Report Project 2

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Part I

aSDAs

Finite Element Formulation

Continous system

We start by considering the continous system of equations, given in the lecture as

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \nabla p = 0 \tag{1}$$

$$\frac{1}{c^2} \frac{\nabla p}{t} + \rho \nabla \cdot \boldsymbol{v} = 0 \tag{2}$$

The normal form, provided in the lecture notes reads

$$\frac{\partial u}{\partial t} + D(f(u)) = 0 \tag{3}$$

One can quickly see, that Equations 1-(2) can be fitted into Eqaution (3) by defining

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{v} \\ p \end{pmatrix}, D := \begin{pmatrix} \nabla \cdot & 0 \\ 0 & \nabla \end{pmatrix}, \boldsymbol{A} := \begin{pmatrix} 0 & \frac{1}{\rho} \\ \rho c^2 * \boldsymbol{I} & 0 \end{pmatrix}, f(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} \tag{4}$$

for the particular simple 1-dimensional case of this problem, one thus gets

$$\boldsymbol{u} = \begin{pmatrix} v \\ p \end{pmatrix}, D := \begin{pmatrix} \frac{\partial}{\partial x} \cdot & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix}, \boldsymbol{A} := \begin{pmatrix} 0 & \frac{1}{\rho} \\ \rho c^2 & 0 \end{pmatrix} \tag{5}$$

1-dimensional discretization

For the all furhther considerations are restricted to the 1D-case described in Equations (3) and (5).
For the Galerkin approximation, test functions are defined as follows

$$\boldsymbol{w} = \begin{pmatrix} w1 \\ w2 \end{pmatrix} \; with \; U = H_0^1(\Omega) = \left\{ w \in H^1(\Omega), \; w = 0 \text{ on } \partial\Omega \right\} \tag{6}$$

One thus gets

$$(u_t, w)_{D^k} + (D(f(u)), w)_{D^k} = 0 \tag{7}$$

partial integration leads to

$$(u_t, w)_{D^k} - (f(u), D(w))_{D^k} + (f * (u), Gw)_{\partial D^k} = 0 \tag{8}$$

$$G = \begin{pmatrix} \hat{n} & 0 \\ 0 & \hat{n} \cdot \end{pmatrix} \tag{9}$$

where f^* is the so-called "numerical flux", which simply is the value for $f(u)$ taken at the interface. Since in the DG context, this value differes from one adjacent element to the other, a rule must be found to decide upo a certain value.

In the 1D case, the above equation can be written as

$$(u,w)_{D^k} - (f(u), Dw)_{D^k} + f^*(u)^T \Big|_{x_k}^{x_{k+1}} \boldsymbol{I} = 0 \tag{10}$$

The problem is now discretized with a Bubnov-Galerkin scheme,so

$$w_h = \sum_{k=1}^{N} \boldsymbol{\phi}_{\mathbf{k}} \mathrm{w_k} \tag{11}$$

$$u_h = \sum_{k=1}^{N} \boldsymbol{\phi}_{\mathbf{k}} \mathrm{u_k} \tag{12}$$

are the approximation to the weighting-function and the test-function.

In the above equations, one still has to take into account, that u consists of the two independent variables v and p. Therefor, they have to be weighted and tested independently.We can thus write:

$$w_h = \sum_{k=1}^{N} \begin{bmatrix} \phi_k^1 & 0 \\ 0 & \phi_k^2 \end{bmatrix} \cdot \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix} \tag{13}$$

$$u_h = \sum_{k=1}^{N} \begin{bmatrix} \phi_k^1 & 0 \\ 0 & \phi_k^2 \end{bmatrix} \cdot \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \tag{14}$$

The shape-functions ϕ_k are taken as lagrange-polynomials based on Gauss-Lobatto points. For the purpose i=of this report, shape-functions up to a degree of 10 have been implemented.

From Equation (**??**), on can thus derive the matrix forms:

$$\mathbf{M_{ij}} = (\boldsymbol{\phi}_{\mathbf{i}}, \boldsymbol{\phi}_{\mathbf{j}}) \tag{15}$$

$$\mathbf{M_k} = \int_{D^k} \mathbf{N}\mathbf{N}^T det(\mathbf{J}) d\xi \tag{16}$$

$$\mathbf{S_k} = \int_{D^k} \mathbf{DNAN}^T \mathbf{J}^{-1} det \mathbf{J} d\xi \tag{17}$$

F-matrix

The derivation of the flux-matrix $\mathbf{F}$ is somewhat more difficult. We consider the last term of the wek form given in (10). Clearly the resulting matrix depends on the definition of f^*. For this report, the Lax-Friedrichs(LF) and the Hydrizable Discontinous Galerkin(HDG) flux were considered.

Lax-Friedrich flux

The Lax Friedrich flux is defined as

$$f^{*,LF}(u^+, u^-) = \frac{f(u^+) + f(u^-)}{2} + \frac{C}{2} \hat{\boldsymbol{n}}^- (u^- - u^+) \tag{18}$$

where u^+ denotes the u value of the current element and u^- denotes the u value of the neigbour.

The Flux matrix shall be exemplarily derived by looking at one elmenet k as described in Figure 2.

Left Node

$$u^+ = \begin{pmatrix} v_k^1 \\ p_k^1 \end{pmatrix}, u^- = \begin{pmatrix} v_{k-1}^2 \\ p_{k-1}^2 \end{pmatrix}, \hat{n}^- = 1 \tag{19}$$

$$f^{*,LF}(x_k) = \frac{\mathbf{A} * \begin{pmatrix} v_{k-1}^2 \\ p_{k-1}^2 \end{pmatrix} + \mathbf{A} \begin{pmatrix} v_k^1 \\ p_k^1 \end{pmatrix}}{2} + \frac{C\left(\begin{pmatrix} v_{k-1}^2 \\ p_{k-1}^2 \end{pmatrix} - \begin{pmatrix} v_k^1 \\ p_k^1 \end{pmatrix}\right)}{2} \tag{20}$$

$$= \frac{1}{2}(\mathbf{A} - C\mathbf{I}) \begin{pmatrix} v_k^1 \\ p_k^1 \end{pmatrix} + \frac{1}{2}(\mathbf{A} + C\mathbf{I}) \begin{pmatrix} v_{k-1}^2 \\ p_{k-1}^2 \end{pmatrix} \tag{21}$$

Right Node

$$u^+ = \begin{pmatrix} v_k^2 \\ p_k^2 \end{pmatrix}, u^- = \begin{pmatrix} v_{k+1}^1 \\ p_{k+1}^1 \end{pmatrix}, \hat{n}^- = -1 \tag{22}$$

$$f^{*,LF}(x_k) = \frac{\mathbf{A}\begin{pmatrix} v_k^2 \\ p_k^2 \end{pmatrix} + \mathbf{A}\begin{pmatrix} v_{k+1}^1 \\ p_{k+1}^1 \end{pmatrix}}{2} + \frac{C\left(\begin{pmatrix} v_{k+1}^1 \\ p_{k+1}^1 \end{pmatrix} - \begin{pmatrix} v_k^2 \\ p_k^2 \end{pmatrix}\right)}{2} \tag{23}$$

$$= \frac{1}{2}(\mathbf{A} + C\mathbf{I})\begin{pmatrix} v_{k+1}^1 \\ p_{k+1}^1 \end{pmatrix} + \frac{1}{2}(\mathbf{A} - C\mathbf{I})\begin{pmatrix} v_k^2 \\ p_k^2 \end{pmatrix} \tag{24}$$

So as a whole, for the boundary integral in Equation (10) the following term can be derived

$$f^{*,LF}(x_{k+1}) - f^{*,LF}(x_k) = \frac{1}{2}(\mathbf{A} + C\mathbf{I})\begin{pmatrix} v_{k+1}^1 \\ p_{k+1}^1 \end{pmatrix} + \frac{1}{2}(\mathbf{A} - C\mathbf{I})\begin{pmatrix} v_k^2 \\ p_k^2 \end{pmatrix} - \frac{1}{2}(\mathbf{A} - C\mathbf{I})\begin{pmatrix} v_k^1 \\ p_k^1 \end{pmatrix} - \frac{1}{2}(\mathbf{A} + C\mathbf{I})\begin{pmatrix} v_{k-1}^2 \\ p_{k-1}^2 \end{pmatrix} \tag{25}$$

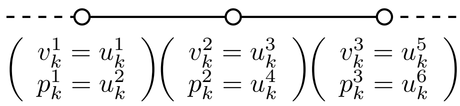

Figure 1: Dof numbering convention for the 1D-example

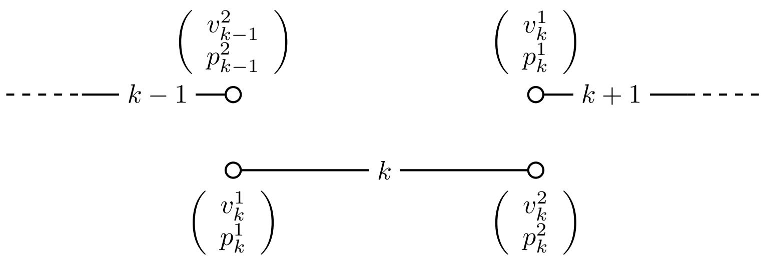

Figure 2: Element notation