EUREKOM 22/06/2009

SSP Statistical Signal Processing

Exam

You can answer the questions in French or in English. If you get stuck in a particular part of a question, do not hesitate to try to answer the rest of the question or start the next question. Results that are available in the course notes can be quoted and do not need to be rederived.

Parameter Estimation

1. ML Estimation of Roundtrip Delay Distribution.

Assume that for the roundtrip delay in a computer network, as considered in the homework, we now consider a truncated exponential distribution:

$$f(y|\lambda,\alpha,\beta) = \left\{ \begin{array}{ll} 0 & , y < \alpha \\ \gamma\, e^{-\lambda\, y} & , \alpha \leq y \leq \beta \\ 0 & , \beta < y \end{array} \right\} = \gamma\, e^{-\lambda\, y}\; 1_{[\alpha,\beta]}(y) \tag{1}$$

where γ is a normalization constant and

$$1_{\mathcal{A}}(y) = \left\{ \begin{array}{ll} 1 & , \;\; y \in \mathcal{A} \\ 0 & , \;\; y \notin \mathcal{A} \end{array} \right.$$

is the indicator function for the set $\mathcal{A}$.

(a) Determine the normalization constant γ as a function of λ, α and β.

In what follows, you need to substitute γ in $f(y|\lambda,\alpha,\beta)$ by this function of λ, α and β. If you cannot find γ, just leave it as γ.

(b) We now collect n i.i.d. measurements y_i into the vector Y. Assume for the moment that $\lambda > 0$ is a given constant.
Find the likelihood function $l(\alpha,\beta|Y,\lambda)$ for α and β given Y and λ.
Note that $1_{[\alpha,\beta]}(y) = 1_{[\alpha,\infty)}(y)\; 1_{(-\infty,\beta]}(y)$.

(c) Maximize this likelihood function to determine the Maximum Likelihood (ML) estimate of α and β on the basis of Y and λ (temporarily assumed to be given).

(d) Consider now also λ as unknown and determine its ML estimate.

In what follows, consider the special case in which $\alpha = 0$, $\beta = \infty$, i.e. the untruncated exponential distribution case. In this case, λ is the only remaining parameter in the distribution $f(y|\lambda)$.

(e) Determine the mean $m_y = \mathrm{E}\; y$ and the variance $\sigma_y^2 = \mathrm{E}\; y^2 - (\mathrm{E}\; y)^2$ as a function of λ.

(f) Determine the log likelihood function from n i.i.d. measurements Y, $L(\lambda|Y) = \ln f(Y|\lambda)$.

(g) Determine the Maximum Likelihood (ML) estimate $\widehat{\lambda}_{ML}$ and express it as a function of the sample mean $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

(h) Show that in this case the ML estimate can be interpreted as an application of the method of moments.

(i) Asymptotically ($n \gg 1$), the estimation error $\overline{y} - m_y$ will be very small. Hence develop $\widehat{\lambda}_{ML}$ up to first order in $\overline{y} - m_y$. From this asymptotic expression of $\widehat{\lambda}_{ML}$, obtain the asymptotic mean $m_{\widehat{\lambda}_{ML}}$ of $\widehat{\lambda}_{ML}$ and asymptotic variance (for large but finite n) $\sigma^2_{\widetilde{\lambda}_{ML}}$ of $\widetilde{\lambda}_{ML} = \lambda - \widehat{\lambda}_{ML}$. Express both $m_{\widehat{\lambda}_{ML}}$ and $\sigma^2_{\widetilde{\lambda}_{ML}}$ in terms of λ.
Is $\widehat{\lambda}_{ML}$ asymptotically unbiased?

(j) Determine the Fisher Information and the Cramer-Rao bound (CRB) for any unbiased estimator $\widehat{\lambda}$.
Is ML asymptotically efficient in this case?

Spectrum Estimation

2. Maximum-Entropy Covariance Extension

Assume that we have measured the following correlations at lags 0,1,2 for a stationary process: $r_0 = 1$, $r_1 = a$ and $r_2 = a^2$. We shall compute the correlations at the remaining lags by performing maximum-entropy covariance extension. Assume that the process has zero mean.

(a) From the above information, can you infer bounds on the possible values for a?

(b) Maximum-entropy covariance extension is based on autoregressive modeling. Since we are given correlations up to lag 2, we shall perform linear prediction of order 2. Compute the prediction coefficients (suggestion: using the Levinson algorithm may be helpful, though is not strictly necessary).

(c) Using these prediction coefficients, find all the remaining correlation coefficients, on the basis of the autoregressive model assumption.

We shall repeat the same steps, but now for the following three correlations:
$r_0 = 1$, $r_1 = a$, $r_2 = 1$.

(d) Repeat question (a).

(e) Repeat question (b).

(f) Repeat question (c).

(g) What is the structure of the stationary process in this case?

Wiener Filtering and Equalization

3. FIR (U)MMSE Linear Equalization of a FIR Channel

Consider a causal FIR equalizer H with N coefficients, $\widehat{x}_k = H^T\, Y_k$. For a FIR channel of length L, the received signal vector Y_k can be written as

$$\begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-N+1} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{L-1} & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & c_{L-2} & c_{L-1} & \ddots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_0 & \cdots & \cdots & c_{L-1} \end{bmatrix} \begin{bmatrix} s_k \\ s_{k-1} \\ \vdots \\ s_{k-N-L+2} \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k-1} \\ \vdots \\ v_{k-N+1} \end{bmatrix}$$

or

$$\underbrace{Y_k}_{N\times 1} = \underbrace{C}_{N\times(N+L-1)} \underbrace{S_k}_{(N+L-1)\times 1} + \underbrace{V_k}_{N\times 1} = [\underline{c}_0\ \underline{c}_1\ \cdots\ \underline{c}_{N+L-2}]\, S_k + V_k = \sum_{i=0}^{N+L-2} \underbrace{\underline{c}_i}_{N\times 1}\, s_{k-i} + V_k\ .$$

The vector $\underline{c}_i$ is column $i{+}1$ of the matrix C. The symbol sequence s_k is considered to be white noise with zero mean and variance σ_s^2. The additive noise v_k is independent of the symbol sequence and white Gaussian with zero mean and variance σ_v^2. We shall see that it may be advantageous to introduce an equalization delay d. Hence consider $x_k^{(d)} = s_{k-d}$, $d \in \{0, 1, \ldots, N{+}L{-}2\}$.
MMSE FIR equalization is a particular instance of FIR Wiener filtering. Hence the MMSE FIR equalizer coefficients $H_{MMSE}^{(d)}$ satisfy the normal equations
$R_{YY}\, H_{MMSE}^{(d)} = R_{Yx^{(d)}}$ or hence $H_{MMSE}^{(d)} = R_{YY}^{-1}\, R_{Yx^{(d)}}$, and
the MMSE is $\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = R_{x^{(d)}x^{(d)}} - R_{x^{(d)}Y} R_{YY}^{-1}\, R_{Yx^{(d)}}$.

(a) Determine R_{YY} in terms of σ_s^2, σ_v^2, the matrix C and the identity matrix I_N, and determine $R_{Yx^{(d)}}$ in terms of σ_s^2 and the vector $\underline{c}_d$.

(b) Express the MMSE $\sigma^2_{\tilde{x}^{(d)}_{MMSE}}$ in terms of these same quantities.

The corresponding (naive) SNR is $\mathrm{SNR}^{(d)}_{MMSE} = \dfrac{\sigma_s^2}{\sigma^2_{\tilde{x}^{(d)}_{MMSE}}}$.

In what follows, we shall consider the specific case of an equalizer with $N = 2$ coefficients, a channel with $L = 2$ coefficients $c_0 = 1$, $c_1 = a$, and no noise $\sigma_v^2 = 0$. The range of possible delays is now limited to $d \in \{0, 1, 2\}$. In the absence of noise, the MSE is determined by intersymbol interference which is unavoidable here with an FIR equalizer.

(c) Compute the MMSE $\sigma^2_{\tilde{x}^{(d)}_{MMSE}}$ for $d = 0, 1, 2$, in terms of σ_s^2 and a.

(d) Compute, for $d = 0, 1, 2$, $\mathrm{SNR}^{(d)}_{UMMSE}$ (via $\mathrm{SNR}^{(d)}_{UMMSE}{=}\mathrm{SNR}^{(d)}_{MMSE}{-}1$) in terms of a.

(e) Determine the optimal delay $d \in \{0, 1, 2\}$ as a function of $|a|$ (optimal in the sense of maximizing $\mathrm{SNR}^{(d)}_{UMMSE}$).

Adaptive Filtering

4. Determination of the FIR Equalizer via the Steepest-Descent/LMS Algorithm

Assume we are applying the Steepest-Descent algorithm to the equalizer problem in question 3, for a FIR equalizer with $N = 2$ coefficients and a FIR channel with $L = 2$ coefficients $c_0 = 1$, $c_1 = a$. Consider again the noiseless case, so

$$R_{YY} = \sigma_s^2 \begin{bmatrix} 1+a^2 & a \\ a & 1+a^2 \end{bmatrix} .$$

(a) Find the eigenvalues λ_1, λ_2 of R_{YY} (order them so that $\lambda_1 \geq \lambda_2$) and associated eigenvectors V_1, V_2.
Hint: express the λ_i in terms of $|a|$ (note well: $|a|$ and not just a).

(b) When running the Steepest-Descent iterative algorithm in this particular case, what are the bounds on the stepsize μ (as a function of $|a|$) so that convergence occurs (in other words, $? < \mu < ?$).

(c) What is the value for μ for fastest convergence?

(d) With the stepsize as in (c), what is the slowest mode?

(e) How does the equalizer delay influence the convergence of the Steepest-Descent algorithm?