Exercise sheet 2

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Exercise 1: Splitting the dollar

Two players bargain over how to split \$10. Each player $i \in \{1, 2\}$ chooses a number $s_i \in [0, 10]$ (which does not need to be an integer). Each player's payoff is the money he receives. We consider two allocation rules. In each case, if $s_1 + s_2 \leq 10$, each player gets his chosen amount s_i and the rest is destroyed.

1. In the first case, if $s_1 + s_2 > 10$, both players get zero. What are the (pure strategy) Nash equilibria?

   Answer: All couples such that $s_1 + s_2 = 10$ and $(s_1 = 10, s_2 = 10)$.

2. In the second case, if $s_1 + s_2 > 10$ and $s_1 \neq s_2$, the player who chose the smallest amount receives this amount and the other gets the rest. If $s_1 + s_2 > 10$ and $s_1 = s_2$, they both get \$5. What are the (pure strategy) Nash equilibria?

   Answer: $(s_1 = 5, s_2 = 5)$.

3. Now suppose that s_1 and s_2 must be integers. Does this change the (pure strategy) Nash equilibria in either case?

   Answer: For 1. the NE are all couples of integers such that $s_1 + s_2 = 10$ and $(s_1 = 10, s_2 = 10)$. For 2. they are $(s_1 = 5, s_2 = 5)$, $(s_1 = 5, s_2 = 6)$, $(s_1 = 6, s_2 = 5)$ and $(s_1 = 6, s_2 = 6)$.

Exercise 2: Partnership game

Two partners jointly own a firm and equally share its revenue. Each partner s_i chooses how much effort to put into the firm. We denote by s_i the effort of partner i (say, in hours) and assume that the action set of each partner is $S_i = [0, 4]$. The firm revenue is given by $4(s_1 + s_2 + bs_1s_2)$, where $b \in (0, 1/4]$ is a parameter measuring the synergy of the two partners. We assume that the cost supported by a partner equals the square of its effort s_i (notice that the cost of providing another unit of effort is therefore increasing in the amount of effort already provided). The payoffs are then

$$u_1(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1s_2)] - s_1^2$$
$$u_2(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1s_2)] - s_2^2$$

1. Find the best response of partner 1 to an effort s_2 of partner 2.

   Answer: Mention FOC and SOC in the maximization. $BR_1(s_2) = 1 + bs_2$.

2. Find the best response of partner 2 to an effort s_1 of partner 1. [Hint: look at the problem's symmetry.]

   Answer: $BR_2(s_1) = 1 + bs_1$.

3. On a single graph with s_1 on the x-axis and s_2 on the y-axis, plot the two best responses. (Such a plot is called a best-response diagram.)

4. Find the Nash equilibrium and show its position on the diagram.

   Answer: $s_1^* = s_2^* = 1/(1-b)$.

5. What are the efforts (s_1, s_2) that would maximize the total net profit of the firm (i.e., the sum of the utility of each partner)? How does it compare to the Nash equilibrium?

   Answer: $s_1^W = s_2^W = 1/(1/2 - b) > s_1^*$. **This is due to externality and has been discussed in class.**