# Statistical Signal Processing

*Lecture 2*

chapter 1: parameter estimation

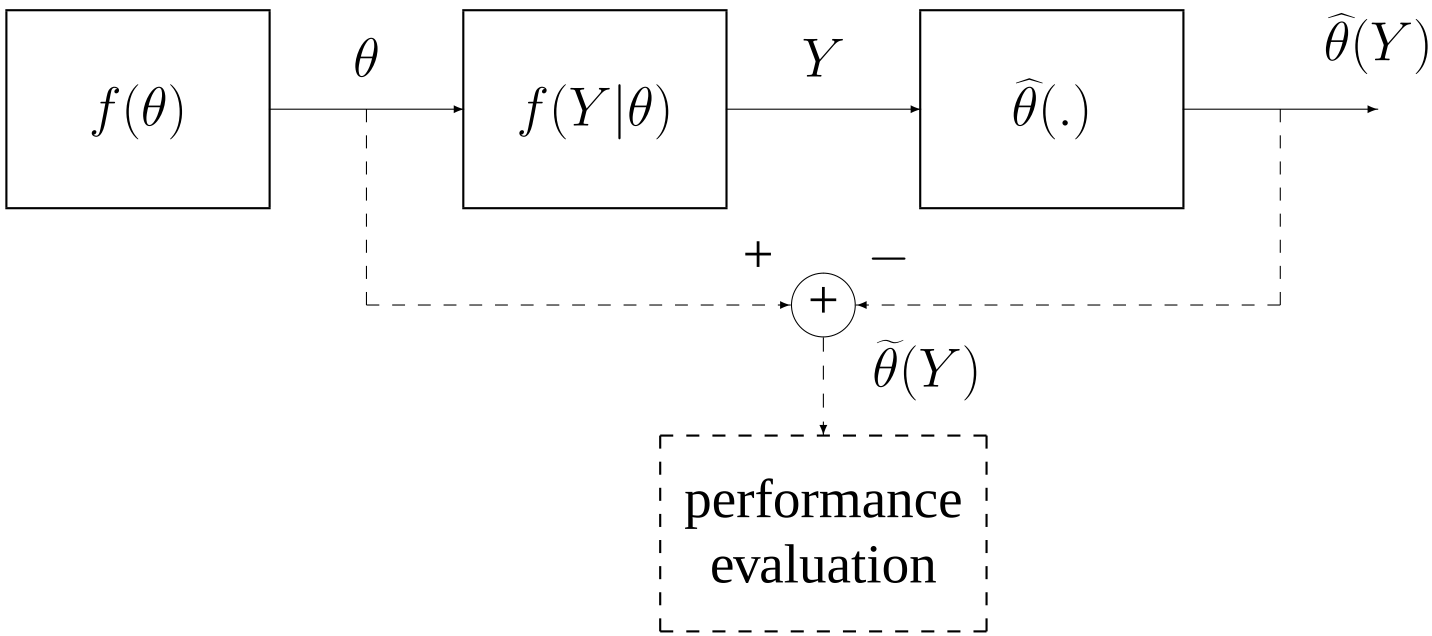
stochastic parameters

- the parameter estimation problem
- Bayes estimation: the MMSE, absolute value and uniform cost functions
- examples: Gaussian mean in Gaussian noise, Poisson process
- vector parameters
- Fischer Information Matrix

# Vector Parameters

- $\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$ , $\widehat{\theta}(Y) = \begin{bmatrix} \widehat{\theta}_1(Y) \\ \vdots \\ \widehat{\theta}_m(Y) \end{bmatrix}$ , $\widetilde{\theta} = \widetilde{\theta}(\theta, Y) = \theta - \widehat{\theta}(Y)$

- problem formulation:
  - a prior distribution $f_{\boldsymbol{\theta}}(\theta)$
  - a conditional distribution $f_{\mathbf{Y}|\boldsymbol{\theta}}(Y|\theta)$
  - Bayes' rule : joint distribution $f_{\mathbf{Y},\boldsymbol{\theta}}(Y, \theta) = f_{\mathbf{Y}|\boldsymbol{\theta}}(Y|\theta) f_{\boldsymbol{\theta}}(\theta)$



# Bayes Risk Function

- cost $\mathcal{C}(\theta, \widehat{\theta}(Y))$
- $\mathcal{C}(\theta, \widehat{\theta}(Y))$ often directly a funtion of the estimation error $\widetilde{\theta}$ and in fact often a function of the length of the estimation error $\|\widetilde{\theta}\| = \sqrt{\widetilde{\theta}^T\widetilde{\theta}} = \sqrt{\Sigma_{i=1}^n \widetilde{\theta}_i^2}$
- We obtain the estimator function $\widehat{\theta}(.)$ by minimizing the risk, which is the expected value of the cost:

$$\begin{aligned}
\min_{\widehat{\theta}(.)} \mathcal{R}(\widehat{\theta}(.)) &= \min_{\widehat{\theta}(.)} E\,\mathcal{C}(\theta, \widehat{\theta}(Y)) = \min_{\widehat{\theta}(.)} E_{\mathbf{Y},\boldsymbol{\theta}}\mathcal{C}(\theta, \widehat{\theta}(Y)) \\
&= \min_{\widehat{\theta}(.)} E_{\mathbf{Y}} E_{\boldsymbol{\theta}|\mathbf{Y}}\mathcal{C}(\theta, \widehat{\theta}(Y)) = E_{\mathbf{Y}}\,[\min_{\widehat{\theta}(Y)} E_{\boldsymbol{\theta}|\mathbf{Y}}\mathcal{C}(\theta, \widehat{\theta}(Y))] \\
&= E_{\mathbf{Y}}\,[\min_{\widehat{\theta}(Y)} \mathcal{R}(\widehat{\theta}(Y)|Y)] \;.
\end{aligned}$$

- $\mathcal{R}(\widehat{\theta}(.))$ is a weighted average of $\mathcal{R}(\widehat{\theta}(Y)|Y)$, weighted by the nonnegative weighting function $f_{\mathbf{Y}}(Y)$. The minimum of $\mathcal{R}(\widehat{\theta}(.))$ w.r.t. $\widehat{\theta}(.)$ will hence be obtained by minimizing $\mathcal{R}(\widehat{\theta}(Y)|Y)$ w.r.t. $\widehat{\theta}(Y)$ for every $Y$.
- again $\mathcal{R}(\widehat{\theta}(Y)|Y)$ depends on the posterior distribution $f_{\boldsymbol{\theta}|\mathbf{Y}}(\theta|Y)$.

# Optimization w.r.t. Vector Parameters

- $g(\theta) = [g_1(\theta) \cdots g_l(\theta)]$: $1 \times l$ row vector function, its gradient w.r.t. $\theta$ :

$$\frac{\partial\, g(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial\, g(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial\, g(\theta)}{\partial \theta_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial\, g_1(\theta)}{\partial \theta_1} & \cdots & \frac{\partial\, g_l(\theta)}{\partial \theta_1} \\ \vdots & & \vdots \\ \frac{\partial\, g_1(\theta)}{\partial \theta_m} & \cdots & \frac{\partial\, g_l(\theta)}{\partial \theta_m} \end{bmatrix} \quad m \times l$$

  If $g(\theta)$ is a scalar ($l = 1$), then $\frac{\partial\, g(\theta)}{\partial \theta}$ is a column vector of the same dimensions as $\theta$.

- in particular: $\frac{\partial\, \theta^T}{\partial \theta} = \left[\frac{\partial\, \theta_j}{\partial \theta_i}\right] = [\delta_{ij}] = I_m$

- The gradient operator commutes with linear operations. Let $X$ be $m \times 1$

$$\frac{\partial}{\partial \theta}\left(\theta^T X\right) = \left(\frac{\partial\, \theta^T}{\partial \theta}\right) X = I_m\, X = X \,.$$

- Since a scalar equals its transpose, we get

$$\frac{\partial}{\partial \theta}\left(X^T \theta\right) = \frac{\partial}{\partial \theta}\left(\theta^T X\right) = X$$

# Optimization w.r.t. Vector Parameters (2)

- If $A$ is $m \times l$ : $\frac{\partial}{\partial \theta}\left(\theta^T A\right) \;=\; \left(\frac{\partial\, \theta^T}{\partial \theta}\right) A \;=\; I_m\, A \;=\; A$
- scalar case: $(uv)' = u'\, v + u\, v'$
- vector case: let $g(\theta)$ and $h(\theta)$ be $l \times 1$. Since

$$g^T(\theta)h(\theta) = \left(g^T(\theta)h(\theta)\right)^T = h^T(\theta)g(\theta)$$

  we get

$$\frac{\partial}{\partial \theta}\left(g^T(\theta)h(\theta)\right) \;=\; \left(\frac{\partial g^T(\theta)}{\partial \theta}\right) h(\theta) + \left(\frac{\partial h^T(\theta)}{\partial \theta}\right) g(\theta)$$

- Particular application with $g(\theta) = \theta$ and $h(\theta) = A\theta$ :

$$\frac{\partial}{\partial \theta}\left(\theta^T A\theta\right) \;=\; \left(\frac{\partial\, \theta^T}{\partial \theta}\right) A\theta + \left(\frac{\partial\, \theta^T\, A^T}{\partial \theta}\right) \theta \;=\; \left(A + A^T\right)\theta$$

- When $A$ is symmetric, this gradient reduces to $2\, A\theta$.

# MMSE Criterion: Vector Parameters

- quadratic cost function $\mathcal{C}_{MMSE}(\theta,\widehat{\theta}) = \|\widetilde{\theta}\|_2^2 = \widetilde{\theta}^T\widetilde{\theta} \;=\; \Sigma_{i=1}^n\; \widetilde{\theta}_i^2$

- minimizing the conditional Bayes risk :

$$\min_{\widehat{\theta}(Y)} \mathcal{R}_{MMSE}(\widehat{\theta}(Y)|Y) \;=\; \min_{\widehat{\theta}(Y)} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\theta|Y)\; \left(\theta-\widehat{\theta}\right)^T \left(\theta-\widehat{\theta}\right) d\theta_1\cdots d\theta_m$$

- extrema:

$$\frac{\partial}{\partial\widehat{\theta}}\mathcal{R}_{MMSE}(\widehat{\theta}|Y) \;=\; \frac{\partial}{\partial\widehat{\theta}}\int f(\theta|Y)\;\left(\theta-\widehat{\theta}\right)^T\left(\theta-\widehat{\theta}\right)\,d\theta$$
$$=\;\int f(\theta|Y)\;\left(\frac{\partial}{\partial\widehat{\theta}}\left(\theta-\widehat{\theta}\right)^T\left(\theta-\widehat{\theta}\right)\right)d\theta \;=\; 2\int f(\theta|Y)\left(\widehat{\theta}-\theta\right)d\theta \;=0$$

- or hence $\widehat{\theta}(Y)\;\underbrace{\int f(\theta|Y)\,d\theta}_{=1} \;=\; \int \theta\, f(\theta|Y)\,d\theta \;\;\Rightarrow\;\; \widehat{\theta}_{MMSE}(Y) \;=\; E(\theta|Y)$

  which is again the *mean* of the a posteriori distribution of $\theta$ given $Y$.

- extremum = minimum?

$$Hessian \;=\; \left[\frac{\partial^2}{\partial\widehat{\theta}_i\,\partial\widehat{\theta}_j}\mathcal{R}_{MMSE}(\widehat{\theta}|Y)\right] \;=\; \frac{\partial}{\partial\widehat{\theta}}\left(\frac{\partial}{\partial\widehat{\theta}}\mathcal{R}_{MMSE}(\widehat{\theta}|Y)\right)^T$$
$$=\;2\int f(\theta|Y)\left[\frac{\partial\widehat{\theta}^T}{\partial\widehat{\theta}}-\frac{\partial\theta^T}{\partial\widehat{\theta}}\right]d\theta \;=\; 2\,I\int f(\theta|Y)\,d\theta \;=\; 2\,I \;>\; 0$$

# MMSE Criterion: Vector Parameters (2)

- MMSE estimation commutes over linear transformations: with $\phi = A\theta$

$$\widehat{\phi}_{MMSE} \;=\; E(\phi|Y) \;=\; E(A\theta|Y) \;=\; A\,E(\theta|Y) \;=\; A\,\widehat{\theta}_{MMSE}$$

- orthogonality property of MMSE estimators:

$$\widehat{\theta}(\mathbf{Y}) \;=\; E(\boldsymbol{\theta}|\mathbf{Y}) \quad \text{iff} \quad E((\boldsymbol{\theta} - \widehat{\theta}(\mathbf{Y}))\,g(\mathbf{Y})) \;=\; 0\,,\;\; \forall\, g(.)$$

  where $g(.)$ is a scalar function. Equivalently :

$$E(\widehat{\theta}(\mathbf{Y})\,g(\mathbf{Y})) \;=\; E(\boldsymbol{\theta}\,g(\mathbf{Y}))\;,\;\; \forall\, g(.)$$

  which represents an alternative way of defining $E(\boldsymbol{\theta}|\mathbf{Y})$.

- use orthogonality to show optimality: let $\widehat{\theta}(Y)$ be any function of $Y$,

$$\begin{aligned}
E\left\|\theta - \widehat{\theta}(Y)\right\|_2^2 \;&=\; E\left\|\theta - E(\theta|Y) + E(\theta|Y) - \widehat{\theta}(Y)\right\|_2^2 \\
=\; E\,\|\theta - E(\theta|Y)\|_2^2 &+ \underbrace{E\left\|E(\theta|Y) - \widehat{\theta}(Y)\right\|_2^2}_{\geq 0} + 2\,\underbrace{E\left((\theta - E(\theta|Y))^T (E(\theta|Y) - \widehat{\theta}(Y))\right)}_{=0} \\
\geq\; E\,\|\theta - E(\theta|Y)\|_2^2 &
\end{aligned}$$

- correlation matrices: $E\,(\theta - E(\theta|Y))(\theta - E(\theta|Y))^T \;\leq\; E\,(\theta - \widehat{\theta})(\theta - \widehat{\theta})^T = R_{\widetilde{\theta}\widetilde{\theta}}$

# MAP Estimators: Vector Parameters

- introduce: a ball centered around $\theta_o$ with radius $\delta$

$$\mathcal{B}_\delta(\theta_o) \;=\; \{\theta \in \Theta : \|\theta - \theta_0\|_2 \le \delta\}$$

- Then the natural extension to the vector case of the uniform cost function is

$$\mathcal{C}_{UNIF}(\theta, \widehat{\theta}) \;=\; \begin{cases} 0 \;,\; \theta \in \mathcal{B}_\delta(\widehat{\theta}) \\ 1 \;,\; \theta \in \Theta \setminus \mathcal{B}_\delta(\widehat{\theta}) \end{cases}$$

- The conditional Bayes risk becomes

$$\mathcal{R}_{UNIF}(\widehat{\theta}(Y)|Y) \;=\; \int_\Theta f(\theta|Y)\, \mathcal{C}_{UNIF}(\theta, \widehat{\theta}) d\theta \;=\; \int_{\Theta\setminus\mathcal{B}_\delta(\widehat{\theta})} f(\theta|Y)\, d\theta$$

$$= \; \int_\Theta f(\theta|Y)\, d\theta \;-\; \int_{\mathcal{B}_\delta(\widehat{\theta})} f(\theta|Y)\, d\theta \;=\; 1 \;-\; \int_{\mathcal{B}_\delta(\widehat{\theta})} f(\theta|Y)\, d\theta$$

- The optimization problem $\min\limits_{\widehat{\theta}(Y)} \mathcal{R}_{UNIF}(\widehat{\theta}(Y)|Y)$ hence leads to

$$\max_{\widehat{\theta}(Y)} \int_{\mathcal{B}_\delta(\widehat{\theta})} f(\theta|Y)\, d\theta \;\approx\; \text{Vol}\,(\mathcal{B}_\delta(0)) \;\; \max_{\widehat{\theta}(Y)} f(\widehat{\theta}|Y)$$

the approximation becomes arbitrarily accurate as $\delta$ becomes small

# MAP Estimators: Vector Parameters (2)

- This leads to the Maximum A Posteriori (likelihood) estimator

$$\widehat{\theta}_{MAP}(Y) \;=\; \arg\max_{\theta\in\Theta} f(\theta|Y) \qquad \textit{posterior likelihood}$$

- The same remarks as in the scalar case hold here also. In particular, one may equivalently obtain $\widehat{\theta}_{MAP}(Y)$ from the optimization problem

$$\widehat{\theta}_{MAP}(Y) \;=\; \arg\max_{\theta\in\Theta} \ln f(\theta|Y) \qquad \textit{posterior log likelihood}$$

- Under certain regularity conditions, $\widehat{\theta}_{MAP}(Y)$ can be found from

$$\frac{\partial}{\partial\theta}\ln f(\theta|Y) \;=\; 0 \;=\; \frac{\partial}{\partial\theta}\ln f(Y|\theta) + \frac{\partial}{\partial\theta}\ln f(\theta)$$

- Also MAP commutes over linear transformations: $\phi = A\theta$ ($A$ invertible)

$$\begin{aligned}\widehat{\phi}_{MAP}(Y) \;&=\; \arg\max_{\phi} f_{\boldsymbol{\phi}|\mathbf{Y}}(\phi|Y) \;=\; A\,\arg\max_{\theta} f_{\boldsymbol{\phi}|\mathbf{Y}}(A\theta|Y) \\ &=\; A\,\arg\max_{\theta} \frac{1}{|\det A|} f_{\boldsymbol{\theta}|\mathbf{Y}}(\theta|Y) \;=\; A\,\widehat{\theta}_{MAP}(Y)\;.\end{aligned}$$

This argument can be extended to the case in which $\dim\phi \neq \dim\theta$

# Fisher Information Matrix

There exists a lower bound on the correlation matrix of the estimator errors. It is independent of the Bayes estimator (cost) used; it depends only on the posterior distribution. The lower bound is specified in terms of the *information matrix*, which should express in quantitive terms the information carried by the posterior distribution about the parameters $\theta$. For such an information measure, the following properties are desirable:

- The information should increase as the *sensitivity* of $f(\theta|Y)$ to changes in $\theta$ increases. Hence, the information should be an increasing function of $\frac{\partial\ f(\theta|Y)}{\partial\theta}$ or of $\frac{\partial\ \ln\ f(\theta|Y)}{\partial\theta}$.
- The information should be *additive* in the sense that it should be the sum of the informations from the prior distribution ($f(\theta)$) and from the data ($f(Y|\theta)$). Furthermore if, given $\theta$, $Y_1$ and $Y_2$ are independent ($f(Y_1, Y_2|\theta) = f(Y_1|\theta)f(Y_2|\theta)$), then the informations in $Y_1$ and $Y_2$ should add up.
- The information should be positive and should be insensitive to a change of sign of $\theta$.
- The information should be a *deterministic* quantity.

# Fisher Information Matrix (2)

- The information matrix is defined as

$$J \;=\; E\left(\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta}\right)\left(\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta}\right)^T$$

  It can be shown to satisfy all the properties mentioned above.

- The perturbation of Mutual Information (Information Theory) w.r.t. a parameter can be expressed in terms of its Fisher Information.

- With $\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta} \;=\; \frac{1}{f(\theta|Y)}\frac{\partial\;f(\theta|Y)}{\partial\theta}$ we can write the Hessian of $\ln\;f(\theta|Y)$ as

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta}\right)^T \;=\; \frac{1}{f^2(\theta|Y)}\left[f(\theta|Y)\,\frac{\partial}{\partial\theta}\left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)^T - \left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)\left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)^T\right]$$

$$= \;\frac{1}{f(\theta|Y)}\,\frac{\partial}{\partial\theta}\left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)^T - \left(\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta}\right)\left(\frac{\partial\;\ln\;f(\theta|Y)}{\partial\theta}\right)^T$$

- For the expectation of the first term, we get

$$E\,\frac{1}{f(\theta|Y)}\,\frac{\partial}{\partial\theta}\left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)^T \;=\; \int d\theta\int dY\;f(Y)\,\frac{\partial}{\partial\theta}\left(\frac{\partial\;f(\theta|Y)}{\partial\theta}\right)^T$$

$$= \;\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\int d\theta\int dY\;f(Y)f(\theta|Y)\right)^T \;=\; \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}1\right)^T \;=\; 0$$

# Fisher Information Matrix (3)

- It follows that we can rewrite the information matrix as

$$J = -E\,\frac{\partial}{\partial\theta}\left(\frac{\partial\,\ln\,f(\theta|Y)}{\partial\theta}\right)^T$$

This expression will often allow us to obtain $J$ more easily.

- Note also that

$$\frac{\partial\,\ln\,f(Y,\theta)}{\partial\theta} = \frac{\partial\,\ln\,f(\theta|Y)}{\partial\theta} + \underbrace{\frac{\partial\,\ln\,f(Y)}{\partial\theta}}_{=0} = \frac{\partial\,\ln\,f(\theta|Y)}{\partial\theta}$$

so that as long as derivatives are taken, we can interchange $f(Y,\theta)$ and $f(\theta|Y)$. Hence

$$\frac{\partial\,\ln\,f(Y,\theta)}{\partial\theta} = \frac{\partial\,\ln\,f(\theta|Y)}{\partial\theta} = \frac{\partial\,\ln\,f(Y|\theta)}{\partial\theta} + \frac{\partial\,\ln\,f(\theta)}{\partial\theta}$$

and

$$J = -E\,\frac{\partial}{\partial\theta}\left(\frac{\partial\,\ln\,f(\theta,Y)}{\partial\theta}\right)^T = \underbrace{-E\,\frac{\partial}{\partial\theta}\left(\frac{\partial\,\ln\,f(Y|\theta)}{\partial\theta}\right)^T}_{J_{data}} \underbrace{-\,E\,\frac{\partial}{\partial\theta}\left(\frac{\partial\,\ln\,f(\theta)}{\partial\theta}\right)^T}_{J_{prior}}$$

# Conditions on the Estimator Bias

- The (conditional) *bias* of an estimator $\widehat{\theta}(Y)$ of $\theta$ is defined as

$$b_{\widehat{\theta}}(\theta) \;=\; -E_{\mathbf{Y}|\boldsymbol{\theta}}\widetilde{\theta} \;=\; E_{\mathbf{Y}|\boldsymbol{\theta}}\left(\widehat{\theta}(Y)-\theta\right) \;=\; E_{\mathbf{Y}|\boldsymbol{\theta}}\widehat{\theta}(Y) \;-\; \theta$$

- An estimator will be called *unbiased* if either

$$E_{\boldsymbol{\theta}} b_{\widehat{\theta}}(\theta) \;=\; 0 \;\Leftrightarrow\; E_{\boldsymbol{\theta},\mathbf{Y}}\,\widetilde{\theta} = 0$$

which means that the unconditional or average bias is zero, or

$$\lim_{\theta\to\partial\Theta} f(\theta) b_{\widehat{\theta}}(\theta) \;=\; 0$$

where $\Theta$ is the domain for $\theta$ and $\partial\Theta$ is its boundary.

- **Lemma 0.1 (Unit Cross Correlation)** *If either condition above is satisfied, then*

$$E\,\frac{\partial\;\ln\; f(Y,\theta)}{\partial\theta}\left(\widehat{\theta}-\theta\right)^T \;=\; I\;.$$

In words, the cross correlation matrix between $\frac{\partial\;\ln\; f(Y,\theta)}{\partial\theta}$ and the estimation error of any unbiased estimator is the identity matrix.

# Inner Products

- An inner product $< .,. >$ associates a real number $< x, y > \in \mathcal{R}$ with two vectors $x$ and $y$ of the vector space $\mathcal{V}$ we are considering, and it has the following properties:

  1. linearity: $\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{R}$, $\forall x, x_1, x_2, y, y_1, y_2 \in \mathcal{V}$:

  $$\begin{aligned} < \alpha_1 x_1 + \alpha_2 x_2, y > &= \alpha_1 < x_1, y > + \alpha_2 < x_2, y > \\ < x, \beta_1 y_1 + \beta_2 y_2 > &= < x, y_1 > \beta_1 + < x, y_2 > \beta_2 \end{aligned} \tag{1}$$

  2. symmetry: $< x, y > = < y, x >$
  3. non-degeneracy (of the norm induced by the inner product): $< x, x > = \|x\|^2 \geq 0$. If $\|x\| = 0$, then $x = 0$.

- One particular example is a space of random variables with the correlation as inner product: $< x, y > = E\,xy$. Non-degeneracy subtlety:

$$E\,x^2 \;=\; 0 \quad \Rightarrow \quad x = 0 \;\text{ in m.s.}$$

$x = 0$ “in mean square”. Indeed, $Ex^2 \;=\; m_x^2 + \sigma_x^2 \;=\; 0 \quad \Rightarrow \quad m_x = 0, \sigma_x^2 = 0$. This often (not always) implies $x = 0$ “almost surely” (a.s.) or “almost everywhere” (a.e.) or “with probability 1” (w.p. 1): $\Pr(x = 0) = 1$.

# Matrix Inner Products

- We now consider a vector space $\mathcal{V}$ in which the vectors have multiple components such that the inner product is a real matrix.
- Example 1: consider a vector space of random vectors with inner product $< X, Y >= E\,XY^T$ where $X$ and $Y$ are column vectors of random variables (not necessarily with the same number of rows).
- Example 2: a vector space in which the "vectors" are $* \times k$ real matrices where $k$ is fixed and $*\,(\geq k)$ is arbitrary. Inner product: $< X, Y >= XY^T$.
- Matrix valued inner products satisfy the following properties, which are natural generalizations of the scalar case.

1. linearity: let $X, X_1, X_2 \in \mathcal{V}$ have $m$ rows and $Y, Y_1, Y_2 \in \mathcal{V}$ have $n$ rows. Then $\forall \alpha_1, \alpha_2 \in \mathcal{R}^{k\times m}$, $\forall \beta_1, \beta_2 \in \mathcal{R}^{l\times n}$, for any $k$ and $l$,

$$\begin{aligned} < \alpha_1 X_1 + \alpha_2 X_2, Y > &= \alpha_1 < X_1, Y > + \alpha_2 < X_2, Y > \\ < X, \beta_1 Y_1 + \beta_2 Y_2 > &= < X, Y_1 > \beta_1^T + < X, Y_2 > \beta_2^T \end{aligned} \qquad (2)$$

2. symmetry: $< X, Y > = < Y, X >^T$
3. non-degeneracy: $< X, X >= \|X\|^2 \geq 0$. If $\|X\|^2 = 0$, then $X = 0$.

# Schur Complements

- **Lemma 0.2 (Schur Complements)** *Let* $X_1$ *and* $X_2$ *be vectors in a certain vector space with a certain inner product and denote* $R_{ij} =< X_i, X_j >$ *,* $i, j = 1, 2$ *so that* $R_{ij} = R_{ji}^T$*. Assume that* $R_{11}$ *is nonsingular. Then, because of property 3 of the inner product (non-degeneracy), we have*

$$\begin{aligned} \|X_2 - R_{21}R_{11}^{-1}X_1\|^2 &= < X_2 - R_{21}R_{11}^{-1}X_1, X_2 - R_{21}R_{11}^{-1}X_1 > \\ &= R_{22} - 2R_{21}R_{11}^{-1}R_{12} + R_{21}R_{11}^{-1}R_{11}R_{11}^{-1}R_{12} \\ &= R_{22} - R_{21}R_{11}^{-1}R_{12} \;\geq\; 0 \end{aligned}$$

*with equality iff* $X_2 = R_{21}R_{11}^{-1}X_1$.

(matrix version of Cauchy-Schwarz inequality)

- The name for this lemma stems from the following congruence relation

$$\begin{aligned} < \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} >= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} &= \begin{bmatrix} I & O \\ R_{21}R_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} - R_{21}R_{11}^{-1}R_{12} \end{bmatrix} \begin{bmatrix} I & R_{11}^{-1}R_{12} \\ O & I \end{bmatrix} \\ &= \begin{bmatrix} I \\ R_{21}R_{11}^{-1} \end{bmatrix} R_{11} \begin{bmatrix} I \\ R_{21}R_{11}^{-1} \end{bmatrix}^T + \begin{bmatrix} 0 & 0 \\ 0 & R_{22} - R_{21}R_{11}^{-1}R_{12} \end{bmatrix} . \end{aligned}$$

(block LDU triangular factorization). The matrix $R_{22} - R_{21}R_{11}^{-1}R_{12}$ is called the *Schur complement* of $R_{11}$ within the big matrix on the LHS .