Technical University of Denmark

Written exam, 21st May 2024

Course name: Discrete mathematics 2: algebra — Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_5, \circ)$ be the group of permutations of $\{1,2,3,4,5\}$. Let f denote the permutation

$$f := (1\,4\,3) \circ (2\,3) \circ (2\,4).$$

a) Write f as a composition of disjoint cycles.

b) What are the order and the cycle type of f?

c) Compute $(1\,2\,3) \circ (4\,5) \circ (1\,2\,5\,4)^{-2}$.

d) Let g be the permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix} \in S_5.$$

Are $f \circ g$ and $g \circ f$ equal?

Question 2

Let n be a natural number and let $(S_n, \circ)$ denote the symmetric group on n letters. An element of S_n is called homocyclic if it is a product of disjoint cycles of the same length. (Thus $(1\,2) \circ (3\,4)$ is a homocyclic element in S_4, but $(1\,2)$ is not, since it has cycles of length 2 and 1).

a) Show that a homocyclic element in S_n has an order dividing n.

b) Show that any power of a homocyclic element in S_n is again homocyclic.

c) Is the product of any two homocyclic elements homocyclic?

d) Suppose that G is a subgroup of S_n. For $1 \leq a \leq n$ let G_a be the stabilizer of a in G, that is, $G_a = \{g \in G \mid g[a] = a\}$. We call G semiregular if $G_a = \{id_n\}$ for all a, $1 \leq a \leq n$. Show that G is semiregular if and only if all elements in G are homocyclic.

Question 3

Let $(K, +_K, \cdot_K)$ be a field. It is given a discrete valuation on K, that is, a function $v : K^* \mapsto \mathbb{Z}$ such that $v(a \cdot_K b) = v(a) + v(b)$ for all $a, b \in K^*$, v is surjective and $v(a +_K b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K^*$ with $a +_K b \neq 0_K$.

a) Consider the set $R := \{x \in K^* \mid v(x) \geq 0\} \cup \{0_K\}$. Prove that $(R, +_K, \cdot_K)$ is a ring.

b) Prove that for each nonzero $x \in K$ either $x \in R$ or $x^{-1} \in R$.

c) Prove that an element x is a unit of R if and only if $v(x) = 0$.

Question 4

As usual, the finite field with 5 elements is denoted by $(\mathbb{F}_5, +, \cdot)$, while $(\mathbb{F}_5[X], +, \cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_5$. Define the quotient ring $(R, +, \cdot)$, where $R := \mathbb{F}_5[X]/\langle X^4 + 3X^3 + 2X^2 + X + 1\rangle$.

a) Compute the standard form of the coset $X^6 + 2X^3 + 1 + \langle X^4 + 3X^3 + 2X^2 + X + 1\rangle$.

b) Write the polynomial $X^4 + 3X^3 + 2X^2 + X + 1 \in \mathbb{F}_5[X]$ as the product of irreducible polynomials.

c) Find 8 distinct zero-divisors in R.

d) Show that $X + \langle X^4 + 3X^3 + 2X^2 + X + 1\rangle$ is a unit of R and compute its multiplicative inverse.

END OF THE EXAM