Statistical Signal Processing

Lecture 10

chapter 3: Optimal Filtering

Wiener filtering

- FIR Wiener filtering
  - iterative solution: steepest-descent algorithm

chapter 4: Adaptive Filtering

- LMS algorithm
- Normalized LMS (NLMS) algorithm
- tracking behavior of LMS and RLS
- optimal tracking via Kalman filtering

chapter 5: Sinusoids in Noise

RLS Algorithm

- LS: replace the statistical averages by a time averages:

$$\xi_k\left(H\right)=\sum_{i=1}^{k}\left(x_i-H^TY_i\right)^2+\left(H-H_0\right)^TR_0\left(H-H_0\right) \quad ,$$

where the second term with $R_0=R_0^T>0$ allows for a proper initialization of the algorithm (the first term alone has a singular Hessian ($=2\sum_{i=1}^{k}Y_iY_i^T$) for $k<N$).

- We can rewrite

$$\begin{aligned}\xi_k\left(H\right) &= H^T\left(\sum_{i=1}^{k}Y_iY_i^T\right)H-2H^T\left(\sum_{i=1}^{k}Y_ix_i\right)+\sum_{i=1}^{k}x_i^2+\left(H-H_0\right)^TR_0\left(H-H_0\right)\\ &= H^T\left(R_0+\sum_{i=1}^{k}Y_iY_i^T\right)H-2H^T\left(R_0H_0+\sum_{i=1}^{k}Y_ix_i\right)+\sum_{i=1}^{k}x_i^2+H_0^TR_0H_0\\ &= H^TR_kH-2H^TP_k+\sum_{i=1}^{k}x_i^2+H_0^TR_0H_0\end{aligned}$$

where

$$\begin{aligned}R_k &= R_0+\sum_{i=1}^{k}Y_iY_i^T &&= R_{k-1}+Y_kY_k^T\\ P_k &= R_0\,H_0+\sum_{i=1}^{k}Y_i\,x_i &&= P_{k-1}+Y_kx_k \quad .\end{aligned}$$

Recursive Least-Squares Algorithm (2)

- By putting the gradient of $\xi_k(H)$ equal to zero and noting that the Hessian $2R_k > 0$, we find that the LS filter H_k that minimizes the LS criterion solves the following normal equations

$$R_k\, H_k = P_k \quad .$$

To solve this set of equations at each time instant k would take $\mathcal{O}(N^3)$ operations at each time instant. In what follows, we shall derive the Recursive LS algorithm, which allows us, using information obtained at time $k{-}1$, to obtain H_k with only $\mathcal{O}(N^2)$ operations.

- we can rewrite $P_k = P_{k-1} + Y_k x_k$ as

$$\begin{aligned} R_k\, H_k &= R_{k-1} H_{k-1} + Y_k x_k \\ &= \left(R_k - Y_k Y_k^T\right) H_{k-1} + Y_k x_k \\ &= R_k H_{k-1} + Y_k \epsilon_k^p \end{aligned}$$

where $\epsilon_k^p = x_k - H_{k-1}^T Y_k$ as in the LMS algorithm. This leads immediately to

$$H_k = H_{k-1} + R_k^{-1} Y_k \epsilon_k^p$$

where $R_k^{-1} Y_k$ is called the Kalman gain (the RLS algorithm is a special case of the so-called Kalman filter).

Recursive Least-Squares Algorithm (3)

- Clearly, the RLS algorithm requires the recursive update of R_k^{-1}. This can be obtained using the Matrix Inversion Lemma:

$$\begin{aligned} R_k^{-1} &= \left(R_{k-1} + Y_k Y_k^T\right)^{-1} \\ &= R_{k-1}^{-1} - R_{k-1}^{-1} Y_k \left(1 + Y_k^T R_{k-1}^{-1} Y_k\right)^{-1} Y_k^T R_{k-1}^{-1} \quad . \end{aligned}$$

  This equation allows us to obtain R_k^{-1} from R_{k-1}^{-1} and Y_k using $\mathcal{O}(N^2)$ operations. When multiplying both sides with Y_k to the right, we obtain

$$R_k^{-1} Y_k = R_{k-1}^{-1} Y_k \left(1 + Y_k^T R_{k-1}^{-1} Y_k\right)^{-1} \quad .$$

  We find for the *a posteriori* error

$$\epsilon_k = x_k - H_k^T Y_k = \left(1 - Y_k^T R_k^{-1} Y_k\right) \epsilon_k^p = \left(1 + Y_k^T R_{k-1}^{-1} Y_k\right)^{-1} \epsilon_k^p \; .$$

- All this can be formulated as the RLS algorithm:

$$\left\{ \begin{aligned} \epsilon_k^p &= x_k - H_{k-1}^T Y_k \\ \epsilon_k &= \epsilon_k^p \left(1 + Y_k^T R_{k-1}^{-1} Y_k\right)^{-1} \\ H_k &= H_{k-1} + R_{k-1}^{-1} Y_k \epsilon_k \\ R_k^{-1} &= R_{k-1}^{-1} - R_{k-1}^{-1} Y_k \left(1 + Y_k^T R_{k-1}^{-1} Y_k\right)^{-1} Y_k^T R_{k-1}^{-1} \quad . \end{aligned} \right.$$

Recursive Least-Squares Algorithm (4)

- The initial values for R_k^{-1} and H_k are R_0^{-1} and H_0. Compared to the LMS algorithm, the scalar stepsize μ gets replaced by a matrix stepsize R_k^{-1}. The RLS algorithm takes $\mathcal{O}(N^2)$ operations while the LMS algorithm takes only $2N$ operations. However, it converges much faster.
- *performance analysis* : with $x_k = H^{oT} Y_k + \tilde{x}_k$ (and $R_0 = 0$), we get $R_k \, H_k = P_k = \sum_{i=1}^{k} Y_i x_i = R_k H^o + \sum_{i=1}^{k} Y_i \tilde{x}_i$. Hence

$$\widetilde{H}_k = H^o - H_k = -R_k^{-1} \sum_{i=1}^{k} Y_i \tilde{x}_i$$

From this, we obtain

$$C_k \stackrel{\triangle}{=} E \, \widetilde{H}_k \widetilde{H}_k^T = \sigma_{\tilde{x}}^2 R_k^{-1} \quad .$$

Since R_k^{-1} behaves as $1/k$, we see that C_k converges to zero as $1/k$.

- *Exponential Weighting* In order to be able to track a possibly time-varying $H^o = H_k^o$, one introduces an exponential forgetting factor $\lambda \in (0,1)$ into the cost function to obtain

$$\xi_k\,(H) = \sum_{i=1}^{k} \lambda^{k-i} \left(x_i - H^T Y_i\right)^2 + \lambda^k \, (H - H_0)^T \, R_0 \, (H - H_0) \ \ .$$

This implies that the past (and in particular the initial conditions H_0, R_0) is forgotten exponentially fast with a window with time constant $1/(1-\lambda)$.

Recursive Least-Squares Algorithm (5)

- Wiener filtering: x_k and y_k are two joint stochastic processes and we're trying to estimate x_k from the y_k using a LMMSE estimator. For an FIR Wiener filter, there are a finite set of coefficients H^o involved in this LMMSE estimator.
  RLS approach: replaced statistical averages with temporal averages.

Parameter estimation interpretation

- Assume now our usual model for the *measurements* x_k,

$$x_k = H^{oT} Y_k + \tilde{x}_k$$

  where the $\tilde{x}_k$ are iid with zero mean and variance $\sigma^2_{\tilde{x}}$. Consider here $\{y_k\}$ as a deterministic signal, so the only randomness comes from the $\{\tilde{x}_k\}$. The H^o are the unknown parameters governing the model.

- The analysis of the RLS algorithm is much simpler than that of the LMS algorithm since for each k the RLS solution H_k coincides with the solution of a Least-Squares problem with a closed-form solution: $H_k = R_k^{-1} P_k$.

- Assume now $k \geq N, H_0 = 0, R_0 = 0$, and that R_k is nonsingular. The performance of the least-squares estimate is simple to analyze and leads to

$$C_k \stackrel{\triangle}{=} E\, \widetilde{H}_k \widetilde{H}_k^T = \sigma^2_{\tilde{x}} R_k^{-1} \quad .$$

- If $\tilde{x}_k$ Gaussian, $\Rightarrow$ H_k = ML estimate of H^o (efficient, C_k = CRB).

Recursive Least-Squares Algorithm (6)

A Bayesian Context - A Priori Information

- Instead of treating the filter coefficients H^o as unknown constant parameters, we could also consider H^o as a stochastic parameter vector about which we have some prior information, possibly from previous adaptive filtering experience. Assume now that, prior to obtaining the measurements $x_1, x_2, \cdots$, we know that H^o has a distribution with mean $E\, H^o = H_0$ and covariance $E\,(H^o - H_0)(H^o - H_0)^T = C_0$. So now the randomness in the x_k comes from both the $\tilde{x}_k$ and H^o.
- The problem formulation can now be recognized to be one of a *Bayesian Linear Model*. The AMMSE estimator can be shown to be the filter estimate resulting from the original RLS criterion with $R_0 = \sigma^2_{\tilde{x}} C_0^{-1}$. $C_k = E\, \widetilde{H}_k \widetilde{H}_k^T$ now satisfies

$$C_k^{-1} = \sigma_{\tilde{x}}^{-2} R_k = \sigma_{\tilde{x}}^{-2} \sum_{i=1}^{k} Y_i\, Y_i^T + C_0^{-1} \quad .$$

  Note that C_k^{-1} is an increasing function of C_0^{-1} and hence C_k is a decreasing function of C_0^{-1} and hence of R_0.
- So we see that H_0 and R_0 in the LS cost function have the interpretation of the prior mean and the inverse of the prior covariance of H^o. We'll choose R_0 small if we don't have a lot of confidence in our prior guess H_0 (C_0 big). In practice, R_0 is often chosen as $R_0 = \eta I_N$.

Other Adaptive Filtering Algorithms

- Fast RLS algorithms: Fast Transversal Filter (FTF) algorithm ($8N$), Fast Lattice/QR Algorithms ($\mathcal{O}(N)$ complexity)
- LMS with prewhitened input
- block processing/frequency domain LMS
- subband structures
- Fast Newton Transversal Filter (FNTF): replace R^{-1} in RLS by a banded matrix (appropriate for AR processes, hence speech)
- projection algorithms (like NLMS) on an extended subspace of L input vectors (FAP: Fast Affine Projection: complexity $2N + \mathcal{O}(L^2)$ or $2N + \mathcal{O}(L)$)
- Fast Subsampled Updating (FSU) versions of LMS and FTF: introduce some delay to reduce complexity below $\mathcal{O}(N)$
- multistage Wiener filter / polynomial expansion:

$$r_0\, R^{-1} = [\frac{1}{r_0}\, R]^{-1} = [\underbrace{I}_{\text{diagonal}} + \underbrace{(\frac{1}{r_0}\, R - I)}_{\text{off-diagonal part}}]^{-1} = \sum_{i=0}^{\infty} (I - \frac{1}{r_0}\, R)^i = \sum_{i=0}^{\infty} \alpha_i\, R^i$$

- convergence speed (RLS best) versus tracking speed (FAP best?)

Initial Convergence RLS

- Consider now $H_0 \neq 0$, $R_0 \neq 0$,
  $R_k = R_0 + R_{1:k}$, $R_{1:k} = Y_{1:k} Y_{1:k}^T$, $Y_{1:k} = [Y_1 \cdots Y_k]$, $P_k = R_0 H_0 + P_{1:k}$
- $\widetilde{H}_k = H^o - H_k = H^o - R_k^{-1} P_k = (R_0 + R_{1:k})^{-1} (R_0 \widetilde{H}_0 - \sum_{i=1}^{k} Y_i \widetilde{x}_i)$
- $C_k = E\, \widetilde{H}_k \widetilde{H}_k^T$ hence

$$\begin{aligned} C_k &= (R_0 + R_{1:k})^{-1} R_0 \widetilde{H}_0 \widetilde{H}_0^T R_0 (R_0 + R_{1:k})^{-1} + \sigma_{\tilde{x}}^2 (R_0 + R_{1:k})^{-1} R_{1:k} (R_0 + R_{1:k})^{-1} \\ &= \underbrace{\sigma_{\tilde{x}}^2 (R_0 + R_{1:k})^{-1}}_{\sim \frac{1}{k}} + \underbrace{(R_0 + R_{1:k})^{-1} (R_0 \widetilde{H}_0 \widetilde{H}_0^T R_0 - \sigma_{\tilde{x}}^2 R_0)(R_0 + R_{1:k})^{-1}}_{\sim \frac{1}{k^2} \text{ due to initialization}} \end{aligned}$$

- noiseless case: $\sigma_{\tilde{x}}^2 = 0$

$$C_k = (R_0 + R_{1:k})^{-1} R_0 \widetilde{H}_0 \widetilde{H}_0^T R_0 (R_0 + R_{1:k})^{-1}$$

- initial convergence: $1 \leq k < N$, consider $R_0 = \eta\, I$

$$C_k = \eta^2\, (\eta\, I + R_{1:k})^{-1} \widetilde{H}_0 \widetilde{H}_0^T (\eta\, I + R_{1:k})^{-1}$$

Initial Convergence RLS (2)

- *Singular Value Decomposition* (SVD): $Y_{1:k},\ N\times k$ assumed full column rank

$$Y_{1:k} = V\Sigma U^T \quad V^TV = I_k, U^{-1} = U^T, \Sigma = \mathrm{diag}\{\sigma_1,\ldots,\sigma_k\}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$ "singular values" full column rank $\leftrightarrow \sigma_k > 0$

- *Moore-Penrose pseudo-inverse*: $Y_{1:k}^+ = U\Sigma^{-1}V^T = (Y_{1:k}^TY_{1:k})^{-1}Y_{1:k}^T$
- projection on column space: $\mathrm{P}_{Y_{1:k}} = Y_{1:k}Y_{1:k}^+ = VV^T$
- $V^+ = V^T$ $\qquad \mathrm{P}_{Y_{1:k}} = VV^T = VV^+ = \mathrm{P}_V \qquad \mathrm{P}_V^+ = \mathrm{P}_V$
- eigendecomposition: $R_{1:k} = Y_{1:k}Y_{1:k}^T = V\Sigma^2V^T$
- let $V^\perp$ be such that $[V\ V^\perp]$ is orthogonal:

$$[V\ V^\perp][V\ V^\perp]^T = I = VV^T + V^\perp V^{\perp\,T} = \mathrm{P}_V + \mathrm{P}_{V^\perp} = \mathrm{P}_V + \mathrm{P}_V^\perp$$

$\mathrm{P}_V^\perp = I - \mathrm{P}_V = \mathrm{P}_{V^\perp}$, $V^\perp$ spans orthogonal complement of V

- SVD alternatively: $Y_{1:k} = [V\ V^\perp]\begin{bmatrix}\Sigma\\0\end{bmatrix}U^T$, $\begin{bmatrix}\Sigma\\0\end{bmatrix}^+ = [\Sigma^+\ \ 0]$, $\sigma^+ = \begin{cases}1/\sigma & ,\sigma>0\\ 0 & ,\sigma=0\end{cases}$
- eigendecomposition projection:

$$\mathrm{P}_V = V\,I\,V^T = [V\ V^\perp]\begin{bmatrix}I & 0\\0 & 0\end{bmatrix}[V\ V^\perp]^T \quad \text{eigenvalues are 1 or 0}$$

Initial Convergence RLS (3)

- let $\eta \ll \sigma_k^2$ be small, then

$$\begin{aligned}\eta(\eta I + R_{1:k})^{-1} &= \eta(\eta V^{\perp}V^{\perp\,T} + \eta V V^T + V\Sigma^2 V^T)^{-1} \approx \eta(\eta V^{\perp}V^{\perp\,T} + V\Sigma^2 V^T)^{-1} \\ &= \eta\left([V\ V^{\perp}]\begin{bmatrix}\Sigma^2 & 0\\ 0 & \eta I\end{bmatrix}[V\ V^{\perp}]^T\right)^{-1} = \eta[V\ V^{\perp}]\begin{bmatrix}\Sigma^2 & 0\\ 0 & \eta I\end{bmatrix}^{-1}[V\ V^{\perp}]^T \\ &= \eta V\Sigma^{-2}V^T + V^{\perp}V^{\perp\,T} = \eta\, R_{1:k}^{+} + \mathrm{P}_{R_{1:k}}^{\perp}\end{aligned}$$

  where $R_{1:k}^{+} = Y_{1:k}(Y_{1:k}^T Y_{1:k})^{-2}Y_{1:k}^T$ and $\mathrm{P}_{R_{1:k}} = \mathrm{P}_{Y_{1:k}}$

- hence

$$C_k = (\eta\, R_{1:k}^{+} + \mathrm{P}_{R_{1:k}}^{\perp})\,\widetilde{H}_0\widetilde{H}_0^T\,(\eta\, R_{1:k}^{+} + \mathrm{P}_{R_{1:k}}^{\perp})$$

  $\eta\, R_{1:k}^{+}\,\widetilde{H}_0$ reduced to $\mathcal{O}(\eta)$ in k-dim. subspace, column space of $Y_{1:k}$

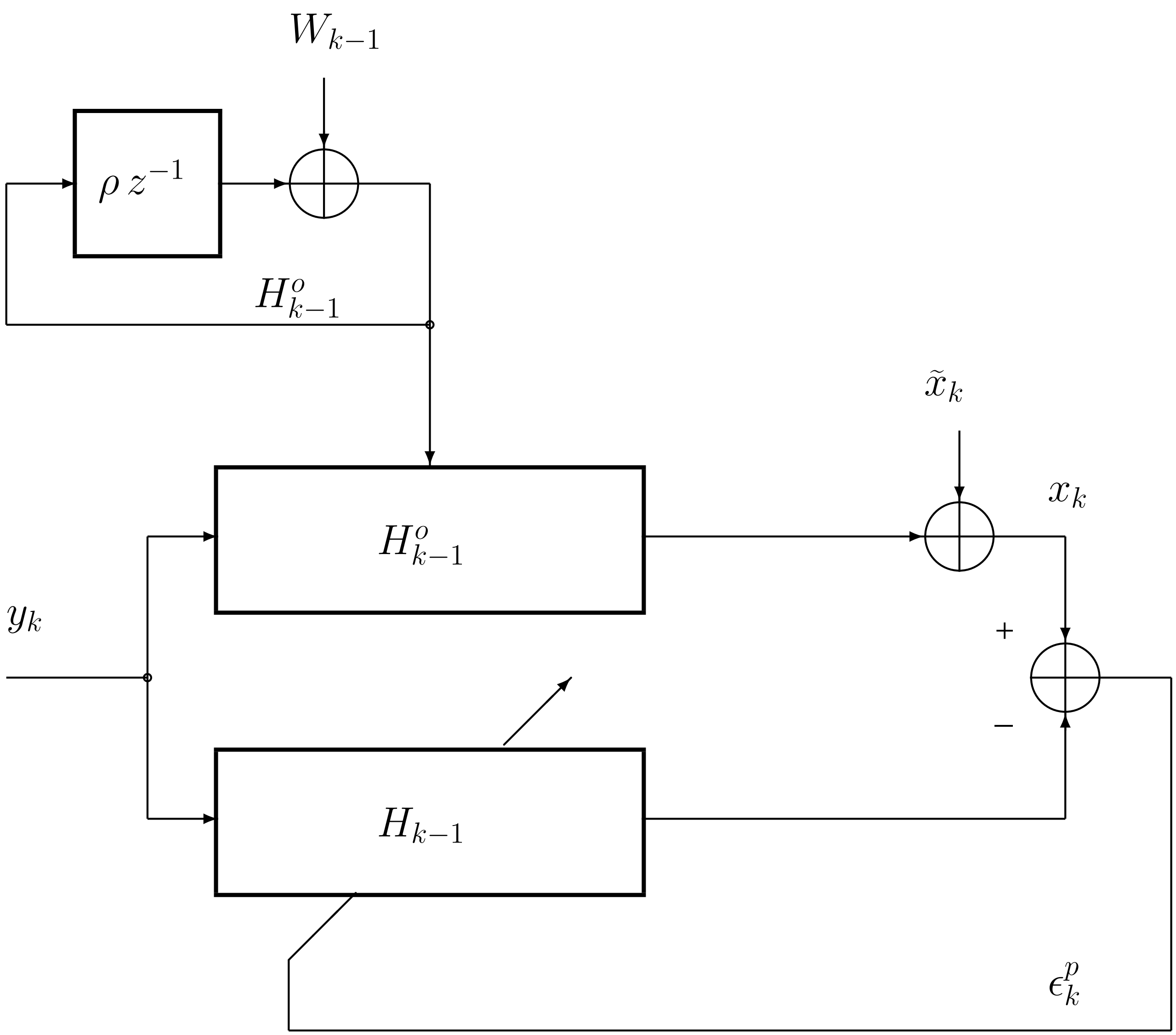
  $\mathrm{P}_{R_{1:k}}^{\perp}\,\widetilde{H}_0$ unchanged in $(N-k)$-dim. orthogonal complement

- C_k rank 1 (noiseless case): only the mean of $\widetilde{H}_k$ needs to converge

- RLS: the mean of $\widetilde{H}_k$ has essentially converged (filter estimate unbiased) after $k = N \;\Rightarrow\;$ very fast (mean dominates initial convergence in general)

  LMS: the mean needs to converge exponentially, dynamics of steepest-descent

Tracking Time-Varying Filters



Time-Varying System Identification Set-Up

- system processes:

$$
\begin{aligned}
x_k &= Y_k^T H_{k-1}^o + \widetilde{x}_k & E\widetilde{x}_k\widetilde{x}_i &= \xi^o \, \delta_{ki} \\
H_k^o &= \rho \, H_{k-1}^o + W_k & E W_k W_i^T &= Q \, \delta_{ki} \\
\widetilde{H}_k &= H_k^o - H_k & E W_k \widetilde{x}_i &= 0
\end{aligned}
$$

- time-varying filter modeled as AR(1) process, requires $|\rho| < 1$ for stationarity $\rightarrow$ stationary case of nonstationarity
- adaptive filter a priori error signal:

$$\epsilon_k^p = x_k - Y_k^T H_{k-1} = Y_k^T \widetilde{H}_{k-1} + \widetilde{x}_k$$

- learning curve: (independence assumption)

$$\xi_k = E(\epsilon_k^p)^2 = \xi^o + \xi_k^e = \xi^o(1+\mathcal{M}) \ , \quad \xi_k^e = \mathrm{tr}\{R_{YY}\, C_{k-1}\} \ , \quad C_k = E\, \widetilde{H}_k \widetilde{H}_k^T$$

- consider $\frac{1}{1-\rho} \gg$ adaptation time constants so that we can take $\rho = 1$ for the analysis

Tracking Analysis LMS

- filter deviation recursion:

$$\begin{aligned}\widetilde{H}_k &= \widetilde{H}_{k-1} - \mu\, \epsilon^p_k\, Y_k + W_k \\ &= (I - \mu\, Y_k Y_k^T)\, \widetilde{H}_{k-1} - \mu\, \widetilde{x}_k\, Y_k + W_k \\ &\approx (I - \mu\, R_{YY})\, \widetilde{H}_{k-1} - \mu\, \widetilde{x}_k\, Y_k + W_k\end{aligned}$$

  where we introduced the averaging approach in the last step
- filter error correlation matrix recursion:

$$C_k = (I - \mu\, R_{YY})\, C_{k-1}\, (I - \mu\, R_{YY}) + \mu^2\, \xi^o\, R_{YY} + Q$$

- the stationary nonstationarity combined with a constant stepsize leads to a steady-state, for which we get (with small μ):

$$R_{YY}\, C_\infty + C_\infty\, R_{YY} = \mu\, \xi^o\, R_{YY} + \frac{1}{\mu}\, Q$$

- steady-state misadjustment: $\mathcal{M}_{LMS} = \underbrace{\frac{\mu}{2}\, \mathrm{tr} R_{YY}}_{\text{estimation noise}} + \underbrace{\frac{1}{2\mu\xi^o}\, \mathrm{tr} Q}_{\text{lag noise}}$

Tracking Analysis RLS

- filter deviation recursion: $\lambda < 1$ to allow tracking

$$\begin{aligned}\widetilde{H}_k &= \widetilde{H}_{k-1} - R_k^{-1}\, Y_k\, \epsilon_k^p + W_k \\ &= (I - R_k^{-1} Y_k Y_k^T)\, \widetilde{H}_{k-1} - R_k^{-1}\, Y_k\, \widetilde{x}_k + W_k\end{aligned}$$

- after averaging in steady-state, assuming small $1-\lambda$ ($I - R_k^{-1} Y_k Y_k^T = \lambda\, R_k^{-1} R_{k-1} \approx \lambda\, I\ ,\quad R_k^{-1} \approx (1-\lambda) R_{YY}^{-1}$)

$$\widetilde{H}_k = \lambda\, \widetilde{H}_{k-1} - (1-\lambda)\, R_{YY}^{-1}\, Y_k\, \widetilde{x}_k + W_k \quad \text{(dynamics indep. of } R_{YY}\text{)}$$

- filter error correlation matrix recursion:

$$C_k = \lambda^2\, C_{k-1} + (1-\lambda)^2\, \xi^o\, R_{YY}^{-1} + Q$$

- which leads to the steady-state value (assuming small $1-\lambda$)

$$C_\infty = \frac{1-\lambda}{2}\, \xi^o\, R_{YY}^{-1} + \frac{1}{2(1-\lambda)}\, Q$$

- steady-state misadj.: $\mathcal{M}_{RLS} = \underbrace{\frac{1-\lambda}{2}\, N}_{\text{estimation noise}} + \underbrace{\frac{1}{2(1-\lambda)\xi^o}\, \mathrm{tr}\{R_{YY} Q\}}_{\text{lag noise}}$

Tracking Optimization & LMS-RLS Comparison

- stepsize μ, $1-\lambda$ design result of compromise between:
  estimation noise: finite stepsize prevents convergence, consistency
  lag noise: small stepsize leads to lowpass filtering and to a filter estimate that lags behind the true filter
- LMS: $\mu^{opt} = \sqrt{\dfrac{\mathrm{tr}Q}{\xi^o\,\mathrm{tr}R_{YY}}}\;,\quad \mathcal{M}^{opt}_{LMS} = \sqrt{\dfrac{\mathrm{tr}R_{YY}\;\mathrm{tr}Q}{\xi^o}}$
- RLS: $\lambda^{opt} = 1 - \sqrt{\dfrac{\mathrm{tr}\{R_{YY}Q\}}{N\,\xi^o}}\;,\quad \mathcal{M}^{opt}_{RLS} = \sqrt{\dfrac{N\,\mathrm{tr}\{R_{YY}Q\}}{\xi^o}}$
- comparison:

$$\frac{\mathcal{M}^{opt}_{LMS}}{\mathcal{M}^{opt}_{RLS}} = \sqrt{\frac{\mathrm{tr}R_{YY}\;\mathrm{tr}Q}{N\,\mathrm{tr}\{R_{YY}Q\}}}$$

$$Q = \begin{cases} q\,I & : \text{equal performance, at least for small } q \\ q\,R_{YY} & : \text{LMS is better} \\ q\,R_{YY}^{-1} & : \text{RLS is better} \end{cases}$$

- faster initial convergence of RLS could be exploited for *jumping* parameters, if windowsize properly adapted

Optimal Tracking via Kalman Filtering

- *state-space model*: state = AR(1) vector process

$$\begin{array}{rrcl} \textit{state}\text{ equation} & H^o_k & = & A_k\; H^o_{k-1} + W_k \\ \textit{measurement}\text{ equation} & x_k & = & Y^T_k\; H^o_{k-1} + \widetilde{x}_k \end{array} \quad , \quad E \begin{bmatrix} W_k \\ \widetilde{x}_k \end{bmatrix} \begin{bmatrix} W_i \\ \widetilde{x}_i \end{bmatrix}^T = \begin{bmatrix} Q_k & P^T_k \\ P_k & \xi^o_k \end{bmatrix} \delta_{ki}$$

  time-varying (at the very least due to Y_k), usually $P_k = 0$
  state noise: W_k, measurement noise: $\widetilde{x}_k$, state transition matrix A_k

- Kalman filter (KF): estimates recursively in time the *state* H^o_{k-1} on the basis of the *measurements* $x_0, \ldots, x_k$ in a LMMSE sense. Sources of randomness: W_k, $\widetilde{x}_k$. Signal y_k treated as deterministic.

- special case: $H^o_k = H^o_{k-1} \;\Rightarrow\;$ Kalman filter $\rightarrow$ RLS algorithm

- RLS with exponential weighting can be interpreted as KF for the case of some non-zero Q_k

- in the time-invariant case (x_k and H^o_k jointly stationary apart from initial conditions), the KF converges to the causal Wiener filter.

Chapter 5: Sinusoids in White Noise

- $x_k = \sum_{i=1}^{M} A_i \cos(\omega_i k + \phi_i)\,, \quad y_k = x_k + v_k \qquad\qquad \omega_i = 2\pi\, f_i$

- the support of $S_{xx}(f) = \sum_{i=1}^{M} \frac{A^2}{4}(\delta(f-f_i)+\delta(f+f_i))$ has measure zero,
  R_{XX} is singular for dimension $> \; 2M$

- $P(q)\, x_k = 0\,, \quad P(z) = \prod_{i=1}^{M} (1-2\cos\omega_i\, z^{-1}+z^{-2})\,, \quad q^{-1}x_k = x_{k-1}$

- Hence, x_k is perfectly predictible from the previous $2M$ samples. $P(z)$ and hence the ω_i can be found by linear prediction: *Prony* method.
  (Baron Prony, 18th century)
  Normal equations:
  $$P\, R_{XX} = [0 \cdots 0\ \sigma^2]\,, \quad \sigma^2 = 0$$
  where
  $$\begin{aligned} R_{XX} &= E\, X X^T \\ X &= [x_0 \cdots x_{2M}]^T \\ P &= [P_{2M} \cdots P_1\ P_0] \\ P_0 &= 1 \\ P_i &= P_{2M-i}\,, \quad i = 0, \ldots, M-1 \end{aligned}$$

Sinusoids in Noise: Signal and Noise Subspaces

- signal structure

$$X_k = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ \cos\omega_1 & \sin\omega_1 & \cdots & \cos\omega_M & \sin\omega_M \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos\omega_1 k & \sin\omega_1 k & \cdots & \cos\omega_M k & \sin\omega_M k \end{bmatrix} \begin{bmatrix} A_1\cos\phi_1 \\ -A_1\sin\phi_1 \\ \vdots \\ A_M\cos\phi_M \\ -A_M\sin\phi_M \end{bmatrix} = \mathcal{V}\, S$$

- one calls

$$\begin{aligned} Range\,\{\mathcal{V}\} &= \text{signal subspace} \\ (Range\,\{\mathcal{V}\})^{\perp} &= \text{noise subspace} \end{aligned}$$

- covariance structure:

$$Y_k \;=\; X_k + V_k \;=\; \mathcal{V}_k\, S + V_k \quad\Rightarrow\quad R_{YY} \;=\; \mathcal{V}\, R_{SS}\, \mathcal{V}^T + \sigma_v^2 I$$

if angles uniform and uncorrelated:

$$R_{SS} = \frac{1}{2}\,\text{diag}\left\{A_1^2, A_1^2, \ldots, A_M^2, A_M^2\right\}$$

Sinusoids in Noise: Signal and Noise Subspaces (2)

- Consider the eigendecomposition of R_{YY} ($\lambda_1 \geq \lambda_2 \geq \cdots$):

$$R_{YY} \;=\; \sum_{i=1}^{2M} \lambda_i V_i\, V_i^T \;+\; \sum_{i=2M+1}^{k+1} \lambda_i V_i\, V_i^T \;=\; V_{\mathcal{S}} \Lambda_{\mathcal{S}} V_{\mathcal{S}}^T \;+\; V_{\mathcal{N}} \Lambda_{\mathcal{N}} V_{\mathcal{N}}^T$$

  where $\Lambda_{\mathcal{N}} = \sigma_v^2 I_{k+1-2M}$.

- Assuming $\mathcal{V}$ and R_{SS} to have full rank, the sets of eigenvectors $V_{\mathcal{S}}$ and $V_{\mathcal{N}}$ are orthogonal: $V_{\mathcal{S}}^T V_{\mathcal{N}} = 0$, and $\lambda_i > \sigma_v^2\,,\; i = 1, \ldots, 2M$.

- Equivalent descriptions of the signal and noise subspaces:

$$Range\,\{V_{\mathcal{S}}\} \;=\; Range\,\{\mathcal{V}\} \quad , \quad V_{\mathcal{N}}^T \mathcal{V} = 0$$

- Linear prediction in the noisy case: minimize variance subject to norm constraint: *Pisarenko* method
  with $k = 2M$: noise subspace dimension = 1

$$\min_{\|P\|=1} P\, R_{YY} P^T = \min_{\|P\|=1} P\, R_{XX} P^T + \sigma_v^2 \;\;\Rightarrow\;\; P\, R_{XX} = [0 \cdots 0]\,,\; P^T = V_{2M+1}$$

Sinusoids in Noise: Signal Subspace Fitting

- two equivalent signal subspace descriptions: $\mathcal{V}$ and $V_{\mathcal{S}}$
- with an estimated covariance matrix, $V_{\mathcal{S}}$ is approximate, so consider

$$\min_{\omega,T} \|\mathcal{V}(\omega) - V_{\mathcal{S}}T\|_F^2 \qquad \|A\|_F^2 = \text{tr}\, AA^T$$

  where $\omega = [\omega_1 \cdots \omega_M]$.

- separable problem: orthogonality of LS: $V_{\mathcal{S}}^T(\mathcal{V} - V_{\mathcal{S}}T) = 0 \;\Rightarrow\; T = V_{\mathcal{S}}^T\mathcal{V}$
- $\mathcal{V} - V_{\mathcal{S}}T = (I - V_{\mathcal{S}}V_{\mathcal{S}}^T)\mathcal{V} = (I - \mathrm{P}_{V_{\mathcal{S}}})\mathcal{V} = \mathrm{P}_{V_{\mathcal{S}}}^{\perp}\mathcal{V}$
- projection on column space of X: $\mathrm{P}_X = X(X^TX)^{-1}X^T$, $\quad \mathrm{P} = \mathrm{P}^T$, $\mathrm{PP} = \mathrm{P}$

$$\|\mathrm{P}_{V_{\mathcal{S}}}^{\perp}\mathcal{V}\|_F^2 \;=\; \text{tr}\,\mathcal{V}^T\mathrm{P}_{V_{\mathcal{S}}}^{\perp}\mathcal{V} = \text{tr}\,\mathcal{V}^T\mathrm{P}_{V_{\mathcal{N}}}\mathcal{V} = \|V_{\mathcal{N}}^T\mathcal{V}\|_F^2$$

- hence multi-D optim.

$$= \sum_{i=2M+1}^{k} \|V_i^T\mathcal{V}\|^2 = \sum_{j=1}^{M}\sum_{i=2M+1}^{k} |V_i(\omega_j)|^2$$

- approximate solution: plot as a function of ω and find M largest peaks of

$$\frac{1}{\sum_{i=2M+1}^{k} |V_i(\omega)|^2} \qquad \text{MUSIC!}$$

Sinusoids in Noise: Noise Subspace Parameterization

- $P(q)\,\cos\omega_i k = 0,\; P(q)\,\sin\omega_i k = 0,\;\Rightarrow\;\; \mathcal{G}(P)^T\mathcal{V} = 0$ where

$$\mathcal{G}^T(P) = \begin{bmatrix} P_0 & P_1 & \cdots & P_{2M} & 0 & \cdots & 0 \\ 0 & P_0 & P_1 & \cdots & P_{2M} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & P_0 & P_1 & \cdots & P_{2M} \end{bmatrix} \quad \text{Toeplitz}\;,\; (k-2M)\times(k+1)$$

- noise subspace fitting:

$$\min_{P,T} \|\mathcal{G}(P) - V_{\mathcal{N}}T\|_F$$

- separable problem $\;\Rightarrow\; T = V_{\mathcal{N}}^T\mathcal{G}\;,\;\; \mathcal{G} - V_{\mathcal{N}}T = \mathrm{P}^{\perp}_{V_{\mathcal{N}}}\mathcal{G}$ and hence

$$\|\mathrm{P}^{\perp}_{V_{\mathcal{N}}}\mathcal{G}\|_F^2 \;=\; \mathrm{tr}\,\mathcal{G}^T\mathrm{P}^{\perp}_{V_{\mathcal{N}}}\mathcal{G} = \mathrm{tr}\,\mathcal{G}^T\mathrm{P}_{V_{\mathcal{S}}}\mathcal{G} = \|V_{\mathcal{S}}^T\mathcal{G}\|_F^2$$

$$= \sum_{i=1}^{2M} \|\mathcal{G}^T V_i\|^2$$

Let $\mathcal{G}^T V_i = \mathcal{W}_i\,P^T$ where $\mathcal{W}_i = \mathcal{W}(V_i)$ is Hankel, then we get (with $P = PJ$)

$$\min_{P}\; P\left[(\sum_{i=1}^{2M}\mathcal{W}_i^T\mathcal{W}_i) + \;J\,(\sum_{i=1}^{2M}\mathcal{W}_i^T\mathcal{W}_i)\,J\right]\;P^T$$

subject to $P_0 = 1$ or $\|P\| = 1$.

Sinusoids in Noise: Maximum Likelihood Estimation

- additive noise v_k white and Gaussian $\rightarrow$ likelihood criterion

$$\min_{\omega,S} \|Y - \mathcal{V}(\omega)\, S\|^2$$

- separable: $\mathcal{V}^T(Y - \mathcal{V}\, S) = 0 \quad \Rightarrow \quad S = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T Y$

$$\Rightarrow \; \|Y - \mathcal{V}\, S\|^2 = Y^T \mathrm{P}^{\perp}_{\mathcal{V}} Y = Y^T \mathrm{P}_{\mathcal{G}(P)} Y = P \mathcal{Y}^T (\mathcal{G}(P)^T \mathcal{G}(P))^{-1} \mathcal{Y} P^T$$

where $\mathcal{G}(P) Y = \mathcal{Y}(Y)\, P^T$ (commutativity of convolution, $\mathcal{Y}$ Hankel)

- IQML (Iterative Quadratic Maximum Likelihood), iteration n:

$$\min_{P^{(n)}} P^{(n)}\, \mathcal{Y}^T (\mathcal{G}(P^{(n-1)})^T \mathcal{G}(P^{(n-1)}))^{-1} \mathcal{Y}\, P^{(n)\,T}$$

subject to $P_0 = 1$ or $\|P\| = 1$

- with a consistent initialization, only one iteration is required to get a BAN estimate at high SNR

- denoised IQML : $Y^T \mathrm{P}_{\mathcal{G}(P)} Y = \mathrm{tr}\left\{\mathrm{P}_{\mathcal{G}(P)}\, Y Y^T\right\} \;\rightarrow\; \mathrm{tr}\left\{\mathrm{P}_{\mathcal{G}(P)}\, (Y Y^T - \widehat{\sigma}^2_v I)\right\}$

- Pseudo-QML (PQML): $\frac{\partial}{\partial P} Y^T \mathrm{P}_{\mathcal{G}(P)} Y = 2Q(P)P \;\rightarrow\; 2Q(\widehat{P})P = \frac{\partial}{\partial P} P^T Q(\widehat{P})\, P$

Sinusoids in Noise: Adaptive Notch Filtering

- notch filter model

$$P(q)x_k = 0 \;\Rightarrow\; P(q)y_k = P(q)v_k \;\Rightarrow\; v_k = \frac{P(q)}{P(q/\rho)}\,y_k \;\text{ as } \rho \to 1$$

- notch filter output

$$\epsilon_k = H(q)\,y_k = H(q)x_k + H(q)v_k\;, \quad H(q) = \frac{P(q)}{P(q/\rho)}$$

- notch filter $H(z)$: zeros = $e^{\pm j\omega_i}$, poles = $\rho\, e^{\pm j\omega_i}$,
- notch filter output variance ($\mathrm{H}(f) = H(e^{j2\pi f})$)

$$E\,\epsilon_k^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathrm{H}(f)|^2 S_{xx}(f)df + \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathrm{H}(f)|^2 S_{vv}(f)df = \sum_{i=1}^{M} \frac{A_i^2}{2}\,|\mathrm{H}(f_i)|^2 + \sigma_v^2$$

- notch adaptation by output variance minimization

$$\min_P \; E\,\epsilon_k^2$$

adaptively: Recursive Prediction Error Method (RPEM): solve ML recursively