Exam

You can answer the questions in French or in English. If you get stuck in a particular part of a question, do not hesitate to try to answer the rest of the question or start the next question. Results that are available in the course notes can be quoted and do not need to be rederived.

Bayesian Parameter Estimation

1. Bayesian Level Estimation in Gaussian Noise with Laplacian Prior

Assume that we obtain the following noisy measurements

$$y_i \;=\; \theta + v_i\;, \quad i = 1, \ldots, n \tag{1}$$

of the unknown level θ with Laplacian prior distribution

$$f_\theta(x) \;=\; \mu\, e^{-\lambda\,|x|} \quad , \quad -\infty < x < \infty \tag{2}$$

and the measurement noise is i.i.d. and Gaussian: $v_i \sim \mathcal{N}(0, \sigma_v^2)$, and independent of θ. The quantities λ and σ_v^2 are known.

(a) Given λ, determine μ.

(b) Determine the prior mean m_θ.

(c) Determine the prior variance σ_θ^2.

(d) Compute the MAP estimator of θ given $Y = [y_1 \cdots y_n]^T$. Express $\widehat{\theta}_{MAP}$ in terms of $\overline{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$.
Hints: express $\overline{y} = |\overline{y}|\,\mathrm{sign}(\overline{y})$, $\theta = |\theta|\,\mathrm{sign}(\theta)$. First find $\mathrm{sign}(\theta)$ and then $|\theta|$, for which two cases can occur.

(e) Is $\widehat{\theta}_{MAP}$ unbiased (in a Bayesian sense)?

Turn the page please.

Deterministic Parameter Estimation

2. Deterministic Level Estimation in Laplacian Noise

Assume that we obtain the following noisy measurements

$$y_i \;=\; \theta + v_i\;, \quad i = 1, \ldots, n \tag{3}$$

of the unknown level θ and the measurement noise is i.i.d. with Laplacian distribution

$$f_v(x) \;=\; \frac{\delta}{2}\, e^{-\delta\,|x|}\;. \tag{4}$$

(a) Determine σ_v^2 as a function of δ.

(b) Show that the maximum likelihood estimator is $\widehat{\theta}_{ML} = \arg\min_{\theta} \sum_{i=1}^{n} |y_i - \theta|$.

(c) We shall find $\widehat{\theta}_{ML}$ explicitly as a function of $Y = [y_1\; y_2 \cdots y_n]^T$ by proceeding in small steps:

  (i) By drawing $g(\theta) = \sum_{i=1}^{n} |y_i - \theta|$, find $\widehat{\theta}_{ML}$ for $n = 1, 2, 3, 4$.
  Hint: let $\{\tilde{y}_i,\; i = 1, \ldots, n\}$ be a reordering of $\{y_i,\; i = 1, \ldots, n\}$ so that $\tilde{y}_1 \le \tilde{y}_2 \le \cdots \le \tilde{y}_n$.

  (ii) Extrapolate now your results from (i) to give $\widehat{\theta}_{ML}$ for general n. Distinguish the cases n even and n odd. In particular for n odd, what is the name of the function $\widehat{\theta}_{ML}(Y)$?

(d) We'll consider now the BLUE estimator.

  (i) Give $\widehat{\theta}_{BLUE}(Y)$.

  (ii) Give $\mathrm{E}\;\widehat{\theta}_{BLUE}(Y)$. Is $\widehat{\theta}_{BLUE}(Y)$ biased?

  (iii) Compute $R_{\widetilde{\theta}_{BLUE}\widetilde{\theta}_{BLUE}}$ and express it in terms of σ_v^2.

(e) Compute the Fisher information and the CRB and express both in terms of σ_v^2.

(f) Is the BLUE estimator efficient?

(g) Asymptotically, as $n \to \infty$, give $R_{\widetilde{\theta}_{ML}\widetilde{\theta}_{ML}}$ as a function of σ_v^2. Explain (remember the asymptotic properties of ML). Compare to $R_{\widetilde{\theta}_{BLUE}\widetilde{\theta}_{BLUE}}$.

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Wiener and Adaptive Filtering

3. Wiener Filtering vs Instantaneous LMMSE Estimation for Signal in Noise

We consider the signal in noise case, in which we estimate x_k from a noisy version $y_k = x_k + v_k$. We have seen that Wiener filter and MMSE are all determined in terms of the signal and noise spectra $S_{xx}(f)$ and $S_{vv}(f)$. In particular, the Wiener filter is

$$H(f) \;=\; \frac{S_{xx}(f)}{S_{xx}(f) + S_{vv}(f)} \;\in\; [0,1]$$

and the associated MMSE is

$$\mathrm{MMSE}_{WF} = \mathrm{E}\; \widetilde{x}_k^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\widetilde{x}\widetilde{x}}(f)\, df \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_{xx}(f)\, S_{vv}(f)}{S_{xx}(f) + S_{vv}(f)} df$$

(a) Now consider the basic instantaneous LMMSE estimation from chapter 1. So here we estimate x_k as $\hat{x}_k \;=\; h\, y_k$ where we have seen that the LMMSE choice for h is $h = r_{xy} r_{yy}^{-1}$.

Express h in terms of $S_{xx}(f)$ and $S_{vv}(f)$.

Remember that e.g. $r_{xy} = r_{xy}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xy}(f)\, df$.

(b) We have seen that the associated MMSE is

$$\mathrm{LMMSE} = r_{\widetilde{x}\widetilde{x}} = r_{xx} - r_{xy} r_{yy}^{-1} r_{yx}\;.$$

Express LMMSE in terms of $S_{xx}(f)$ and $S_{vv}(f)$.

(c) Can you find an inequality between MMSE_{WF} and LMMSE? Prove the inequality.

For what follows, consider the specific case in which the signal is lowpass and the noise is complementary highpass:

$$S_{xx}(f) \;=\; \begin{cases} \sigma^2 & , \; |f| \le f_c \\ 0 & , \; f_c \le |f| \le \frac{1}{2} \end{cases}$$

and

$$S_{vv}(f) \;=\; \begin{cases} 0 & , \; |f| \le f_c \\ \sigma^2 & , \; f_c \le |f| \le \frac{1}{2} \end{cases}$$

(d) Compute the LMMSE coefficient h in this case.

(e) Compute the LMMSE $r_{\widetilde{x}\widetilde{x}}$ in this case.

(f) Compute the Wiener filter $H(f)$ in this case.

(g) Compute MMSE_{WF} in this case.

Turn the page please.

4. FIR (U)MMSE Linear Equalization of a FIR Channel

Consider a causal FIR equalizer H with N coefficients, $\widehat{x}_k = H^T\, Y_k$. For a FIR channel of length L, the received signal vector Y_k can be written as

$$\begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-N+1} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{L-1} & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & c_{L-2} & c_{L-1} & \ddots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_0 & \cdots & \cdots & c_{L-1} \end{bmatrix} \begin{bmatrix} s_k \\ s_{k-1} \\ \vdots \\ s_{k-N-L+2} \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k-1} \\ \vdots \\ v_{k-N+1} \end{bmatrix}$$

or $\underbrace{Y_k}_{N\times 1} = \underbrace{C}_{N\times(N+L-1)}\ \underbrace{S_k}_{(N+L-1)\times 1} + \underbrace{V_k}_{N\times 1} = [\underline{c}_0\ \underline{c}_1\ \cdots\ \underline{c}_{N+L-2}]\, S_k + V_k = \sum_{i=0}^{N+L-2} \underbrace{\underline{c}_i}_{N\times 1}\ s_{k-i} + V_k$. The vector $\underline{c}_i$ is column $i{+}1$ of the matrix C, so $\underline{c}_i = C\,\underline{e}_i$ where $\underline{e}_i$ is a standard unit vector with all zeros and a 1 in position $i{+}1$. The symbol sequence s_k is considered to be white noise with zero mean and variance σ_s^2. The additive noise v_k is independent of the symbol sequence and white Gaussian with zero mean and variance σ_v^2. We shall see that it may be advantageous to introduce an equalization delay d. Hence consider $x_k^{(d)} = s_{k-d},\ d \in \{0, 1, \ldots, N{+}L{-}2\}$. MMSE FIR equalization is a particular instance of FIR Wiener filtering. Hence the MMSE FIR equalizer coefficients $H_{MMSE}^{(d)}$ satisfy the normal equations

$R_{YY}\ H_{MMSE}^{(d)} = R_{Yx^{(d)}}$ or hence $H_{MMSE}^{(d)} = R_{YY}^{-1}\ R_{Yx^{(d)}}$, and

the MMSE is $\sigma^2_{\widetilde{x}^{(d)}_{MMSE}} = R_{x^{(d)}x^{(d)}} - R_{x^{(d)}Y} R_{YY}^{-1}\ R_{Yx^{(d)}}$.

(a) Determine R_{YY} in terms of σ_s^2, σ_v^2, the matrix C and the identity matrix I_N, and determine $R_{Yx^{(d)}}$ in terms of σ_s^2 and the vector $\underline{c}_d$.

(b) Express the MMSE $\sigma^2_{\widetilde{x}^{(d)}_{MMSE}}$ in terms of these same quantities.

The corresponding (naive) SNR is $\text{SNR}_{MMSE}^{(d)} = \dfrac{\sigma_s^2}{\sigma^2_{\widetilde{x}^{(d)}_{MMSE}}}$.

In what follows, we shall consider the specific case of a channel with $L = 2$ coefficients $c_0 = 1$, $c_1 = -a$, and no noise: $\sigma_v^2 = 0$. The range of possible delays is now limited to $d \in \{0, 1, \ldots, N\}$. In the absence of noise, the MSE is determined by intersymbol interference which is unavoidable here with an FIR equalizer, so the MSE is nonzero even in absence of noise.

(c) Show that MMSE $\sigma^2_{\widetilde{x}^{(d)}_{MMSE}} = \sigma_s^2\ \underline{e}_d^T P^{\perp}_{C^T} \underline{e}_d$ where the projection matrices are $P_X^{\perp} = I - P_X$ and $P_X = X(X^TX)^{-1}X^T$.

(d) Consider the channel transfer function $\mathrm{C}(z) = 1 - a\,z^{-1}$. Hence $\mathrm{C}(z = a) = 0$. As a result, show that

$$C\ V = 0 \tag{5}$$

where $V = [1\ \frac{1}{a}\ \frac{1}{a^2} \cdots \frac{1}{a^N}]^T$. So, since C^T has full column rank, we get $P^{\perp}_{C^T} = P_V$.

(e) Now compute again MMSE $\sigma^2_{\widetilde{x}^{(d)}_{MMSE}}$ for $d \in \{0, 1, \ldots, N\}$, as a function of a, N and d.

(f) Which is the optimal delay d as a function of a?