5. Group actions and Burnside's Lemma

5.1 Group actions

Definition 5.1.1: Group action (p109)

Let $(G, \cdot)$ be a group and A a set. A **group action *of* G *on* A** is a **map**

$$\boxed{\varphi : G \to S_A}$$

such that

1. $\boxed{\varphi(e) = \mathrm{id}_A}$
   - $e \in G$ is the identity element of G
   - $\mathrm{id} \in S_A$ is the identity permutation on A.
2. $\forall f, g \in G : \boxed{\varphi(f \cdot g) = \varphi(f) \circ \varphi(g)}$

We say that "the **group G *acts* on the set A via the group action φ**".

On occasion we would like to evaluate the permutation $\varphi(f) \in S_A$ in an element a of A. This would give rise to a notation such as $\varphi(f)[a]$. To avoid having to use parentheses and brackets in this way, we will often **write φ_f instead of $\varphi(f)$**.

In this notation, a **group action of G on A satisfies**

1. $\boxed{\varphi_e = \mathrm{id}_A}$
2. $\forall f, g \in G : \boxed{\varphi_{f \cdot g} = \varphi_f \circ \varphi_g}$

Lemma 5.1.2: Properties of group actions (p110)

Let $(G, \cdot)$ be a group that acts on a set A via the group action $\varphi : G \to S_A$. Then

$$\forall f \in G : \boxed{\varphi_{f^{-1}} = (\varphi_f)^{-1}}$$

So the permutation induced by the inverse of an element f of G is the inverse of the permutation induced by f.

Examples of group actions:

- For the group $G = \mathrm{GL}(2, \mathbb{R})$, the set $A = \mathbb{R}^2$ and the group action $\varphi : \mathrm{GL}(2, \mathbb{R}) \to S_{\mathbb{R}^2}$ defined by

  $$\forall M \in \mathrm{GL}(2, \mathbb{R}), \forall x \in \mathbb{R}^2 : \quad \varphi_M[x] := Mx$$

  we have a group action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$.
- The symmetric group on n letters, denoted S_n, acts on the set $\{1, 2, \ldots, n\}$ by defining

  $$\boxed{\varphi : S_n \to S_{\{1,2,\ldots,n\}} : \quad \varphi_f \mapsto f}$$

Conjugate of an element in a group (p110)

Let $(G, \cdot)$ be a group and let $f, g \in G$. The element

$$f \cdot g \cdot f^{-1}$$

is called the **conjugate of g**.

- A group $(G, \cdot)$ **acts on itself via conjugation** by defining the group action $\varphi : G \to S_G$ as

$$\forall f, g \in G : \quad \varphi_f[g] := f \cdot g \cdot f^{-1}.$$

So why is fgf^{-1} a permutation of G? Because the map $\varphi_f : G \to G : g \mapsto fgf^{-1}$ has the inverse map $\varphi_{f^{-1}} : G \to G : g \mapsto f^{-1}gf$ and by Lemma 2.1.4 it is thus a bijection. Also, a bijection from the set to itself is a permutation, by definition of permutation (Section 2.2, p44)

Exercise 5.12: A group acts on itself via conjugation.

Let $(G, \cdot)$ be a group. Then the map

$$\varphi : G \to S_G : \quad \varphi_f[g] = f \cdot g \cdot f^{-1}$$

is a **group action of G on itself**.

Exercise 5.11: Center $Z(G)$ of a group G

Let $(G, \cdot)$ be a group. The **center** of G is the set

$$\boxed{Z(G) = \{g \in G \mid \forall f \in G : g \cdot f = f \cdot g\}} = \{f \in G \mid \varphi_f = \mathrm{id}_G\}$$

For any $g \in Z(G)$, we have $O_g = \{g\}$ and $G_g = G$.

Exercise 5.20: $Z(G)$ is a normal subgroup of G

Let $(G, \cdot)$ be a group. Then $Z(G)$ is a **normal subgroup** of G.
In other words, the left and right cosets of $Z(G)$ in G coincide:

$$\forall g \in G : \boxed{g \cdot Z(G) = Z(G) \cdot g}$$

- For any $f \in G, g \in Z(G)$, we can use the rule $fg = gf$.
- $Z(G)$ is the union of all singleton orbits, i.e. $Z(G) = \bigcup_{g \in Z(G)} O_g = \{g\}$.
- $f \cdot Z(G) = Z(G) \cdot f$ for all $f \in G$.
- $g \in Z(G) \iff \{g\}$ is an orbit under the conjugation action of G on itself.

5.2 Orbits and stabilizers

Definition 5.2.1: Orbit and stabilizer (p111)

Let $(G, \cdot)$ be a group that acts on a set A via the group action $\varphi : G \to S_A$. Then we define for each $a \in A$:

- The **orbit of** a under the action of G is the *set*

$$\boxed{O_a := \{\varphi_g[a] \mid g \in G\}} \subseteq A$$

- The **stabilizer of** a in G is the **subgroup**

$$\boxed{G_a := \{g \in G \mid \varphi_g[a] = a\}} \subseteq G$$

Exercise 6 (II): Stabilizer is a subgroup

Let $(G, \cdot)$ be a group that acts on a set A via the group action $\varphi : G \to S_A$. Further let $a \in A$. Then **all <u>stabilizers are subgroups</u>**:

$$G_a = \{g \in G \mid \varphi_g[a] = a\} \text{ is a } \mathbf{suroup} \text{ of } (G, \cdot).$$

Just like cosets are equivalence classes of a certain equivalence relation (recall that $\boxed{f \sim_H g \iff f^{-1}g \in H}$), **orbits are also equivalence classes** of a certain equivalence relation.

Lemma 5.2.5: $\sim_\varphi$ is an equivalence relation with equivalence classes given by orbits (p112)

Let $\varphi : G \to S_A$ be a group action of G on A. Define the relation $\sim_\varphi$ on A as

$$\forall a, b \in A : \quad \boxed{a \sim_\varphi b \iff \exists f \in G : b = \varphi_f[a]}$$

Then $\sim_\varphi$ is an **equivalence relation on** A. Moreover, for any $a \in A$, the equivalence class of a with respect to $\sim_\varphi$ is

$$\boxed{[a]_{\sim_\varphi} = O_a}$$

Now, just like for cosets, we apply Theorem 1.3.3 (properties of equivalence classes) to the orbits O_a:

Proposition 5.2.6: Properties of orbits (p112)

Let $(G, \cdot)$ be a group that acts on a set A via the group action $\varphi : G \to S_A$. Then

1. $\forall a \in A : \boxed{a \in O_a}$
2. The set A is covered by all orbits: $\boxed{\bigcup_{a \in A} O_a = A}$.
3. $\forall a, b \in A$, **either**
   - $O_a = O_b$
   - $O_a \cap O_b = \emptyset$
4. $\forall a, b \in A : \boxed{O_a = O_b \iff \exists f \in G : b = \varphi_f[a]} \iff a \sim_\varphi b.$

Proposition 5.2.6 says that A **is the disjoint union of its <u>distinct</u> orbits**. This means that $|A|$ is the sum of the sizes of its distinct orbits:

Size of A as sum of sizes of orbits

Let $(G, \cdot)$ be a group that acts on a set A via the group action $\varphi : G \to S_A$.
Let $\{O_{a_i} \mid i \in I\}$ be the set of **distinct** orbits of A under the action of G. Then

$$\boxed{|A| = \sum_{i \in I} |O_{a_i}|}$$

This follows from Proposition 5.2.6, which says that
A **is the disjoint union of its <u>distinct</u> orbits**.

Now we will study stabilizers.

⚙ **Lemma 5.2.7 (p113)**

Lert $\varphi : G \to S_A$ be a group action of $(G, \cdot)$ on a set A. Further let $a \in A$. Then

- G_a is a **subgroup** of $(G, \cdot)$.
- If $b \in O_a$ and $f \in G$ satisfies $b = \varphi_f[a]$, then

  $$f \cdot G_a = \{g \in G \mid \varphi_g[a] = b\}.$$

  "If you know one element f that moves a to b, then **all** other elements g that also move a to b, look like f times something in the stabilizer G_a of a"

ORBIT-STABILIZER THEOREM

*(Remember that G_a is a **subgroup** of G by Lemma 5.2.7 and we can thus consider the index $[G : G_a]$, i.e. the number of left cosets $f \cdot G_a$ of G_a in G.)*

Theorem 5.2.8: Orbit-stabilizer theorem (p114)

Let $(G, \cdot)$ be a group and suppose that $\varphi : G \to S_A$ is a group action of G on a set A. Then for any $a \in A$, we have

$$\boxed{[G : G_a] = |O_a|}$$

In case G is **finite**, we have

$$|O_a| = [G : G_a] \overset{\text{Lagrange}}{=} \frac{|G|}{|G_a|} \implies \boxed{|G| = |G_a| \cdot |O_a|}$$

For any $a \in A$, the number of distinct left cosets of the stabilizer G_a in G is equal to the size of the orbit O_a. In case G is ***finite***, we can apply Lagrange's theorem to conclude that **the size of the orbit O_a times the size of the stabilizer G_a is equal to the size of the group G.**

The above **orbit-stabilizer theorem** is very useful to solve combinatorial problems.

Size of an orbit divides size of the group

Let $(G, \cdot)$ be a **finite group** that acts on a set A via the group action $\varphi : G \to S_A$. Then for any $a \in A$, the **size of the orbit O_a divides the size of the group G**:

$$\boxed{|O_a| \text{ divides } |G|}$$

TODO: EXAMPLE COMBINATORIAL PROBLEM

5.3 Burnside's lemma

Now we will use the theory of group actions to solve the following problem:

- we take a **cube**
- give **each of the six sides** (also called *faces* or *facets*) of the cube a **color**
- Let us assume that we can choose between **2 colors**.
- Then a priori there are 2^6 **possible colorings**
- "Two colorings are the same if one can be obtained from the other by a rotation symmetry of the cube"
- **How many distinct colorings** does the cube have?

Definition 5.3.1: Burnside's lemma (p115)

Let $(G, \cdot)$ be a **finite group** that acts on a set A via the group action $\varphi : G \to S_A$. Define

$$\boxed{\mathrm{i}(g) := \{a \in A \mid \varphi_g[a] = a\}}$$

for each $g \in G$. Then the **number of distinct orbits** of A under the action of G is equal to

$$\boxed{\frac{1}{|G|} \sum_{g \in G} |\mathrm{i}(g)|}$$

In case of the cube coloring problem, we have model the colorings as a set $A = \{\text{coloring}_1, \text{coloring}_2, \ldots \text{coloring}_{2^6}\}$.

We consider the group action

$$\varphi : G \to S_A$$

of the group G of rotational symmetries of the cube (not a regular n-gon, so no C_n or D_n) on the set A of colorings. An orbit $O_a \subseteq A$ of a coloring a represents other elements (=*colorings*) in A that can be reached from $a \in A$ by a permutation φ_g of the six side colors for **some** rotation $g \in G$. To find the number of **distinct colorings**, we thus need to find the number of **distinct orbits** of A under the action of G.

We apply **Burnside's lemma** to find the number of distinct orbits. For this, we need to find $|\mathrm{i}(g)|$ for each rotation $g \in G$.

Let's compute the number of rotational symmetries of the cube:

- A cube has 8 vertices, 12 edges, and 6 faces.
- **Identity rotation:**
  - 1 such rotation
  - $1 \Rightarrow 1$ rotation.
  - $|\mathrm{i}(e)| = 2^6 = 64$ colorings.
- **Axis 1: through the centers of opposite faces**
  - There are 3 such axes, each allowing for 3 rotations (**excluding** identity e!)
    - a 0° rotation
    - a 180° rotation
    - a 270° rotation
    - a 360° rotation (= identity, already counted) (II)
  - $3 \Rightarrow 3 \cdot 3 =$ rotations.
- **Axis 2: through the midpoints of opposite edges.**
  - There are 6 such axes, each allowing for 1 rotation (**excluding** identity e!):
    - a 180° rotation
    - a 360° rotation (= identity, already counted)
  - $6 \Rightarrow 6 \cdot 1 = 6$ rotations.
- **Axis 3: through opposite vertices/corners.**
  - There are 4 such axes, each allowing for 2 rotations (**excluding** identity e!)
    - a 120° rotation
    - a 240° rotation
    - a 360° rotation (= identity, already counted)
  - $4 \Rightarrow 4 \cdot 2 = 8$ rotations.
- In total, there are $1 + \ + 6 + 8 = 24$ rotational symmetries of the cube, so $|G| = 24$.

For all 24 rotations $g \in G$, we need to compute $|\mathrm{i}(g)|$, i.e. the number of colorings that remain unchanged under the rotation g.

- **Identity rotation**: $|\,\mathrm{i}(e)| = 2^6 = 64$ colorings.
- **Axis 1, angle** $pi/2$ **or** $3pi/2$: (3 axes, 2 angles each = 6 rotations)
  - Top and bottom faces remain fixed (2^2 choices).
  - The 4 side faces cycle among themselves, so they must all be the same color (2 choices).
  - TOTAL: 6×2^3 colorings remaining fixed.
- **Axis 1, angle** pi: (3 axes, 1 angle each = 3 rotations)
  - Top and bottom faces remain fixed (2^2 choices).
  - The 4 side faces form 2 pairs that swap, so each pair must be the same color (2 choices for each of the two pairs).
  - TOTAL: $3 \times 2^2 \times 2^2 = 3 \times 2^4$ colorings remaining fixed.
- **Axis 2, angle** pi: (6 axes, 1 angle each = 6 rotations)
  - The 6 faces form 3 pairs that swap, so each pair must be the same color (2 choices for each of the three pairs).
  - TOTAL: 6×2^3 colorings remaining fixed.
- **Axis 3, angle** $2pi/3$ **or** $4pi/3$: (4 axes, 2 angles each = 8 rotations)
  - Each rotation groups the 6 faces into 2 cycles of 3 faces each, so each cycle must be the same color (2 choices for each of the two cycles).
  - TOTAL: 8×2^2 colorings remaining fixed.

Now we can apply Burnside's lemma to find the number of distinct colorings:

$$\frac{1}{|G|}\sum_{g \in G} |\,\mathrm{i}(g)| = \frac{1}{24}(64 + 48 + 48 + 48 + 32) = \frac{240}{24} = 10.$$

Exercises

Exercise 5.12: Transitive group actions

Let $(G, \cdot)$ be a finite group that acts on a set A via the group action $\varphi : G \to S_A$. The action is called **transitive** if

$$\boxed{\forall a, b \in A : \exists f \in G : \varphi_f[a] = b}$$

Or, in other words, if there is **only one orbit**, i.e. $\boxed{O_a = A}$ for any $a \in A$.

Why is there only one orbit? Because by definition of transitive action, for any $a, b \in A$, there exists $f \in G$ such that $b = \varphi_f[a]$. By Lemma 5.2.5, this means that $a \sim_\varphi b$, so $O_a = O_b$. Since this is true for any $a, b \in A$, all orbits are equal, and thus there is only one orbit, which must be the entire set A, since orbits cover A (Proposition 5.2.6, point 2).

Exercise 5.19: Double transitive group actions

Let $(G, \cdot)$ be a finite group that acts on a set A via the group action $\varphi : G \to S_A$. The action is called **doubly transitive** if

$$\boxed{\forall a_1, a_2, b_1, b_2 \in A, a_1 \neq a_2, b_1 \neq b_2 : \exists f \in G : \varphi_f[a_1] = b_1 \text{ and } \varphi_f[a_2] = b_2}$$

We can define the group action $: G \to S_{A \times A}$ on $A \times A$ as

$$\forall f \in G, \forall (a_1, a_2) \in A \times A : \quad {}_f[(a_1, a_2)] = (\varphi_f[a_1], \varphi_f[a_2])$$

And we see that being doubly transitive is equivalent to saying that the group action is transitive on the set

$$\{(a_1, a_2) \in A \times A \mid a_1 \neq a_2\} = A^2 \setminus \{(a, a) \mid a \in A\}.$$

This in turn means that there is **only one orbit**

$$\boxed{O_{(a_1, a_2)} = A^2 \setminus \{(a, a) \mid a \in A\}}$$

for the action on the set of distinct pairs.

Why is on $A \times A$ transitive?

Because for any two pairs (a_1, a_2) and (b_1, b_2) in $A \times A$ with $a_1 \neq a_2$ and $b_1 \neq b_2$, by the definition of doubly transitive action, there exists $f \in G$ such that $\varphi_f[a_1] = b_1$ and $\varphi_f[a_2] = b_2$. Therefore, ${}_f[(a_1, a_2)] = (b_1, b_2)$, showing that any pair can be mapped to any other pair, which is the definition of transitivity for the action on the set of distinct pairs.

Why only one orbit?

Because for any two distinct pairs (a_1, a_2) and (b_1, b_2) in $A \times A$, there exists $f \in G$ such that ${}_f[(a_1, a_2)] = (b_1, b_2)$. This means that the action can map any distinct pair to any other distinct pair, so all distinct pairs are in the same orbit under the action .

We can see this in another way, too. By Proposition 5.2.6, point 4, two elements are in the same orbit if and only if they are related by the equivalence relation $\sim$. Since for any two distinct pairs (a_1, a_2) and (b_1, b_2), there exists $f \in G$ such that ${}_f[(a_1, a_2)] = (b_1, b_2)$, it follows that $(a_1, a_2) \sim (b_1, b_2)$. Thus, all distinct pairs are equivalent under $\sim$, meaning they all belong to the same equivalence class, which is the orbit $O_{(a_1, a_2)}$. Therefore, there is only one orbit for the action on the set of distinct pairs.