Analytical and Numerical Approaches for the Computation of Aeroelastic Sensitivities Using the Direct and Adjoint Methods

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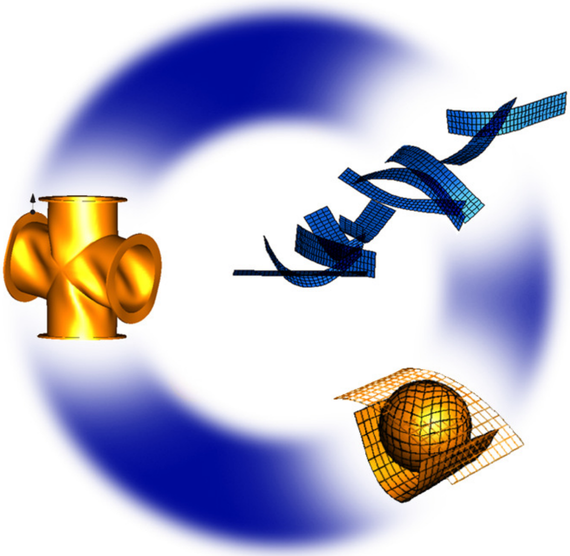
Master Thesis



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Own work declaration

Hereby I confirm that this is my own work and I referenced all sources and auxiliary means used and acknowledged any help that I have received from others as appropriate.

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Abstract

This is the most often read section. It will determine if someone actually bothers to read more of what you've written. Only write this section after you have completed your report. The abstract consist of only one paragraph, mostly limited to 200-300 words. Literally, a summary of your work. Write a sentence or two about each of the main sections of your report, as discussed in this section. When summarizing results, make the reader aware of the most important results (including numbers when applicable) and important conclusions or questions that follow from these.

Zusammenfassung

german version of your abstract.

Contents

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Nomenclature

Operators

Algebraic operations

A^T	Transpose of a tensor	`T`
A^{-1}	Inverse of a tensor	`inv`
$\bar{a}$	Average component of a quantity	`av`
a'	Fluctuating component of a qunatity	`fluc`
$\lVert a \rVert$	Norm of a quatntity	`norm`

Analytic operations

$\ddot{A}$	Second time derivative at a fixed refrence position	`dderivtime`
$\dot{A}$	First time derivative at a fixed reference position	`derivtime`
$\frac{\partial A}{\partial B}$	partial derivative of one argument with respect to the other	`pdfrac`
$\frac{\partial^2 A}{\partial B^2}$	Second partial derivative of one argument with respect to the other	`ppdfrac`
$\frac{DA}{DB}$	Material time derivative	`mfrac`
$\frac{\mathrm{d}A}{\mathrm{d}B}$	First total derivative of one argument with respect to the other	`tfrac`

Representation of scalars, tensors and other quantities

a	Scalar quantity	`s`
$\boldsymbol{a}$	Vector quantity	`v`
$\boldsymbol{A}$	Matrix quantity	`m`
A, a	continous quantity	`c`
A, a	discrete quantity	`d`
$a^{(k)}$	Iteration index in the optimization loop	`it`
$a^{(k)}$	Iteration index in the optimization loop	`ito`
$a^{(n)}$	Iteration index in thestaggered algorithm	`its`
$\bar{A}$	Fictious entity	`fic`
$\mathcal{O}(A)$	Order of magnitude	`order`

Symbols

3-field formulations

$\mathbf{T}_p$	Interface projection matrix from fluid to structure mesh	`ifaceprojFt`
$\boldsymbol{H}_2$	Second order Jacobian	`jactwo`
$\mathbf{S}$	Turbulence term	`turbmat`
$\mathbf{T}_u$	Interface projection matrix from structure to fluid mesh	`ifaceprojSt`
$\mathbf{F}$	Convective part of the flux matrix	`fluxmatconv`
$\mathbf{w}$	Discrete fsluid state vector	`dfstate`
$\mathcal{D}_{EOS}$	State equation of the mesh	`EOSmesh`
$\boldsymbol{P}$	Fluid load TODO	`load`
$\boldsymbol{a_u}$	Adjoint structure displacement	`structdispa`
$\boldsymbol{w}$	Fluid state vector	`fstate`
$\boldsymbol{P}_F$	Fluid load	`fload`
$\mathcal{S}_{EOS}$	State equation of the structure	`EOSstruct`
$\boldsymbol{a_w}$	Adjoint fluid state vector	`fstatead`
$\tilde{\boldsymbol{w}}$	Primitive ffluid state vector	`fstateprim`
$\boldsymbol{x}$	Fluid mesh motion	`mms`
$\mathbf{G}$	diffusive part of the flux matrix	`fluxmatdiff`
$\mathcal{F}_{EOS}$	State equation of the fluid	`EOSfluid`
S	Source term in the Reynolds Averaged Navier-Stokes (RANS) equations	`turbulences`
$\boldsymbol{a_x}$	Adjoint fluid mesh motion	`mmsad`
$\boldsymbol{w}_{RANS}$	Augumentes fluid state vector in the RANS formulation	`fstaterans`
$\boldsymbol{P}_T$	Structure load	`sload`
$\boldsymbol{u}$	Structure displacement	`structdisp`
$\mathbf{A}$	Diagonal matrix with vell volumes	`cellvolmat`
$\boldsymbol{\chi}$	Additional fluid state variable introduced by the turbulence model	`turbparamve`

Fluid Structure Interaction

$\mathcal{P}$	State equation of the structure	`strucstateq`
$\mathcal{D}$	State equation of the mesh motion	`mmsstateeq`
$\mathcal{F}$	State equation of the fluid	`fluidstateq`

Sturctural Analysis

$\mathbf{K}$	Finite Elements (FE) stiffness matrix	`stiffmat`
u	Displacement vector	`disp`
$\mathbf{d}$	Interface displacement	`ifacedispvec`
$\mathbf{u}$	Discrete displacement vector	`dispvec`
$\boldsymbol{x}$	Mesh motion	`motion`

Optimization

s	Abstract optimazation variable	`absvar`
q	Optimization criterium	`optcrit`
ϵ^{SA}	Specified tolerance in the Sensitivity analysis	`tolsa`
$\boldsymbol{\eta}$	Lagrange multipliers of the equality constraints	`lagmultseq`
L	Lagrangian function of the optimization problem	`Lagfunc`
$\boldsymbol{s}$	Vector of abstract optimization variables	`absvars`
$\boldsymbol{g}$	Non-equality constraints	`neqctr`
$\boldsymbol{\gamma}$	Lagrange multipliers of the inequality constraints	`lagmultsneq`
$\boldsymbol{q}$	Vector of optimization criteria	`optcrits`
$\boldsymbol{d}$	Physical design parameters	`physvars`
z	Target cost function	`costfunc`
$n_{\boldsymbol{g}}$	Number of non-equality constraints	`numneqctr`
$\boldsymbol{h}$	Equality constraints	`eqctr`
$\boldsymbol{a}$	Adjoint solutions	`adjoints`
$n_{\boldsymbol{h}}$	Number of equality constraints	`numeqctr`

Fluid Mechanics

p	Pressure	`pres`
$\mathcal{F}$	Convective fluxes	`fluxesconv`
$\boldsymbol{P}$	Matrix that contains the eigenvectors of the jacobian matrix of $\mathbf{F}$	`jaceigvecs`
T	Fluid temperature	`temp`
v_3	Fluid velocity in z-direction	`fluidvelz`
$\boldsymbol{\Lambda}$	Diagonal matrix that contains the eigenvalues of the jacobian matrix of $\mathbf{F}$	`jaceigvals`
$\mathcal{H}$	Jacoian matrix	`jac`
ρ	Density	`dens`
μ	Dynamic viscosity	`viscosdyn`
$\mathbf{I}$	Identity matrix	`eye`
ν	Kinematic viscosity	`viscoskin`
$\mathcal{A}$	Flux Jacobian	`fluxjac`
τ	Deviatoric fluid stress tensor	`fluidstressc`
$\mathcal{G}$	Diffusive fluxes	`fluxesdiff`
q	Heat flux comopnenent	`heatfluxcomp`
k	Thermal conductivity of the fluid	`thermcond`
E	Total energies	`energytot`
RE	Reynolds number	`reynolds`
$\boldsymbol{q}$	Heat flux vector	`heatflux`
γ	Specific heat ratio	`specheatrati`
v_2	Fluid velocity in y-direction	`fluidvely`
e	Internal energy	`energyint`
v	Fluid velocity vector	`fluidvelcomp`
v_1	Fluid velocity in x-direction	`fluidvelx`
$\boldsymbol{M}$	Averaging function associated with the Roe flux	`roeavgfunc`

Abbrevations

SQP Sequential Quadratic Programming

GSE Global Sensitivity Equations

SA Sensitivity Analysis

VOF Volume of Fluid Method

GFM Ghost Fluid Method

GFMP Ghost Fluid Method of the Poor

EOS Equation of State

JWL Jones-Wilkins-Lee

PG Perfect Gas

SG Stiffened Gas

MUSCL Monotonic Upwind scheme for Conservation Laws

FIVER Finite Volume Method with exact two-phase Riemann Integrals

RANS Reynolds Averaged Navier-Stokes

FV Finite Volumes

FD Finite Differences

FE Finite Elements

SA Sensitivity Analysis

LNM Lehrstuhl für Numerische Mechanik(Institute for Computational Mechanics)

TUM Technical University of Munich

CFD Computational Fluid Dynamics

ALE Arbitrary Lagrangian Eulerian

VOF Volume of Fluid Methods

DFP Davidon-Fletcher Powell formula

BFGS Broyden-Fletcher-Goldfarb-Shanno algorithm

LDR Lift over Drag Ratio

NSE Navier Stokes Equations

NSME Momentum Equation of the Navier Stokes Equations

NSCE Continuity Equation of the Navier Stokes Equations

NSEE Energy Equation of the Navier Stokes Equations

PDE Partial Differential Equation

CG Conjugate Gradient

PCG Preconditioned Conjugate Gradient

SA Sensitivity Analysis

CFD Computational Fluid Dynamics

FSI Fluid Structure Interaction

FRG Farhat Research Group

1 Introduction

1.1 Motivation

Thanks to the advent of massively parallel high-performance computers, Computational Fluid Dynamics (CFD) is nowadays capable of accurately predicting the flow characteristics of a large variety of real-world problems. Typically, it involves numerical analysis and algorithms in order to analyze the characteristics of a given input setting.
In contrast to a pure scientific point of view, which put the main objective in gaining fundamental insight into the flow phenomenon, an engineer typically sees the computational methods as a mean of designing and improving his product. Therefore he is not only satisfied with an sufficiently accurate approximation of the flow variables around his design objective, one could imagine an aircraft or an automobile, but he seeks to use that information to improve his product. For trivial cases it might be intuitive to identify opportunities for improvement, in a highly complex system with mutual interactions, however, this is not only time consuming but it might even be not possible at all.

The effort thus is to automatically determine the gradient of specific variables of interest with respect to design parameters. More vividly speaking, this could be the change of lift of a given airfoil with respect to its front edge curvature or its twist. In fact there are not restrictions when it comes to the choice of the design parameters. In fact one can also think of purely abstract ones, as we will describe in Section **??**. However, the appropriate choice of this variables is the key-point in getting a satisfactory results. Details of this will be discussed in Section **??**. The numerical methodology of calculating this gradients is known as Sensitivity Analysis (SA).

There are different approaches to do SA. Most intuitively, one can approximate the sensitivity of a target value with respect to some design parameter by a simple Finite Differences (FD) approach. Figuratively speaking, in the airfoil example above, this would mean, that on could run two(or more) simulations with slightly different edge curvatures and then obtain the sensitivity of the lift with respect to that design decision as the difference of the absolute lift values of both simulation divided by the difference in absolute curvature. This approach is appealing, since it can be done with a standard flow solver. However, to get one sensitivity result, at least two simulations now have to be run. What is more, finite differencing not only introduces an approximation error, the appropriate choice of the stencil size is also difficult. Moreover, two slightly different meshes are required, or at least a mean to perturb the original one according to the design parameter choice.
Another approach is automatic differentiation. It is different form numerical differentiation in the sense that its error is only determined by the finite accuracy of the machine, but like finite differencing, automatic differentiation of a function works on a specific input parameter set. Therefore it has to be done repeatedly over and over again. An elegant alternative that solves both issues it symbolic derivation, which

means that the equations of interest are derived by specific variables in a fully general manner, leading to a symbolic formula for the derivatives. Once obtained, this formula can then simply be evaluated by the code. We therefore no longer need multiple function evaluations, nor is it necessary to do the derivation process over and over again. The payoff lies in the complexity. On can imagine that in a complex 3D fluid simulation, potentially with viscous, inviscid and turbulence-closure terms, this derivation is cumbersome and involves multiple chain rules.

The main purpose of SA lies in optimization. Given a set of design parameters one can imagine that the methodology above is able to give reliable values for the target function, e.g. airfoil lift, with respect to these parameters like airfoil thickness at specific locations. Based on this information it is now possible to slightly modify the airfoil, according to these new results. After that the whole process starts again until it finally converges to the optimal configuration.

1.2 Objective

The main purpose of this project is the implementation of the SA method into the CFD code AERO-F [4]. Both analytical methods and numerical schemes for the computation of aeroelastic sensitivities in the presence of turbulent viscous flows will be considered. The scope of this thesis covers both, the direct **??** and adjoint approaches **??**, within the context of Eulerian, Arbitrary Lagrangian Eulerian (ALE) and Embedded formulations **??**.. The main effort of this thesis concentrates on the computation of the contributions of the viscous terms to the aforementioned sensitivities, as well as a generalization of the technique to embedded methods.

2 Manual derivation of Mach sensitivity for ideal gas

The prupose of this chapter is exemplarily manual derivation of the sensitivity of the Lift over Drag ratio with respect to the Mach number. This example is chosen over shape sensitivity since it promises to be much simpler and straightforward to apply.

3 Non Dimensional State Variables

$$\text{length:} \qquad \bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L_{ref}} \tag{1}$$

$$\text{time:} \qquad \bar{t} = \frac{t u_{ref}}{L_{ref}} \tag{2}$$

$$\text{density:} \qquad \bar{\rho} = \frac{\rho}{\rho_{ref}} \tag{3}$$

$$\text{velocity:} \qquad \bar{u} = \frac{u}{u_{ref}} \tag{4}$$

$$\text{pressure:} \qquad \bar{p} = \frac{p}{\rho_{ref} u_{ref}^2} \tag{5}$$

$$\text{temperature:} \qquad \bar{T} = \frac{c_v T}{u_{ref}^2} \tag{6}$$

$$\text{force:} \qquad \bar{F} = \frac{F}{u_{ref}^2 L_{ref}^2 \rho_{ref}} \tag{7}$$

The gas models therefore comes to: Ideal gas:

$$p = \rho R T \quad \rightarrow \quad \bar{p} = \bar{\rho}\bar{T}(\gamma - 1) \tag{8}$$

$$p = (\gamma - 1)\rho e - \gamma p_{SG} \quad \rightarrow \tag{9}$$

The derivatives ot the mach number can be written as

Dimensional	**Non-Dimensional**
$M_a = \sqrt{\frac{\rho \boldsymbol{v}^T \boldsymbol{v}}{\gamma p}}$	$\bar{M}_a = \sqrt{\frac{\bar{\rho} \bar{\boldsymbol{v}}^T \bar{\boldsymbol{v}}}{\gamma \bar{p}}}$
$\frac{\partial M_a}{\partial \rho} = \frac{1}{2}\frac{M_a}{\rho}$	$\frac{\partial \bar{M}_a}{\partial \bar{\rho}} = \frac{1}{2}\frac{\bar{M}_a}{\rho}$
$\frac{\partial M_a}{\partial \boldsymbol{v}} = \frac{\rho}{M_a \gamma p}\boldsymbol{v}$	$\frac{\partial \bar{M}_a}{\partial \bar{\boldsymbol{v}}} = \frac{\bar{\rho}}{\bar{M}_a \gamma \bar{p}}\bar{\boldsymbol{v}}$
$\frac{\partial M_a}{\partial p} = -\frac{1}{2}\frac{M_a}{p}$	$\frac{\partial \bar{M}_a}{\partial \bar{p}} = -\frac{1}{2}\frac{\bar{M}_a}{\bar{p}}$

The derivatives of the temperature can be written as:

Dimensional	**Non-Dimensional**
$T = \frac{p}{\rho R}$ $\frac{\partial T}{\partial \rho} = -\frac{T}{\rho} \quad \frac{\partial T}{\partial v_i} = 0 \quad \frac{\partial T}{\partial p} = \frac{T}{p}$	$\bar{T} = \frac{\bar{p}}{\bar{\rho}(\gamma - 1)}$ $\frac{\partial \bar{T}}{\partial \bar{\rho}} = -\frac{\bar{T}}{\bar{\rho}} \quad \frac{\partial \bar{T}}{\partial \bar{v}_i} = 0 \quad \frac{\partial \bar{T}}{\partial \bar{p}} = \frac{\bar{T}}{\bar{p}}$

The derivatives of the total pressure can be written as:

Dimensional	**Non-Dimensional**
$p_{tot} = p(1 + \frac{\gamma - 1}{2} M_a^2)^{\frac{\gamma}{\gamma-1}} = p(\sim)^{\frac{\gamma}{\gamma-1}}$ $\frac{\partial p_{tot}}{\partial \rho} = \frac{p\gamma M_a^2}{2\rho}(\sim)^{\frac{1}{\gamma-1}}$ $\frac{\partial p_{tot}}{\partial v_i} = \rho v_i(\sim)^{\frac{1}{\gamma-1}}$ $\frac{\partial p_{tot}}{\partial p} = (\sim)^{\frac{\gamma}{\gamma-1}} - \frac{M_a^2\gamma}{2}(\sim)^{\frac{1}{\gamma-1}}$	$\bar{p_{tot}} = \bar{p}(1 + \frac{\gamma - 1}{2} \bar{M_a}^2)^{\frac{\gamma}{\gamma-1}} = \bar{p}(\sim)^{\frac{\gamma}{\gamma-1}}$ $\frac{\partial \bar{p_{tot}}}{\partial \bar{\rho}} = \frac{\bar{p}\gamma \bar{M_a}^2}{2\bar{\rho}}(\sim)^{\frac{1}{\gamma-1}}$ $\frac{\partial \bar{p_{tot}}}{\partial \bar{v}_i} = \bar{\rho}\bar{v}_i(\sim)^{\frac{1}{\gamma-1}}$ $\frac{\partial \bar{p_{tot}}}{\partial \bar{p}} = (\sim)^{\frac{\gamma}{\gamma-1}} - \frac{\bar{M_a}^2\gamma}{2}(\sim)^{\frac{1}{\gamma-1}}$

The derivatives of the sound-speed follow as:

Dimensional	**Non-Dimensional**
$c = \sqrt{\frac{\gamma p}{\rho}}$ $\frac{\partial c}{\partial \rho} = \frac{\gamma}{sc\rho}$ $\frac{\partial c}{\partial v_i} = 0$ $\frac{\partial c}{\partial p} = -\frac{\gamma}{sc\rho}\frac{p}{\rho}$	$\bar{c} = \sqrt{\frac{\gamma \bar{p}}{\bar{\rho}}}$ $\frac{\partial \bar{c}}{\partial \bar{\rho}} = \frac{\gamma}{2\bar{c}\bar{\rho}}$ $\frac{\partial \bar{c}}{\partial \bar{v}_i} = 0$ $\frac{\partial \bar{c}}{\partial \bar{p}} = -\frac{\gamma}{2\bar{c}\bar{\rho}}\frac{\bar{p}}{\bar{\rho}}$

3.1 Formulation of the objective function

The first step will be the formulation of the objective function and constraints. For simplicity, we ignore the equality and inequailty constraints here, since we are only interested in the derivation of the sensitivity terms and not in the optimiazation routines themeselves.

Also, we are considering the sensitivity analsyis wit respect to a rigid structure, as explained in REF, rather then the fully aeroelastic one.

If TODO denotes the fluid structure interface, which in the ALE context coincides with the airfoil surface, one can formulate the lift and drag of an airfoil in a steady state as:

$$
\begin{aligned}
L &= \int_S p(\boldsymbol{w}, \boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{e}_2 dS && (10)\\
D &= \int_S p(\boldsymbol{w}, \boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{e}_1 dS && (11)\\
& && (12)
\end{aligned}
$$

The optimization criterion as introduced in **REF**thus becomes

$$q = \frac{L(\boldsymbol{w})}{D\boldsymbol{w}} \tag{13}$$

3.2 Formulation of the sensitivity equation

Now, we recall from equation (**??**) ,that if the abstract variable is not a shape parameter, we end up with the simple relation

$$\left.\frac{\mathrm{d}q}{\mathrm{d}s_i}\right|_{\boldsymbol{w}_0} = -\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0} \left[\left.\frac{\partial \mathcal{F}_{EOS}}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}_0}\right]^{-1} \left.\frac{\partial \mathcal{F}_{EOS}}{\partial s_i}\right|_{\boldsymbol{w}_0} \tag{14}$$

Now we can substitute the last term as

$$\left.\frac{\partial \mathcal{F}_{EOS}}{\partial s_i}\right|_{\boldsymbol{w}_0} = \left.\frac{\partial \mathcal{F}_{EOS}}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial s_i}\right|_{\boldsymbol{w}_0} + \left.\cancelto{0}{\frac{\partial \mathcal{F}_{EOS}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial s_i}}\right|_{\boldsymbol{w}_0} \tag{15}$$

where the last term cancels again, since the mesh motion at the fluid interface only depends on s_i if s_i is a shape variable.

Inserting Equation (15) into Equation (14) gives

$$
\begin{aligned}
\left.\frac{\mathrm{d}q}{\mathrm{d}s_i}\right|_{\boldsymbol{w}_0} &= -\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0} \cancelto{1}{\left[\left.\frac{\partial \mathcal{F}_{EOS}}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}_0}\right]^{-1} \left.\frac{\partial \mathcal{F}_{EOS}}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}_0}} \left.\frac{\partial \boldsymbol{w}}{\partial s_i}\right|_{\boldsymbol{w}_0}\\
&= -\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0} \left.\frac{\partial \boldsymbol{w}}{\partial s_i}\right|_{\boldsymbol{w}_0}\\
&= -\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0} \left.\frac{\partial \boldsymbol{w}}{\partial M_a}\right|_{\boldsymbol{w}_0} && (16)
\end{aligned}
$$

We will therefor denote the following two paragraphs to the derivation of the two terms $\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0}$ and $\left.\frac{\partial \boldsymbol{w}}{\partial M_a}\right|_{\boldsymbol{w}_0}$. It should also be pointed out, that the first one is independent of s_i and thus can be efficently re-used in the case of multiple abstract parameters, e.g. a range of shape variables.

3.3 Derivatiopn of the sensitivity equation terms

3.3.1 Derivation of $\left.\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}}\right|_{\boldsymbol{w}_0}$

As a first step, the chain rule gives

$$\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}} = \left.\cancelto{0}{\frac{\partial q}{\partial \boldsymbol{w}}}\right|_{\boldsymbol{w}_0} + \left.\frac{\partial q}{\partial L}\frac{\partial L}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}_0} + \left.\frac{\partial q}{\partial D}\frac{\partial D}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}_0} \tag{17}$$

where

$$\frac{\partial q}{\partial L} = \frac{1}{D} \tag{18}$$

$$\frac{\partial q}{\partial D} = -\frac{L}{D} \tag{19}$$

$$\frac{\mathrm{d}L}{\mathrm{d}\boldsymbol{w}} = \int_S \frac{\partial p}{\partial \boldsymbol{w}}\boldsymbol{n}(\boldsymbol{x})\cdot \boldsymbol{e}_2 dS \tag{20}$$

$$\frac{\mathrm{d}D}{\mathrm{d}\boldsymbol{w}} = \int_S \frac{\partial p}{\partial \boldsymbol{w}}\boldsymbol{n}(\boldsymbol{x})\cdot \boldsymbol{e}_1 dS \tag{21}$$

$$\tag{22}$$

which inserted into **REF**finally gives

$$\frac{\mathrm{d}q}{\mathrm{d}\boldsymbol{w}} = \frac{D\int \frac{\partial p}{\partial \boldsymbol{w}}\boldsymbol{n}(\boldsymbol{x})\cdot \boldsymbol{e}_2 - L\int \frac{\partial p}{\partial \boldsymbol{w}}\boldsymbol{n}(\boldsymbol{x})\cdot \boldsymbol{e}_1}{D^2} \tag{23}$$

Everything is in the continous setting here! Check if I can to the derivation here and just do the disceretization at the end

3.3.2 Derivation of $\left.\frac{\partial \boldsymbol{w}}{\partial M_a}\right|_{\boldsymbol{w}_0}$

If we define our primitive variable set to be $(\rho, \boldsymbol{v}, p)$, one can write

$$\frac{\partial \boldsymbol{w}}{\partial M_a} = \underbrace{\frac{\partial \boldsymbol{w}}{\partial \rho}}_{②}\underbrace{\frac{\partial \rho}{\partial M_a}}_{①} + \underbrace{\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{v}}}_{②}\underbrace{\frac{\partial \boldsymbol{v}}{\partial M_a}}_{①} + \underbrace{\frac{\partial \boldsymbol{w}}{\partial p}}_{②}\underbrace{\frac{\partial p}{\partial M_a}}_{①} \tag{24}$$

① **Mach number derivatives** The Mach number is defined in terms of the speed of sound as

$$M_a = \frac{\|\boldsymbol{v}\|_2}{c} = \frac{\|\boldsymbol{v}\|_2}{\sqrt{\gamma RT}} \tag{25}$$

where the temperature is related to the pressure through the perfect gas equation as

Inserting the perfect gas relation $p = \rho RT$ into Equation (25) leads to the following expression for the machnumber, which is solemnly dependent on primitive variables:

$$M_a = \sqrt{\frac{\rho \boldsymbol{v}^T \boldsymbol{v}}{\gamma p}} \tag{26}$$

The derivatives of the machnumber can therefor be easily obtained as

$$\frac{\partial M_a}{\partial \rho} = \frac{1}{2}\left(\frac{\boldsymbol{v}^T \boldsymbol{v} \rho}{\gamma p}\right)^{-\frac{1}{2}} \frac{\boldsymbol{v}^T \boldsymbol{v}}{\gamma p} = \qquad \dots \qquad = \frac{1}{2}\frac{M_a}{\rho} \tag{27}$$

$$\frac{\partial M_a}{\partial \boldsymbol{v}} = \qquad \dots \qquad = \frac{\rho}{M_a \gamma p}\boldsymbol{v} \tag{28}$$

$$\frac{\partial M_a}{\partial p} = \qquad \dots \qquad = -\frac{1}{2}\frac{M_a}{p} \tag{29}$$

② $\dfrac{\partial \boldsymbol{w}}{\partial \rho}$ for the derivatives of the fluid state vector, we first recall the definition of the state vector as

$$\boldsymbol{w} = \begin{bmatrix} \rho \\ \rho \boldsymbol{v} \\ E \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad E = \rho e + \frac{1}{2}\boldsymbol{v}^T \boldsymbol{v} \tag{30}$$

This gives the following for the derivatives:

$$\frac{\partial \boldsymbol{w}}{\partial \rho} = \begin{bmatrix} 1 \\ \boldsymbol{v} \\ \frac{\partial E}{\partial \rho} \end{bmatrix} \quad \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{v}} = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \\ \frac{\partial E}{\partial v_1} & \frac{\partial E}{\partial v_2} & \frac{\partial E}{\partial v_3} \end{bmatrix} \quad \frac{\partial \boldsymbol{w}}{\partial E} = \begin{bmatrix} 0 \\ \boldsymbol{0} \\ \frac{\partial E}{\partial p} \end{bmatrix} \tag{31}$$

Where, taking into account Equation **REF**for the total energy, one gets:

$$\frac{\partial E}{\partial \rho} = e + \rho \frac{\partial e}{\partial \rho} \qquad \frac{\partial E}{\partial \boldsymbol{v}} = \rho \frac{\partial e}{\partial \boldsymbol{v}} + \boldsymbol{v} \qquad \frac{\partial E}{\partial p} = \rho \frac{\partial e}{\partial p} \tag{32}$$

Since we are considering an ideal gas, one can write

$$e = \frac{RT}{\gamma - 1} = \frac{p}{\rho(\gamma - 1)} \qquad R, \gamma = const. \tag{33}$$

Thus the sought derivatives are

$$\frac{\partial e}{\partial \rho} = \frac{-p}{\rho^2(\gamma - 1)} \qquad \frac{\partial e}{\partial \rho} = \mathbf{0} \qquad \frac{\partial e}{\partial \rho} = \frac{1}{\rho(\gamma - 1)} \tag{34}$$

Therefor, backward substitution of Eqautions **REF**into ref finally gives

$$\frac{\partial e}{\partial \rho} = \frac{-p}{\rho^2(\gamma - 1)} \qquad \frac{\partial e}{\partial \boldsymbol{v}} = \mathbf{0} \qquad \frac{\partial e}{\partial p} = \frac{1}{\rho(\gamma - 1)} \tag{35}$$

$$\frac{\partial E}{\partial \rho} = 0 \qquad \frac{\partial E}{\partial \boldsymbol{v}} = \boldsymbol{v} \qquad \frac{\partial E}{\partial p} = \frac{1}{\gamma - 1} \tag{36}$$

$$\frac{\partial \boldsymbol{w}}{\partial \rho} = \begin{bmatrix} 0 \\ \boldsymbol{v} \\ \frac{1}{\gamma - 1} \end{bmatrix} \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{v}} = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \\ v_1 & v_2 & v_3 \end{bmatrix} \frac{\partial \boldsymbol{w}}{\partial p} = \begin{bmatrix} 0 \\ \mathbf{0} \\ \frac{1}{\gamma - 1} \end{bmatrix} \tag{37}$$

Putting everything together Equation **REF**finally becomes

$$\frac{\partial \boldsymbol{w}}{\partial M_a} = \begin{bmatrix} \frac{2p}{M_a} \\ \frac{2p}{M_a} v_1 + \frac{M_a \gamma p}{v_1} \\ \frac{2p}{M_a} v_2 + \frac{M_a \gamma p}{v_2} \\ \frac{2p}{M_a} v_3 + \frac{M_a \gamma p}{v_3} \\ 3 - \frac{2p}{(\gamma - 1) M_a} \end{bmatrix} \tag{38}$$

TODO do a dimensionanalysis here and check whether the sums even make sense

3.4 Final result

Putting Equations **REF**and **REF**together, one can finally obtain the follwoing expression for $\frac{\partial q}{\partial M_a}$:

References

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