Technical University of Denmark

Written exam, 21st May 2025

Course name: Discrete mathematics 2: algebra — Course nr. 01018
Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_6, \circ)$ be the group of permutations of $\{1,2,3,4,5,6\}$. Let g_1 and g_2 denote the following permutations from S_6

$$g_1 := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 1 & 5 & 3 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 4 & 6 & 1 & 2 \end{pmatrix}.$$

a) Write $g_1 \circ g_2$ as a composition of disjoint cycles.

b) What are the orders and the cycle types of g_1 and g_2?

c) Let $\tau \in S_6$ such that $\tau g_1 = g_1 \tau$. Prove that $\tau(5) = 5$ and $\tau(\{1,4\}) = \{1,4\}$.

d) Prove that $\tau = g_1^i$ for some positive integer i.

Question 2

Let $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$ be two groups and let $f, g : G_1 \to G_2$ be two group homomorphisms. Consider the set

$$A := \{x \in G_1 \mid f(x) = g(x)\}.$$

a) Prove that A is a subgroup of G_1.

b) Prove that if G_2 is abelian then A is a normal subgroup of G_1.

c) Prove that if the order of G_1 is 36, the order of G_2 is 3 and $|A|$ is divisible by 9 then $f = g$.

Question 3

Let S be a set and denote with $\mathscr{P}(S)$ the set of subsets of S. We define the operation Δ on $\mathscr{P}(S)$ as follows: for $X, Y \in \mathscr{P}(S)$ let

$$X \Delta Y := (X \cup Y) \setminus (X \cap Y),$$

that is, the difference between the union and the intersection of X and Y. The triple $(\mathscr{P}(S), \Delta, \cap)$ is a commutative ring. You can use this fact without proving it.

a) What are the zero-element and the one-element in this ring?

b) Prove that every proper subset of S is a zero-divisor in $\mathscr{P}(S)$.

c) Let $Y \in \mathscr{P}(S)$. How does the principal ideal $\langle Y \rangle$ look like?

d) If S is finite prove that every ideal of $\mathscr{P}(S)$ is principal.

Question 4

As usual, the finite field with 5 elements is denoted by $(\mathbb{F}_5,+,\cdot)$, where $\mathbb{F}_5 = \{0,1,\dots,4\}$, while $(\mathbb{F}_5[X],+,\cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_5$. Define the quotient ring $(R,+,\cdot)$, where $R := \mathbb{F}_5[X]/\langle X^4+3X^2+X+2\rangle$.

a) Compute the standard form of the coset $X^6+X^5+3X^4+1+\langle X^4+3X^2+X+2\rangle$.

b) Write the polynomial $X^4+3X^2+X+2 \in \mathbb{F}_5[X]$ as the product of irreducible polynomials.

c) Show that $X^2+2+\langle X^4+3X^2+X+2\rangle$ is a unit of R and compute its multiplicative inverse.

d) Find 20 zero-divisors in R.

END OF THE EXAM