Pb3 :

$$C(z) = 1 - a z^{-1} = \frac{z-a}{z} \Rightarrow C^{+}(z) = 1 - a^{*} z$$

$$\boxed{H_{ZF}(z) = \frac{1}{C(z)} = \frac{z}{z-a}} \quad \text{(lecture Note)}$$

$$MSE_{ZF} = \frac{\sigma_v^2}{2\pi j} \oint_{|z|=1} \frac{dz}{z} \frac{1}{C(z)\,C(\frac{1}{z})}$$

$$= \frac{\sigma_v^2}{2\pi j} \oint_{|z|=1} \frac{dz}{z} \frac{z}{(z-a)\,(1-a^{*}z)}$$

$$= \frac{\sigma_v^2}{2\pi j} \oint_{|z|=1} \frac{dz}{(z-a)(1-a^{*}z)}$$

$$= \frac{-\sigma_v^2}{2\pi j\, a^{*}} \oint_{|z|=1} \frac{dz}{(z-a)\,(z-1/a^{*})}$$

$$= \frac{-\sigma_v^2}{2\pi j\, a^{*}} \oint_{|z|=1} \frac{f(z)}{(z-a)}\, dz \quad \text{with } f(z) = \frac{1}{z - \frac{1}{a^{*}}}$$

we use the rule : $\oint_{|z|=1} dz \frac{f(z)}{z-z_0} = 2j\pi f(z_0)$ } $z_0$ should be inside the domain of integration (here it's the circle unity)

Then :

$$MSE_{ZF} = \frac{-\sigma_v^2}{2\pi j\, a^{*}} \times 2\pi j\, f(a)$$

→ here we assumed $|a| < 1$ so inside the domain of integration.

$$= -\frac{\sigma_v^2}{a^{*}} \frac{1}{a - \frac{1}{a^{*}}}$$

$$\boxed{MSE_{ZF} = \frac{\sigma_v^2}{1-|a|^2}}$$

when $|a| \rightarrow 1$, $MSE_{ZF} \longrightarrow \infty$ as the zero of $C(z)$ get close to the unit circle.

→ the phenomenon of noise enhancement.

$$H_{MMSE}(z) = \frac{\sigma_x^2 C^+(z)}{\sigma_x^2 C^+(z) C(z) + \sigma_v^2}$$

$$= \frac{C^+(z)}{C^+(z)\, C(z) + \frac{\sigma_v^2}{\sigma_x^2}}$$

Let's denote $\gamma = \frac{\sigma_x^2}{\sigma_v^2}$

$$\longrightarrow H_{MMSE}(z) = \frac{1 - a^* z}{(1 - a^* z)(1 - a z^{-1}) + \frac{1}{\gamma}}$$

$$= \frac{(1 - a^* z)}{z^{-1}(1 - a^* z)(z - a) + \frac{1}{\gamma}}$$

$$= \frac{z(1 - a^* z)}{(1 - a^* z)(z - a) + \frac{1}{\gamma} z}$$

$$= \frac{z(1 - a^* z)}{z - a - a^* z^2 + |a|^2 z + \frac{1}{\gamma} z}$$

$$= \frac{(-a^*) \times z(z - 1/a^*)}{-a^* z^2 + (1 + |a|^2 + 1/\gamma) z - a}$$

$$= \frac{z(z - 1/a^*)}{z^2 - \left[\frac{1 + |a|^2 + 1/\gamma}{a^*}\right] z + \frac{a}{a^*}}$$

$$\Longrightarrow \boxed{H_{MMSE}(z) = \frac{z(z - 1/a^*)}{z^2 - (a + (1 + 1/\gamma)/a^*) z + \frac{a}{a^*}}}$$

$$MSE_{MMSE} = \frac{\sigma_v^2}{2\pi j} \oint_{|z|=1} \frac{dz}{z} \frac{1}{C^{\dagger}(z)\, C(z) + \frac{\sigma_v^2}{\sigma_x^2}}$$

$$= \frac{\sigma_v^2}{2\pi j} \oint_{|z|=1} \frac{dz}{z} \frac{1}{(1-a^* z)(1-a z^{-1}) + \frac{1}{\gamma}}$$

$$= \frac{-\sigma_v^2}{2\pi j a^*} \oint_{|z|=1} \frac{1}{z^2 - (a + (1+1/\gamma)/a^*) z + \frac{a}{a^*}}\, dz$$

Let's consider: $P(z) = z^2 - (a + (1+1/\gamma)/a^*) z + \frac{a}{a^*}$

$$\Delta = (a + (1+1/\gamma)/a^*)^2 - 4\frac{a}{a^*} = \frac{(|a|^2 + 1 + 1/\gamma)^2 - 4|a|^2}{a^{*2}}$$

we are dealing with polynom on $z$, so the notion of $(+)$ or $(-)$ does not exist! (for Rim Zaher).

$f(\alpha) = (\alpha + 1 + \frac{1}{\gamma})^2 - 4\alpha > 0$

$(\alpha = |a|^2 < 1$ since $|a| < 1$ since it is stable).

hence the 2 square roots are:

$$P_1 = \frac{(|a|^2 + 1 + 1/\gamma) - \sqrt{(|a|^2 + 1 + 1/\gamma)^2 - 4|a|^2}}{2a^*}$$

$$P_2 = \frac{(|a|^2 + 1 + 1/\gamma) + \sqrt{(|a|^2 + 1 + 1/\gamma)^2 - 4|a|^2}}{2a^*}$$

so:

$$MSE_{MMSE} = \frac{-\sigma_v^2}{2\pi j a^*} \oint_{|z|=1} \frac{1}{(z-P_1)(z-P_2)}\, dz$$

$$= \frac{-\sigma_v^2}{2\pi j a^*} \oint_{|z|=1} \frac{F(z)}{(z-P_1)}\, dz = \frac{-\sigma_v^2}{2\pi j a^*} \times 2\pi j F(P_1)$$

where $F(z) = \frac{1}{(z-P_2)}$

$$\Longrightarrow MSE_{MMSE} = \frac{-\sigma_v^2}{2\pi j a^*} \frac{2\pi j}{(P_1 - P_2)} = \frac{-\sigma_v^2}{a^*} \cdot \frac{-a^*}{\sqrt{(|a|^2 + 1 + \frac{1}{\gamma})^2 - 4|a|^2}}$$

$$\boxed{MSE_{MMSE} = \frac{\sigma_v^2}{\sqrt{(|a|^2 + 1 + 1/\gamma)^2 - 4|a|^2}}}$$

when $|a| \to 1$, $MSE_{MMSE} = \dfrac{\sigma_v^2}{\sqrt{(2+\frac{1}{\gamma})^2 - 4}} = \dfrac{\sigma_v^2}{\sqrt{\frac{2}{\gamma} - \frac{1}{\gamma^2}}}$

$$= \frac{\sigma_x^2}{\sqrt{2\gamma - 1}}$$

$$\longrightarrow \boxed{MSE_{MMSE} = \frac{\sigma_x^2}{\sqrt{2\frac{\sigma_x^2}{\sigma_v^2} - 1}}}$$

the MSE doesn't diverge like it is the case for ZF.

we notice that when $\gamma \to \infty$, $MSE_{MMSE} = \dfrac{\sigma_v^2}{(1-|a|^2)} = MSE_{ZF}$ which is expected.

$$H_{UMMSE}(z) = \left(\frac{1}{2\pi j}\oint \frac{dz}{z} C^{+}(z) S_{yy}^{-1}(z) C(z)\right)^{-1} C^{+}(z) S_{yy}^{-1}(z)$$

Or $H_{MMSE}(z) = \sigma_x^2 C^{+}(z) S_{yy}^{-1}(z)$ (see your lecture notes)

$$\Rightarrow H_{UMMSE}(z) = \left(\frac{1}{2\pi j}\oint \frac{dz}{z} C^{+}(z) S_{yy}^{-1}(z) C(z)\right)^{-1} \frac{H_{MMSE}(z)}{\sigma_x^2}$$

$$= L \cdot H_{MMSE}(z)$$

where $L = \dfrac{1}{\sigma_x^2}\left[\dfrac{1}{2\pi j}\oint_{|z|=1} \dfrac{dz}{z} C^{+}(z) S_{yy}^{-1}(z) C(z)\right]^{-1}$

or according to the lecture notes:

$$MSE_{MMSE} = \sigma_x^2 \Bigg[1 - \underbrace{\frac{\sigma_x^2}{2\pi j}\oint \frac{dz}{z} C^{+}(z) S_{yy}^{-1}(z) C^{+}(z)}_{= L^{-1}}\Bigg]$$

$$\Rightarrow L = \frac{1}{1 - \frac{MSE_{MMSE}}{\sigma_x^2}} \quad (MSE_{MMSE} \text{ computed previously})$$

$$= \frac{1}{1 - \dfrac{1/\gamma}{\sqrt{(|a|^2+1+1/\gamma)^2 - 4|a|^2}}}$$

and so:

$$H_{UMMSE}(z) = L \, H_{MMSE}(z)$$

$$\Rightarrow \boxed{H_{UMMSE}(z) = \frac{\sqrt{(|a|^2+1+\frac{1}{\gamma})^2 - 4|a|^2}}{\sqrt{(|a|^2+1+\frac{1}{\gamma})^2 - 4|a|^2} - \frac{1}{\gamma}} \cdot H_{MMSE}(z)}$$

according to the lecture note:

$$MSE_{UMMSE} = \left[\frac{1}{2\pi j} \oint \frac{dz}{z} \, C^+(z) \, S_{yy}^{-1}(z) \, C(z)\right]^{-1} - \sigma_x^2$$

$$= \sigma_x^2 \left[ L - 1 \right]$$

$$= \sigma_x^2 \left[ \frac{1}{1 - \frac{MSE_{MMSE}}{\sigma_x^2}} - 1 \right]$$

$$= \sigma_x^2 \left[ \frac{MSE_{MMSE}/\sigma_x^2}{1 - \frac{MSE_{MMSE}}{\sigma_x^2}} \right]$$

$$\boxed{MSE_{UMMSE} = \frac{MSE_{MMSE}}{1 - \frac{MSE_{MMSE}}{\sigma_x^2}}}$$

when $|a| \to 1$:

$$MSE_{UMMSE} = \frac{\sigma_x^2}{\sqrt{2\frac{\sigma_x^2}{\sigma_v^2} - 1}} \cdot \frac{1}{1 - \frac{1}{\sqrt{2\frac{\sigma_x^2}{\sigma_v^2} - 1}}}$$

$$\boxed{MSE_{UMMSE} = \frac{\sigma_x^2}{\sqrt{2\gamma - 1} - 1}}$$

bP4:

$y_k = x_k + v_k$ , $v_k \sim \mathcal{N}(0, \sigma_v^2) \rightarrow S_{vv}(f) = \sigma_v^2$

$$y_k = x_k + v_k \longrightarrow \boxed{H(z)} \longrightarrow \hat{x}_k$$

a) $\text{MMSE} = E\left[(\tilde{x}_k)^2\right]$

$$\tilde{x}_k = x_k - \hat{x}_k$$

according to the lecture note:

$$\text{MMSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_{xx}(f)\, S_{vv}(f)}{S_{xx}(f) + S_{vv}(f)}\, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sigma_v^2\, S_{xx}(f)}{S_{vv}(f) + S_{xx}(f)}\, df$$

or $\dfrac{S_{xx}(f)}{S_{vv}(f) + S_{xx}(f)} = H(f)$ (see lecture)

$$\Longrightarrow \text{MMSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_v^2\, H(f)\, df$$

$$= \sigma_v^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)\, df$$

$$= \sigma_v^2 \cdot h_0$$

In fact, $H(f) = \sum_{m=-\infty}^{+\infty} h_m e^{-2j\pi f m}$

$$= h_0 + \sum_{m \in \mathbb{Z}\setminus\{0\}} h_m e^{-2j\pi f m}$$

$$\longrightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)\, df = \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} h_0\, df}_{= h_0} + \sum_{m \in \mathbb{Z}\setminus\{0\}} h_m \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2j\pi f m}\, df$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2j\pi f m}\, df = \left[\frac{e^{-2j\pi f m}}{-2j\pi m}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{e^{-j\pi m} - e^{+j\pi m}}{-2j\pi m} = \frac{1-1}{-2j\pi m} = 0$$

$$\Rightarrow \boxed{MMSE = \sigma_v^2 h_0}$$

b)-
$$H_{ZF}(z) = \frac{1}{C(z)} = \frac{1}{z^{-2} - \frac{5}{2} z^{-1} + 1}$$
$$= \frac{z^2}{z^2 - \frac{5}{2} z + 1}$$

we have:

$$C(z) = z^{-2} - \frac{5}{2} z^{-1} + 1$$

$$\Delta = \frac{25}{4} - 4 = \frac{9}{4}$$

$$\Rightarrow \begin{cases} z_1 = \dfrac{\frac{5}{2} - \frac{3}{2}}{2} = \frac{1}{2} \\ z_2 = \dfrac{\frac{5}{2} + \frac{3}{2}}{2} = 2 \end{cases}$$

$$\Rightarrow C(z) = \left(z^{-1} - \frac{1}{2}\right)\left(z^{-1} - 2\right)$$
$$= \left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2 z^{-1}\right)$$

$$\Rightarrow H_{ZF}(z) = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2 z^{-1}\right)}$$
$$= \frac{\alpha}{1 - \frac{1}{2} z^{-1}} + \frac{\beta}{1 - 2 z^{-1}}$$

$$\lim_{\substack{z \to \frac{1}{2} \\ (z^{-1} \to 2)}} \left(1 - \frac{1}{2} z^{-1}\right) H_{ZF}(z) = \alpha = \frac{-1}{3}$$

$$\lim_{\substack{z \to 2 \\ (z^{-1} \to \frac{1}{2})}} \left(1 - 2 z^{-1}\right) H_{ZF}(z) = \beta = \frac{4}{3}$$

$$\Rightarrow H_{ZF}(z) = -\frac{1}{3} \underbrace{\frac{1}{1 - \frac{1}{2} z^{-1}}}_{\text{minimum phase factor}} + \frac{4}{3} \underbrace{\frac{1}{1 - 2 z^{-1}}}_{\text{maximum phase factor}} = \frac{-1}{3} \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{2}{3} \frac{z}{1 - \frac{1}{2} z}$$
$$= \frac{-1}{3} H_1(z) + \frac{4}{3} H_2(z)$$

or notice that:

$$\star \quad H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \lim_{K\to+\infty} \frac{1-\left(\frac{1}{2}z^{-1}\right)^K}{1-\frac{1}{2}z^{-1}} = \sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k z^{-k}$$

$\longrightarrow \quad h_1(n) = \left(\frac{1}{2}\right)^n u(n)$ where $u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{else.} \end{cases}$

notice that $\lim_{K\to+\infty}\left(\frac{1}{2}z^{-1}\right)^K$ is finit (convergence) when $\left|\frac{1}{2}z^{-1}\right| < 1$

$\longrightarrow \quad \boxed{\frac{1}{2} < |z|}$ (it contains the unit circle)

$$\star \quad H_2(z) = \frac{1}{1-2z^{-1}} = \lim_{K\to-\infty} \frac{1-\left(2z^{-1}\right)^K}{1-2z^{-1}} = -\sum_{k=-\infty}^{-1} 2^k z^{-k}$$

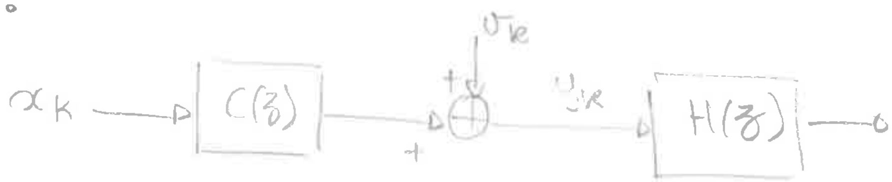
$$= -\sum_{k=-\infty}^{-1}\left(\frac{1}{2}\right)^{-k} z^{-k}$$

$\longrightarrow \quad h_2(n) = -\left(\frac{1}{2}\right)^{-n} u(-n-1)$ $\left(\begin{array}{c}\text{domain of convergence:} \\ |z| < 2\end{array}\right) \Rightarrow$ (it contains the unit circle)

$$\Rightarrow \quad \boxed{h(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3}\left(\frac{1}{2}\right)^{-n} u(-n-1)}$$

c) $$MSE_{ZF} = E\left[\tilde{x}_k^2\right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\tilde{x}\tilde{x}}(f)\, df$$

or according to the previous questions, we can deduce this schema:




$$\hat{x}_k = \underbrace{H(q)C(q)}_{1} x_k + H(q)v_k$$
$$= x_k + H(q)v_k.$$
$$= x_k + \tilde{x}_k$$

hence: $S_{\tilde{x}\tilde{x}}(f) = |H(f)|^2 \sigma_v^2$

$$\longrightarrow \quad MSE_{ZF} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2\, df\right)\sigma_v^2$$

$$= \sum_{k=-\infty}^{+\infty} h^2(n) \cdot \sigma_v^2.$$

$$= \sigma_v^2\left[\left(\frac{1}{3}\right)^2 \sum_{m=0}^{+\infty}\left(\frac{1}{2}\right)^{2n} + \left(\frac{4}{3}\right)^2 \sum_{m=-\infty}^{-1}\left(\frac{1}{2}\right)^{-2n}\right] = \left(\frac{1}{9}\,\frac{1}{1-\frac{1}{4}} + \frac{16}{9}\cdot\frac{1}{4}\,\frac{1}{1-\frac{1}{4}}\right)\sigma_v^2$$

$$\boxed{MSE_{ZF} = \frac{20}{27}\sigma_V^2}$$

d) According to the lecture notes:

$$SNR_{ZF} = \frac{\sigma_x^2}{MSE_{ZF}} = \frac{\sigma_x^2}{\frac{20}{27}\sigma_V^2}$$

$$\Rightarrow \boxed{SNR_{ZF} = \frac{27}{20}\frac{\sigma_x^2}{\sigma_V^2}}$$

According to the lecture notes:

$$SNR_{MBF} = \frac{\sigma_x^2}{\sigma_V^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} |C(f)|^2 df$$

$$= \frac{\sigma_x^2}{\sigma_V^2}\sum_{k=-\infty}^{+\infty} c_k^2$$

or since $C(z) = 1 - \frac{5}{2}z^{-1} + z^{-2} \rightarrow \begin{cases} C_0 = 1 \\ C_{-1} = -5/2 \\ C_{-2} = 1 \end{cases}$

hence $SNR_{MBF} = \frac{\sigma_x^2}{\sigma_V^2}\Big(\overbrace{1 + 1 + \tfrac{25}{4}}^{33/4}\Big)$

$$\boxed{SNR_{MBF} = \frac{33}{4}\frac{\sigma_x^2}{\sigma_V^2}}$$

$$\Rightarrow \frac{SNR_{MBF}}{SNR_{ZF}} = \frac{33/4}{27/20} = 6.11 \Rightarrow$$ MBF is 6.11 better than ZF.

Pb5

a) we have $E[Y_k] = C\,E[\overset{0}{\overset{\shortparallel}{S_k}}] + E[\overset{0}{\overset{\shortparallel}{V_k}}] = 0$

hence $R_{YY} = E[Y_k Y_k^T] = E[(CS_k+V_k)(CS_k+V_k)^T]$

$$= C\,\overbrace{E[S_kS_k^T]}^{\sigma_s^2 I_{N+L-1}}C^T + E[V_kV_k^T] + C\,E[\underbrace{S_kV_k^T}_{0}] + E[\underbrace{V_kS_k^T}_{0}]C^T$$

$$= \sigma_s^2 CC^T + \sigma_v^2 I_N$$

$\rightarrow$ $\boxed{R_{YY} = \sigma_s^2 CC^T + \sigma_v^2 I_N}$

$$R_{Yx^{(d)}} = E\Big[(CS_k+V_k)\underbrace{s_{k-d}^T}_{s_{k-d}\ \text{(scalar)}}\Big] = C\,E[S_k s_{k-d}] + E[\underbrace{V_k s_{k-d}}_{0}]$$

$\{s_k\}$ and $\{V_k\}$ independent process.

we have: 
$$E[S_k s_{k-d}] = \begin{bmatrix} E[s_k s_{k-d}] \\ \vdots \\ E[s_{k-d}^2] \\ \vdots \\ E[s_{k-N-L+2}\, s_{k-d}] \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E[s_{k-d}^2] \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (d+1)^{th}\ \text{row}$$

($\{s_k\}$ is white noise process)

$$= \sigma_s^2 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow R_{Yx^{(d)}} = \sigma_s^2\, C \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_s^2 \begin{array}{c} \phantom{c_0\cdots}(d+1)^{th}\ \text{column} \\ \left[\underline{c}_0 \cdots \underline{c}_d \cdots \underline{c}_{N+L-2}\right] \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \sigma_s^2\, \underline{c}_d$$

$\rightarrow$ $\boxed{R_{Yx^{(d)}} = \sigma_s^2\, \underline{c}_d}$ $\leftarrow$ columnvector $L\times 1$

and $R_{x^{(d)}Y} = R_{Yx^{(d)}}^T = \sigma_s^2\, \underline{c}_d^T \leftarrow$ row-vector $1\times L$

b) $$\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = R_{x^{(d)}x^{(d)}} - R_{x^{(d)}y} R_{yy}^{-1} R_{yx^{(d)}}$$

$$= E[s_{k-d}^2] - \sigma_s^2 \underline{c}_d^T \left[\sigma_s^2 CC^T + \sigma_v^2 I_N\right]^{-1} \sigma_s^2 \underline{c}_d$$

$$\boxed{\sigma_{\tilde{x}^{(d)}_{MMSE}} = \sigma_s^2 \left[1 - \underline{c}_d^T \left[CC^T + \frac{\sigma_v^2}{\sigma_x^2} I_N\right]^{-1} \underline{c}_d\right]}$$

$$\boxed{SNR^{(d)}_{MMSE} = \frac{1}{1 - \underline{c}_d^T \left[CC^T + \frac{\sigma_v^2}{\sigma_x^2} I_N\right]^{-1} \underline{c}_d}}$$

c) $$R_{yy} = \sigma_s^2 \underbrace{CC^T}_{N\times N} + \sigma_v^2 I_N$$

we have $C$ Toeplitz matrix, henc $CC^T$ is Block Toeplitz (instead of repeated elements along diagonals like in Toeplitz matrix, we have here blocks that are repeated along diagonals), the scalar $\sigma_s^2$ and the diagonal matrix $\sigma_v^2 I_N$ doesn't influence the resulting shape of $R_{yy}$.

- first row of $R_{yy} \Rightarrow R_{yy}(1,:)$

$$R_{yy}(1,:) = \sigma_s^2 [\overbrace{c_0\ c_1 \cdots c_{L-1}}^{L}\ \overbrace{0 \cdots 0}^{N-1}] \begin{bmatrix} \underline{c}_0^T \\ \underline{c}_1^T \\ \vdots \\ \underline{c}_{N+L-2}^T \end{bmatrix} + \sigma_v^2 [1\ 0 \cdots 0]$$

$$R_{yy}(1,1) = \sigma_s^2 \sum_{i=0}^{L-1} c_i^2 + \sigma_v^2$$

$$R_{yy}(1,2) = \sigma_s^2 \sum_{i=0}^{L-2} c_i c_{i+1}$$

$$\vdots$$

$$R_{yy}(1,k+1) = \sigma_s^2 \sum_{i=0}^{L-k-1} c_i c_{i+k}$$

$$\vdots$$

$$R_{yy}(1,L) = \sigma_s^2 c_{L-1} c_0$$

$$\Rightarrow \boxed{R_{yy}(1,k+1) = \sigma_s^2 \sum_{i=0}^{L-k-1} c_i c_{i+k} + \delta_{k0}\sigma_v^2}$$

delay $0 \leq k \leq L-1$ $$= \sigma_s^2 \sum_{i=0}^{L-1} c_i c_{i+k} + \delta_{k0}\sigma_v^2$$

d) $\sigma_V^2 = 0$

$$\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = \sigma_\Delta^2 \left( 1 - \underline{c}_d^T \left[ C C^T \right]^{-1} \underline{c}_d \right)$$

we have: $\underline{c}_d = C \cdot \underline{e}_d \Longrightarrow \sigma^2_{\tilde{x}^{(d)}_{MMSE}} = \sigma_\Delta^2 \Big[ \underbrace{1}_{\underline{e}_d^T I \underline{e}_d} - \underline{e}_d^T \overbrace{C^T \underbrace{(C C^T)^{-1}}_{N \times N} C}^{N+L-1 \times N+L-1}_{= P_{C^T}} \underline{e}_d \Big]$

$$= \sigma_\Delta^2 \, \underline{e}_d^T \underbrace{\left[ I_{N+L-1} - C^T (C C^T)^{-1} C \right]}_{P^{\perp}_{C^T}} \underline{e}_d$$

$$\longrightarrow \boxed{\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = \sigma_\Delta^2 \, \underline{e}_d^T P^{\perp}_{C^T} \underline{e}_d}$$

e) $C(z) = \sum_{k=0}^{L-1} c_k z^{+k} \Longrightarrow C(z_i) = \sum_{k=0}^{L-1} c_k z_i^{+k} = 0, \quad i = 1 \ldots L-1$

we have to show that: $P_{C^T} + P_V = I \qquad \left( P_V = P^{\perp}_{C^T} \right)$

$$P_{C^T} + P_V = I \iff C^T (C C^T)^{-1} C + V (V^T V)^{-1} V^T = I$$

$$\iff \begin{bmatrix} C^T & V \end{bmatrix} \begin{bmatrix} (C C^T)^{-1} & 0 \\ 0 & (V^T V)^{-1} \end{bmatrix} \begin{bmatrix} C \\ V^T \end{bmatrix} = I$$

so if we prove the last line, we show that $P_V = P^{\perp}_{C^T}$.

we notice that:

$$C \cdot V = \begin{bmatrix} c_0 & c_1 & \cdots & c_{L-1} & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & & c_{L-1} & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \; c_0 & c_1 & \cdots & c_{L-1} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_{L-1} \\ \vdots & & \vdots \\ z_1^{N+L-2} & \cdots & z_{L-1}^{N+L-2} \end{bmatrix}$$

$$= \begin{bmatrix} C(z_1) & C(z_2) & \cdots & C(z_{L-1}) \\ z_1 C(z_1) & z_2 C(z_2) & \cdots & z_{L-1} C(z_{L-1}) \\ z_1^2 C(z_1) & z_2^2 C(z_2) & \cdots & z_{L-1}^2 C(z_{L-1}) \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} C(z_1) & z_2^{N-1} C(z_2) & \cdots & z_{L-1}^{N-1} C(z_{L-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_{L-1}^{N-1} \end{bmatrix} \underbrace{\begin{bmatrix} C(z_1) & 0 & \cdots & 0 \\ 0 & C(z_2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C(z_{L-1}) \end{bmatrix}}_{= \underline{0}_{L\times L-1}}$$

$$= \underline{0}_{N\times L-1}$$

$$\Longrightarrow \quad C.V = 0 \quad \text{and so } V^T C^T = 0 .$$

We notice then that $[C^T V]$ is a square matrix of full rank $(N+L-1)\times(N+L-1)$, in fact $V_{(N+L-1)\times(L-1)}$ and $C^T_{(N+L-1)\times N}$ :

$$\begin{array}{c|c} & \overset{N}{\longleftrightarrow} \quad \overset{L-1}{\longleftrightarrow} \\ N+L-1 \updownarrow & \left[\begin{array}{c|c} C^T & V \end{array}\right] \end{array}$$

full rank since $V$ is full rank column (the roots of $C$ are distinct) and $C^T$ is full rank column too.

This property permits us to write :

$$[C^T V] \underbrace{\left([C^T V]^T [C^T V]\right)^{-1}} [C^T V]^T = I_{N+L-1}$$

$$= \begin{bmatrix} C C^T & \overset{=0}{C V} \\ \underset{=0}{V^T C^T} & V^T V \end{bmatrix}^{-1} = \begin{bmatrix} (CC^T)^{-1} & 0 \\ 0 & (V^T V)^{-1} \end{bmatrix}$$

$$\Rightarrow \quad I_{N+L-1} = [C^T V] \begin{bmatrix} (CC^T)^{-1} & 0 \\ 0 & (V^T V)^{-1} \end{bmatrix} \begin{bmatrix} C \\ V^T \end{bmatrix}$$

$$\Rightarrow \quad \boxed{I_{N+L-1} = P_V + P_{C^T}} \quad \Rightarrow \quad P_V = P_{C^T}^{\perp}$$

and so $\boxed{\sigma^2_{\hat{x}(d)\,MMSE} = \sigma_s^2\, \underline{e}_d^T P_V\, \underline{e}_d}$

f) $$\text{av-MSE} = \frac{1}{N+L-1} \sum_{d=0}^{N+L-2} \sigma^2_{\tilde{x}(d)_{MMSE}}$$

$$= \frac{1}{N+L-1} \sum_{d=0}^{N+L-2} \sigma_s^2 \, \underline{e}_d^T P_V \underline{e}_d$$

$$= \frac{\sigma_s^2}{N+L-1} \sum_{d=0}^{N+L-2} \operatorname{tr}\left(P_V \underline{e}_d \underline{e}_d^T\right)$$

$$= \frac{\sigma_s^2}{N+L-1} \sum_{d=0}^{N+L-2} P_V(d,d)$$

$$= \frac{\sigma_s^2}{N+L-1} \operatorname{tr}\left(P_V\right)$$

$$= \frac{\sigma_s^2}{N+L-1} \operatorname{tr}\left(V(V^T V)^{-1} V^T\right)$$

$$= \frac{\sigma_s^2}{N+L-1} \operatorname{tr}\left[\underbrace{(V^T V)^{-1}(V^T V)}_{= I_{L-1}}\right]$$

$$= \frac{L-1}{N+L-1} \sigma_s^2$$

$$\Rightarrow \boxed{\text{average MSE} = \frac{L-1}{N+L-1} \sigma_s^2}$$

we notice that when $\frac{N}{L} \longrightarrow \infty$, $MSE \longrightarrow 0$

$\rightarrow$ it converge to ZF in absence of Noise.