1 The Finite Volume method

1.1 General

Like **FD!** (**FD!**) or **FE!** (**FE!**), the **FV!** (**FV!**) method is a mean to solving a **PDE!** (**PDE!**) by transorming it into a discrete algebraic form. In the field of fluid meachnics, thr finite volume method is the most popular approach. Other than **FD!** or **FE!**, the finite volume is conservative by construction(we will explain what that means in Section REF). Other than **FD!** it can easily be implemented for unstructured grids. Compared to the **FE!** method, where boundary conditions come naturally from the formulation, this causes some difficulties in **FV!**.
It shall also be mentioned that the introduction of stabilization schemes(like streamline upwinding), is much easier in an **FV!** formulation. Overall, for **CFD!** (**CFD!**), **FV!** has so far shown to be the best compromise between accuracy, stability and efficiency.

Finite volume methods are typically derived from the so called conservative form of a **PDE!**. It is shown in Section **??**, that the **NSG!** (**NSG!**) equation can be brought to the same form. In general, the conservative form can be written as:

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} + \nabla \cdot f(\boldsymbol{\xi}) = \mathbf{0} \tag{1}$$

$$\tag{2}$$

where $\boldsymbol{\xi}$ represents a vector of states and f is the so-called flux tensor.
After subdiving the domain into finite volumes, also called cells, on can write for each particulat cell i

$$\int_{V_i} \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} dV_i + \int_{V_i} \nabla \cdot f(\boldsymbol{\xi}) dV_i \tag{3}$$

After allpying the divergence theorem to the second term this gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_i} \boldsymbol{\xi} dV_i + \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \boldsymbol{n} \; dS_i \tag{4}$$

And after integration the first term to get the volume average

$$V_i \frac{\mathrm{d}\bar{\boldsymbol{\xi}}}{\mathrm{d}t} + \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \boldsymbol{n} \; dS_i \tag{5}$$

So that finally, the equation can be written as

$$\frac{\mathrm{d}\bar{\boldsymbol{\xi}}}{\mathrm{d}t} + \frac{1}{V_i} \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \boldsymbol{n} \; dS_i \tag{6}$$

which can easily be interpreted. The cell average of the conserved properties changes according to the total flux through the cells surface. Of course, the conservative value is defined as beeing constant within one cell, so there will be different values on faces or edges, depending on wich side is one is looking at. There aredifferent appoaches on how to choose an appropriate value. And the choice made greatly effects the numerical properties. **Is this accurate? If so, provide more details.**

1.2 Finite Volume method for fluid mechanics

It has already been shown in Section REF, that the **NSG!** can be brought into the conservative form(Equation REF). The flux has been schown to consist of a convective and a diffusive part(Equation REF). For this thesis, a **MUSCL!** (**MUSCL!**) type **FV!** framework for unstructured three-dimensional grids, as described in [**?**] has been used.

The formulation is derived by re-writing equation **??** in variational form by multiplying with a test function:

$$\int_\Omega \frac{\partial \boldsymbol{w}}{\partial t}\phi_i d\Omega + \int_\Omega \nabla \cdot \mathcal{F}(\boldsymbol{w})\phi_i d\Omega + \int_\Omega \nabla \cdot \mathcal{G}(\boldsymbol{w})\phi_i d\Omega = \mathbf{0} \tag{7}$$

where $\phi_i \in C^0(\Omega)$. For this formulation ϕ_i is chosen to be piecewise linear. Particularly, the testfunctions fulfill

$$\phi_i(\boldsymbol{X}_j) = \delta_{ij} \tag{8}$$

Now, Gauss diuvergence theorem is applied to the last part, giving

$$\int_\Omega \frac{\partial \boldsymbol{w}}{\partial t}\phi_i \; d\Omega + \int_\Omega \nabla \cdot \mathcal{F}(\boldsymbol{w})\phi_i \; d\Omega + (\int_\Gamma \mathcal{G}(\boldsymbol{w}) \cdot \boldsymbol{n} \; d\Gamma - \int_\Omega \mathcal{G}(\boldsymbol{w}) \cdot \nabla\phi_i \; d\Omega) = \mathbf{0} \tag{9}$$

In contrast to a real **FE!** formulation, the contribution of the viscous flux at the far fiels boundary is now neglected, as ecplained in **REF Main 109**. How the boundary conditions at the far fiels are taken care of will be explained at a proper place. We finally get the mixed **FV!-FE!** form by mass lumpoing the first two terms. Since we are using a vertex-centered approach here, masslumping is equivalent to using a constant test function of ϕ_i on the dual cell. We therefor get:

$$\int_{\Omega_i} \frac{\partial \boldsymbol{w}}{\partial t} \; d\Omega + \int_{\Omega_i} \nabla \cdot \mathcal{F}(\boldsymbol{w}) \; d\Omega - \int_\Omega \mathcal{G}(\boldsymbol{w}) \cdot \nabla\phi_i \; d\Omega = \mathbf{0} \tag{10}$$

Please note that we have switched from an integral over the whole domain to an integral across the dual cells here. Finnaly by averaging $\boldsymbol{w}$ over the cell in the first term, we can derive the **FV!** formulation as

$$\frac{\partial \boldsymbol{w}_i}{\partial t} + \frac{1}{\|\Omega_i\|}\int_{\Omega_i} \nabla \cdot \mathcal{F}(\boldsymbol{w}) \; d\Omega_i - \frac{1}{\|\Omega_i\|}\int_\Omega \mathcal{G}(\boldsymbol{w}) \cdot \nabla\phi_i \; d\Omega = \mathbf{0} \tag{11}$$

Where now, $\boldsymbol{w}_i$ denotes the average of $\boldsymbol{w}$ in the dual cell Ω_i, which is by constraction the value of $\boldsymbol{w}$ at vertex i. The vloume of cell Ω_i is denoted as $\|\Omega_i\|$ here. Finnally, gauss divergence theorem is applied to the convective term, resulting in

$$\frac{\partial \boldsymbol{w}_i}{\partial t} + \frac{1}{\|\Omega_i\|}\int_{\partial\Omega_i} \mathcal{F}(\boldsymbol{w}) \cdot dS - \frac{1}{\|\Omega_i\|}\int_\Omega \mathcal{G}(\boldsymbol{w}) \cdot \nabla\phi_i \; d\Omega = \mathbf{0} \tag{12}$$

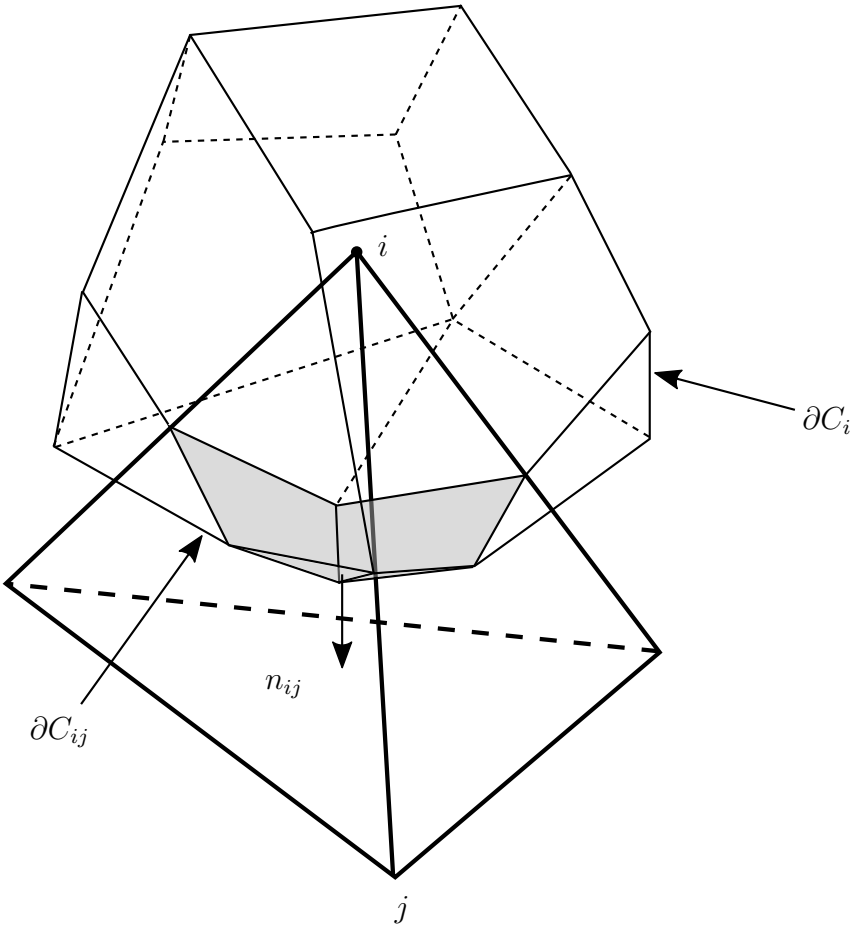

Figure 1: ***FV!*** *semi-discretization of an unstrucured mesh. Vertex i is the center of dual cell C_i. The boundary of the dual cell is denoted as ∂C_{ij}*

As can be seen from picture REF, the dual cells themselves can have very random shapes. This makes the integration over the boundary of the second term in Equation REF more cumbersome than in a primal approach. Also, the volume integral in the **FE!** like expression has to be splitted into regular shapes subdomains, e.g. tetrahedra, such that standard integration rules like gauss rule can be applied.

For a closer look into the convective fluxes integral, we decompose the boundary as $\partial\Omega_i = \sum_{i\in\kappa(i)} \partial\Omega_{ij}$, where $\kappa(i)$ denotes the set of vertices connected by an edge to vertex i