
1 The Finite Volume method

1.1 General

Like **FD!** (**FD!**) or **FE!** (**FE!**), the **FV!** (**FV!**) method is a mean to solving a **PDE!** (**PDE!**) by transforming it into a discrete algebraic form. In the field of fluid mechanics, the finite volume method is the most popular approach. Other than **FD!** or **FE!**, the finite volume is conservative by construction. Other than **FD!** it can easily be implemented for unstructured grids. Compared to the **FE!** method, where boundary conditions come naturally from the formulation, this causes some difficulties in **FV!**.

It shall also be mentioned that the introduction of stabilization schemes (like stream-line upwinding), is much easier in an **FV!** formulation. Overall, for **CFD!** (**CFD!**), **FV!** has so far shown to be the best compromise between accuracy, stability and efficiency.

Finite volume methods are typically derived from the so called conservative form of a **PDE!**. It is shown in Section ??, that the **NSE!** (**NSE!**) equation can be brought to the same form. In general, the conservative form can be written as:

$$\frac{d\boldsymbol{\xi}}{dt} + \nabla \cdot f(\boldsymbol{\xi}) = \mathbf{0} \quad (1)$$

(2)

where $\boldsymbol{\xi}$ represents a vector of states and f is the so-called flux tensor.

After subdividing the domain into finite volumes, also called cells, one can write for each particular cell i

$$\int_{V_i} \frac{d\boldsymbol{\xi}}{dt} dV_i + \int_{V_i} \nabla \cdot f(\boldsymbol{\xi}) dV_i \quad (3)$$

After applying the divergence theorem to the second term this gives:

$$\frac{d}{dt} \int_{V_i} \boldsymbol{\xi} dV_i + \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \mathbf{n} dS_i \quad (4)$$

And after integration the first term to get the volume average

$$V_i \frac{d\bar{\boldsymbol{\xi}}}{dt} + \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \mathbf{n} dS_i \quad (5)$$

So that finally, the equation can be written as

$$\frac{d\bar{\boldsymbol{\xi}}}{dt} + \frac{1}{V_i} \int_{\partial V_i} f(\boldsymbol{\xi}) \cdot \mathbf{n} dS_i \quad (6)$$

which can easily be interpreted. The cell average of the conserved properties changes according to the total flux through the cells surface. Of course, the conservative

value is defined as being constant within one cell, so there will be different values on faces or edges, depending on which side one is looking at. There are different approaches on how to choose an appropriate value. And the choice may greatly effect the numerical properties. In a **FV!** solver, the numerical flux at the interface is typically constructed such, that upwinding is achieved.

1.2 Finite Volume method for fluid mechanics

It has already been shown in Section ??, that the **NSE!** can be brought into the conservative form (??). The flux has been shown to consist of a convective and a diffusive part(??)(??). For this thesis, a **MUSCL!** (**MUSCL!**) type **FV!** framework for unstructured three-dimensional grids, as described in [?] has been used.

The formulation is derived by re-writing equation ?? in variational form by multiplying with a test function:

$$\int_{\Omega} \frac{\partial \mathbf{w}}{\partial t} \phi_i d\Omega + \int_{\Omega} \nabla \cdot \mathcal{F}(\mathbf{w}) \phi_i d\Omega + \int_{\Omega} \nabla \cdot \mathcal{G}(\mathbf{w}) \phi_i d\Omega = 0 \quad (7)$$

where $\phi_i \in C^0(\Omega)$. For this formulation ϕ_i is chosen to be piecewise linear. Particularly, the test functions fulfill

$$\phi_i(\mathbf{X}_j) = \delta_{ij} \quad (8)$$

Now, Gauss divergence theorem is applied to the last part, giving

$$\int_{\Omega} \frac{\partial \mathbf{w}}{\partial t} \phi_i d\Omega + \int_{\Omega} \nabla \cdot \mathcal{F}(\mathbf{w}) \phi_i d\Omega + \left(\int_{\Gamma} \mathcal{G}(\mathbf{w}) \cdot \mathbf{n} d\Gamma - \int_{\Omega} \mathcal{G}(\mathbf{w}) \cdot \nabla \phi_i d\Omega \right) = 0 \quad (9)$$

In contrast to a real **FE!** formulation, the contribution of the viscous flux at the far field boundary is now neglected, as explained in [?]. How the boundary conditions at the far fields are taken care of will be explained at a proper place. We finally get the mixed **FV!-FE!** form by mass lumping the first two terms. Since we are using a vertex-centered approach here, mass lumping is equivalent to using a constant test function of ϕ_i on the dual cell. We therefore get:

$$\int_{\Omega_i} \frac{\partial \mathbf{w}}{\partial t} d\Omega + \int_{\Omega_i} \nabla \cdot \mathcal{F}(\mathbf{w}) d\Omega - \int_{\Omega} \mathcal{G}(\mathbf{w}) \cdot \nabla \phi_i d\Omega = 0 \quad (10)$$

Please note that we have switched from an integral over the whole domain to an integral across the dual cells here. Finally by averaging \mathbf{w} over the cell in the first term, we can derive the **FV!** formulation as

$$\frac{\partial \mathbf{w}_i}{\partial t} + \frac{1}{\|\Omega_i\|} \int_{\Omega_i} \nabla \cdot \mathcal{F}(\mathbf{w}) d\Omega_i - \frac{1}{\|\Omega_i\|} \int_{\Omega} \mathcal{G}(\mathbf{w}) \cdot \nabla \phi_i d\Omega = 0 \quad (11)$$

Where now, \mathbf{w}_i denotes the average of \mathbf{w} in the dual cell Ω_i , which is by construction the value of \mathbf{w} at vertex i . The volume of cell Ω_i is denoted as $\|\Omega_i\|$ here. Finally,

gauss divergence theorem is applied to the convective term, resulting in

$$\frac{\partial \mathbf{w}_i}{\partial t} + \frac{1}{\|\Omega_i\|} \int_{\partial\Omega_i} \mathcal{F}(\mathbf{w}) \cdot d\mathbf{S} - \frac{1}{\|\Omega_i\|} \int_{\Omega} \mathcal{G}(\mathbf{w}) \cdot \nabla_{\phi_i} d\Omega \stackrel{\text{eq:nsg-basic-fv-form}}{=} \mathbf{0} \quad (12)$$

As can be seen from figure 1, the dual cells themselves can have very random shapes. This makes the integration over the boundary of the second term in Equation (12) more cumbersome than in a primal approach. Also, the volume integral in the **FE!**-like expression has to be splitted into regular shaped subdomains, e.g. tetrahedra, such that standard integration rules, like gauss rule, can be applied.

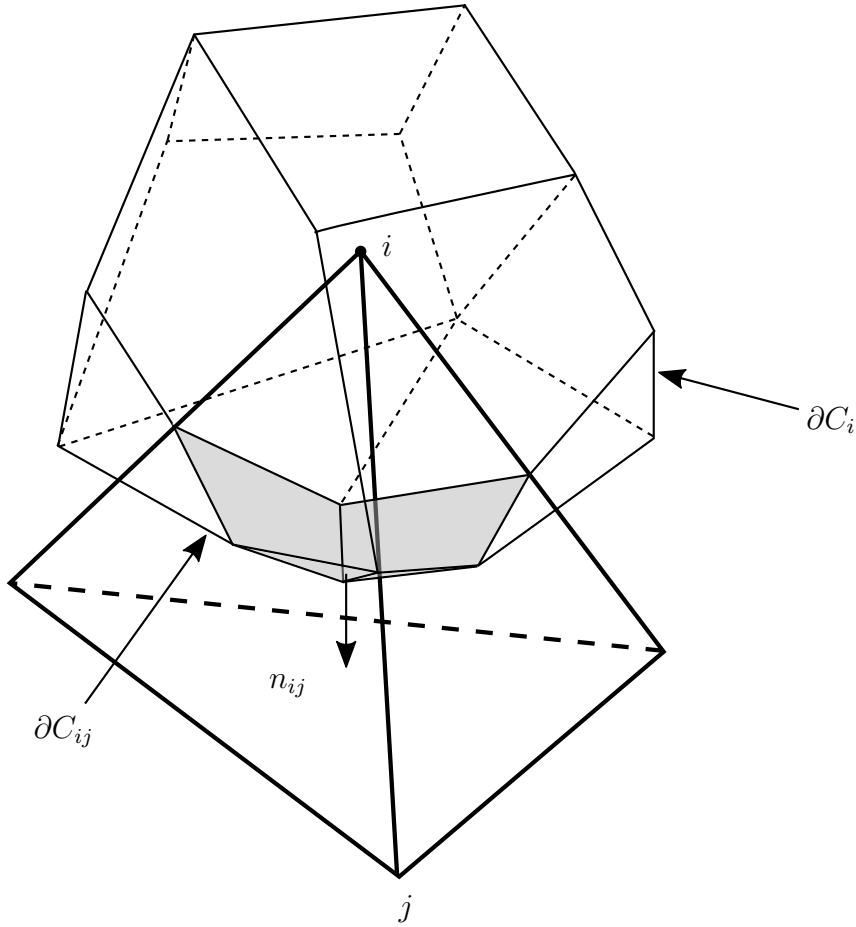


Figure 1: **FV!** semi-discretization of an unstructured mesh. Vertex i is the center of dual cell C_i . The boundary of the dual cell is denoted as ∂C_i

For a closer look into the convective fluxes integral, we decompose the boundary as $\partial\Omega_i = \sum_{i \in \kappa(i)} \partial\Omega_{ij}$, where $\kappa(i)$ denotes the set of vertices connected by an edge to vertex i .

In this thesis, the surface integral in equation (12) is than approximated using a Riemann solver and a **MUSCL!** [?] technique. This approximation can be written

as

$$\int_{\partial\Omega_i} \mathcal{F}(\mathbf{w}) \cdot d\mathbf{S} \approx \sum_{j \in \kappa(i)} \phi_{ij}(\mathbf{w}_{ij}, \mathbf{w}_{ji}, \mathbf{n}_{ij}) \quad (13)$$

where ϕ_{ij} denotes the chosen numerical flux function along the edge $i - j$ and the two extrapolated fluid states at the i and the j -side of the intersection of the cell boundary $\partial\Omega_{ij}$ and edge $i - j$ are denoted by \mathbf{w}_{ij} and \mathbf{w}_{ji} respectively. The area-weighted normal of edge $i - j$ is denoted as \mathbf{n}_{ij} .

As for the volume integral of the diffusive term in Equation (12), it shall be noted that the shape function is still denoted in the primal cell. Since the gradient of the test function is constant, as is the diffusive flux itself, the integral becomes a summation of the primal sub-tetrahedral contributions.

1.3 Treatment of the viscous term

The equivalence of the finite volume representation of the viscous term, with the finite element-like representation provided in Equation **TODO** might not be obvious to the reader. This section is therefore devoted to a **FE!** discretization of the governing equations, that can then be reformulated such that the mixed **FV!-FE!** formulation is obtained.

First, we start with the full **NSE!** in conservative form, as provided in Equation **TODO12**. Next, a test function is defined. We chose the function $(\chi_i)_{i \in \mathcal{P}_h}$ where \mathcal{P}_h represents the primal mesh, as step functions over the dual cells of \mathcal{D}_h . A multiplication with the test-functions and subsequent integration thus gives the following weak form

$$\int_{\Omega} \frac{\partial \mathbf{w}}{\partial t} \chi_i dx + \int_{\Omega} \nabla \cdot \mathcal{F}(\mathbf{w}) \chi_i dx + \int_{\Omega} \nabla \cdot \mathcal{G}(\mathbf{w}, \nabla \mathbf{w}) \chi_i dx = 0 \quad \text{fv_weak_form_felike} \quad (14)$$

(15)

where by using the divergence theorem, we obtain

$$\begin{aligned} \int_{\Omega} \nabla \cdot \mathcal{F}(\mathbf{w}) \chi_i dx &= \int_{\partial C_i} \mathcal{F}(\mathbf{w}) \cdot \mathbf{n} ds \quad \text{eq:divtheorem_felike} \quad (16) \\ \int_{\Omega} \nabla \cdot \mathcal{G}(\mathbf{w}, \nabla \mathbf{w}) \chi_i dx &= \underbrace{\int_{\partial C_i} \mathbb{K} \nabla \mathbf{w} \cdot \mathbf{n} ds}_{II} \quad (17) \end{aligned}$$

Where the first term is already equivalent to what we have derived by the **FV!** approach in **TODO**.

It the diffusive tensor \mathbb{K} is approximated to be constant over the triangle, which is justified in **TODO**, the product $\mathbb{K} \mathbf{w}$ is constant on can now write the above integral as

$$II = \sum_{T \in \mathcal{D}_i} \mathbb{K} \nabla \mathbf{w} \int_{\partial C_i \cap T} \mathbf{n}_T ds \quad (18)$$

Now, we use a particular geometric relation, that holds for primitive elements like triangles and tetrahedra, and will be derived in Section 1.3.1

$$\int_{C_i \cap T} \mathbf{n} ds = \frac{\mathbf{n}_i^T}{2} \quad (19)$$

where \mathbf{n}_i^T is the normal to the primal mesh triangle T on the edge opposite of node i . This is where the geometric considerations of section 1.3.1 come into place. As shown in equations (28), the integral over the dual face edge can be written via the normal of the primal mesh triangle as

$$- \sum_{T \in \mathcal{D}_i} \mathbb{K} \nabla \mathbf{w} \frac{\mathbf{n}_i^T}{2} \quad (20)$$

Also, we have shown in Section 1.3.1, that for the case of a simple triangle (but equally applicable to tetrahedra), the last term of the above equation can be written as

$$-\sum_{T \in \mathcal{D}_i} \mathbb{K} \nabla \mathbf{w} \int_T \nabla \phi_i dx \quad (21)$$

where ϕ_i is the P1 shape-function defined in (23). Finally we can go the opposite way, and join the summations over integrals over triangular contributions back to an overall integral over the whole dual cell. Doing this finally leads to

$$II = - \int_{\mathcal{D}_i} \mathbb{K} \nabla \mathbf{w} \cdot \nabla \phi_i dx \quad (22)$$

which is exactly the term provided in equation **TODO**. Since we used the same step-function based test-functions for all terms in equation (14), this approach is not only consistent, for the special case, of linear triangles we have shown that our finite element term is equivalent to a FV approximation.

1.3.1 Geometric considerations

This section is dedicated to the the derivation of a particular link between the shape-funfion of a linear triangle/tetrahedra and its outward facing normals that we use in chapter ?? to proof the equivalence of the FE-formualtion of the finite volume term with the **FV!** formulation.

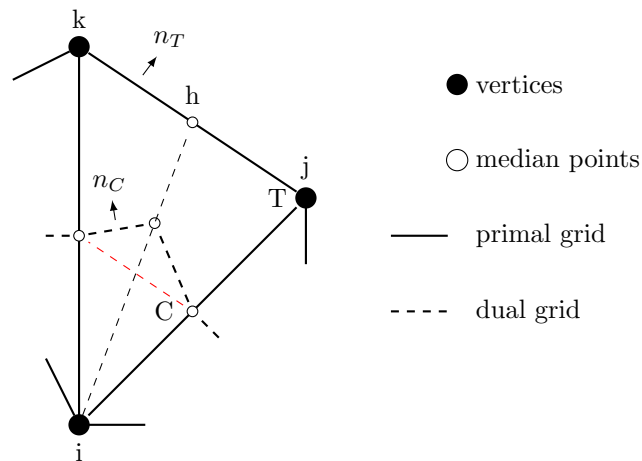


Figure 2: One Triangle T out of the set of triangles Σ_T , that are connected to vertex i . C denotes the dual cell related to vertex i .

We first look at one triangle of the set of triangles \sum_T that belong to node i . The setup is visualized in Figure 2. The reader can easily verify, that a linear shape-function of node i can be written as:

$$\phi_i(x) = \frac{hx \cdot hi}{\|hi\|^2} \quad \text{eq:p1_triangle_eq1}_{(23)}$$

where hx is the vector from point h to any point x in the triangle, and hi is the vector from point h to point i . For this particular representation of the shape function, the gradient can be elegantly written as

$$\nabla\phi_i(\mathbf{x}) = \frac{\mathbf{h}i}{\|\mathbf{h}i\|^2} = \frac{\hat{\mathbf{h}}i}{\|\mathbf{h}i\|} \quad \text{eq:p1_triangle_eq2} \quad (24)$$

where $\hat{\mathbf{h}}i$ is the normalized vector in $h - i$ direction. Moreover, simple geometry tells us that

$$\|\mathbf{h}i\| \|\mathbf{k}j\| = 2|T| \quad \text{eq:p1_triangle_eq3} \quad (25)$$

where $|T|$ is the area of the triangle T

Substituting equation 25 into equation 24 therefore leads to

$$\nabla\phi_i(\mathbf{x}) = \mathbf{h}i \cdot \frac{\|\mathbf{k}j\|}{2|T|} = -\frac{\boldsymbol{\nu}}{2|T|} \quad \text{eq:p1_triangle_eq4} \quad (26)$$

where $\boldsymbol{\nu}$ is the weighted outward facing normal to edge jk .

$$\boldsymbol{\nu} = -\hat{\mathbf{h}}i \|\mathbf{k}j\| = -\int_{[jk]} \mathbf{n}_T d\Gamma \quad (27)$$

Referring to previous chapter we remind the reader that in the vertex based **FV!** approach, and integration over the dual face interface (dashed line in Figure 2) via the weighted interface normal is typically performed.

By simple geometrical considerations we can now determine, that the following relation holds

$$\int_{\Gamma_C \cap T} \text{const. } \mathbf{n}_c = \frac{1}{2} \int_{\Gamma_T} \text{const. } \mathbf{n}_T \quad \text{eq:final_relation} \quad (28)$$