v_1,v_1}(z)}{S_{v_1,v_1}(z) + S_{v_2,v_2}(z)}$$

We see therefore that these filters have the form that we indicated from the outset.

2 Equalization and Wiener Filtering

Problem 3. Equalization of a First-Order FIR Channel

Here we want to equalize a channel with response $C(z) = 1 - az^{-1}$. The zero forcing equalizer is simply $H_{\mathrm{ZF-LE}}(z) = 1/C(z) = z/(z-a)$. The associated MSE is given by

$$\begin{aligned}\mathrm{MSE}_{\mathrm{ZF-LE}} &= \frac{\sigma_v^2}{2\pi j}\oint_{|z|=1}\frac{dz}{zC^\dagger(z)C(z)} \qquad (2)\\ &= -\frac{\sigma_v^2}{2\pi j a^*}\oint_{|z|=1}\frac{dz}{(z-a)(z-1/a^*)} \qquad (3)\end{aligned}$$

Assuming $|a| < 1$ this is simply $-\sigma_v^2/a^*$ times the residue at $z = a$ which is $1/(a - 1/a^*)$ so that

$$\mathrm{MSE}_{\mathrm{ZF-LE}} = \frac{\sigma_v^2}{1-|a|^2}$$

We see that as $|a| \to 1$, the MSE tends to infinity. This is because the channel response has a zero close to $|z| = 1$ so that the equalizer has a pole close to $|z| = 1$. This has the effect of amplifying the noise at the corresponding frequency by a large amount.

We now consider the MMSE equalizer which is simply the Wiener filter

$$H_{\mathrm{MMSE-LE}}(z) = \frac{C^\dagger(z)}{C(z)C^\dagger(z)+1/\gamma} = \frac{1-a^*z}{(1-a^*z)(1-a/z)+1/\gamma} = \frac{z(z-1/a^*)}{z^2-(a+(1+\gamma)/a^*)z+a/a^*} \qquad (4)$$

where $\gamma = \sigma_x^2/\sigma_v^2$ is the SNR (signal–to–noise ratio). The MSE for this case is given by

$$\mathrm{MSE}_{\mathrm{MMSE-LE}} = \frac{\sigma_v^2}{2\pi j}\oint_{|z|=1}\frac{dz}{z(C(z)C^\dagger(z)+1/\gamma)} = -\frac{\sigma_v^2}{2\pi j a^*}\oint_{|z|=1}\frac{dz}{z^2-(a+(1+1/\gamma)/a^*)z+a/a^*} \qquad (5)$$

The poles are given by $p_{1,2} = \frac{1}{2a^*}\left(|a|^2+1+1/\gamma \pm \sqrt{(|a|^2+1+1/\gamma)^2-4|a|^2}\right)$ so that

$$\mathrm{MSE}_{\mathrm{MMSE-LE}} = \frac{\sigma_v^2}{\sqrt{\left(|a|^2+1+1/\gamma\right)^2-4|a|^2}}$$

We see that as the SNR tends to infinity, $\mathrm{MSE}_{\mathrm{MMSE-LE}} = \mathrm{MSE}_{\mathrm{ZF-LE}}$ which is to be expected.

The UMMSE equalizer is simply the MMSE equalizer scaled by the factor

$$L = \frac{1}{\sigma_x^2}\underbrace{\left(\frac{1}{2\pi j}\oint_{|z|=1}\frac{dz}{z}C^\dagger(z)S_{yy}^{-1}(z)C(z)\right)^{-1}}_{K}$$

which can be expressed in terms of $\mathrm{MSE}_{\mathrm{MMSE-LE}}$ as

$$L = \frac{1}{1-\mathrm{MSE}_{\mathrm{MMSE-LE}}/\sigma_x^2}$$

so that $H_{\mathrm{UMMSE-LE}} = LH_{\mathrm{MMSE-LE}}$. The MSE is given by

$$MSE_{\mathrm{UMMSE-LE}} = K - \sigma_x^2 = \sigma_x^2(L-1) = \frac{MSE_{\mathrm{MMSE-LE}}}{1-\frac{MSE_{\mathrm{MMSE-LE}}}{\sigma_x^2}}$$

Problem 4. Wiener Filtering and Zero-Forcing Linear Equalization of a Second-Order FIR Channel

(a) We get from the course notes

$$
\begin{aligned}
\text{MMSE} &= E\,\tilde{x}_k^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_{xx}(f)S_{vv}(f)}{S_{xx}(f)+S_{vv}(f)}df \\
&= \sigma_v^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_{xx}(f)}{S_{xx}(f)+S_{vv}(f)}df \;=\; \sigma_v^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)df \;=\; \sigma_v^2\, h_0 \;.
\end{aligned}
$$

(b) We have $\mathrm{H}_{ZF}(z) = \dfrac{1}{\mathrm{C}(z)}$. As stated in the course notes, we have to factor $\mathrm{C}(z)$ into its minimum-phase and maximum-phase factors since we need to take the causal inverse for the minimum-phase factor and the anticausal inverse for the maximum-phase factor in order to have a stable inverse. Now,

$$
\mathrm{C}(z) = 1 - \frac{5}{2}z^{-1} + z^{-2} = (1-\frac{1}{2}z^{-1})(1-2z^{-1}) = -2z^{-1}(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z) \;.
$$

So we get

$$
\begin{aligned}
\mathrm{H}_{ZF}(z) &= \frac{1}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{-\frac{1}{2}z}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{Bz}{1-\frac{1}{2}z} = \frac{(A-\frac{1}{2}B)+(B-\frac{1}{2}A)z}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}
\end{aligned}
$$

from which we find

$$
\begin{cases} A - \frac{1}{2}B = 0 \\ B - \frac{1}{2}A = -\frac{1}{2} \end{cases} \quad \Rightarrow \quad \begin{cases} A = -\frac{1}{3} \\ B = -\frac{2}{3} \end{cases}
$$

Hence

$$
\mathrm{H}_{ZF}(z) = -\frac{1}{3}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{2}{3}\frac{z}{1-\frac{1}{2}z}
$$

from which we can find the impulse response

$$
h_k^{ZF} = \begin{cases} -\frac{1}{3}\left(\frac{1}{2}\right)^k & ,\; k \geq 0 \\ -\frac{4}{3}\left(\frac{1}{2}\right)^{-k} & ,\; k < 0 \end{cases}
$$

(c) As the term Zero-Forcing tells us, there is no ISI at the equalizer output, thus we have the following scheme :

```
              --------      v_k      --------
x_k          |        |     |+      |        |     x_k+w_k
-------|  C(z)  |-----O------|  H(z)  |_____
             |        |             |        |
             |________|             |_______|
```

where $w_k = \text{H}(q)v_k$, hence $\widetilde{x}_k = w_k$ and MSE$=Ew_k^2$. From $S_{ww}(f) = |H(f)|^2 S_{vv}(f) = \sigma_v^2 |H(f)|^2$, we get

$$
\begin{aligned}
MSE_{ZF} &= E\,w_k^2 = \int_{-1/2}^{1/2} S_{ww}(f)df = \sigma_v^2 \int_{-1/2}^{1/2} |H(f)|^2 df = \sigma_v^2 \sum_{k=-\infty}^{\infty} h_k^2 \\
&= \sigma_v^2 \left(\frac{1}{9} \frac{1}{1-\frac{1}{4}} + \frac{4}{9} \frac{1}{1-\frac{1}{4}} \right) = \sigma_v^2\, \frac{5}{9} \frac{1}{1-\frac{1}{4}} = \sigma_v^2\, \frac{20}{27}\,.
\end{aligned}
$$

(d) With the MSE, we find immediately the $SNR_{ZF} = \sigma_x^2 / MSE_{ZF} = \dfrac{27}{20} \dfrac{\sigma_x^2}{\sigma_v^2} = 1.35\, \dfrac{\sigma_x^2}{\sigma_v^2}$. For the MFB on the other hand,

$$
MFB = \frac{\sigma_x^2}{\sigma_v^2} \int_{-1/2}^{1/2} |C(f)|^2 df = \frac{\sigma_x^2}{\sigma_v^2} \sum_{k=-\infty}^{\infty} c_k^2 = \frac{33}{4} \frac{\sigma_x^2}{\sigma_v^2} = 8.25\, \frac{\sigma_x^2}{\sigma_v^2}\,.
$$

So the MFB is $\dfrac{8.25}{1.35} = 6.11$ times better than the ZF-LE SNR.