Technical University of Denmark

Written exam, 13th December 2021

Course name: Discrete mathematics 2: algebra — Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: The exercises need to be solved by hand. All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_{14}, \circ)$ be the symmetric group on 14 letters and consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 13 & 12 & 3 & 8 & 2 & 14 & 4 & 10 & 6 & 5 & 11 & 9 & 1 \end{pmatrix}$$

from S_{14}.

a) Write f as a composition of mutually disjoint cycles.

b) What is the order of f?

c) Let $G = \langle f \rangle$ be the subgroup of $(S_{14}, \circ)$ generated by f. Further, let $H = \{g \in G \mid \text{the order of } g \text{ is odd}\}$. Prove that H is a cyclic subgroup of $(G, \circ)$.

Question 2

Let $\phi : D_4 \to S_4$ be a group homomorphism, where $(D_4, \circ)$ denotes the dihedral group of order 8 and where $(S_4, \circ)$ denotes the symmetric group on 4 letters. It is given that $\phi(r) = (1\,2\,3\,4)$ and $\phi(s) = (2\,4)$.

a) Compute $\phi(rs)$ and $\phi(r^2)$.

b) Let $C_4 = \{e, r, r^2, r^3\} \subseteq D_4$. Compute the image of C_4 under ϕ, i.e. compute $\phi(C_4) = \{\phi(f) \mid f \in C_4\}$, and determine whether or not $\phi(C_4)$ is a normal subgroup of $(S_4, \circ)$. Hint: for the latter part, consider the cosets $(1\,2) \circ \phi(C_4)$ and $\phi(C_4) \circ (1\,2)$.

c) Let $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$ be two groups and suppose that $\psi : G_1 \to G_2$ is a surjective group homomorphism. Show that if H is a normal subgroup of $(G_1, \cdot_1)$, then $\psi(H)$ is a normal subgroup of $(G_2, \cdot_2)$. Here $\psi(H) = \{\psi(h) \mid h \in H\}$ is the image of H under ψ.

Question 3

A commutative ring $(R, +, \cdot)$ is called regular if for all $x \in R$ there exists a $y \in R$ such that $x = x^2 \cdot y$. Now prove the following:

a) If $(R, +, \cdot)$ is a field, then $(R, +, \cdot)$ is regular.

b) If $(R, +, \cdot)$ is a domain and regular, then $(R, +, \cdot)$ is a field.

c) If I is an ideal of R and $(R, +, \cdot)$ is regular, then the quotient ring $(R/I, +, \cdot)$ is also regular.

Question 4

As usual, the finite field with 3 elements is denoted by $(\mathbb{F}_3,+,\cdot)$, while $(\mathbb{F}_3[X],+,\cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_3$. Define the quotient ring $(R,+,\cdot)$, where $R := \mathbb{F}_3[X]/\langle X^4+X^3+X^2+2X+1\rangle$.

a) Compute the standard form of the coset $X^7+X^6+2X^5+X^4+2+\langle X^4+X^3+X^2+2X+1\rangle$.

b) Find 4 distinct zero-divisors in R.

c) Show that $X+\langle X^4+X^3+X^2+2X+1\rangle$ is a unit of R and compute its multiplicative inverse.

d) Can $R\setminus\{0+\langle X^4+X^3+X^2+2X+1\rangle\}$ contain an element which is neither a unit nor a zero-divisor?

END OF THE EXAM