Technical University of Denmark

Written exam, 13th December 2024

Course name: Discrete mathematics 2: algebra   Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_5, \circ)$ be the group of permutations of $\{1,2,3,4,5\}$. Let g_1 and g_2 denote the following permutations from S_5

$$g_1 := (1\,2\,3), \quad g_2 := (1\,2) \circ (4\,5).$$

a) Write $g_1 \circ g_2$ as a composition of disjoint cycles.

b) What are the orders and the cycle types of g_1 and g_2?

c) Is it possible to find nonnegative integers i and j such that $(1\,2) = g_1^i \circ g_2^j$?

d) It is given that the set of permutations

$$H := \{\mathrm{id}, (1\,2) \circ (4\,5), (2\,3) \circ (4\,5), (1\,3) \circ (4\,5), (1\,2\,3), (1\,3\,2)\} \subset \mathrm{S}_5$$

is a subgroup of $(S_5, \circ)$. You may use this fact without proving it. The group H acts on the set $A := \{1,2,3,4,5\}$ via the group action $\varphi : H \to S_5$ defined by $\varphi(h) := h$. Use Burnside's lemma to compute the number of orbits in A under this group action.

Question 2

Let $(D_6, \circ)$ be the dihedral group consisting of all rotation and reflection symmetries of the regular 6-gon. Further let $H \subset D_6$ be the set $H := \{e, r^2, r^4\}$, where $r \in D_6$ as usual denotes the counter-clockwise rotation over an angle of $2\pi/6$ radians.

a) Show that H is a subgroup of $(D_6, \circ)$.

b) Is H a normal subgroup of $(D_6, \circ)$? Motivate your answer.

c) Show that the quotient group $(D_6/H, \circ)$ is not isomorphic to $(\mathbb{Z}_4, +_4)$.

Question 3

Let $(R, +_R, \cdot_R)$ be a commutative ring with zero-element 0_R and one-element 1_R. An element $x \in R$ is called *nilpotent* if there exists $m \in \mathbb{Z}_{>0}$ positive integer such that $x^m = 0_R$.

a) Prove that if $x \in R$ is nilpotent then either x is the zero-element 0_R or a zero-divisor.

b) Prove that if $x \in R$ is nilpotent then $r \cdot_R x$ is also nilpotent for all $r \in R$.

c) Prove that if $x \in R$ is nilpotent then $1_R +_R x$ is a unit. [**Hint:** Note that $1_R +_R x^m = 1_R$ and $1_R +_R (-x^m) = 1_R$. Here as usual $-x$ denotes the additive inverse of x in R. Show that either $1_R +_R x^m$ or $1_R +_R (-x^m)$ can be written as a multiple of $1_R +_R x$].

Question 4

As usual, the finite field with 7 elements is denoted by $(\mathbb{F}_7, +, \cdot)$, where $\mathbb{F}_7 = \{0, 1, \ldots, 6\}$, while $(\mathbb{F}_7[X], +, \cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_7$. Define the quotient ring $(R, +, \cdot)$, where $R := \mathbb{F}_7[X]/\langle X^4 + 3X^2 + 2X + 5\rangle$.

a) Compute the standard form of the coset $X^6 + X^5 + 3X^4 + 5X^3 + 5X + 1 + \langle X^4 + 3X^2 + 2X + 5\rangle$.

b) Write the polynomial $X^4 + 3X^2 + 2X + 5 \in \mathbb{F}_7[X]$ as the product of irreducible polynomials.

c) Show that $X + \langle X^4 + 3X^2 + 2X + 5\rangle$ is a unit of R and compute its multiplicative inverse.

d) Find 19 zero-divisors in R.

END OF THE EXAM