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Convergence of a substructuring method with Lagrange multipliers*

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Summary. We analyze the convergence of a substructuring iterative method with Lagrange multipliers, proposed recently by Farhat and Roux. The method decomposes finite element discretization of an elliptic boundary value problem into Neumann problems on the subdomains plus a coarse problem for the subdomain nullspace components. For linear conforming elements and preconditioning by the Dirichlet problems on the subdomains, we prove the asymptotic bound on the condition number $C(1 + \log(H/h))^{\gamma}$, $\gamma = 2$ or 3,where h is the characteristic element size and H subdomain size.

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1. Introduction

We analyze the convergence of a substructuring method with Lagrange multipliers, proposed by Farhat and Roux [10] under the name Finite Element Tearing and Interconnecting (FETI) method. The main idea of the FETI method is to decompose the problem domain into non-overlapping subdomains and to enforce continuity on subdomain interfaces by Lagrange multipliers. Eliminating the subdomain variables yields a dual problem for the Lagrange multipliers, which is solved by preconditioned conjugate gradients. This idea is related to the fictitious domain method where the Lagrange multipliers enforce boundary conditions as in Dinh et al. [4].

Elimination of the subdomain variables is implemented by solving Neumann problems on all subdomains in every iteration, which can be done completely in parallel. However, the subdomain problems are singular, so a small auxiliary problem for the nullspace components of the subdomain solutions needs to be

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solved in every iteration. This is an added complication, but also a blessing. Farhat, Mandel, and Roux [9] have shown numerically, and proved for the FETI method without preconditioning, that the auxiliary problem plays the role of a coarse problem, namely, it causes the condition number to be bounded independently of the number of subdomains. The method was further extended to time-dependent problems, which lack the naturally occurring coarse problem, by Farhat, Chen, and Mandel [8].

In this paper, we show that the condition number of the preconditioned FETI method is bounded independently of the number of subdomains and polylogarithmically in terms of subdomain size, as is the case for other optimal non-overlapping domain decomposition methods [3, 5, 7, 15, 16]. We refer to [9] for numerical results that confirm the theory and for parallel implementation and performance.

The FETI method is in a sense dual to the Neumann-Neumann method with a coarse problem, developed by Mandel under the name Balancing Domain Decomposition [14] based on an earlier method of de Roeck and LeTallec [18]. A modified method was analyzed by Dryja and Widlund [7].

Analysis of domain decomposition methods typically proceeds by demonstrating spectral equivalence of the quadratic form that defines the problem in a variational setting and the quadratic form that defines the preconditioner, often by way of P.L. Lions lemma [1, 5, 6, 13]. Since the preconditioner in the FETI method is quite complicated and it is not defined in terms of a quadratic form, we proceed differently and find a bound on the norm of the product of the system operator and the preconditioner to bound the maximal eigenvalue, and a bound on the inverse to bound the minimal eigenvalue. Related analyses were previously done for methods without crosspoints between the subdomains, or done formally in functional spaces, cf., for example, Glowinski and Wheeler [11]. In this paper, we present a complete analysis in terms of upper and lower bound on the preconditioned operator for decompositions with crosspoints in 2D and edges and crosspoints in 3D.

2. Formulation of the method

In this section, we briefly review formulation of the FETI method according to [9], where one can find more details about the algorithmic side. At the same time, we introduce the spaces and operators that will be used in our analysis.

We consider iterative solution of a system of linear equations $Lx = b$ arising from a finite element discretization of an elliptic boundary value problem on a bounded domain Ω, which is decomposed into non-overlapping subdomains Ω_i, $i = 1, \ldots, n_s$. The matrix A is assumed to be symmetric and positive definite. Let

$$W_i = V_h(\partial\Omega_i) \tag{1}$$

be the space of local vectors of degrees of freedom associated with the boundary of Ω_i, and let

(2) $$Y = V_h\Big(\bigcup_{i=1}^{n_s} \partial\Omega_i\Big)$$

be the space of global vectors of degrees of freedom associated with all subdomain boundaries. The correspondence of the local and global vectors of degrees of freedom is given by zero-one matrices $N_i : W_i \to Y$.

We find it convenient to identify vectors of degrees of freedom, which are in some spaces $\mathbb{R}^n$, with the associated finite element functions. Operators between the spaces are represented as matrices, and we frequently commit an abuse of notations by using matrices and operators interchangeably. The l^2 inner product is denoted by $\langle \cdot, \cdot \rangle$ on all spaces. The associated norm is $\|u\|^2 = \langle u, u \rangle$. The transpose of a matrix M is denoted by M'.

After elimination of the interior degrees of freedom in all subdomains Ω_i, we obtain the reduced system of linear equations for the vectors $w_i \in W_i$ of degrees of freedom on subdomain boundaries, which we write in subassembly form as

(3) $$\sum_{i=1}^{n_s} N_i S_i w_i = f$$

(4) $$\sum_{i=1}^{n_s} B_i w_i = 0$$

Here, S_i are the Schur complements of the subdomain stiffness matrices obtained by elimination of the interior degrees of freedom, and B_i are matrices with entries $0, 1, -1$ such that (4) expresses the continuity of the solution between subdomains, that is, the requirement that the values of degrees of freedom common to more than one subdomain coincide.

To describe the method in a concise form, we need to define the following spaces. W is a space of all boundary degrees of freedom on all subdomains:

(5) $$W = \bigotimes_{i=1}^{n_s} W_i$$

X is a space of vectors with entries corresponding to pairs of degrees of freedom on the interfaces where we enforce continuity:

(6) $$X \subset \bigotimes_{\partial\Omega_i \cap \partial\Omega_j \neq \emptyset} V_h(\partial\Omega_i \cap \partial\Omega_j).$$

Denote the block matrix

(7) $$B : W \to X = (B_1, \ldots, B_{n_s}),$$

and the space of Lagrange multipliers

(8) $$U = \text{Range } B.$$

This is all detail we need for the purpose of describing the method. A more specific description of B will be given in the next section. Finally, denote the symmetric block diagonal matrix

$$S : W \to W, \quad S = \begin{pmatrix} S_1 & 0 & & & & 0 \\ 0 & S_2 & 0 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & 0 \\ 0 & & & & 0 & S_{n_s} \end{pmatrix} \tag{9}$$

The problem (3), (4) can be now written as minimization of total subdomain energy subject to the continuity condition:

$$\mathscr{E}(w) = \frac{1}{2}\langle Sw, w\rangle + \langle f, w\rangle \to \min, \quad \text{subject to} \quad w \in W,\ Bw = 0. \tag{10}$$

Writing the Lagrangean of this minimization problem

$$\mathscr{L}(w, \lambda) = \frac{1}{2}\langle Sw, w\rangle + \langle f, w\rangle + \langle \lambda, Bw\rangle, \quad w \in W,\ \lambda \in U,$$

we solve the dual problem

$$\max_{\lambda \in U} \inf_{w \in W} \mathscr{L}(w, \lambda) \equiv \max_{\lambda \in U} \mathscr{C}(\lambda). \tag{11}$$

By a direct computation,

$$\mathscr{C}(\lambda) = \begin{cases} -\infty & \text{if } \langle f, w\rangle + \langle \lambda, Bw\rangle \neq 0 \ \text{ for some } w \in \operatorname{Ker} S, \\ -\frac{1}{2}\langle S^+(f - B'\lambda), f - B'\lambda\rangle & \text{otherwise,} \end{cases} \tag{12}$$

where $S^+ : W \to W$ is any pseudoinverse of S, that is, an operator such that $w = S^+ g$ solves $Sw = g$ if $g \perp \operatorname{Ker} S$. It is easy to see from (12) that the choice of S^+ does not change the value of $\mathscr{C}$. Without loss of generality, assume that S^+ is given by the spectral decomposition,

$$S^+ = \sum_{t>0} \frac{1}{t} v_t v_t', \tag{13}$$

where

$$S = \sum_t t v_t v_t', \qquad S v_t = t v_t, \quad v_t' v_t = 1. \tag{14}$$

The dual problem (11) is equivalent to maximizing $\mathscr{C}(\lambda)$ on the admissible set

$$\mathscr{A} = \{\lambda \in U \mid \mathscr{C}(\lambda) > -\infty\}.$$

Define the space of admissible increments

$$\begin{aligned} V &= \{\lambda_1 - \lambda_2 \mid \lambda_1 \in \mathscr{A}, \lambda_2 \in \mathscr{A}\} \\ &= \{\mu \in U \mid \langle \mu, Bw\rangle = 0 \quad \forall w \in \operatorname{Ker} S\}. \end{aligned} \tag{15}$$

At the maximum of $\mathscr{C}(\lambda)$, $\lambda \in \mathscr{A}$, the derivative of $\mathscr{C}$ is zero in all directions in V:

$$D\mathscr{C}(\lambda; \mu) = 0 \quad \forall \mu \in V.$$

By a straightforward computation, this becomes

$$\lambda \in \mathscr{A}, \quad \langle -BS^+B'\lambda + BS^+f, \mu\rangle = 0, \quad \forall \mu \in V. \tag{16}$$

In order to express (16) as a linear equation in the space V, let $P_V : U \to V$ be the projection onto V orthogonal in the l_2-inner product $\langle ., .\rangle$. Then for $\mu \in V$,

$$\langle -BS^+B'\lambda + BS^+f, \mu\rangle = \langle -BS^+B'\lambda + BS^+f, P_V\mu\rangle = \langle P_V(-BS^+B'\lambda + BS^+f), \mu\rangle$$

since P_V is orthogonal, so $P_V = P_V'$. Therefore, the dual problem (11) is equivalent to the linear equation in V for the unknown μ,

$$\mu \in V, \qquad P_V(-BS^+B'(\mu + \lambda_0) + BS^+f) = 0, \tag{17}$$

where λ_0 is an arbitrary starting feasible solution, that is, $\lambda_0 \in \mathscr{A}$.

The FETI method is the method of preconditioned conjugate gradients in the space V applied to the linear equation (17). The linear part of the operator in (17) is $P_V F$, where

$$F = BS^+B' \tag{18}$$

and we consider the preconditioner $P_V M$, where

$$M = A'SA, \qquad A = \frac{1}{2}B'. \tag{19}$$

That is, in each iteration of the preconditioned conjugate gradients algorithm, $z = P_V Mr$ is evaluated as an approximate solution of the residual equation $P_V Fz = r$. The preconditioner (19) was proposed in [9]. Note that the evaluation of the matrix-vector product Su can be implemented by solving a Dirichlet problem in each subdomain; therefore it is called the Dirichlet preconditioner in [9].

3. Analysis

A well known bound on the reduction of the error in k iterations of the method of preconditioned conjugate gradients in the norm $|||e||| = \langle P_V Fe, e\rangle^{1/2}$ on V is [12]

$$2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k,$$

where κ is the condition number

$$\kappa = \frac{\lambda_{\max}(P_V F P_V M|_V)}{\lambda_{\min}(P_V F P_V M|_V)} \tag{20}$$

with $\lambda_{\max}$ and $\lambda_{\min}$ being the maximum and minimum eigenvalues of operators on V.

3.1. Abstract framework

The main idea of our convergence analysis is summarized in the following lemma, which we will apply to F and M from (18) and (19).

Lemma 3.1. *Let U be a finite dimensional linear space with the inner product $\langle\cdot,\cdot\rangle$. Let V be a subspace of U, $\|\cdot\|_V$ a norm on V induced by an inner product, and the dual norm defined by $\|v\|_{V'} = \sup_{\tilde{v}\in V}\langle v,\tilde{v}\rangle/\|\tilde{v}\|_V$. Let $P_V : U \to V$ be the $\langle\cdot,\cdot\rangle$ orthogonal projection onto V, and $F, M : U \to V$ linear operators symmetric on V*

$$\begin{aligned}\langle\tilde{\lambda}, F\lambda\rangle &= \langle\lambda, F\tilde{\lambda}\rangle \quad \forall\lambda,\tilde{\lambda}\in V\\ \langle\tilde{v}, Mv\rangle &= \langle v, M\tilde{v}\rangle \quad \forall v,\tilde{v}\in V,\end{aligned}$$

and such that

$$c_1\|\lambda\|_{V'}^2 \le \langle\lambda, F\lambda\rangle \le c_2\|\lambda\|_{V'}^2, \qquad \forall\lambda\in V \tag{21}$$

$$c_3\|v\|_V^2 \le \langle v, Mv\rangle \le c_4\|v\|_V^2 \qquad \forall v\in V \tag{22}$$

with constants $c_1, c_2, c_3, c_4 > 0$. Then

$$\kappa = \frac{\lambda_{\max}(P_V M P_V F)}{\lambda_{\min}(P_V M P_V F)} \le \frac{c_2 c_4}{c_1 c_3} \tag{23}$$

Proof. Since $\lambda\in V$, we can replace in (21) F by $P_V F$. From (21), the operator norm of the mapping $P_V F : V \to V$ and its inverse satisfies

$$\|P_V F\|_{V'\to V} \le c_2, \qquad \|(P_V F)^{-1}\|_{V\to V'} \le \frac{1}{c_1}. \tag{24}$$

Similarly (22) implies

$$\|P_V M\|_{V\to V'} \le c_4, \qquad \|(P_V M)^{-1}\|_{V'\to V} \le \frac{1}{c_3}. \tag{25}$$

Consequently,

$$\lambda_{\max}(P_V M P_V F) \le \|P_V M P_V F\|_{V'\to V'} \le c_2 c_4$$

and

$$\lambda_{\max}((P_V F)^{-1}(P_V M)^{-1}) \le \|(P_V F)^{-1}(P_V M)^{-1})\|_{V'\to V'} \le \frac{1}{c_1 c_3},$$

which gives (23). □

The rest of this paper is concerned with estimating the condition number κ from (23). We will specify a suitable norm $\|\cdot\|_V$ and estimate the constants in Lemma 3.1 for the finite element problem below.

3.2. Assumptions

We need more specific assumptions in order to be able to prove a bound on the condition number κ. So, we are solving the boundary value problem

$$\mathscr{A}u = g \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

where

$$\mathscr{A}v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\alpha(x)\frac{\partial v(x)}{\partial x_j}\right),$$

with $\alpha(x)$ a measurable function such that $0 < \alpha_0 \le \alpha(x) \le \alpha_1$ a.e. in Ω.

The domain Ω is assumed to be divided into non-overlapping subdomains Ω_i, $i = 1, ..., n_s$, which can be generated from a reference domain (square or cube) $\hat{\Omega}$ of unit diameter as $\Omega_i = F_i(\hat{\Omega}_i)$ by mappings F_i, which are assumed to satisfy

$$\|\partial F_i\| \le CH, \qquad \|\partial F_i^{-1}\| \le CH^{-1}$$

with ∂F_i the Jacobian and $\|.\|$ the Euclidean $\mathbb{R}^d$ matrix norm. That is, the subdomains are shape regular and of diameter $O(H)$.

Assume that $V_h(\Omega)$ is a conforming P1 or Q1 finite element space on a triangulation of Ω, which satisfies the standard regularity and inverse assumptions. Denote by h the characteristic element size. Each subdomain Ω_i is assumed to be a union of some of the elements, and all functions in $V_h(\Omega)$ are zero on $\partial\Omega$.

In particular, the degrees of freedom are values at nodes of the triangulation. We assume that B is defined as follows. For a pair of degrees of freedom $w_r(x_\alpha)$ on $\partial\Omega_r$ and $w_s(x_\alpha)$ on $\partial\Omega_s$, such that the node x_α does not belong to any other subdomain, let

$$(Bw)_{rs}(x_\alpha) = \sigma_{rs}(w_r(x_\alpha) - w_s(x_\alpha)), \tag{26}$$

where $\sigma_{rs} = 1$ or $\sigma_{rs} = -1$.

When node x_β belongs to more than two subdomains $\partial\Omega_i$, $i = s_1, s_2, \ldots, s_{n_\beta}$, we assume that $(Bw)_{rs}(x_\beta)$ is defined so that B is full rank and so that the coefficients are ± 1 and determined uniquely by the indices $(s_1, s_2, \ldots, s_{n_\beta})$. For example,

$$(Bw)_{k,k+1}(x_\beta) = (-1)^k w_{s_k}(x_\beta) - (-1)^k w_{s_{k+1}}(x_\beta), \quad k = 1, .., n_\beta - 1. \tag{27}$$

For an example of the definition of the values of B from (27) with $(s_1, s_2, s_3) = (1, 3, 2)$ in 2D around a crosspoint, see Fig. 1.

Remark 3.2. The essential property here is that there are no redundant constraints in enforcing the continuity of the solution at the nodes where more than two subdomains meet, and that the constraints do not change along the edges (in 3D). Only the improved estimate in statement 3 of Lemma 3.8 will require the specific definition (27).

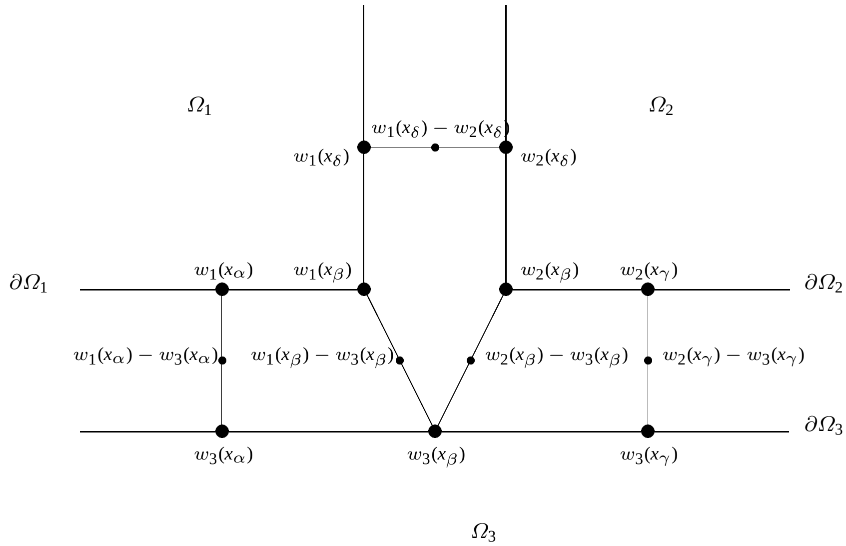

Fig. 1. Definition of B

3.3. Discrete norm bounds

The key to our analysis is a proper choice of norms. We equip the space W with the seminorm and the norm

$$|w|_W^2 = \sum_{i=1}^{n_s} |w_i|^2_{1/2,\partial\Omega_i}, \qquad \|w\|_W^2 = |w|_W^2 + \frac{1}{H}\sum_{i=1}^{n_s} \|w_i\|^2_{0,\partial\Omega_i} \tag{28}$$

and the space V with the norm $\|\cdot\|_V$ and the dual norm $\|\cdot\|_{V'}$

$$\|v\|_V = \|Av\|_W, \qquad \|v\|_{V'} = \sup_{\tilde{v}\in V} \frac{\langle v, \tilde{v}\rangle}{\|\tilde{v}\|_V}, \quad v \in V \tag{29}$$

For the definition and properties of the Sobolev seminorms $|\cdot|_{k,O}$, see, e.g., [17]. The space U is identified with some space $\mathbb{R}^n$. We use the l^2 inner product $\langle\cdot,\cdot\rangle$ as duality pairing.

In the following, we use $a \approx b$ to indicate that $ca \leq b \leq Ca$ with some positive generic constants c, C independent of the characteristic mesh size h and the subdomain diameter H. First we need to relate our discrete norm to a Sobolev norm and to establish equivalence of the norm and seminorm on the complement of the kernel of S.

Lemma 3.3. $|w|_W^2 \approx \langle w, Sw\rangle$, $w \in W$.

Proof. The lemma follows from the standard result [2, 19]

$$|w_i|^2_{H^{1/2}(\partial\Omega_i)} \approx \langle w_i, S_i w_i\rangle$$

by summation over all subdomains Ω_i and using (28). $\square$

Lemma 3.4. $|w|_W \approx \|w\|_W$, $w \in W$, $w \perp \operatorname{Ker} S$.

Proof. From the equivalence of the H^1 norm and seminorm on the factorspace modulo constants [17] or from the Poincaré inequality, and scaling from a reference domain to subdomain Ω_i,

$$\|w_i\|^2_{0,\partial\Omega_i} \le CH|w_i|^2_{1/2,\partial\Omega_i}$$

for all w_i if $\partial\Omega_i$ contains a part of $\partial\Omega$, and for all w_i such that $\int_{\partial\Omega_i} w_i = 0$ otherwise. The lemma follows by summation over the subdomains and from (28). $\square$

We also need the equivalence of the norm $\|Av\|_W$ and the seminorm $|Av|_W$.

Lemma 3.5. $|Av|_W \approx \|Av\|_W$, $v \in V$.

Proof. Let $v \in V$. Since $A = \frac{1}{2}B'$, by definition of V, we have $\langle Au, w\rangle = 0\ \forall w \in \operatorname{Ker} S$, or $Au \perp \operatorname{Ker} S$, which yields the result using Lemma 3.4. $\square$

Our norm on V was chosen so that the preconditioner is coercive and bounded, that is, so that (22) holds with c_1 and c_2 independent of H and h. This is shown in the following lemma.

Lemma 3.6. $\langle v, Mv\rangle \approx \|v\|^2_V$, $\forall v \in V$,

Proof. For $v \in V$, by definition of the preconditioner M, Lemma 3.3 and Lemma 3.5,

$$\langle v, Mv\rangle = \langle v, A'SAv\rangle = \langle Av, SAv\rangle \approx \|v\|_V \quad \square$$

The following lemmas lead to an estimate of coercivity and ellipticity of F. We first summarize some well known results and inequalities in a form suitable for our purposes.

Lemma 3.7. *Let G be a vertex, edge, or face (if $d = 3$) of subdomain Ω_i. A face is understood not to contain adjacent edges, and an edge does not contains its endpoints. For $z \in V_h(\partial\Omega_i)$, define $w \in V_h(\partial\Omega_i)$ by $w(x) = z(x)$ on all nodes $x \in G$, $w(x) = 0$ on all other nodes of $\partial\Omega_i$. Then*

$$\|w\|^2_{H^{1/2}(\partial\Omega_i)} \le C\left(1 + \log\frac{H}{h}\right)^\beta \left(\|z\|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H}\|z\|^2_{L^2(\partial\Omega_i)}\right)$$

where

$\beta = 1$ if $d = 2$ and G is a vertex, or $d = 3$ and G is an edge or a vertex
$\beta = 2$ if $d = 2$ and G is an edge, or $d = 3$ and G is a face.

Proof. The inequality for $d = 2$ was proved in [15, 16]. The case when $d = 3$ follows from Lemmas 4.1 and 4.2 in [3] if G is an edge or a vertex, and Lemma 4.3 in [3] if G is a face. Cf., also [5]. $\square$

Lemma 3.8. *It holds that*

$$\inf_{\substack{\tilde{w} \in W \\ B\tilde{w} = Bw}} \|\tilde{w}\|_W^2 \le C(1 + \log(H/h))^\alpha \|ABw\|_W^2, \quad w \in W,$$

where $\alpha = 1$, *and* $\alpha = 0$ *in the following special cases:*

1. $BA = I$, *which means that there are no nodes shared by more than two subdomains*
2. $d = 2$, *and the matrix* A *has the following property: If* $\bar{w} \in \operatorname{Range} A$, x *is a crosspoint (node shared by more than two subdomains),* $\bar{w}_i(x) = w_i(y)$ *for all* i *such that* $x \in \partial\Omega_i$ *and all nodes* y *that are adjacent to* x *on* $\partial\Omega_i$, *then* $\bar{w}_i(x) = 0$ *for all* i *such that* $x \in \partial\Omega_i$.
3. $d = 2$, B *is defined by (26), (27), and all nodes in the triangulation belong to either one, two, or an odd number of subdomains.*

Proof. Let us first prove that in the general case we obtain $\alpha \le 1$. Let $w \in W$ and $u = Bw$ throughout this proof. From the fact that $BA(BA)^{-1}u = u$, and by the triangle inequality,

$$\inf_{\substack{\tilde{w} \in W \\ B\tilde{w} = u}} \|\tilde{w}\|_W \le \|A(BA)^{-1}u\|_W \le \|Au\|_W + \|A(I - (BA)^{-1})u\|_W. \tag{30}$$

Denote $z = A(I - (BA)^{-1})u$. From the definition of B in (26), z is zero at all nodes that belong to at most two subdomains. The remaining nodes lie on crosspoints or edges (in the 3D case) of subdomains. From the definition of B, at every such node, $z_i(x)$ is a linear combination of the entries of Au that correspond to the same node x and the coefficients of the linear combinations are bounded only in terms of the number of subdomains the node belongs to. Using Lemma 3.7 for the crosspoints of subdomains, we obtain for the 2D case that

$$\|A(I - (BA)^{-1})u\|_W^2 \le C \sum_{\substack{x \ \text{crosspoint} \\ x \in \partial\Omega_i}} ((Au)_i(x))^2 \le C(1 + \log(H/h))\|Au\|_W^2 \tag{31}$$

In the 3D case, the argument for subdomain crosspoints is same. In addition, we note that the coefficients of the linear combination do not change along a subdomain edge, so it remains to apply Lemma 3.7 on every edge.

Let us now turn to the special cases that give $\alpha = 0$.

If $BA = I$, we choose $\tilde{w} = Au$ in the following and get

$$\inf_{\substack{\tilde{w} \in W \\ B\tilde{w} = u}} \|\tilde{w}\|_W \le \|Au\|_W \quad \text{as} \quad B(Au) = u,$$

which proves the special case 1.

Now we prove special case 2. From the definition of the $H^{1/2}$ norm [17] and the fact that Au is a piecewise linear function, it follows that

$$\|Au\|_W^2 \ge \sum_{i=1}^{n_s} |Au|_{1/2,\partial\Omega_i}^2 \ge \sum_{\substack{x \ \text{crosspoint},\ x \in \partial\Omega_i \\ y \ \text{adjacent to x},\ y \in \partial\Omega_i}} ((Au)_i(x) - (Au)_i(y))^2 \tag{32}$$

For any crosspoint x, it follows from the assumption that for every $\bar{w} \in \text{Range } A$,

$$\sum_{\substack{i,\, \partial\Omega_i \ni x \\ y \text{ adjacent to } x,\ y \in \partial\Omega_i}} (\bar{w}_i(x) - \bar{w}_i(y))^2 = 0 \Rightarrow \sum_{i, \partial\Omega_i \ni x} (\bar{w}_i(x))^2 = 0$$

Consequently, by compactness, and since there are only finitely many different numbers of subdomains sharing a crosspoint,

$$\sum_{i,\, \partial\Omega_i \ni x} (\bar{w}_i(x))^2 \leq C \sum_{\substack{i,\, \partial\Omega_i \ni x \\ y \text{ adjacent to } x,\ y \in \partial\Omega_i}} (\bar{w}_i(x) - \bar{w}_i(y))^2\,, \qquad \forall \bar{w} \in \text{Range } A$$

By summation over all crosspoints x and using (31) and (32), we get

$$\|A(I - (BA)^{-1})u\|_W^2 \leq C\|Au\|_W^2\,,$$

which concludes the proof of this case.

In order to prove case 3, we verify the assumptions of case 2. We formulate only the proof for a crosspoint shared by three subdomains (Fig. 1). The proof is similar for a different odd number of subdomains. Let $\bar{w} \in \text{Range } A$. Since $\bar{w}_1(x_\beta) - \bar{w}_1(x_\alpha) = 0$, and $\bar{w}_1(x_\beta) - \bar{w}_1(x_\delta) = 0$, we have $\bar{w}_1(x_\alpha) = \bar{w}_1(x_\delta)$. Similarly, we obtain $\bar{w}_2(x_\alpha) = \bar{w}_2(x_\gamma)$, and $\bar{w}_3(x_\delta) = \bar{w}_3(x_\gamma)$. Now $\bar{w} \in \text{Range } A$ implies $\bar{w}_1(x_\alpha) = -\bar{w}_2(x_\alpha)$, $\bar{w}_2(x_\gamma) = -\bar{w}_3(x_\gamma)$, and $\bar{w}_3(x_\delta) = -\bar{w}_1(x_\delta)$,which can be satisfied only if $\bar{w}_1(x_\alpha) = \bar{w}_1(x_\delta) = \ldots = 0$. □

Remark 3.9. In general, the exponent $\alpha = 1$ in Lemma 3.8 cannot be improved. To see that, let us consider the configuration with values of u and Au in the neighborhood of a crosspoint as in Fig. 2, which violate the assumptions of special case 2. Extending the values of u in Fig. 2 to decay as $\log^\gamma(t/H)$, $\gamma < 1/2$, where t is the distance from the crosspoint, we obtain a function $u \in U$ such that

$$\|Au\|_W \approx C\,, \qquad \|u\|_{H^{1/2}(\partial\Omega_1 \cap \partial\Omega_2)} \approx |\log h/H|^\gamma.$$

If $u = Bw$, then on $\partial\Omega_1 \cap \partial\Omega_2$, $u = w_2 - w_1$, giving

$$\begin{aligned} |u|_{H^{1/2}(\partial\Omega_1 \cap \partial\Omega_2)} &\leq |w_1|_{H^{1/2}(\partial\Omega_1 \cap \partial\Omega_2)} + |w_2|_{H^{1/2}(\partial\Omega_1 \cap \partial\Omega_2)} \\ &\leq |w_1|_{H^{1/2}(\partial\Omega_1)} + |w_2|_{H^{1/2}(\partial\Omega_2)} \\ &\leq \|w\|_W \end{aligned}$$

so $\inf_{Bw=u} \|w\|_W \geq C(\gamma)|\log h/H|^\gamma$ for all $\gamma < 1/2$.

Lemma 3.10. *Let $\lambda \in V$. Then for all $w \in W$, there is a $\tilde{w} \in W$ such that $AB\tilde{w} \perp \text{Ker } S$ and*

$$\frac{\langle \lambda, Bw\rangle^2}{\|w\|_W^2} \leq C(1 + \log H/h)^2 \frac{\langle \lambda, B\tilde{w}\rangle^2}{\|AB\tilde{w}\|_W^2}\,.$$

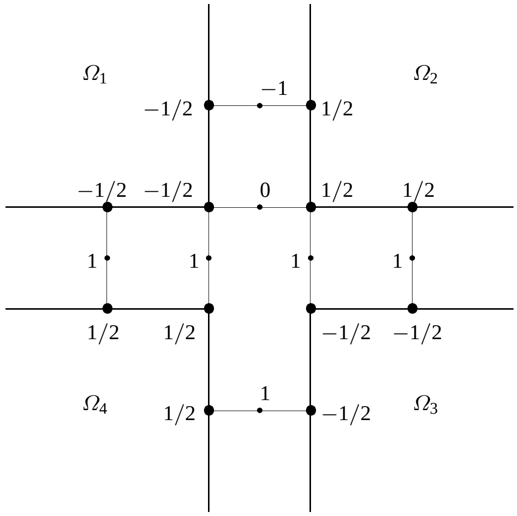

Fig. 2. Counter-example

Proof. Let $w \in W$ be arbitrary, and put $\tilde{w} = w + z$ where $z \in \text{Ker } S$. Since $\lambda \in V$, we have

$$\langle \lambda, Bw \rangle = \langle \lambda, B\tilde{w} \rangle \tag{33}$$

We would like to have $AB\tilde{w} \perp \text{Ker } S$ which can be also written as

$$\langle Bz, B\tilde{z} \rangle = -\langle Bw, B\tilde{z} \rangle \quad \forall \tilde{z} \in \text{Ker } S.$$

The bilinear form $\langle B\cdot, B\cdot \rangle$ is an inner product on the factorspace Ker $S/(\text{Ker } S \cap \text{Ker } B)$, so by Riesz representation theorem we may conclude that there exists $z \in \text{Ker } S$ satisfying $\|Bz\| \le \|Bw\|$.

Now, from the definition of B and the norm in W, we obtain

$$\|Bw\|^2 \le C\|w\|^2 \le CH\|w\|_W^2.$$

Also, since $z \in \text{Ker } S$, it is constant on each $\partial\Omega_i$, and we have the following by Lemma 3.7

$$\|ABz\|_W^2 \le C/H\|Bz\|^2(1 + \log H/h)^2.$$

Together this yields

$$\|ABz\|_W^2 \le C(1 + \log H/h)^2\|w\|_W^2.$$

By the definition of A and B, $(ABw)_i$ on $\partial\Omega_i \cup \partial\Omega_j$ is a linear combination (with bounded coefficients) of (a bounded number of) w_k from all $\partial\Omega_k$ adjacent to $\partial\Omega_i \cup \partial\Omega_j$. From Lemma 3.7,

$$\|ABw\|_W \le C(1 + \log(H/h))\|w\|_W, \quad \forall w \in W.$$

Finally, summarizing,

$$\|AB\tilde{w}\|_W \le \|ABw\|_W + \|ABz\|_W \le C(1 + \log H/h)\|w\|_W.$$

From this and (33), the result follows. □

We have now everything ready to prove the estimate (21).

Lemma 3.11. $c(1+\log(H/h))^{-\alpha}\|\lambda\|^2_{V'} \le \langle\lambda, F\lambda\rangle \le C(1+\log(H/h))^2\|\lambda\|^2_{V'}$, $\forall\lambda\in V$, *with* α *defined in Lemma 3.8.*

Proof. From the spectral decomposition (14), define $S^{-1/2} = \sum_{t>0} t^{-1/2} v_t v_t'$. Then $S^+ = S^{-1/2}S^{-1/2}$, and for $\lambda \in V$,

$$\begin{aligned}\langle\lambda, F\lambda\rangle &= \langle S^+B'\lambda, B'\lambda\rangle = \langle S^{-1/2}B'\lambda, S^{-1/2}B'\lambda\rangle \\ &= \|S^{-1/2}B'\lambda\|^2 = \sup_{x\in W}\frac{\langle S^{-1/2}B'\lambda, x\rangle^2}{\|x\|^2} = \sup_{\substack{x\in W,\ x=x_1+x_2\\ x_1\in\operatorname{Ker} S\ x_2\perp\operatorname{Ker} S}}\frac{\langle B'\lambda, S^{-1/2}x\rangle^2}{\|x_1+x_2\|^2} \\ &= \sup_{x_2\in W,\ x_2\perp\operatorname{Ker} S}\frac{\langle B'\lambda, S^{-1/2}x_2\rangle^2}{\|x_2\|^2}\end{aligned}$$

since $S^{-1/2}x_1 = 0$ and $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Now write any $w \in W$ as

$$w = w_1 + w_2, \qquad w_1 \in \operatorname{Ker} S, \qquad w_2 = S^{-1/2}x_2 \perp \operatorname{Ker} S.$$

From the definition of V in (15), $\lambda \in V$ implies that

$$\langle B'\lambda, w_1\rangle = 0.$$

Since

$$\|x_2\|^2 = \langle x_2, x_2\rangle = \langle w_2, Sw_2\rangle \approx |w_2|_W \approx \|w_2\|_W$$

from Lemma 3.3 and Lemma 3.4, it follows that

$$\langle\lambda, F\lambda\rangle = \sup_{w_2\in W,\ w_2\perp\operatorname{Ker} S}\frac{\langle B'\lambda, w_2\rangle^2}{\langle w_2, Sw_2\rangle} \approx \sup_{w\in W}\frac{\langle\lambda, Bw\rangle^2}{\|w\|^2_W}.$$

Lemma 3.8 shows that

$$\begin{aligned}\sup_{w\in W}\frac{\langle\lambda, Bw\rangle^2}{\|w\|^2_W} &= \sup_{w\in W}\frac{\langle\lambda, Bw\rangle^2}{\inf_{Bv=Bw}\|v\|^2_W} \ge \frac{1}{C(1+\log H/h)^\alpha}\sup_{w\in W}\frac{\langle\lambda, Bw\rangle^2}{\|ABw\|^2_W} \\ &\ge \frac{1}{C(1+\log H/h)^\alpha}\sup_{\substack{w\in W\\ ABw\perp\operatorname{Ker} S}}\frac{\langle\lambda, Bw\rangle^2}{\|ABw\|^2_W}.\end{aligned}$$

Lemma 3.10 yields an upper bound

$$\sup_{w\in W}\frac{\langle\lambda, Bw\rangle^2}{\|w\|^2_W} \le C(1+\log H/h)^2 \sup_{\substack{w\in W\\ ABw\perp\operatorname{Ker} S}}\frac{\langle\lambda, Bw\rangle^2}{\|ABw\|^2_W}.$$

Finally, by definition of the norm $\|\cdot\|_{V'}$,

$$\sup_{\substack{w\in W\\ ABw\perp\operatorname{Ker} S}}\frac{\langle\lambda, Bw\rangle}{\|ABw\|_W} = \sup_{v\in V}\frac{\langle\lambda, v\rangle}{\|Av\|_W} = \|\lambda\|_{V'}$$

since B spans V. $\square$

3.4. Condition number estimate

The final result now follows from the abstract estimate in Lemma 3.1 with the assumptions verified by Lemma 3.6 and Lemma 3.11.

Theorem 3.12. *The condition number of the FETI method with the Dirichlet preconditioner satisfies*

$$\kappa = \frac{\lambda_{\max}(P_V M P_V F)}{\lambda_{\min}(P_V M P_V F)} \leq C\left(1 + \log \frac{H}{h}\right)^{\gamma}$$

with $\gamma = 3$*, and* $\gamma = 2$ *in the special cases listed in Lemma 3.8.*

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