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Preconditioning the FETI Method for Problems with Intra- and Inter-Subdomain Coefficient Jumps

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1 Introduction

The FETI method [FR94, MTF, FCR96] and related Balancing algorithm [Man93, LMV95] are two domain decomposition (DD) based iterative solvers that have gained popularity in the last few years. When applied to the solution of problems where the subdomains (a) do not feature neither inter nor intra coefficient jumps, and (b) have good and/or comparable aspect ratios, these DD methods are scalable and quasi-optimal. In order to extend the range of applications where these solvers excel, a simple scaling procedure was described in [LeT94] to address the issue of inter-subdomain coefficient jumps, and a mesh partitioning optimizer was proposed in [FMB95] to remedy the subdomain aspect ratio problem. In this paper, we revisit both issues and present a preconditioning algorithm that addresses the problems of arbitrary subdomain aspect ratios, and large inter- *as well as* intra-subdomain coefficient jumps (so far, most authors have addressed only the problem of inter-subdomain coefficient jumps [LeT94]). The proposed preconditioner is derived from sound energy principles that were initially introduced in [RF96] for improving the accuracy of the solution of subdomain problems by polynomial and piece-wise polynomial Lagrange multipliers. It can be equally used with the FETI and Balanced algorithms. However, because of space limitation, we limit our presentation to the case of the FETI method. We do not offer a mathematical proof of the optimality of our preconditioner, but we demonstrate numerically its scalability with the solution of highly heterogeneous structural mechanics problems.

2 The Focus Problem

The solution of a problem of the form $Ku = f$, where K is a symmetric positive definite matrix arising from the discretization of some second- or fourth-order elliptic

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problem on a domain Ω, can be obtained by partitioning Ω into N_s subdomains $\Omega^{(s)}$, and gluing these with discrete Lagrange multipliers λ:

$$K^{(s)}u^{(s)} = f^{(s)} - B^{(s)^T}\lambda \qquad s = 1, \ldots, N_s \tag{2.1}$$

$$\sum_{s=1}^{s=N_s} B^{(s)}u^{(s)} = 0 \tag{2.2}$$

Here, $B^{(s)}$ is a signed subdomain Boolean matrix that extracts and signs the interface components of a vector or a matrix related to $\Omega^{(s)}$. Eliminating $u^{(s)}$ from Eqs. (2.1–2.2) leads to the so-called dual interface problem

$$\begin{bmatrix} F_I & -G \\ -G^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix} \tag{2.3}$$

$$F_I = \sum_{s=1}^{s=N_s} B^{(s)}K^{(s)^+}B^{(s)^T}; \quad G = \begin{bmatrix} B^{(1)}R^{(1)} & \ldots & B^{(N_f)}R^{(N_f)} \end{bmatrix}$$

$$d = \sum_{s=1}^{s=N_s} B^{(s)}K^{(s)^+}f^{(s)}; \quad c = -\begin{bmatrix} f^{(1)^T}R^{(1)} & \ldots & f^{(N_s)^T}R^{(N_s)} \end{bmatrix}^T$$

where $K^{(s)^+}$ denotes the inverse of $K^{(s)}$ if $\Omega^{(s)}$ is not a floating subdomain, or a generalized inverse if $K^{(s)}$ is singular. In the latter case, $R^{(s)} = Ker(K^{(s)})$ (rigid body modes in structural mechanics), $\alpha = [\alpha^{(1)}...\alpha^{(N_f)}]^T$ where N_f denotes the total number of floating subdomains, and $\alpha^{(s)}$ stores the amplitude coefficients of $R^{(s)}$.

The FETI method consists in constructing the dual interface problem (2.3) and solving this interface problem by a preconditioned conjugate *projected* gradient (PCPG) algorithm where the projector is set to $P = I - G\ (G^TG)^{-1}G^T$. If W is a diagonal matrix which stores for each interface unknown the number of subdomains it belongs to, and $\overline{F_I^{-1}}$ denotes the chosen preconditioner, the FETI algorithm for second-order elasticity problems can be written as summarized in Table 1. (see [FCR96] for an extension to fourth-order elasticity and shell problems).

Two preconditioners have been previously developed for the FETI method: a mathematically optimal Dirichlet preconditioner $\overline{F}_I^{D^{-1}}$, and a computationally economical "lumped" preconditioner $\overline{F}_I^{L^{-1}}$

$$\overline{F}_I^{D^{-1}} = \sum_{s=1}^{s=N_s} B^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & S_{bb}^{(s)} \end{bmatrix} B^{(s)^T} \qquad \overline{F}_I^{L^{-1}} = \sum_{s=1}^{s=N_s} B^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & K_{bb}^{(s)} \end{bmatrix} B^{(s)^T} \tag{2.4}$$

Here, $S_{bb}^{(s)}$ denotes the primal Schur complement associated with subdomain $\Omega^{(s)}$, and the subscripts i and b designate the interior and interface boundary unknowns, respectively.

It is well known that the performance of many DD methods including FETI can deteriorate when either material or geometrical heterogeneities are present in the

Table 1 The FETI PCPG method

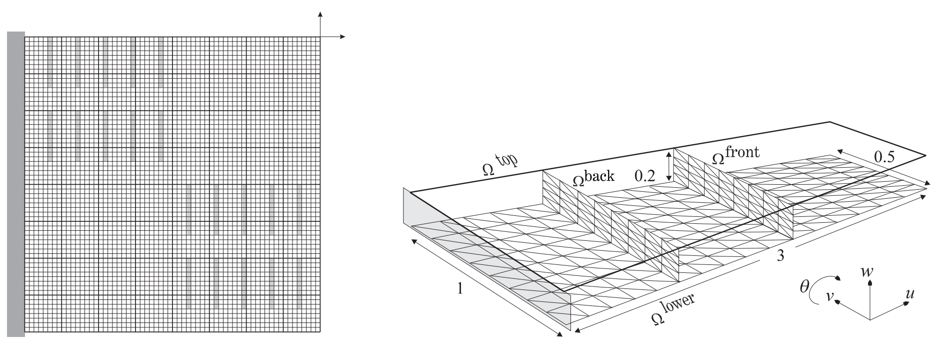
1. Initialize

$$\lambda^0 = -G\ (G^T G)^{-1} c\ , \quad r^0 = d - F_I \lambda^0$$

2. Iterate $k = 1,\ 2,\ ...$ until convergence

$$\begin{array}{rrcl}
Project - Scale & w^{k-1} & = & W^{-1} P^T\ r^{k-1} \\
Precondition & z^{k-1} & = & \overline{F_I^{-1}} w^{k-1} \\
Re - scale - Project & y^{k-1} & = & W^{-1} P\ z^{k-1} \\
 & \zeta^k & = & {y^{k-1}}^T w^{k-1} / {y^{k-2}}^T w^{k-2} \quad (\zeta^1 = 0) \\
 & p^k & = & y^{k-1} + \zeta^k p^{k-1} \quad (p^1 = y^0) \\
 & \nu^k & = & {y^{k-1}}^T w^{k-1} / {p^k}^T F_I p^k \\
 & \lambda^k & = & \lambda^{k-1} + \nu^k p^k\ , \quad r^k = r^{k-1} - \nu^k F_I p^k
\end{array}$$

Figure 1 Two examples of heterogeneous structures



(a) (b)

vicinity of the subdomain interfaces. Two examples of such problems in structural mechanics are depicted in Fig. 1: (a) a 2D clamped structure featuring inserts of a material that is 1000 times softer than the main material, discretized with 64×64 plane stress elements (second-order elasticity) and successively decomposed into 4, 8 and 64 square subdomains; (b) a 3D model of a wing-box structure constructed with DKT plate elements (fourth-order elasticity) and decomposed into subdomains whose interfaces coincide with the intersection of the skin and the stiffeners. Each subdomain with soft inserts in problem (a) is heterogeneous (intra-subdomain heterogeneity) whereas in problem (b) all subdomains are homogeneous, but the mechanical properties are very different depending on the domain orientation (inter-subdomain heterogeneity). In both examples, the subdomain stiffness cannot be characterized by a single coefficient, and therefore the scaling procedure proposed in [LeT94] cannot be applied.

3 Preconditioning with an Energy-based Smoothing Procedure

The Two-subdomain Problem

For the sake of clarity, we consider first the case of a two-subdomain heterogeneous problem. At each iteration of the FETI PCPG algorithm, the matrix-vector product $F_I p^k$ produces a jump across the subdomain interfaces of the iterate solution u^k. In the sequel, we drop the superscript k for simplicity. Elementary mechanics theory suggests that the solution $u^{(s)}$ on the interface boundary of the stiffer subdomain will be closer to the converged solution than the solution on the softer side. This in turn suggests that the computed solution u should be smoothed after each PCPG iteration as follows

$$\tilde{u}_b^{(1)} = \tilde{u}_b^{(2)} = \tilde{u}_I = (1-a)u_b^{(1)} + a u_b^{(2)} \tag{3.5}$$

$$\tilde{u}_i^{(s)} = u_i^{(s)} - K_{ii}^{(s)^{-1}} K_{ib}^{(s)} (\tilde{u}_b^{(s)} - u_b^{(s)}) \qquad s = 1,2 \tag{3.6}$$

which indicates that when the interface solution has been smoothed, a Dirichlet problem must be solved in each subdomain. Of course, the important question is how to select an optimal value of the smoothing parameter a? Let $\delta_I = u_b^{(2)} - u_b^{(1)}$ denote the jump of the solution on the interface $?_I$. After smoothing, the governing equations (2.1) can be written as

$$\begin{bmatrix} K_{ii}^{(1)} & K_{ib}^{(1)} & 0 \\ K_{ib}^{(1)^T} & K_{bb}^{(1)} + K_{bb}^{(2)} & K_{ib}^{(2)} \\ 0 & K_{ib}^{(2)^T} & K_{ii}^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{u}_i^{(1)} \\ \tilde{u}_I \\ \tilde{u}_i^{(2)} \end{bmatrix} = \begin{bmatrix} f_i^{(1)} \\ f_b^{(1)} + f_b^{(2)} \\ f_i^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ r_b \\ 0 \end{bmatrix} \tag{3.7}$$

where r_b is the interface residual induced by smoothing. From (3.5–3.6) and from (2.1), it follows that $r_b(a) = \left(aS_{bb}^{(1)} + (a-1)S_{bb}^{(2)}\right)\delta_I$. Hence, an optimal value of a is one which minimizes r_b. However, rather than minimizing directly some norm of r_b, we propose to adopt a Rayleigh-Ritz approach where the smoothed solutions are viewed as kinematically admissible fields. In view of Eqs. (3.5–3.6–3.7), the total energy can be written as

$$\mathcal{E}(a) = C - 2a\delta_I^T S_{bb}^{(2)} \delta_I + a^2 \delta_I^T (S_{bb}^{(1)} + S_{bb}^{(2)})\delta_I \tag{3.8}$$

where C is an expression that does not depend on a. Minimizing $\mathcal{E}(a)$ yields

$$a^D = \frac{k^{(2)^D}}{k^{(1)^D} + k^{(2)^D}} \ , \qquad k^{(1)^D} = \delta_I^T S_{bb}^{(1)} \delta_I \ , \qquad k^{(2)^D} = \delta_I^T S_{bb}^{(2)} \delta_I \tag{3.9}$$

Here, the superscript D is used to highlight the fact that computing the smoothing parameter a^D requires solving two subdomain Dirichlet problems. Since in general the corrections (3.5–3.6) will create an interface residual $r_b = \Delta f_b^{(1)} + \Delta f_b^{(2)}$ we also propose to correct the Lagrange multipliers iterates as follows

$$\Delta\lambda = -a^D \Delta f_b^{(1)} + (1-a^D)\Delta f_b^{(2)} = -\big(a^D\ S_{bb}^{(1)} a^D + (1-a^D)\ S_{bb}^{(2)} (1-a^D)\big)\delta_I \tag{3.10}$$

which guarantees the symmetry of our solution method.

¿From a physical viewpoint and in a structural mechanics context, the smoothing procedure proposed here consists in treating two subdomains as two linear springs connected in series, computing the jump of the displacement field at their connection, and redistributing this jump among both springs according to their "relative stiffnesses" $k^{(1)}$ and $k^{(2)}$. While this idea is not new [FR94], the derivation of the smoother yields for the first time a rational estimate of the local measure of a subdomain stiffness.

The Multiple Subdomain Problem - a New Coarse Problem

In order to generalize the smoothing procedure discussed above to the case of an arbitrary number of subdomains, we denote by $b^{(s),j}$ the restriction of the Boolean operator $B^{(s)}$ to the j-th edge of the interface boundary $?_I^{(s)}$. Using the interior/interface boundary partitioning of the subdomain unknowns we can write

$$B^{(s)} = \left[\begin{array}{cccccc} 0 & b^{(s),i} & b^{(s),j} & \cdots & b^{(s),l} \end{array} \right] \tag{3.11}$$

$b^{(s),j}$ can be further decomposed into square submatrices $b^{(sr),j}$ that describe the connectivity of subdomains $\Omega^{(s)}$ and $\Omega^{(r)}$ along edge j. Designating by $r, l \ldots$ the subdomains interconnected with $\Omega^{(s)}$ along $?_I^j$, we have

$$b^{(s),j^T} = \left[\begin{array}{cccccccc} 0 & \cdots & b^{(sr),j^T} & 0 & \cdots & b^{(sl),j^T} & \cdots \end{array} \right] \tag{3.12}$$

Next, we designate the unsigned equivalents of $b^{(sr),j}$ by a hat, and introduce the operator $\hat{b}^{(sr),j^T}\hat{b}^{(rs),j}$ which gives the correspondence between the numberings of the unknowns on both sides of the interface. Of course, we have $\hat{b}^{(sr),j^T}\hat{b}^{(sr),j} = I$. The generalization to an arbitrary number of subdomains of the smoothing procedure (3.5–3.6) then goes as follows:

$$\tilde{u}_b^{(s),j} = \beta^{(s),j} u_b^{(s),j} + \sum_{r:\Gamma_I^{(r)} \supset \Gamma_I^j}^{r \neq s} \hat{b}^{(sr),j^T}\hat{b}^{(rs),j}\beta^{(r),j} u_b^{(r),j} \qquad \forall \text{ edge } \ j \tag{3.13}$$

$$\tilde{u}_i^{(s)} = u_i^{(s)} - K_{ii}^{(s)^{-1}} K_{ib}^{(s)} (\tilde{u}_b^{(s)} - u_b^{(s)}) \qquad s = 1, \cdots N_s \tag{3.14}$$

where $\beta^{(s),j}$ are scalar smoothing parameters. If the $\beta^{(s),j}$ are constrained to have a unit sum

$$\sum_{\Gamma_I^{(s)} \supset \Gamma_I^j} \beta^{(s),j} = 1 \qquad \forall \text{ edge } \ j \tag{3.15}$$

the corrections of the subdomain interface solutions can be written as

$$\Delta u_b^{(s),j} = \tilde{u}_b^{(s),j} - u_b^{(s),j} = - \sum_{r:\Gamma_I^{(r)} \supset \Gamma_I^j}^{r \neq s} \beta^{(r),j} \ \hat{b}^{(sr),j^T} \ \delta_I^{(sr),j} \tag{3.16}$$

where $\delta_I^{(sr),j} = \hat{b}^{(sr),j} u_b^{(s),j} - \hat{b}^{(rs),j} u_b^{(r),j}$. To determine the edge coefficients $\beta^{(s),j}$ we follow conceptually the same Rayleigh-Ritz approach as presented in Section 3. If the unit sum condition is enforced by a set of multipliers τ_j, the minimization of the total energy leads to the following *auxiliary coarse problem*

$$\sum_{s:\Gamma_I^{(s)}\supset\Gamma_I^j}^{s\neq q} \quad \sum_{i:\Gamma_I^{(s)}\supset\Gamma_I^i} \quad \sum_{p:\Gamma_I^{(p)}\supset\Gamma_I^i}^{p\neq s} k_s^{(q),j;(p),i} \quad \beta^{(p),i} \;=\; \tau_j \qquad \forall[(q),j] : ?_I^{(q)} \ni j \tag{3.17}$$

$$\text{where} \quad k_s^{(q),j;(p),i} = \left(\hat{b}^{(sq),j}\delta_I^{(qs)}\right)^T [S_{bb}^{(s)}]_{j,i} \left(\hat{b}^{(sp),i}\delta_I^{(ps)}\right) \tag{3.18}$$

and $[S_{bb}^{(s)}]_{j,i}$ is the Schur-complement of $K^{(s)}$ associated with the edges j and i.

For symmetry, the correction of the Lagrange multipliers introduced by between $\Omega^{(s)}$ and $\Omega^{(r)}$ along edge j is then computed as

$$\Delta\lambda^{(sr),j} = b^{(sr),j}\beta^{(r),j}\Delta f_b^{(s),j} + b^{(rs),j}\beta^{(s),j}\Delta f_b^{(r),j} \tag{3.19}$$

where $\Delta f_b^{(s)} = S_{bb}^{(s)}(\tilde{u}_b^{(s)} - \tilde{u}_b^{(s)})$. In summary, using the notation of Table 1, this preconditioning step can be written as

$$z^{k-1} = \left\{ \sum_{s=1}^{s=N_s} \beta^{(s)} \; B^{(s)} \; \begin{bmatrix} 0 & 0 \\ 0 & S_{bb}^{(s)} \end{bmatrix} \; B^{(s)^T} \; \beta^{(s)^T} \right\} \; w^{k-1} \tag{3.20}$$

Cost-effective Alternatives — Lumping and "Superlumping"

The smoothing procedure presented in Sections 3 and 3 requires solving in each subdomain several Dirichlet problems. A first economical variant can be designed by replacing $\tilde{u}_i^{(s)} = u_i^{(s)}$ in (3.14), which has the effect of not propagating the correction of interface smoothing to the subdomain interior unknowns. It can be shown that such a strategy leads to similar expressions of the smoothing coefficients, but replaces the expensive Dirichlet operators $S_{bb}^{(s)}$ by the more economical lumping matrices $K_{bb}^{(s)}$ in the expression (3.20) and in the computation of the interface stiffnesses (3.18). This lumped preconditioner does no longer take into account the intra-subdomain heterogeneities associated with internal nodes. Nevertheless, the heterogeneities associated with the elements on the interface are still treated correctly.

Noting that the auxiliary coarse problem must be reconstructed at each iteration, an even more economical variant for computing the smoothing parameters $\beta^{(s)}$ in the lumped preconditioner can be constructed by assuming that the total energy of the system can be "superlumped" and written as

$$\mathcal{E}(\beta^{(s),j}) = C + \frac{1}{2}\sum_{s=1}^{s=N_s} \Delta u_b^{(s)^T} K_{bb,diag}^{(s)} \; \Delta u_b^{(s)} \tag{3.21}$$

where $K_{bb,diag}^{(s)}$ denotes the diagonal part of $K_{bb}^{(s)}$. In that case, the smoothed interface solution is still given by (3.13), but $\beta^{(r),j}$ is now understood as the diagonal matrix of the interface smoothing parameters (one coefficient per unknown). The unit sum constraint is then expressed at the unknown level. Noting $c^{(sr),j^T} = \hat{b}^{(sr),j^T}\hat{b}^{(rs),j}$ the correspondence between interface numberings, the generalization of the unit sum constraint and (3.16) can be written as

$$\beta^{(s),j} + \sum_{r:\Gamma_I^{(r)} \supset \Gamma_I^j}^{r \neq q} c^{(sr),j^T} \beta^{(r),j} c^{(sr),j} = I \qquad \forall \text{ edge } \; j \tag{3.22}$$

$$\Delta u_b^{(s),j} = - \sum_{r:\Gamma_I^{(r)} \supset \Gamma_I^j}^{r \neq s} \left(c^{(sr),j^T} \beta^{(r),j} c^{(sr),j} \right) \hat{b}^{(sr),j^T} \delta_I^{(sr),j} \tag{3.23}$$

¿From (3.21), it follows that

$$\beta^{(s),j} = \left[K_{bb,diag}^{(s)} \right]_j \left\{ \sum_{r:\Gamma_I^{(r)} \supset \Gamma_I^j} c^{(sr),j^T} \left[K_{bb,diag}^{(r)} \right]_j c^{(sr),j} \right\}^{-1} \tag{3.24}$$

where $c^{(ss),j} = I$. Hence, for this second smoothing alternative referred to here as the superlumped one, the auxiliary coarse problem is diagonal and needs to be constructed only once. Therefore, implementing it is trivial and solving it is inexpensive.

4 Numerical Results

We consider again problems (a) and (b) depicted in Fig 1, and perform their linear static analysis using the FETI method with the Dirichlet and lumped preconditioners, as well as the various smoothing procedures presented in this paper. We report in Table 2 the number of FETI PCPG iterations.

Table 2 Performance results ($||Ku - f||_2 \leq 10^{-6} ||f||_2$)

		Nbr. of PCPG iterations					
			Dirichlet			*lumped*	
Decomposition	$N \times M$	–	smooth.	hyper.	–	smooth.	hyper.
plane stress	2×2	18	16	17	35	35	36
	4×4	63	23	26	80	48	47
	8×8	83	27	25	89	46	44
stiffened panel with 4 subdomains		122	115	25	128	116	50

For the 2D plane stress problem (a), the full smoothing method and its superlumped alternative yield very similar convergence rates, and both of them improve dramatically the performances of the Dirichlet and lumped preconditioners. Table 2 also demonstrates the scalability of the overall solution method with respect to the number of subdomains.

For the stiffened panel problem, the full smoothing procedure improves only slightly the convergence of the FETI method, whereas the superlumped variant reduces the number of iterations by a factor of 5 (Dirichlet preconditioner), and by a factor greater than two (lumped preconditioner). For this problem, the poor efficiency of the full smoothing method can be explained by the fact that one coefficient cannot characterize an interface stiffness, because the relative interface stiffnesses at the intersection between the stiffeners and the skin depend strongly on the direction of the displacement unknown.

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