Technical University of Denmark

Written exam, 22nd May 2023

Course name: Discrete mathematics 2: algebra | Course nr. 01018

Exam duration: 4 hours

Aid: All Aid

“Weighting”: All questions are equally important

Additional information: All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Question 1

Let $(S_6, \circ)$ be the group of permutations of $\{1,2,3,4,5,6\}$. Let f denote the permutation

$$f := (1\,2) \circ (1\,2\,3) \circ (1\,2\,3\,6) \circ (1\,2\,3\,4\,5).$$

a) Write f as a composition of disjoint cycles.

b) What are the order and the cycle type of f?

c) What is the smallest natural number n such that S_n contains a permutation of order 12? Motivate your answer.

d) Let g be the permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 6 & 1 & 3 \end{pmatrix} \in S_6.$$

Are $f \circ g$ and $g \circ f$ equal?

Question 2

Let $(G, \cdot)$ be a group with identity element e.

a) Let p be a prime number dividing $|G|$. Let S denote the set of p-tuples of elements of G where the composition (with the group operation $\cdot$) of the coordinates is e, that is:

$$S = \{(g_1, g_2, \ldots, g_p) \mid g_i \in G \text{ and } g_1 \cdot g_2 \cdot \ldots \cdot g_p = e\}.$$

Show that S contains exactly $|G|^{p-1}$ elements and hence it has cardinality divisible by p.

b) Define the relation $\sim$ on S by letting $\alpha \sim \beta$ if β is a cyclic permutation of α (that is, the coordinates of β are a cyclic shift of the coordinates in α). Show that a cyclic permutation of an element of S is again an element of S and that $\sim$ is an equivalence relation on S.

c) Prove that an equivalence class contains a single element if and only if it is of the form $(g, g, \ldots, g)$ with $g^p = e$.

d) Prove that every equivalence class has cardinality 1 or p (this uses the fact that p is a prime). Deduce that $|G|^{p-1} = k + pd$, where k is the number of classes of size 1 and d is the number of classes of size p.

e) Since $\{(e,e,\dots,e)\}$ is an equivalence class of size 1, conclude from d) that G contains an element of order p.

Question 3

Let
$$R = \left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \mid u,v \in \mathbb{Q} \right\}.$$

a) Show that $(R,+,\cdot)$ is a ring where $+$ and $\cdot$ denote the standard addition and multiplication of matrices. In this exercise you can assume that the distributive and associative laws are true for the addition and multiplication of matrices.

b) Show that
$$I = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \mid v \in \mathbb{Q} \right\}$$
is an ideal in R. Is the same true for
$$J = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in \mathbb{Q} \right\}?$$

c) Let $\varphi : R \to \mathbb{Q}$ with
$$\varphi \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} = u.$$
Prove that φ is a ring homomorphism and compute its kernel and image.

d) Prove that the quotient ring R/I is isomorphic to $\mathbb{Q}$.

Question 4

As usual, the finite field with 7 elements is denoted by $(\mathbb{F}_7,+,\cdot)$, while $(\mathbb{F}_7[X],+,\cdot)$ denotes the ring of polynomials with coefficients in $\mathbb{F}_7$. Define the quotient ring $(R,+,\cdot)$, where $R := \mathbb{F}_7[X]/\langle X^4+3X^3+6X^2+X+5\rangle$.

a) Compute the standard form of the coset $X^7+X^6+5X^5+X^3+2+\langle X^4+3X^3+6X^2+X+5\rangle$.

b) Write the polynomial $X^4+3X^3+6X^2+X+5 \in \mathbb{F}_7[X]$ as the product of irreducible polynomials.

c) Find 4 distinct zero-divisors in R.

d) Show that $X+5+\langle X^4+3X^3+6X^2+X+5\rangle$ is a unit of R and compute its multiplicative inverse.

END OF THE EXAM