Institut Eurécom
Digital Communications

Final Examination

Date: Feb. 15, 2006

Duration: 2 hours

Answer any 3 out of 5 questions. All questions will be graded equally, All documents are allowed.

1 Pulse-Position Modulation

Consider the set of 4 signals in Figure 1. This is an example of 4-ary pulse-position modulation where a basic signaling pulse $p(t)$ is shifted in time by $mT_s/4, m = 0, 1, 2, 3$ where T_s is parameter to be chosen. The value of m indicates the information that is transmitted. The signaling pulse $p(t)$ is time-limited to T_p seconds, in the sense that $p(t) = 0, t \notin [0, T_p)$. The transmitted signal is sent across a finite-duration time-invariant linear channel $h(t)$ with duration T_c seconds. At the receiver we have

$$r(t) = \sqrt{\alpha E_s} p(t - mT_s/4) * h(t) + z(t)$$

where $z(t)$ is additive white Gaussian noise with power-spectral density N_0. We assume that the receiver has perfect knowledge of both $p(t)$ and $h(t)$ (i.e. they are not random). Note that due to time-dispersion the received signal has a duration

1. What is the value of α which guarantees that the average signal energy per symbol is E_s?
2. What is the relationship between T_p, T_c and T_s that makes the 4 signals orthogonal for any channel $h(t)$?
3. For an orthogonal configuration, what is the maximum bit rate that this signal set can achieve. This should be expressed in bits/s as a function of T_p and T_c.

4. What is the maximum-likelihood receiver for this signal set?

5. What is the minimum Euclidean distance between the signals as a function of the auto-correlation function of $h(t) * p(t)$, denoted by $\rho(t)$ and what is the resulting upper-bound (i.e. union-bound) on the probability of symbol error.

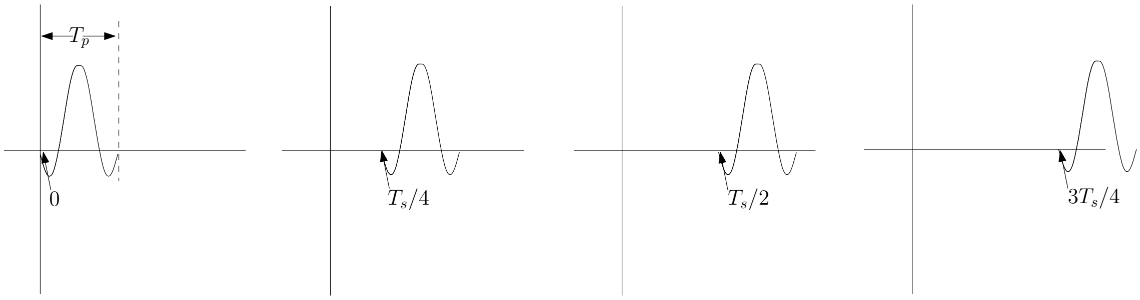

Figure 1: Signal Set for Problem 1

2 Generalized Non-Coherent Detection

Consider the following $N-$dimensional detection problem

$$\mathbf{y} = \sqrt{2E_s}\mathbf{h}x + \mathbf{z}$$

where $\mathbf{y}$ is an N-dimensional observation column vector, $\mathbf{h}$ is an N-dimensional zero-mean real Gaussian column vector with covariance matrix $\mathbf{K}_h = \mathrm{E}\mathbf{h}\mathbf{h}^t$, x is an equally-likely information bit which takes on the value 0 when the bit to be transmitted is zero and 1 when the bit to be transmitted is one, and z is an N-dimensional zero-mean white Gaussian column vector (i.e. diagonal covariance matrix) with variance N_0 in each component. The value of $\mathbf{h}$ is assumed to be unknown to the receiver which observes $\mathbf{y}$ and thus this is a form of non-coherent detection, although more than just phases are unknown to the receiver!

1. What are the likelihood functions under the two hypotheses (i.e. 0 and 1) as a function of K_h.

2. What is the maximum-likelihood detection rule?

3. Since K_h is a covariance matrix, it admits the following diagonalization $K_h = U\Lambda U^t$, where U is an N-dimensional unitary matrix, and Λ is an N-dimensional

diagonal matrix with positive entries (eigenvalues). Suppose we create the transformed observation $\mathbf{y}' = U^t\mathbf{y}$ (this is known as a Karhunen-Loève Transform, or decorrelating transform). Show first that the transformed problem is

$$\mathbf{y}' = \sqrt{2E_s}\mathbf{h}'x + \mathbf{z}'$$

where $\mathbf{h}'$ has $\mathrm{E}h_ih_j = \lambda_i\delta_{ij}$, where $\delta_{ij} = 1$, if $i = j$, and $\delta_{ij} = 0$, $i \neq j$. In other words, the covariance matrix of $\mathbf{h}'$ is $K_{h'} = \Lambda$. Next show that the ML detection rule in terms of $\mathbf{y}'$ is

$$\text{choose } 1 \text{ if} : \sum_{i=1}^{N} \frac{\lambda_i}{\lambda_i + \frac{N_0}{2E\lambda_i}} y_i'^2 \geq \sum_{i=1}^{N} \ln\left(1 + \frac{2E}{N_0}\lambda_i\right) \tag{1}$$

3 OFDM

Consider a generic OFDM system to be designed on a wireless channel with maximum channel duration of 1μ s. The sampling rate of the system is 30.72 Ms/s and occupied channel bandwidth is 20 MHz. The number of carriers per OFDM symbol is denoted N_c, the length of the cyclic-prefix is N_p and the number of useful carriers (i.e. those that are non-zero) is N_u.

1. Assuming the system should have an efficiency of 90%, where the efficiency is the ratio of information samples (N_c) to total samples per symbol (N_s + N_p), what should the number of carriers be to the closest power of 2.
2. What should the number of useful carriers be to occupy the channel bandwidth?
3. Assuming we use 16-QAM modulation what is the raw bit rate of the system?

4 Trellis Diagrams and the Viterbi Algorithm

A BPSK (2-AM) signal with symbol energy E_s is generated using a square-pulse of duration T seconds,

$$p_T(t) = \begin{cases} \sqrt{\frac{1}{T}}, t \in [0, T) \\ 0, t \notin [0, T). \end{cases}$$

It is transmitted across a dispersive channel $h(t) = h_0\delta(t) + h_1\delta(t - .25T)$ yielding the received signal

$$r(t) = \sqrt{E_s}\sum_n a_n p(t - nT) * h(t) + z(t)$$

where a_n is the BPSK information sequence (i.e. $a_n \in \{-1, 1\}$).

1. What is the autocorrelation sequence (g_n) of the cascaded channel $p_T(t) * h(t)$.
2. How many states does the correspoding state-space representation (Ungerboeck form) have?
3. Draw the trellis
4. What is the maximum-likelihood update rule in the Viterbi algorithm for this example?

5 BCJR Algorithm

Suppose we have a 2-state trellis representation for a BPSK signal transmitted over a dispersive channel. Recall the following forward and backward recursions in the Ungerboeck formation of the BCJR algoritm (sum-product algorithm)

$$\alpha(\sigma_{n+1}) = \sum_{\tilde{}\{\sigma_{n+1}\}} \alpha(\sigma_n)T_n(a_n, \sigma_n, \sigma_{n+1})G_n(a_n, \sigma_n)P(a_n)$$
$$\beta(\sigma_n) = \sum_{\tilde{}\{\sigma_n\}} \beta(\sigma_{n+1})T_n(a_n, \sigma_n, \sigma_{n+1})G_n(a_n, \sigma_n)P(a_n)$$

with $G_n(a_n, \sigma_n) = \exp\left\{\frac{1}{N_0}\mathrm{Re}\left[y_n a_n^* - .5|a_n|^2 g_0 - a_n^* a_{n-1} g_1\right]\right\}$ for a two-state trellis, and $a_n \in -1, +1$. $T_n(a_n, \sigma_n, \sigma_{n+1})$ is the trellis indicator function. Under the assumption that the information symbols are equally-likely (i.e. $\Pr(a_n = -1) = \Pr(a_n = 1) = .5$), show that the forward-backward recursions amounts to

$$\alpha_{n+1} = \mathbf{G}_n \alpha_n$$
$$\beta_n = \mathbf{G}_n^t \beta_{n+1}$$

where $\alpha_n = \begin{pmatrix}\alpha(\sigma_n = -1)\\ \alpha(\sigma_n = 1)\end{pmatrix}$, $\beta_n = \begin{pmatrix}\beta(\sigma_n = -1)\\ \beta(\sigma_n = 1)\end{pmatrix}$, and $\mathbf{G}_n$ is a 2×2 matrix. Find $\mathbf{G}_n$.