Technical University of Denmark

Written exam, 13th December 2023

Course name: Discrete mathematics 2: algebra Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_6, \circ)$ be the group of permutations of $\{1,2,3,4,5,6\}$. Let f denote the permutation

$$f := (1\ 3) \circ (1\ 2\ 4\ 5) \circ (1\ 6\ 4\ 2\ 5\ 3).$$

a) Write f as a composition of disjoint cycles.

b) What are the order and the cycle type of f? Compute the permutation f^{70}.

c) Does S_6 contain a permutation with cycle type $(0,1,1,1,0,0)$? Motivate your answer.

d) Let g be the permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 2 & 6 \end{pmatrix} \in S_6.$$

Compute $f \circ g$ and $g \circ f$.

Question 2

Let n be an integer with $n \geq 2$ and consider the symmetric group on n letters $(S_n, \circ)$. Let U be the subset of S_n given by $U := \{f \in S_n \mid f[n] \neq n\}$.

a) Is U a subgroup of $(S_n, \circ)$? Compute $|U|$.

b) Prove that a subgroup H of $(S_n, \circ)$ contains U if and only if $H = S_n$. [**Hint:** use Lagrange's Theorem]

c) Consider the following relation on S_n: for $g, h \in S_n$ we say that $g \sim h$ if $g[n] = h[n]$. Prove that $\sim$ is an equivalence relation on S_n.

d) What is the cardinality of an equivalence class of $\sim$?

e) Prove that S_n is the disjoint union $S_n = U \cup [id_n]_\sim$. Here $[id_n]_\sim$ denotes the equivalence class of the identity permutation id_n with respect to $\sim$.

Question 3 Let $(\mathbb{Z}[X], +, \cdot)$ denote the ring of polynomials with coefficients in $\mathbb{Z}$, and let $I \subseteq \mathbb{Z}[X]$ be the ideal generated by $X - 3$ and 7, that is, $I = \langle X - 3, 7 \rangle$.

a) Show that for every $p(X) \in \mathbb{Z}[X]$, there is an integer $0 \leq \alpha_p \leq 6$ such that $p(X) - \alpha_p \in I$.

b) Prove that $\varphi : \mathbb{Z}[X] \to \mathbb{Z}_7$ with $\varphi(p(X)) = \alpha_p$ is a ring homomorphism.

c) Compute the kernel and the image of φ.

d) Prove that the quotient ring $\mathbb{Z}[X]/I$ is isomorphic to $\mathbb{Z}_7$.

Question 4

As usual, the finite field with 5 elements is denoted by $(\mathbb{F}_5,+,\cdot)$, while $(\mathbb{F}_5[X],+,\cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_5$. Define the quotient ring $(R,+,\cdot)$, where $R := \mathbb{F}_5[X]/\langle X^4+2X^3+4X+3\rangle$.

a) Compute the standard form of the coset $X^6+3X^5+X^4+1+\langle X^4+2X^3+4X+3\rangle$.

b) Write the polynomial $X^4+2X^3+4X+3 \in \mathbb{F}_5[X]$ as a product of irreducible polynomials.

c) Prove that R contains at least 13 distinct zero-divisors.

d) Show that $X+\langle X^4+2X^3+4X+3\rangle$ is a unit of R and compute its multiplicative inverse.

END OF THE EXAM