Technical University of Denmark

Written exam, 14th December 2022

Course name: Discrete mathematics 2: algebra Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_5, \circ)$ be the group of permutations of $\{1,2,3,4,5\}$. Let f denote the permutation

$$f := (1\,2) \circ (1\,2\,3) \circ (1\,2\,3\,4) \circ (1\,2\,3\,4\,5).$$

a) Write f as a composition of disjoint cycles.

b) What are the order and the cycle type of f?

c) What is the smallest natural number n such that S_n contains a permutation of order 10? Motivate your answer.

d) Let g be the permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix} \in S_5.$$

Are $f \circ g$ and $g \circ f$ equal?

Question 2

For a fixed integer $n \geq 3$, consider the symmetric group on n letters $(S_n, \circ)$. Let $H := \{f \in S_n \mid \ f[1] = 1,\ f[n] = n\} \subset S_n$ and define $X_{i,j} := \{f \in S_n \mid f[1] = i,\ f[n] = j\} \subset S_n$, for $i \neq j$ and $1 \leq i, j \leq n$.

a) Prove that H is a subgroup of S_n.

b) Prove that the left cosets of H are precisely the sets $X_{i,j}$, with $i \neq j$ and $1 \leq i, j \leq n$.

c) What is the order of H?

d) For $n \geq 4$, show that H is not a normal subgroup of S_n. What happens for $n = 3$? [**Hint:** for $n \geq 4$ consider the cosets $(1\,2) \circ H$ and $H \circ (1\,2)$]

Question 3

Let $S \subset \mathbb{Q}$ be the subset of rational numbers with odd denominators (when expressed with relatively prime numerator and denominator), that is

$$S = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0,\ b \textit{ is odd and} \gcd(a,b) = 1 \right\} \subset \mathbb{Q}.$$

Denote with $+$ and $\cdot$ the usual addition and multiplication of rational numbers. In this exercise you may use that $(\mathbb{Q}, +, \cdot)$ satisfies the ring axioms.

a) Prove that $(S,+,\cdot)$ is a ring.

b) Let $I \subset S$ be the subset of rational numbers with odd denominator and even numerator (when expressed with relatively prime numerator and denominator). Prove that I is an ideal of S.

c) Show that the quotient ring $(S/I,+,\cdot)$ is isomorphic to $(\mathbb{Z}_2,+_2,\cdot_2)$. [**Hint**: Consider the map $\varphi : S \to \mathbb{Z}_2$ with $\varphi(a/b) = a \ mod\ 2$].

Question 4

As usual, the finite field with 5 elements is denoted by $(\mathbb{F}_5,+,\cdot)$, while $(\mathbb{F}_5[X],+,\cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_5$. Define the quotient ring $(R,+,\cdot)$, where $R := \mathbb{F}_5[X]/\langle X^4+2X^3+X+2\rangle$.

a) Compute the standard form of the coset $X^7+X^6+2X^5+X^4+2+\langle X^4+2X^3+X+2\rangle$.

b) Write the polynomial $X^4+2X^3+X+2 \in \mathbb{F}_5[X]$ as the product of irreducible polynomials.

c) Find 4 distinct zero-divisors in R.

d) Show that $X+3+\langle X^4+2X^3+X+2\rangle$ is a unit of R and compute its multiplicative inverse.

END OF THE EXAM