TD2: Optimal and Adaptive Filtering, Equalization

1 Wiener Filtering

Problem 1. A Wiener filtering problem

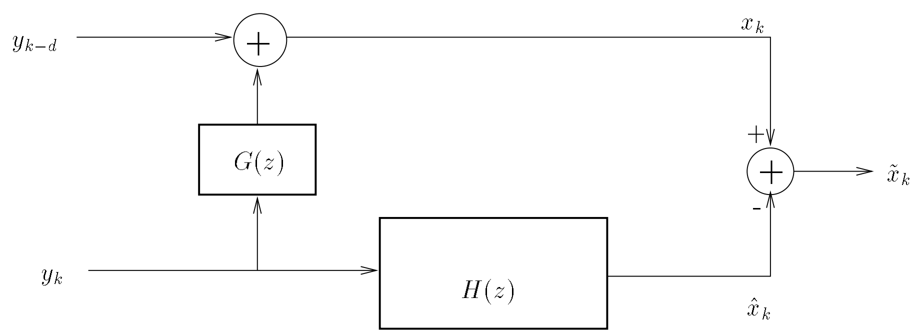


Figure 1: Wiener filtering problem.

In Fig. 1, a Wiener filtering problem is sketched. We can measure a signal y_k but we are interested in a related signal x_k that is indicated in the figure. $G(z)$ is the transfer function of some linear time-invariant filter and d is some delay. What is the Wiener filter $H(z)$ (in terms of the quantities indicated in the figure) for optimally estimating x_k from y_k and what is the associated MMSE?

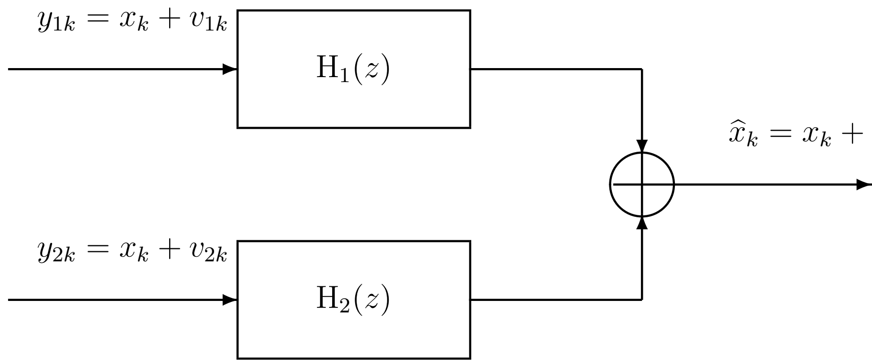
Problem 2. Constrained and Unconstrained Two-Channel Wiener Filtering

Consider the problem of combining two independent noisy measurements of the same signal as depicted in the figure below.

(a) Often a spectral description of x_k is not available, or it may not be properly modeled as a random process. Furthermore, one may require that the signal be filtered without distortion. We consider x_k to be deterministic in this case, and constrain the two filters to satisfy

$$\mathrm{H}_1(z) + \mathrm{H}_2(z) = 1, \quad \forall z \quad , \tag{1}$$

so that x_k passes undistortedly (that is, $\widehat{x}_k = x_k +$ noise). Due to the presence of more than one filter, we can take here a point of view that is intermediate between the classical filtering point of view of (no) distortion and the statistical MSE point of view. We can



constrain the filters in order to have no signal distortion. But due to the presence of more than one filter, this no distortion requirement does not fix all the degrees of freedom. We can use the remaining degrees of freedom to minimize the MSE.
Show how to choose $\mathrm{H}_1(z)$ so as to minimize the MSE, the variance of the output noise $v_k = -\widetilde{x}_k$. Calculate the associated MMSE.

(b) Consider now on the contrary that x_k is a random process and suppose that we have its spectral description $S_{xx}(f)$ available after all. We can now formulate a Wiener filtering problem for the generalized case where we have a vector process available, namely in this case

$$y_k \;=\; \begin{bmatrix} y_{1k} \\ y_{2k} \end{bmatrix} . \tag{2}$$

Using appropriately a vector valued transfer function

$$\mathrm{H}(z) \;=\; [\mathrm{H}_1(z) \;\; \mathrm{H}_2(z)] \; , \tag{3}$$

we can form a linear estimator

$$\widehat{x}_k \;=\; \mathrm{H}(q)\, y_k \;=\; \mathrm{H}_1(q)\, y_{1k} + \mathrm{H}_2(q)\, y_{2k} \; . \tag{4}$$

Using the short-cut frequency domain derivation for the optimal Wiener filter, in which at every frequency f the filter $H(f)$ is just a vector of (complex) coefficients to form the LMMSE estimate of the scalar $X(f)$ given the vector $Y(f)$, we find straightforwardly the following generalization

$$H(f) \;=\; S_{xy}(f)\, S_{yy}^{-1}(f) \;=\; R_{X(f)Y(f)}\, R_{Y(f)Y(f)}^{-1} \tag{5}$$

where in this example

$$S_{xy}(f) = [S_{xy_1}(f) \;\; S_{xy_2}(f)] \; , \quad S_{yy}(f) = \begin{bmatrix} S_{y_1y_1}(f) & S_{y_1y_2}(f) \\ S_{y_2y_1}(f) & S_{y_2y_2}(f) \end{bmatrix} . \tag{6}$$

Show that for the signal in noise problem at hand (with n_{1k} and n_{2k} being uncorrelated), we get

$$S_{xy}(f) = S_{xx}(f)\,[1 \;\; 1] \; , \quad S_{yy}(f) = S_{xx}(f) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} S_{n_1n_1}(f) & 0 \\ 0 & S_{n_2n_2}(f) \end{bmatrix} . \tag{7}$$

Find the Wiener filter $H(f)$ and compare to the solution of (a). Find the associated MMSE, the variance of $\widetilde{x}_k = x_k - \widehat{x}_k$ when the Wiener filter is used for $\widehat{x}_k$, and compare to the solution of (a).

2 Equalization and Wiener Filtering

Problem 3. Equalization of a First-Order FIR Channel

Consider the following simple channel $\mathrm{C}(z) = 1 - az^{-1}$. Compute the linear equalizer transfer function $\mathrm{H}(z)$ and the associated MSE for the ZF, MMSE and UMMSE criteria. Comment especially on what happens when $|a| \to 1$.

Problem 4. Wiener Filtering and Zero-Forcing Linear Equalization of a Second-Order FIR Channel

(a) Consider the signal in noise case ($y_k = x_k + v_k$). Show that when the noise v_k is white with variance σ_v^2, the MMSE of the Wiener filter turns out to be

$$\text{MMSE } = E\,\widetilde{x}_k^2 = \sigma_v^2\, h_0$$

where h_k is the impulse response of the non-causal Wiener filter.

(b) Consider the following discrete-time channel

$$\mathrm{C}(z) = 1 - \frac{5}{2} z^{-1} + z^{-2}\;.$$

Compute the impulse response h_k of the (non-causal) zero-forcing linear equalizer.

(c) For the same channel, in which we assume the variance of the additive white noise to be σ_v^2, compute the MSE of the zero-forcing (ZF) linear equalizer (LE).

(d) Compute the SNR for the ZF-LE, and the Matched Filter Bound (MFB) and compare.

Problem 5. FIR MMSE Linear Equalization of a FIR Channel

Consider a causal FIR equalizer H with N coefficients, $\widehat{x}_k = H^T\, Y_k$. For a FIR channel of length L, the received signal vector Y_k can be written as

$$\begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-N+1} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{L-1} & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & c_{L-2} & c_{L-1} & \ddots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_0 & \cdots & \cdots & c_{L-1} \end{bmatrix} \begin{bmatrix} s_k \\ s_{k-1} \\ \vdots \\ s_{k-N-L+2} \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k-1} \\ \vdots \\ v_{k-N+1} \end{bmatrix}$$

or

$$\underbrace{Y_k}_{N\times 1} = \underbrace{C}_{N\times(N+L-1)}\ \underbrace{S_k}_{(N+L-1)\times 1} + \underbrace{V_k}_{N\times 1} = [\underline{c}_0\ \underline{c}_1\ \cdots\ \underline{c}_{N+L-2}]\; S_k + V_k = \sum_{i=0}^{N+L-2} \underbrace{\underline{c}_i}_{N\times 1}\ s_{k-i} + V_k\;.$$

The vector $\underline{c}_i$ is column $i{+}1$ of the matrix C. The symbol sequence s_k is considered to be white noise with zero mean and variance σ_s^2. The additive noise v_k is independent of the symbol sequence and white Gaussian with zero mean and variance σ_v^2. We know that it may be advantageous to introduce an equalization delay d. Hence consider $x_k^{(d)} = s_{k-d},\ d \in \{0, 1, \ldots, N{+}L{-}2\}$.

MMSE FIR equalization is a particular instance of FIR Wiener filtering. Hence the MMSE FIR equalizer coefficients $H^{(d)}_{MMSE}$ satisfy the normal equations

$R_{YY}\ H^{(d)}_{MMSE} = R_{Yx^{(d)}}$ or hence $H^{(d)}_{MMSE} = R^{-1}_{YY}\ R_{Yx^{(d)}}$, and

the MMSE is $\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = R_{x^{(d)}x^{(d)}} - R_{x^{(d)}Y} R^{-1}_{YY}\ R_{Yx^{(d)}}$.

(a) Determine R_{YY} in terms of σ^2_s, σ^2_v, the matrix C and the identity matrix I_N, and determine $R_{Yx^{(d)}}$ in terms of σ^2_s and the vector $\underline{c}_d$.

(b) Express the MMSE $\sigma^2_{\tilde{x}^{(d)}_{MMSE}}$ in terms of these same quantities.

The corresponding (naive) SNR is $\text{SNR}^{(d)}_{MMSE} = \dfrac{\sigma^2_s}{\sigma^2_{\tilde{x}^{(d)}_{MMSE}}}$.

(c) The matrix R_{YY} has which structure? Determine the entries of the first row of R_{YY}.

(d) In what follows, consider the case of no noise: $\sigma^2_v = 0$. In the absence of noise, the MSE is determined by intersymbol interference which is unavoidable here with an FIR equalizer. Let $\underline{e}_i$ be the standard unit vector with a 1 in the $(i+1)^{\text{st}}$ row and zeros elsewhere, $P^{\perp}_X = I - P_X$ and $P_X = X(X^TX)^{-1}X^T$ for a matrix X with full column rank.

Show that $\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = \sigma^2_s\ \underline{e}^T_d\ P^{\perp}_{C^T}\ \underline{e}_d$.

(e) Let z_i be the roots of $C(z)$: $\ C(z_i) = 0\,,\ i = 1, ..., L-1$, and assume all roots to be different. Consider the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{L-1} \\ \vdots & \vdots & & \vdots \\ z_1^{N+L-2} & z_2^{N+L-2} & \cdots & z_{L-1}^{N+L-2} \end{bmatrix} .$$

Show that $P^{\perp}_{C^T} = P_V$ and hence that $\sigma^2_{\tilde{x}^{(d)}_{MMSE}} = \sigma^2_s\ \underline{e}^T_d\ P_V\ \underline{e}_d$.

(f) Compute the average MSE $\frac{1}{N+L-1} \sum_{d=0}^{N+L-2} \sigma^2_{\tilde{x}^{(d)}_{MMSE}}$.

3 Steepest-Descent and Adaptive Filtering Algorithms

Problem 6. Steepest-descent algorithm

Consider the steepest-descent algorithm for iterating towards the Wiener FIR filter for the case of two filter coefficients and the input having the covariance sequence $r_{yy}(k) = \rho^{|k|}$. Determine both the maximum stepsize μ for convergence and the stepsize that gives the fastest convergence as a function of ρ. With this stepsize, what is the slowest mode ?

Problem 7. An application of the LMS algorithm

Assume we are applying the LMS algorithm to a problem in which the input signal is of the form $y_k = a + w_k$ where $a \sim \mathcal{N}(0, \sigma_a^2)$ is a random dc component, the $w_k \sim \mathcal{N}(0, \sigma_w^2)$ are white noise samples and a is independent of the w_k.

(a) Find the mean E y_k.

(b) Find the covariance matrix R_{YY}.

(c) Find the eigenvalues of R_{YY} (order them so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$) and associated eigenvectors $V_1, \ldots, V_N$.

(d) Consider the analysis obtained by using the averaging theorem. When running the LMS algorithm in this particular case, what are the bounds on the stepsize μ so that convergence occurs (in other words, $? < \mu < ?$).

(e) What is the value for μ for fastest convergence?