Game Theory
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Lecture 5

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Lecture 3-4 recap

- Defined mixed strategy Nash equilibrium
- Proved existence of mixed strategy Nash equilibrium in finite games
- Discussed computation and interpretation of mixed strategies Nash equilibrium
- Defined another concept of equilibrium from evolutionary game theory

→Today: introduce other solution concepts for simultaneous moves games

→Introduce solutions for sequential moves games

Outline

- Other solution concepts for simultaneous moves
  - Stability of equilibrium
    - Trembling-hand perfect equilibrium
  - Correlated equilibrium
  - Minimax theorem and zero-sum games
  - ε-Nash equilibrium
- The lender and borrower game: introduction and concepts from sequential moves

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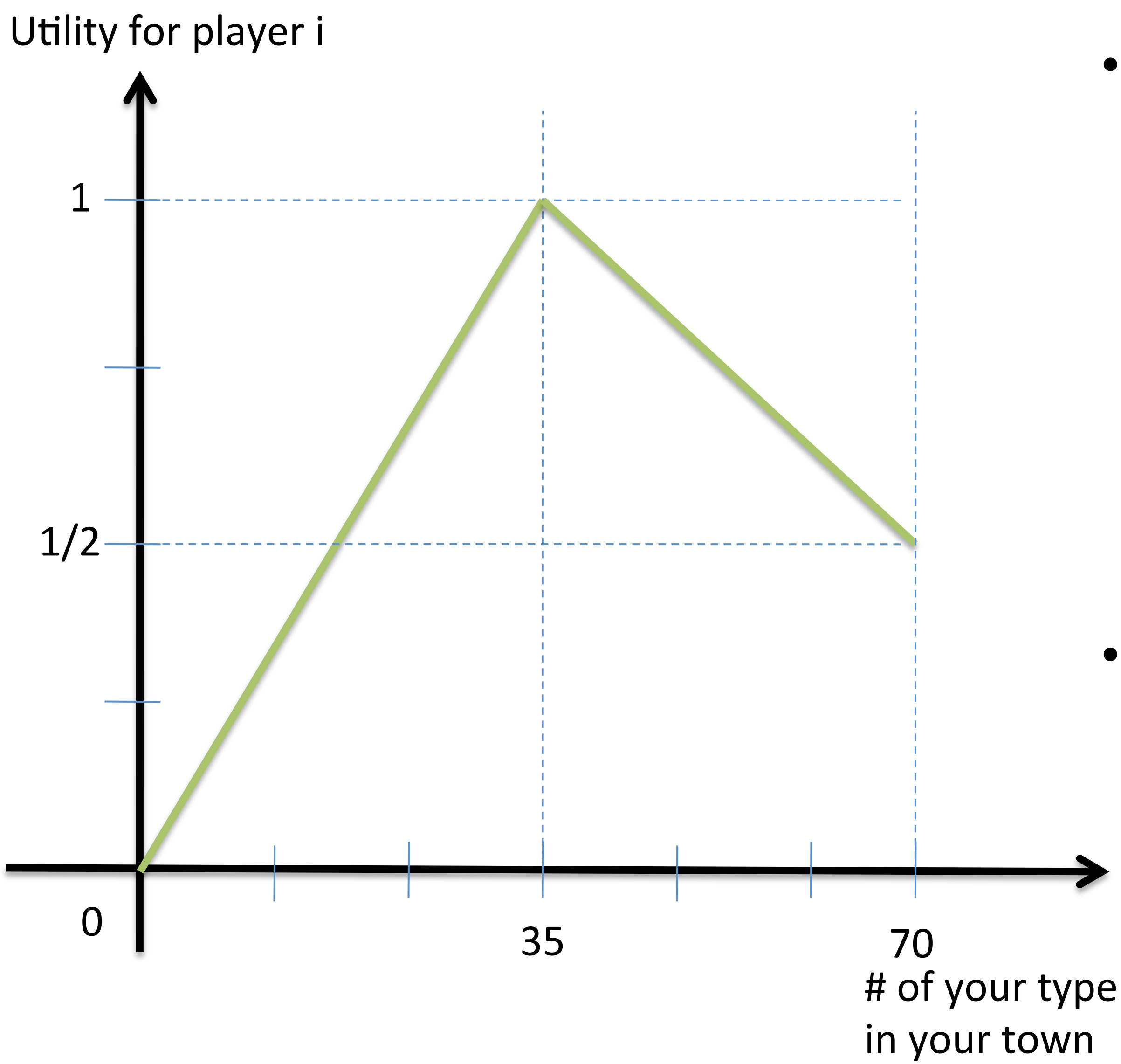
The Location Model

- Assume we have *2N* players in this game (e.g., N=70)
  - Players have two types: tall and short
  - There are *N* tall players and *N* short players
- Players are people who need to decide in which town to live
- There are two towns: East town and West town
  - Each town can host no more than *N* players

- Assume:
  - If the number of people choosing a particular town is larger than the town capacity, the surplus will be redistributed randomly

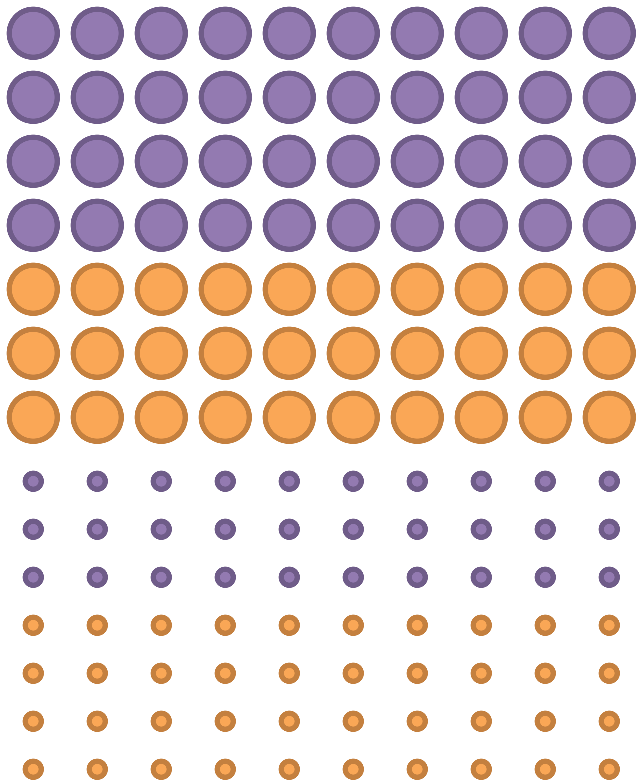
- Game:
  - Players: *2N* people
  - Strategies: East or West town
  - Payoffs

The Location Model: payoffs



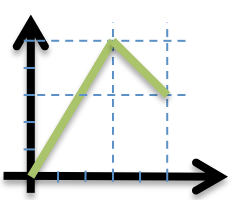
- The idea is:
  - If you are a small ***minority*** in your town you get a payoff of zero
  - If you are in large ***majority*** in your town you get a payoff of ½
  - If you are well ***integrated*** you get a payoff of 1
- People would like to live in mixed towns, but if they cannot, then they prefer to live in the majority town

Initial state



- Assume the initial picture is this one

- What will players do?



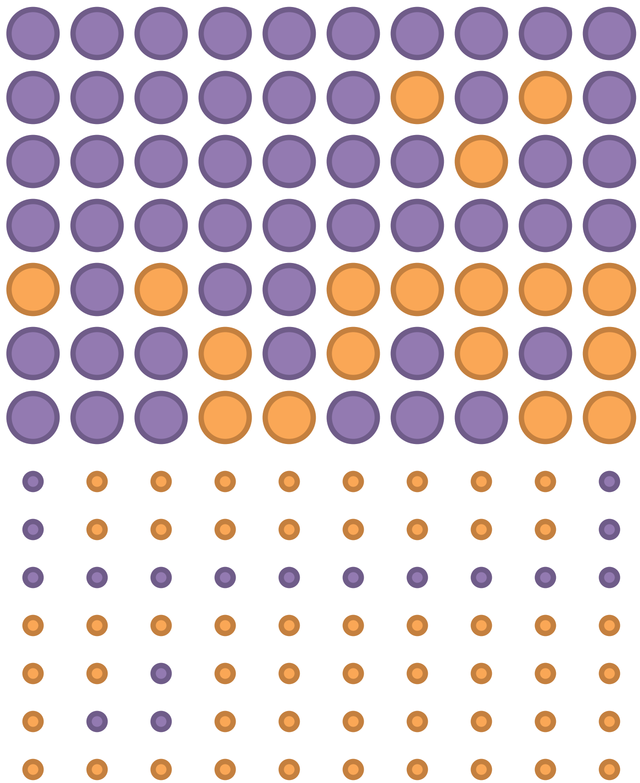
Tall player

Short player

West Town

East Town

First iteration

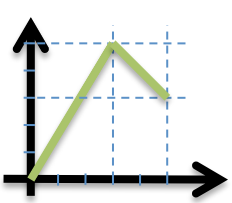


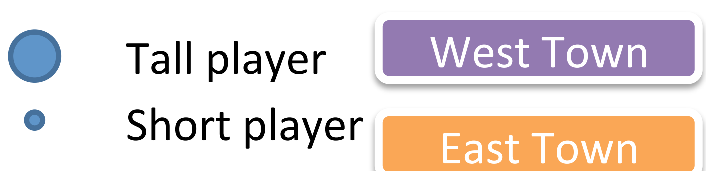
- For tall players
  - There's a minority of east town "giants" to begin with

  → switch to West town

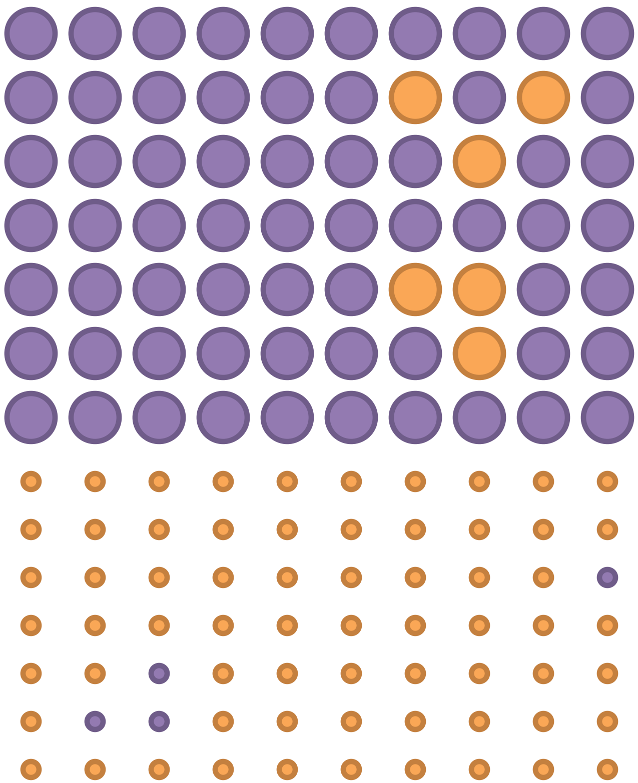
- For short players
  - There's a minority of west town "dwarfs" to begin with

  →switch to East town

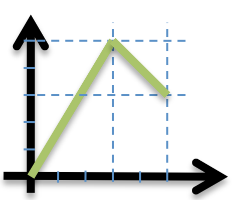




Second iteration



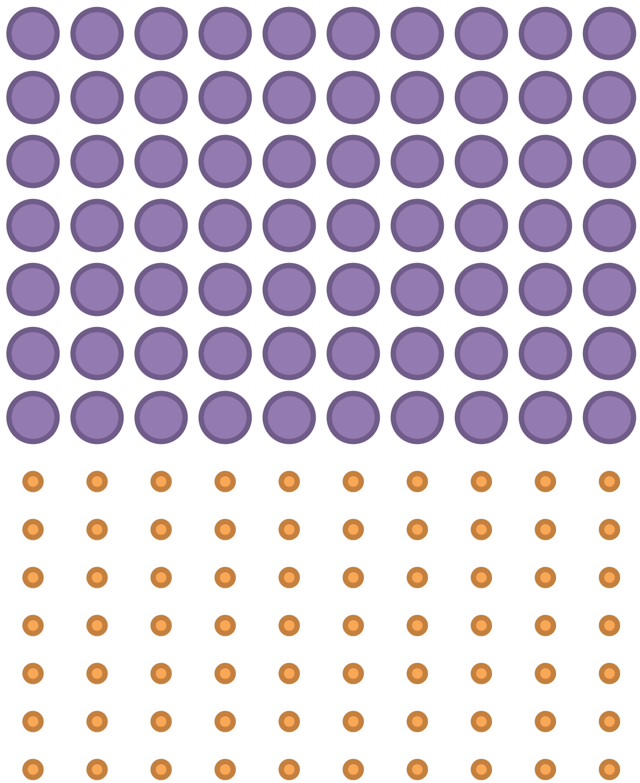
- Same trend
- Still a few players who did not understand
  - What is their payoff?



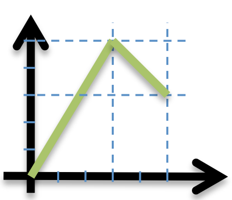
Tall player
Short player

West Town
East Town

Last iteration



- People got segregated
- But they would have preferred integrated towns!
  - Why? What happened?
  - People that started in a minority (even though not a “bad” minority) had incentives to deviate



Tall player

Short player

West Town

East Town

The Location Model: Nash equilibria

- Two segregated NE:
  - Short, E ; Tall, W
  - Short, W; Tall, E

- Is there any other NE?

Stability of equilibria

- The integrated equilibrium is not stable
  - If we move away from the 50% ratio, even a little bit, players have an incentive to deviate even more
  - We end up in one of the segregated equilibrium
- The segregated equilibria are stable
  - Introduce a small perturbation: players come back to segregation quickly

- Notion of stability in Physics: if you introduce a small perturbation, you come back to the initial state

- Tipping point:
  - Introduced by Grodzins (White flights in America)
  - Extended by Shelling (Nobel prize in 2005)

Trembling-hand perfect equilibrium

Definition: Trembling-hand perfect equilibrium

A (mixed) strategy profile s is a trembling-hand perfect equilibrium if there exists a sequence $s^{(0)}, s^{(1)}, \ldots$ of fully mixed strategy profiles that converges towards s and such that for all k and all player i, s_i is a best response to $s^{(k)}_{-i}$.

- Fully-mixed strategy: positive probability on each action
- Informally: a player's action s_i must be BR not only to opponents equilibrium strategies s_{-i} but also to small perturbations of those $s^{(k)}_{-i}$.

The Location Model

- The segregated equilibria are trembling-hand perfect
- The integrated equilibrium is not trembling-hand perfect

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Example: battle of the sexes

		Player 2	
		Opera	Soccer
Player 1	Opera	**2,1**	**0,0**
	Soccer	0,0	1,2

- NE: (O, O), (S, S) and ((1/3, 2/3), (2/3, 1/3))
  - The mixed equilibrium has payoff 2/3 each
- Suppose the players can observe the outcome of a fair toss coin and condition their strategies on this outcome
  - New strategies possible: O if head, S if tails
  - Payoff 1.5 each
- The fair coin acts as a correlating device

Correlated equilibrium: general case

- In the previous example: both players observe the exact same signal (outcome of the coin toss random variable)
- General case: each player receives a signal which can be correlated to the random variable (coin toss) and to the other players signal
- Model:
  - n random variables (one per player)
  - A joint distribution over the n RVs
  - Nature chooses according to the joint distribution and reveals to each player only his RV

  → Agent can condition his action to his RV (his signal)

Correlated equilibrium: definition

Definition: Correlated equilibrium

A correlated equilibrium of the game $(N, (A_i), (u_i))$ is a tuple (v, π, σ) where

- $v=(v_1, \ldots, v_n)$ is a tuple of random variables with domains $(D_1, \ldots, D_n)$
- π is a joint distribution over v
- $\sigma=(\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i: D_i \rightarrow A_i$

such that for all i and any mapping $\sigma_i': D_i \rightarrow A_i$,

$$\sum_{d \in D_1 \times \cdots \times D_n} \pi(d) u(\sigma_1(d_1), \cdots, \sigma_i(d_i), \cdots, \sigma_n(d_n)) \geq \sum_{d \in D_1 \times \cdots \times D_n} \pi(d) u(\sigma_1(d_1), \cdots, \sigma_i'(d_i), \cdots, \sigma_n(d_n))$$

Correlated vs Nash equilibrium

- The set of correlated equilibria contains the set of Nash equilibria

Theorem:

For every Nash equilibrium σ^*, there exists a correlated equilibrium (v, π, σ) such that for each player i, the distribution induced on A_i is σ_i.

- Proof: construct it with $D_i = A_i$, independent signals ($\pi(d) = \sigma^*_1(d_1) \times \ldots \times \sigma^*_n(d_n)$) and identity mappings σ_i

Correlated vs Nash equilibrium (2)

- Not all correlated equilibria correspond to a Nash equilibrium
- Example, the correlated equilibrium in the battle-of-sex game

→ Correlated equilibrium is a strictly weaker notion than NE

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Maxmin strategy

- Maximize "worst-case payoff"

Definition: Maxmin strategy

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

- Example
  - Attacker: Not attack
  - Defender: Defend
- This is not a Nash equilibrium!

		defender	
		Defend	Not def
attacker	Attack	-2,1	2,-2
	Not att	0,-1	0,0

Maxmin strategy: intuition

- Player i commits to strategy s_i (possibly mixed)
- Player –i observe s_i and choose s_{-i} to minimize i's payoff

- Player i guarantees payoff at least equal to the maxmin value $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

Two players zero-sum games

- Definition: a 2-players zero-sum game is a game where $u_1(s)=-u_2(s)$ for all strategy profile s
  - Sum of payoffs constant equal to 0
- Example: Matching pennies
- Define $u(s)=u_1(s)$
  - Player 1: maximizer
  - Player 2: minimizer

		Player 2	
		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

Minimax theorem

Theorem: Minimax theorem (Von Neumann 1928)

For any two-player zero-sum game with finite action space: $\max_{s_1} \min_{s_2} u(s_1, s_2) = \min_{s_2} \max_{s_1} u(s_1, s_2)$

- This quantity is called the value of the game
  - corresponds to the payoff of player 1 at NE
- Maxmin strategies ⇔ NE strategies
- Can be computed in polynomial time (through linear programming)

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ε-Nash equilibrium

Definition: ε-Nash equilibrium

For ε>0, a strategy profile (s_1*, s_2*,…, s_N*) is an ε-Nash equilibrium if, for each player i,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) - \varepsilon \text{ for all } s_i \neq s_i^*$$

- It is an approximate Nash equilibrium
  - Agents indifferent to small gains (could not gain more than ε by unilateral deviation)
- A Nash equilibrium is an ε-Nash equilibrium for all ε!

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“Cash in a Hat” game (1)

- Two players, 1 and 2
- Player 1 strategies: put \$0, \$1 or \$3 in a hat

- Then, the hat is passed to player 2

- Player 2 strategies: either “match” (i.e., add the same amount of money in the hat) or take the cash

“Cash in a Hat” game (2)

Payoffs:

- Player 1:
  - \$0 → \$0
  - \$1 → if match net profit \$1, -\$1 if not
  - \$3 → if match net profit \$3, -\$3 if not
- Player 2:
  - Match \$1 → Net profit \$1.5
  - Match \$3 → Net profit \$2
  - Take the cash → \$ in the hat

Lender & Borrower game

- The "cash in a hat" game is a toy version of the more general "lender and borrower" game:
  - Lenders: Banks, VC Firms, …
  - Borrowers: entrepreneurs with project ideas
- The lender has to decide <u>***how much money to invest***</u> in the project
- After the money has been invested, the borrower could
  - Go forward with the project and work hard
  - Shirk, and run to Mexico with the money

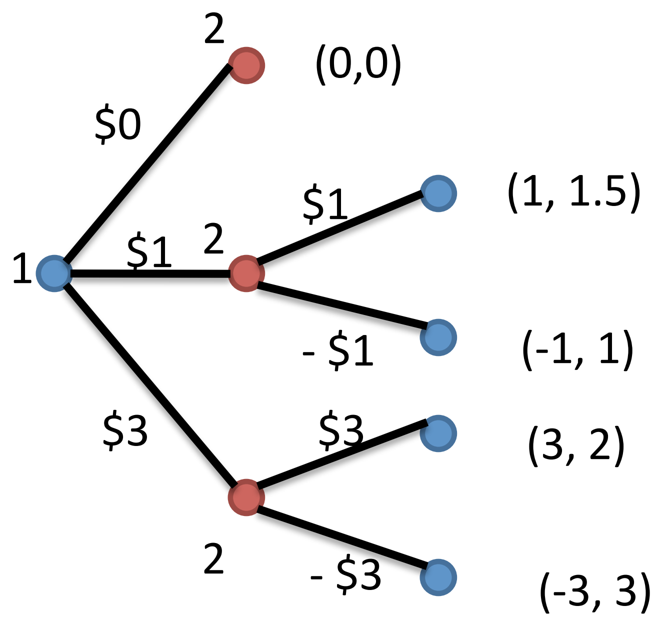
Simultaneous vs. Sequential Moves

- What is different about this game wrt games studied until now?
- It is a sequential move game
  - Player chooses first, then player 2
- Timing is not the key
  - The key is that P2 observes P1's choice before choosing
  - And P1 knows that this is going to be the case

Extensive form games

- A useful representation of such games is ***game trees*** also known as the ***extensive form***
  - Each internal node of the tree will represent the ability of a player to make choices at a certain stage, and they are called ***decision nodes***
  - Leafs of the tree are called ***end nodes*** and represent payoffs to both players
- Normal form games → matrices
- Extensive form games → trees

“Cash in a hat” representation



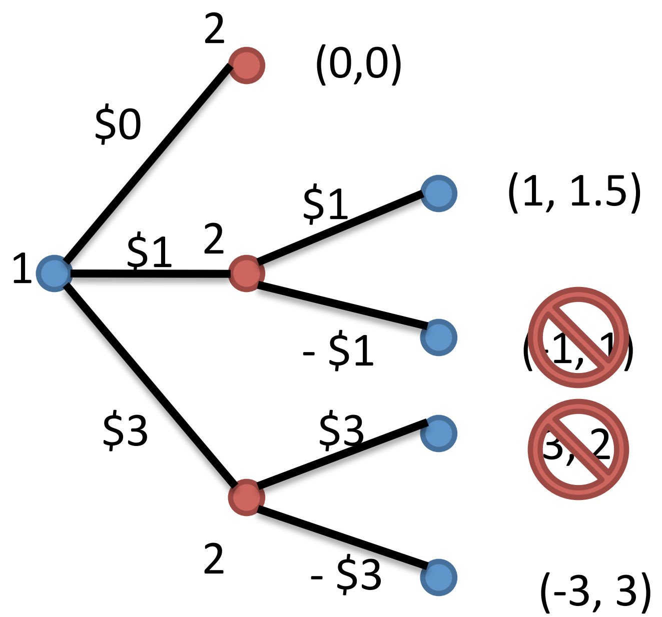
How to analyze such game?

Backward Induction

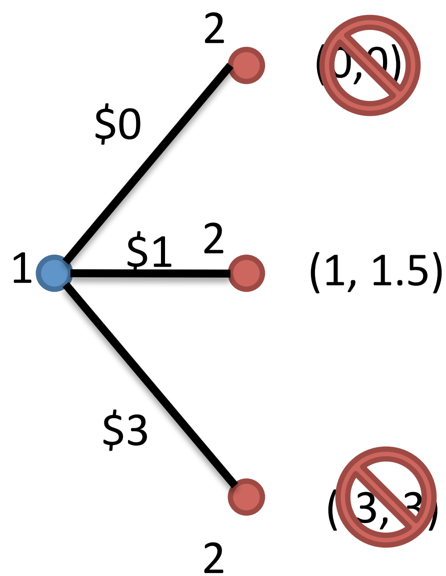
- Fundamental concept in game theory
- Idea: players that move early on in the game should ***put themselves in the shoes of other players playing later***

  → ***anticipation***
- Look at the end of the tree and work back towards the root
  - Start with the last player and chose the strategies yielding higher payoff
  - This simplifies the tree
  - Continue with the before-last player and do the same thing
  - Repeat until you get to the root

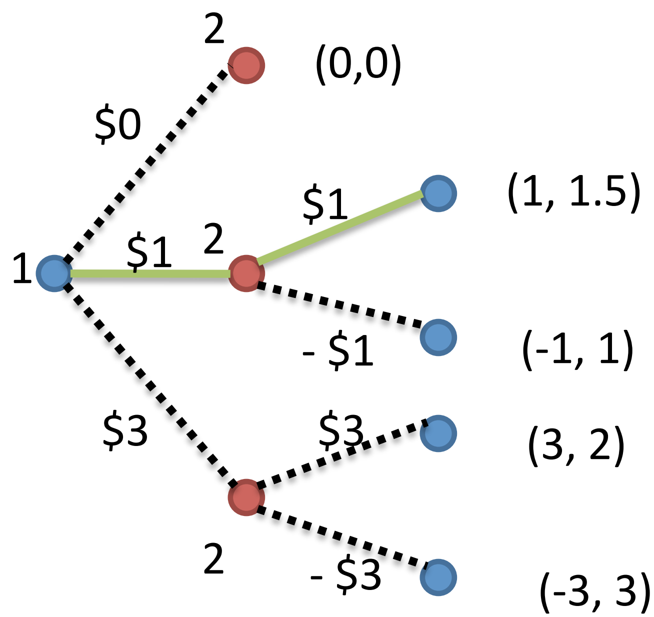
Backward Induction in practice (1)



Backward Induction in practice (2)



Backward Induction in practice (3)



Outcome:

Player 1 chooses to invest $1, Player 2 matches

The problem with the “lenders and borrowers” game

- It is not a disaster:
  - The lender doubled her money
  - The borrower was able to go ahead with a small scale project and make some money
- But, we would have liked to end up in another branch:
  - Larger project funded with \$3 and an outcome better for both the lender and the borrower
- Very similar to prisoner’s dilemna
- What prevents us from getting to this latter good outcome?

Moral Hazard

- One player (the borrower) has incentives to do things that are not in the interests of the other player (the lender)
  - By giving a too big loan, the incentives for the borrower will be such that they will not be aligned with the incentives on the lender
  - Notice that **moral hazard** has also disadvantages for the borrower

- Example: Insurance companies offers “full-risk” policies
  - People subscribing for this policies may have no incentives to take care!
  - In practice, insurance companies force me to bear some deductible costs (“franchise”)

- One party has incentive to take a risk because the cost is felt by another party

- How can we solve the Moral Hazard problem?

Solution (1): Introduce laws

- Today we have such laws: ***bankruptcy laws***

- But, there are limits to the degree to which borrowers can be punished
  - The borrower can say: I can't repay, I'm bankrupt
  - And he/she's more or less allowed to have a fresh start

Solution (2): Limits/restrictions on money

- Ask the borrowers a concrete plan (***<u>business plan</u>***) on how he/she will spend the money
- This boils down to changing the order of play!

- Also faces some issues:
  - Lack of flexibility, which is the motivation to be an entrepreneur in the first place!
  - Problem of timing: it is sometimes hard to predict up-front all the expenses of a project

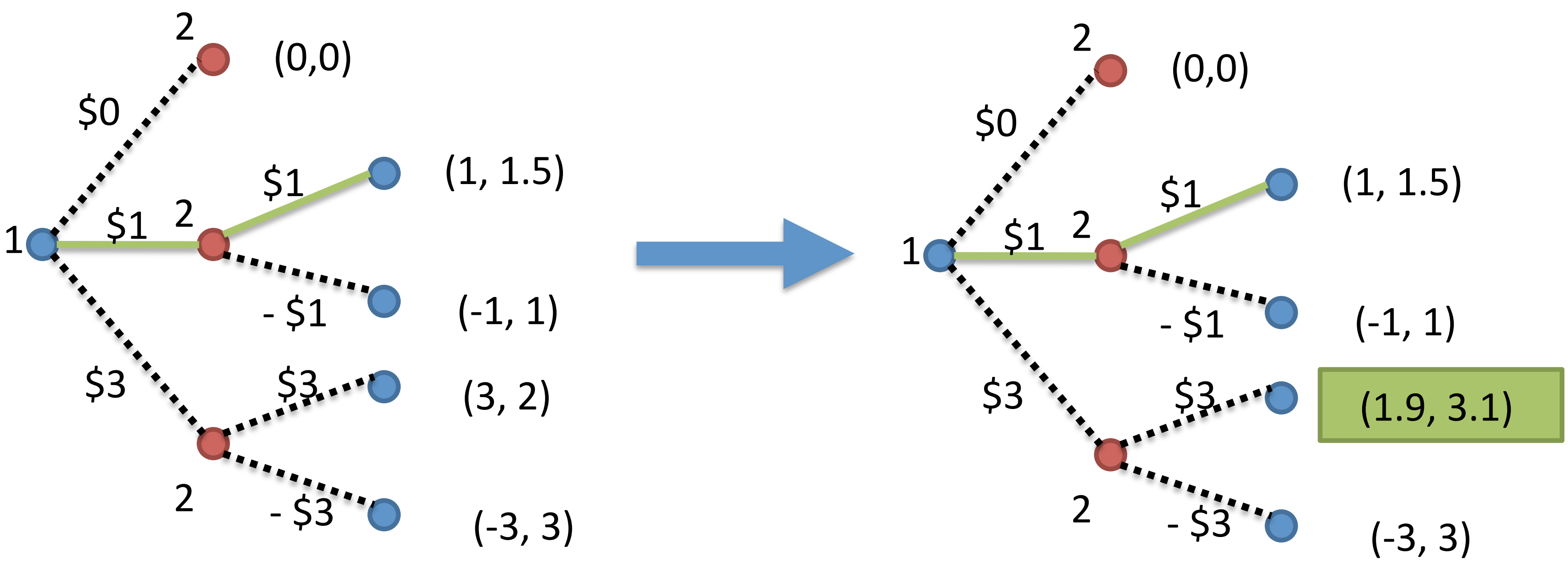
Solution (3): Break the loan up

- Let the loan come in small installments
- If a borrower does well on the first installment, the lender will give a bigger installment next time

- It is similar to taking this one-shot game and turn it into a ***<u>repeated game</u>***

Solution (4): Change contract to avoid shirk -- **Incentives**

- The borrower could re-design the payoffs of the game in case the project is successful



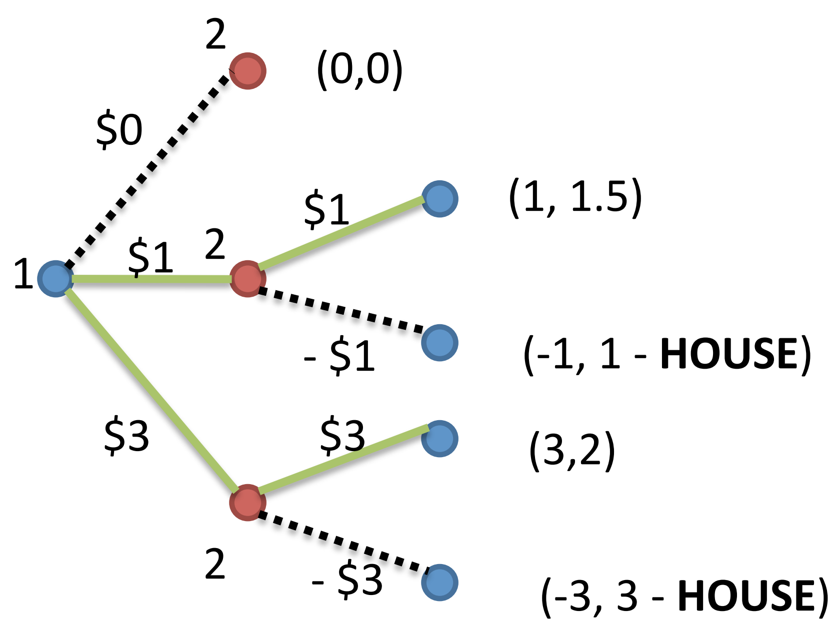
- Profit doesn't match investment but the outcome is better
  - Sometimes a smaller share of a larger pie can be bigger than a larger share of a smaller pie

Absolute payoff vs ROI

- Previous example: larger absolute payoff in the new game on the right, but smaller return on investment (ROI)
- Which metric (absolute payoff or ROI) should an investment bank look at?

Solution (5): Beyond incentives, **collaterals**

- The borrower could re-design the payoffs of the game in case the project is successful
  - Example: subtract house from run away payoffs



  - Lowers the payoffs to borrower at some tree points, yet makes the borrower better off!

Collaterals

- They do hurt a player enough to change his/ her behavior

➔Lowering the payoffs at certain points of the game, does not mean that a player will be worse off!!

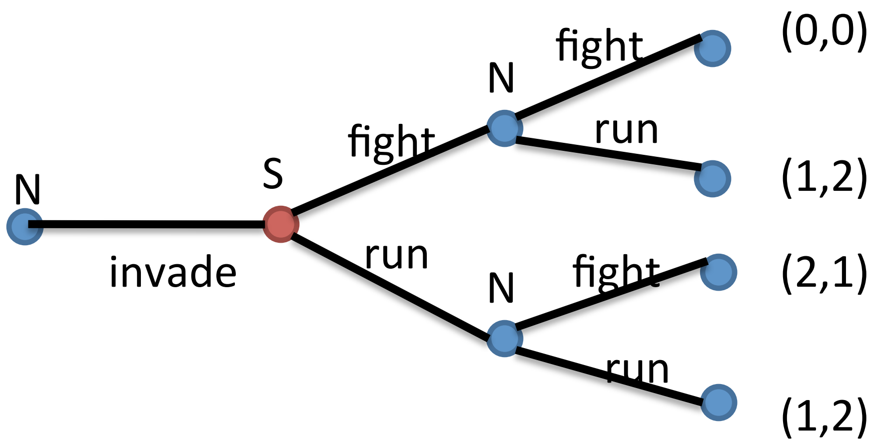
- Collaterals are part of a larger branch called ***commitment strategies***
  - Next, an example of commitment strategies

Norman Army vs. Saxon Army Game

- Collaterals are part of a larger branch called ***commitment strategies***

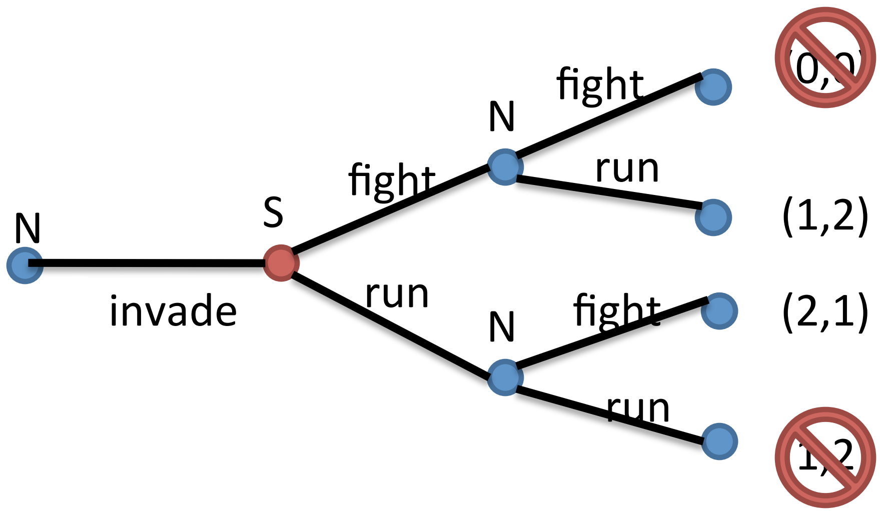
- Back in 1066, William the Conqueror lead an invasion from Normandy on the Sussex beaches
- We're talking about ***military strategy***
- So basically we have two players (the armies) and the strategies available to the players are whether to "fight" or "run"

Norman Army vs. Saxon Army Game

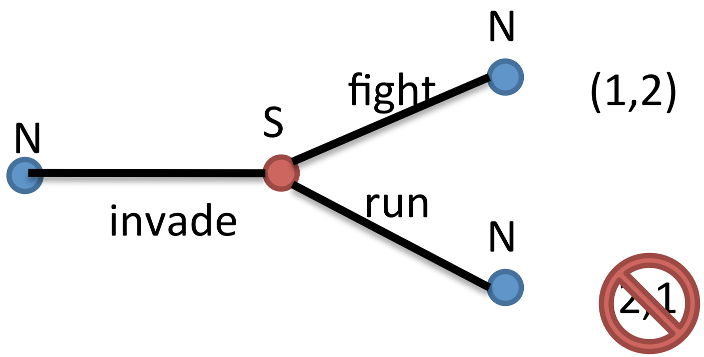


Let's analyze the game with Backward Induction

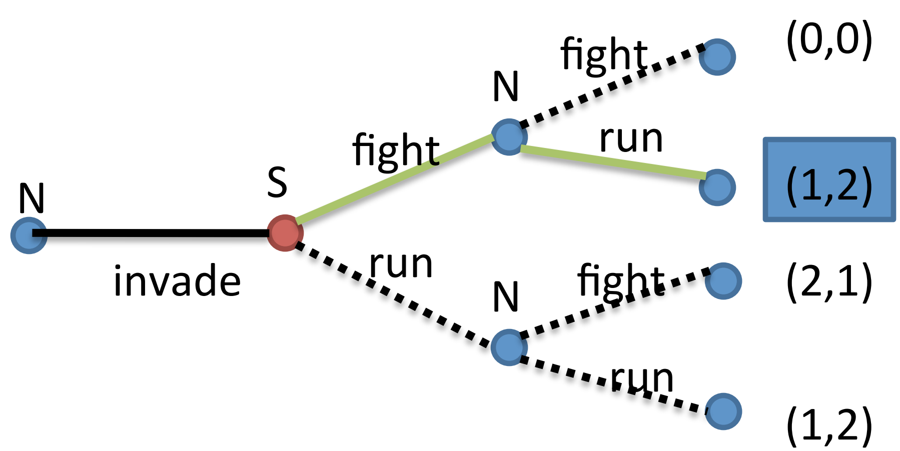
Norman Army vs. Saxon Army Game



Norman Army vs. Saxon Army Game



Norman Army vs. Saxon Army Game

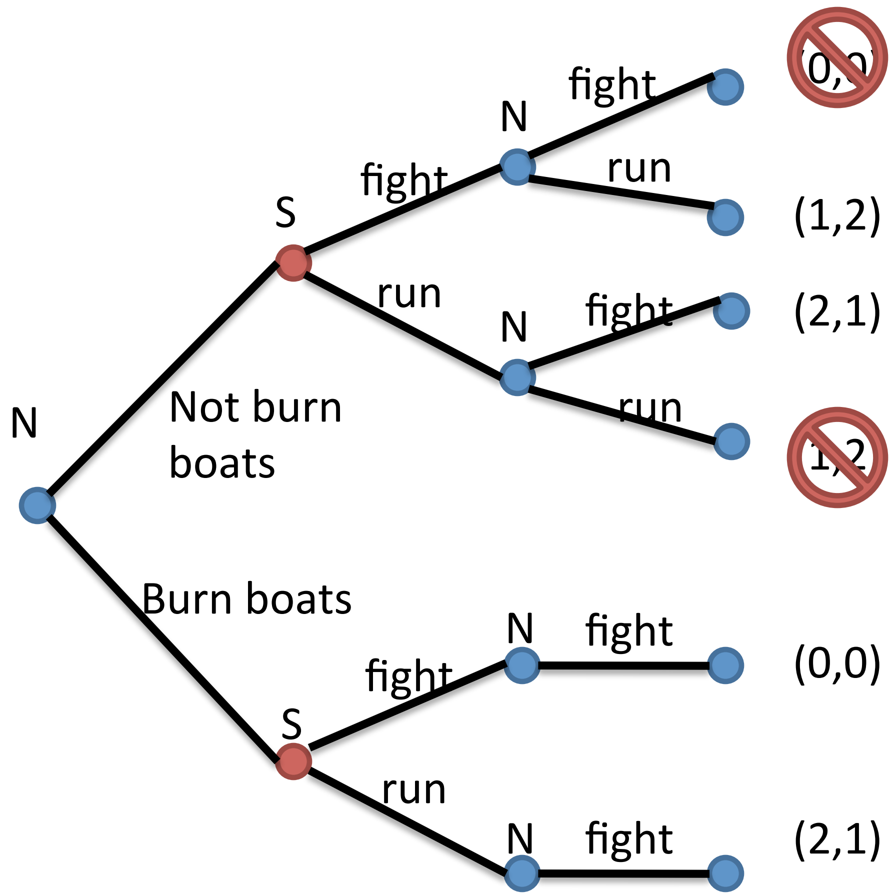


Backward Induction tells us:

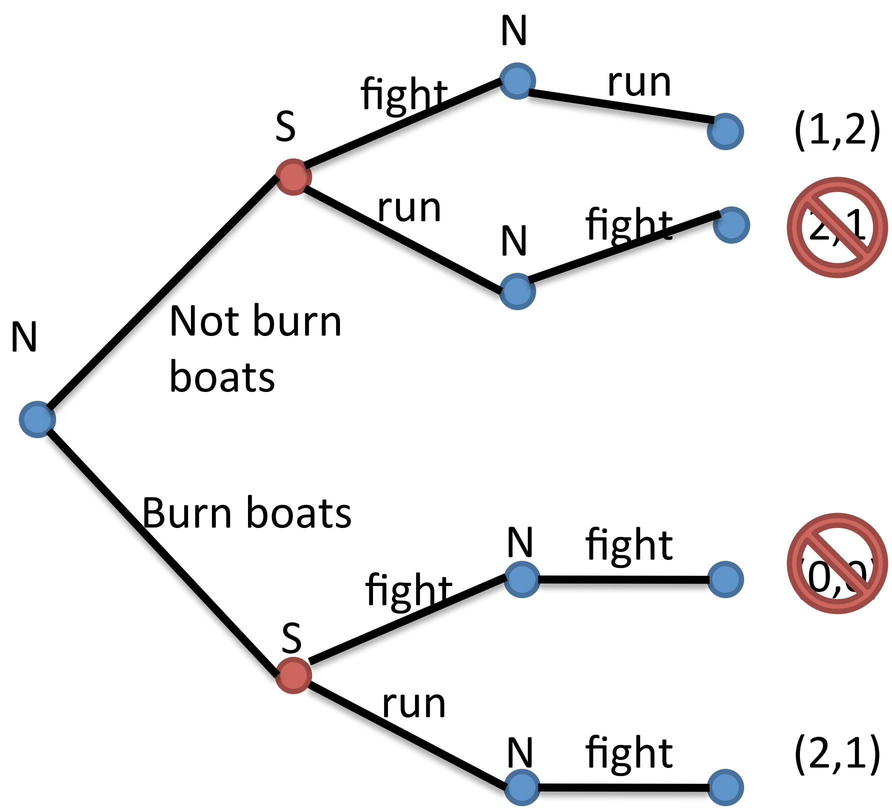
- Saxons will fight
- Normans will run away

What did William the Conqueror do?

Norman Army vs. Saxon Army Game



Norman Army vs. Saxon Army Game



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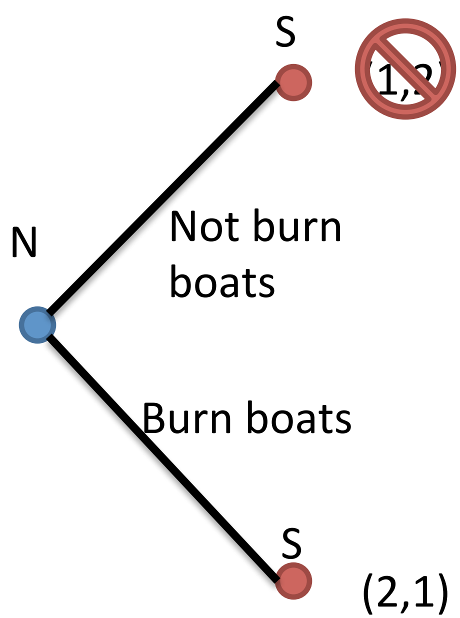
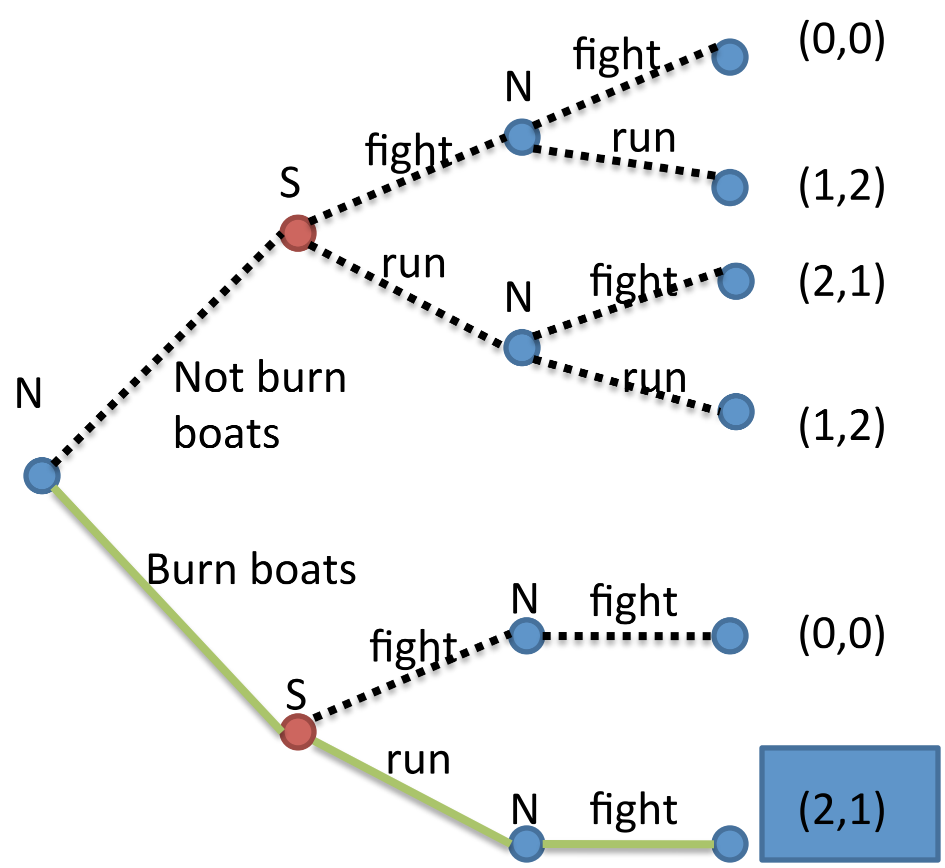


Norman Army vs. Saxon Army Game



Commitment

- Sometimes, getting rid of choices can make me better off!

- ***<u>Commitment</u>***:
  – Fewer options change the behavior of others

- The other players ***<u>must know</u>*** about your commitments
  – Example: Dr. Strangelove movie