FETI and Neumann-Neumann Iterative Substructuring Methods: Connections and New Results

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Abstract

The FETI and Neumann-Neumann families of algorithms are among the best known and most severely tested domain decomposition methods for elliptic partial differential equations. They are iterative substructuring methods and have many algorithmic components in common, but there are also differences. The purpose of this paper is to further unify the theory for these two families of methods and to introduce a new family of FETI algorithms. Bounds on the rate of convergence, which are uniform with respect to the coefficients of a family of elliptic problems with heterogeneous coefficients, are established for these new algorithms. The theory for a variant of the Neumann-Neumann algorithm is also redeveloped stressing similarities to that for the FETI methods. © 2001 John Wiley & Sons, Inc.

1 Introduction

The FETI and Neumann-Neumann families of algorithms are among the best known and most severely tested domain decomposition methods for elliptic partial differential equations; cf., e.g., [1]. They are iterative substructuring methods and they share many algorithmic components, such as local solvers for both Neumann and Dirichlet problems on the subregions into which the region of the original problem has been partitioned. However, there are also differences, and we have seen a need to extend our understanding of the FETI algorithms.

The finite element tearing and interconnecting (FETI) methods were first introduced by Farhat and Roux [15]. An important advance, which made the rate of convergence of the iteration less sensitive to the number of unknowns of the local problems, was made by Farhat, Mandel, and Roux a few years later [14]. Our own work is based on the pioneering work by Mandel and Tezaur [24], who fully analyzed a variant of that algorithm. For a detailed introduction, see [16] or [33].

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For early work on the Neumann-Neumann algorithms and their predecessors, see [2, 3, 4, 5, 9, 17, 21]. For a fine introduction, see [20].

The purpose of this paper is to extend, simplify, and unify the theory for the FETI and Neumann-Neumann algorithms. We introduce a new one-parameter family of FETI preconditioners and prove bounds on the rate of convergence that are independent of possible jumps of the coefficients of an elliptic model problem previously considered in the theory of Neumann-Neumann and other iterative substructuring algorithms; see [8, 11, 23, 30, 31]. In fact, we have found it possible to reduce the analytic core of the theory for the new class of FETI methods to a variant of an estimate that is central in the Neumann-Neumann theory. We will write an arbitrary element u in a product space of traces of finite element functions as the sum of two terms $P_D u$ and $E_D u$; see Lemma 4.3. One of them, $P_D u$, is central in the FETI theory; the other, $E_D u$, in the Neumann-Neumann theory. The norm of each of the two terms is bounded by a factor $C(1 + \log(H/h))$ times that of the given function u. Here and from now on, C is a generic constant that may depend on the aspect ratios of the elements and subregions but is independent of the mesh parameters h and H and the coefficients of the elliptic problem; h is the diameter of a typical element into which the subregions have been divided; and H is the typical subregion diameter. We note that $(H/h)^d$, $d = 2, 3$, measures the number of degrees of freedom associated with a subregion. We note that our bounds are developed locally for a single subregion and its neighbors and that we therefore can interpret H/h in the logarithmic bound as the maximum value of the diameter of any subregion divided by that of its smallest element.

The results for the new family of FETI algorithms become possible because of two special scalings. One of them, for the preconditioner, is closely related to an important algorithmic idea used in the best of the Neumann-Neumann methods; this scaling goes back at least to De Roeck and Le Tallec [4]. A proof of one of the two spectral bounds that are required then becomes just as elementary as for the Neumann-Neumann case. We note that our family of scalings of the preconditioner was apparently first introduced by Sarkis [30, 31]; see also [7]. The other scaling affects the choice of the projection that is used in each step of the FETI iteration, whether preconditioned or not. It is given in terms of an operator Q.

We will show that, for a certain choice of the two scalings, our preconditioned operator is the same as one recently tested successfully for very difficult problems; see Bhardwaj et al. [1]. In that paper, Q is chosen to be equal to the preconditioner. In an important earlier paper by Rixen and Farhat [28], which successfully addresses problems with heterogeneous coefficients, $Q = I$. In this paper, we will select Q either as the preconditioner or as a particular diagonal matrix. We will also consider two different sets of Lagrange multipliers in Sections 4 and 5, respectively; Lagrange multipliers are used in the FETI methods to enforce the continuity of the finite element solutions across the interface defined by the subdomain boundaries. We note that our algorithms are also defined for the class of problems considered in [1], but in our analysis we have to impose certain restrictions on the coefficients and on the geometry of the subregions.

We note that, by now, many variants of the FETI algorithms have been designed and that a number of them have been tested extensively; see in particular the discussion in [29].

This is the second paper on the FETI algorithms by the present authors; cf. [19]. Our present work has already been extended to Maxwell's equation in two dimensions by Toselli and Klawonn [34].

The remainder of this paper is organized as follows. In Section 2, we introduce our elliptic problems and the basic geometry of the decomposition; we have chosen to work only with the more interesting three-dimensional case. In Section 3, we give a short introduction to the FETI methods. In Section 4, we introduce our family of preconditioners and prove two of our main results. They should also be extendable to certain other elliptic problems such as those in [19], our recent study of problems of linear elasticity, and the use of inexact solvers for the FETI algorithm. A connection between one element of our family of preconditioners and the method recently developed in [1, 28] is established in Section 5. In Section 6 we summarize the essence of the balancing Neumann-Neumann iterative substructuring method stressing the similarities to the analysis of the FETI algorithms. In an appendix, we collect some auxiliary technical results that are needed, e.g., in the proofs of Lemmas 4.7 and 4.10. They are borrowed almost directly from Dryja, Smith, and Widlund [8] and from Dryja and Widlund [10, 11].

2 Elliptic Model Problem, Finite Elements, and Geometry

Let $\Omega \subset \mathbb{R}^3$ be a bounded, polyhedral region, let $\partial\Omega_D \subset \partial\Omega$ be a closed set of positive measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. We impose Dirichlet and Neumann boundary conditions, respectively, on these two subsets and introduce the standard Sobolev space $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$.

For simplicity, we will only consider a first-order, conforming finite element approximation of the following scalar, second-order model problem:

Find $u \in H_0^1(\Omega, \partial\Omega_D)$ such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D)\,, \tag{2.1}$$

where

$$a(u, v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx\,, \qquad f(v) := \int_\Omega f v \, dx\,. \tag{2.2}$$

Here $\rho(x) > 0$ for $x \in \Omega$. For simplicity, we have chosen zero Neumann boundary data on $\partial\Omega_N$.

We decompose Ω into nonoverlapping subdomains Ω_i, $i = 1, \ldots, N$, also known as substructures, and each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := \overline{(\bigcup_{i=1}^{N} \partial\Omega_i) \setminus \partial\Omega}$. We denote the standard finite element space of continuous, piecewise linear functions on Ω_i, which vanish at the nodes of $\partial\Omega_D$, by $W^h(\Omega_i)$. For simplicity, we assume that the triangulation of each subdomain is quasi-uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $W_i := W^h(\partial\Omega_i)$, $i = 1, \dots, N$, and by $W := \prod_{i=1}^{N} W_i$ the associated product space. We note that we will often consider elements of W that are discontinuous across the interface. The finite element approximation of the elliptic problem is continuous across the interface, and we denote the corresponding subspace of W by $\widehat{W}$. We note that all the iterates of the Neumann-Neumann methods belong to $\widehat{W}$, while those of the FETI methods normally do not. To simplify our notation, we will identify any element in the product space $\widehat{W}$ with the corresponding continuous, piecewise, discrete harmonic finite element function in $W^h(\Omega)$.

We assume that possible jumps of $\rho(x)$ are aligned with the subdomain boundaries and, for simplicity, that in each subregion Ω_i, $\rho(x)$ has the constant value ρ_i. Our bilinear form and load vector can then be written, in terms of contributions from individual subregions, as

$$a(u, v) = \sum_{i=1}^{N} \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \,, \qquad f(v) = \sum_{i=1}^{N} \int_{\Omega_i} f v \, dx \,. \tag{2.3}$$

In our theoretical analysis, we assume that the subregions Ω_i are tetrahedra or parallelepipeds and that they are shape-regular; i.e., the subdomains are images of a reference domain under affine mappings with a well-conditioned Jacobian. We assume that a nonempty intersection of the closure of any pair of subregions is the closure of an entire face, or an entire edge, or just a vertex. We also assume that if a face of a subdomain intersects $\partial\Omega_D$, then the measure of this set is comparable to that of the face. Similarly, if only an edge of a subdomain intersects $\partial\Omega_D$, we assume that the length of this intersection is bounded from below in terms of the length of the edge as a whole. For the FETI methods and the case of arbitrary coefficients, we also have to make a further assumption that is introduced just before Lemma 4.7.

We next introduce notation related to certain geometrical objects. A face of the substructure Ω_i will be called $\mathcal{F}^{ij}$, $\mathcal{E}^{ik}$ represents an edge, $\mathcal{V}^{i\ell}$ a vertex, and $\mathcal{W}^i$ the wire basket, i.e., the union of the edges and the vertices of the substructure. All the substructures, faces, and edges are regarded as open sets. We note that a face in the interior of the region Ω is common to exactly two substructures, an interior edge is shared by more than two, and a vertex is typically common to still more substructures. The sets of nodes on $\partial\Omega_i$, Γ, and $\partial\Omega$ are denoted by $\partial\Omega_{i,h}$, Γ_h, and $\partial\Omega_h$, respectively.

As in previous work on Neumann-Neumann algorithms, a crucial role is played by the *weighted counting functions* μ_i, which are associated with the individual

$\partial\Omega_i$; cf. [7, 11, 23, 31]. They are defined, for $\gamma \in [1/2, \infty)$ and for $x \in \Gamma_h \cup \partial\Omega_h$, by a sum of contributions from Ω_i and its relevant next neighbors,

$$\mu_i(x) = \begin{cases} \sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)\,, & x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}\,, \\ \rho_i^\gamma(x)\,, & x \in \partial\Omega_{i,h} \cap (\partial\Omega_h \setminus \Gamma_h)\,, \\ 0\,, & x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h}\,. \end{cases} \tag{2.4}$$

Here $\mathcal{N}_x$ is the set of indices of all the subregions that have x on their boundaries. The μ_i are continuous, piecewise discrete harmonic functions. The pseudoinverses $\mu_i^\dagger$, which belong to the same class of functions, are defined, for $x \in \Gamma_h \cup \partial\Omega_h$, by

$$\mu_i^\dagger(x) = \begin{cases} \mu_i^{-1}(x) & \text{if } \mu_i(x) \neq 0\,, \\ 0 & \text{if } \mu_i(x) = 0\,. \end{cases}$$

We note that these functions provide a partition of unity:

$$\sum_i \rho_i^\gamma(x)\mu_i^\dagger(x) \equiv 1 \quad \forall x \in \Gamma_h \cup \partial\Omega_h\,. \tag{2.5}$$

3 A Review of the FETI Method

In this section, we review the original FETI method of Farhat and Roux [15, 16] and the variant with a Dirichlet preconditioner introduced in Farhat, Mandel, and Roux [14]. We will also introduce a general family of projections that was first introduced for heterogeneous problems in [16]. Such methods have recently been tested in very large scale numerical experiments; see [1]. For a more detailed description and extensions beyond scalar elliptic problems, see [12, 13, 25, 27, 33]. Let us point out that there are also other variants of the FETI methods; see, e.g., Park, Justino, and Felippa [26]. The relation of one of them to the FETI methods developed by Farhat and Roux is discussed in [29], and a convergence analysis of this method can be found in Tezaur's dissertation [33].

For a chosen finite element method and for each subdomain Ω_i, we first assemble the local stiffness matrix $K^{(i)}$ and the local load vector corresponding to single, appropriate terms in the sums of (2.3). Any nodal variable not associated with Γ_h is called interior, and it only belongs to one substructure. The interior variables of any subdomain can be eliminated by a step of block Gaussian elimination. This work can clearly be parallelized across the subdomains. The resulting matrices are the Schur complements

$$S^{(i)} = K^{(i)}_{\Gamma\Gamma} - K^{(i)}_{\Gamma I}(K^{(i)}_{II})^{-1}K^{(i)}_{I\Gamma}\,, \quad i = 1, \ldots, N\,.$$

Here Γ and I represent the interface and interior, respectively. We note that the $S^{(i)}$ are only needed in terms of matrix-vector products, and therefore the elements of these matrices need not be explicitly computed.

The elimination of the interior variables of a substructure can also be viewed in terms of an orthogonal projection with respect to the bilinear form $\langle K^{(i)}\cdot, \cdot\rangle$ onto the subspace of vectors with components that vanish at all the nodes of $\overline{\partial\Omega_i \setminus \partial\Omega_N}$.

Here $\langle\cdot,\cdot\rangle$ denotes the ℓ_2 inner product. We note that these vectors represent elements of $W^h(\Omega_i) \cap H_0^1(\Omega_i, \overline{\partial\Omega_i \setminus \partial\Omega_N})$. These local subspaces are orthogonal, in this energy inner product, to the space of discrete harmonic vectors that represent discrete harmonic finite element functions. With v_Γ and w_Γ vectors of interface values, such a vector, $w = (w_I, w_\Gamma)$, is defined, on the subdomain Ω_i, by

$$\langle K^{(i)} w, v\rangle = 0 \quad \forall v \text{ such that } v_\Gamma = 0, \tag{3.1}$$

or, equivalently, by

$$K_{II}^{(i)} w_I + K_{I\Gamma}^{(i)} w_\Gamma = 0. \tag{3.2}$$

We can regard w_Γ as a vector of Dirichlet data given on $\partial\Omega_{i,h} \cap \Gamma_h$ and note that a piecewise discrete harmonic function is completely defined by its values on the interface.

In what follows, we will work almost exclusively with functions in the trace spaces W_i and, whenever convenient, consider such an element as representing a discrete harmonic function in Ω_i. We will denote the discrete harmonic extension of a function w_i in the trace space W_i to the interior of Ω_i by $\mathcal{H}_i(w_i)$. We can easily show that $|w_i|^2_{S^{(i)}} = |\mathcal{H}_i(w_i)|^2_{K^{(i)}}$, where $|w_i|^2_{S^{(i)}} = \langle S^{(i)} w_i, w_i\rangle$ and $|\mathcal{H}_i(w_i)|^2_{K^{(i)}} = \langle K^{(i)}\mathcal{H}_i(w_i), \mathcal{H}_i(w_i)\rangle$.

For $w \in W$, $\mathcal{H}(w)$ denotes the piecewise discrete harmonic extension into all the Ω_i. We also note that it is this piecewise discrete harmonic part of the solution, representing an element of $\widehat{W}$, that is determined by any iterative substructuring method. The other, interior parts of the solution are computed locally as indicated above.

The values of the right-hand vectors also change when the interior variables are eliminated. We denote the resulting vectors, representing the modified load originating in Ω_i, by f_i, and the local vectors of interface nodal values by u_i.

We can now reformulate the finite element problem, reduced to the interface Γ, as a minimization problem with constraints given by the requirement of continuity across Γ:

Find $u \in W$ such that

$$\left.\begin{aligned} J(u) := \tfrac{1}{2}\langle Su, u\rangle - \langle f, u\rangle &\to \min \\ Bu &= 0 \end{aligned}\right\} \tag{3.3}$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} S^{(1)} & O & \cdots & O \\ O & S^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & S^{(N)} \end{bmatrix}.$$

The matrix

$$B = [B^{(1)}, B^{(2)}, \ldots, B^{(N)}] \tag{3.4}$$

is constructed from $\{0, 1, -1\}$ such that the values of the solution u associated with more than one subdomain coincide when $Bu = 0$. We note that the choice of B is far from unique. The local Schur complements $S^{(i)}$ are positive semidefinite, and they are singular for any subregion with a boundary that does not intersect $\partial\Omega_D$. The problem (3.3) is uniquely solvable if and only if $\ker(S) \cap \ker(B) = \{0\}$, i.e., if and only if S is invertible on $\ker(B)$.

By introducing a vector of Lagrange multipliers λ to enforce the constraints $Bu = 0$, we obtain a saddle point formulation of (3.3):

Find $(u, \lambda) \in W \times U$ such that

$$\left.\begin{aligned} Su + B^{\mathsf{T}}\lambda &= f \\ Bu &= 0 \end{aligned}\right\} . \tag{3.5}$$

We note that the solution λ of (3.5) is unique only up to an additive vector of $\ker(B^{\mathsf{T}})$. The space of Lagrange multipliers U is therefore chosen as $\mathrm{range}(B)$.

We will also use a full-column-rank matrix built from all of the null space elements of S; these elements are associated with individual subdomains

$$R = \begin{bmatrix} R^{(1)} & O & \cdots & O \\ O & R^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & R^{(N)} \end{bmatrix} .$$

Thus, $\mathrm{range}(R) = \ker(S)$. In fact, the subdomains that intersect $\partial\Omega_D$ do not contribute to $\ker(S)$, and therefore those columns of R are void. We note that the case of linear elasticity is somewhat more complicated. For the interior subregions, there are full six-dimensional null spaces of rigid body motions. There can also be contributions to R from subdomains with boundaries intersecting $\partial\Omega$ for which there are not enough essential boundary conditions to fully eliminate the entire space of rigid body motions.

A solution of the first equation in (3.5) exists if and only if $f - B^{\mathsf{T}}\lambda \in \mathrm{range}(S)$; this constraint will lead to the introduction of a projection P. We obtain

$$u = S^{\dagger}(f - B^{\mathsf{T}}\lambda) + R\alpha \quad \text{if } f - B^{\mathsf{T}}\lambda \perp \ker(S) ,$$

where $S^{\dagger}$ is a pseudoinverse of S. We will see that α can be determined easily once λ has been found. The pseudoinverse is generally not uniquely determined, but it can easily be seen that our algorithms are invariant to the specific choice. Thus, without loss of generality, we can assume in our analysis that $S^{\dagger}$ is symmetric, e.g., the Moore-Penrose generalized inverse. We note that another, computationally less expensive alternative has been implemented in Farhat and Roux [15].

Substituting u into the second equation of (3.5) gives

$$BS^{\dagger}B^{\mathsf{T}}\lambda = BS^{\dagger}f + BR\alpha . \tag{3.6}$$

We now introduce a symmetric, positive definite matrix Q and define an inner product $\langle \lambda, \mu \rangle_Q := \langle \lambda, Q\mu \rangle$ on U. By considering the component Q orthogonal to $G := BR$, we find that

$$\left.\begin{aligned} P^{\top} F\lambda &= P^{\top} d \\ G^{\top}\lambda &= e \end{aligned}\right\} \tag{3.7}$$

with $F := BS^{\dagger}B^{\top}$, $d := BS^{\dagger}f$, $P := I - QG(G^{\top}QG)^{-1}G^{\top}$, and $e := R^{\top}f$. We note that P is an orthogonal projection from U onto $\ker(G^{\top})$; this projection is orthogonal in the Q^{-1} inner product, i.e., the inner product defined by $\langle\lambda, Q^{-1}\mu\rangle$. By multiplying (3.6) by $(G^{\top}QG)^{-1}G^{\top}Q$, we find that $\alpha := (G^{\top}QG)^{-1}G^{\top}Q(F\lambda - d)$, which then fully determines the primal variables in terms of λ.

There are different successful choices for Q. In the case of homogeneous coefficients, it is sufficient to use $Q = I$, while for problems with jumps in the coefficients, we have to make more elaborate choices to make our proofs work satisfactorily. In our analysis, Q will be either a diagonal scaling matrix or the FETI Dirichlet preconditioner; see Sections 4 and 5 and [1, 16]. We note that we could view the introduction of a nontrivial Q in terms of a scaling of the matrix B from the left by the operator $Q^{1/2}$.

We introduce the spaces

$$V := \{\lambda \in U : \langle\lambda, Bz\rangle = 0 \;\; \forall z \in \ker(S)\} = \ker(G^{\top}) = \mathrm{range}(P)\,,$$

and

$$V' := \{\mu \in U : \langle\mu, Bz\rangle_Q = 0 \;\; \forall z \in \ker(S)\} = \mathrm{range}(P^{\top})\,.$$

It can easily be shown that V' is isomorphic to the dual space of V. Following Farhat, Chen, and Mandel [12], we call V the space of admissible increments. The original FETI method is a conjugate gradient method in the space V applied to

$$P^{\top}F\lambda = P^{\top}d\,, \quad \lambda \in \lambda_0 + V\,, \tag{3.8}$$

with an initial approximation λ_0 chosen such that $G^{\top}\lambda_0 = e$.

The most basic FETI preconditioner, as introduced in Farhat, Mandel, and Roux [14] for $Q = I$, is of the form

$$M^{-1} := BSB^{\top}\,.$$

To apply M^{-1} to a vector, N independent Dirichlet problems have to be solved, one on each subregion; it is therefore called the Dirichlet preconditioner.

We note that the matrix M^{-1} does not have an inverse, but we will show in Lemma 4.4 that $P\widehat{M}^{-1}$ is a one-to-one mapping of V' to V; here $\widehat{M}^{-1}$ is our modified preconditioner defined in the next section. In fact, to keep the search directions of this preconditioned conjugate gradient method in the space V, the application of the preconditioner M^{-1} is followed by an application of the projection P. Hence, the Dirichlet variant of the FETI method is the conjugate gradient algorithm applied to the equation

$$PM^{-1}P^{\top}F\lambda = PM^{-1}P^{\top}d\,, \quad \lambda \in \lambda_0 + V\,. \tag{3.9}$$

We note that for $\lambda \in V$, $PM^{-1}P^{\top}F\lambda = PM^{-1}P^{\top}P^{\top}FP\lambda$, and we can therefore view the operator on the left-hand side of (3.9) as the product of two symmetric matrices.

It is well-known that an appropriate norm of the iteration error of the conjugate gradient method will decrease at least by a factor

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k$$

in k steps; cf., e.g., [18]. Here κ is the ratio of the largest and smallest eigenvalues of the iteration operator. The main task in the theory is therefore always to obtain a good bound for the condition number κ.

We note that several different possibilities of improving the FETI preconditioner M^{-1} have already been explored. Some interesting variants are discussed by Rixen and Farhat [28] in a framework of mechanically consistent preconditioners and redundant Lagrange multipliers; see the discussion and analysis in Section 5. A new family of improved FETI preconditioners, with nonredundant Lagrange multipliers, is introduced and analyzed in Section 4.

4 New FETI Methods with Nonredundant Lagrange Multipliers

In this section, we present and analyze a family of new FETI preconditioners with an improved condition number estimate compared to that of Mandel and Tezaur [24]; the bound in their paper involves three powers of $(1 + \log(H/h))$ in the general case; ours, only two. In addition, we obtain uniform bounds for arbitrary positive values of the ρ_i if the operator Q, which enters the definition of P, is chosen carefully. In our proofs, we use a number of arguments developed in [24], but our presentation also differs considerably in several respects. We remark that for the FETI method described in Park, Justino, and Felippa [26] and for the case of continuous coefficients, a bound involving only two powers of $(1 + \log(H/h))$ is given in Tezaur [33].

We now assume, for the rest of this section, that B has full row rank; i.e., the constraints are linearly independent and there are no redundant Lagrange multipliers.

Our new preconditioner is defined, for any diagonal matrix D with positive elements, as

$$\widehat{M}^{-1} := (BD^{-1}B^{\top})^{-1}BD^{-1}SD^{-1}B^{\top}(BD^{-1}B^{\top})^{-1}.$$

It can easily be seen that $BD^{-1}B^{\top}$ is a block-diagonal matrix, and thus its inverse can be computed at essentially no extra cost; the block sizes are n_x, where n_x is the number of Lagrange multipliers employed to enforce continuity at the point x. To obtain a method which converges at a rate that is independent of the coefficient jumps, we now choose a special family of matrices D; a careful choice of the operator Q will also be required. As in previous work on Neumann-Neumann algorithms, a crucial role is played by the weighted counting functions μ_i associated with the individual $\partial\Omega_i$ and already introduced in (2.4) in Section 2. The diagonal matrix $D^{(i)}$ has the diagonal entry $\rho_i^{\gamma}(x)\mu_i^{\dagger}(x)$ corresponding to the point $x \in \partial\Omega_{i,h}$. Finally, we set

$$D := \begin{bmatrix} D^{(1)} & O & \cdots & O \\ O & D^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & D^{(N)} \end{bmatrix}.$$

We note that this matrix operates on elements in the product space W. This can be regarded as providing a scaling from the right, by $D^{-1/2}$, of the matrix B.

An important role will be played by

$$P_D := D^{-1}B^\top(BD^{-1}B^\top)^{-1}B\,. \tag{4.1}$$

This is a projection that is orthogonal in the scaled ℓ_2 inner product $x^\top Dy$, where $x, y \in W$. We note that this operator is invariant if we replace B by $Q^{1/2}B$. The next two lemmas essentially follow by noting that $BP_D = B$.

LEMMA 4.1 *For any $\mu \in U$, there exists a $\widehat{w} \in \mathrm{range}(P_D)$ such that $\mu = B\widehat{w}$.*

PROOF: For any $\mu \in U = \mathrm{range}(B)$, there exists a $\tilde{w} \in W$ such that $\mu = B\tilde{w}$. We then select $\widehat{w} = P_D\tilde{w} \in W$, since $B\widehat{w} = B\tilde{w} = \mu$ by a simple computation. $\square$

LEMMA 4.2 *The projection operator P_D preserves jumps in the sense that*

$$w - P_D w \in \widehat{W}\,;$$

i.e., this function is continuous across Γ for all $w \in W$.

PROOF: Use that $BP_D = B$. $\square$

We now show that $P_D u$ can be computed very easily.

LEMMA 4.3 *Let $u \in W$. Then*

$$P_D u = u - E_D u\,,$$

where $E_D u \in \widehat{W}$, i.e., a function that is continuous across Γ. The value of $E_D u$ at any $x \in \Gamma_h$ equals the D-weighted average of the values of u at that point.

PROOF: By Lemma 4.2, $u - P_D u$ is continuous across the interface. Let $e_x \in \widehat{W}$ be equal to 1 at a point $x \in \Gamma_h$ and vanish at all other points of Γ_h. Then, since e_x is continuous across Γ, $Be_x = 0$. We find, by using the definition of P_D given in (4.1), that the D-weighted average $P_D u$ at x, which is equal to $e_x^\top D P_D u$, vanishes. Thus the D-weighted averages of u and $E_D u$ coincide. $\square$

To prepare for the analysis of the new preconditioner, we equip V' with the norm

$$\|\mu\|_{V'}^2 := \left|D^{-1}B^{\mathsf{T}}(BD^{-1}B^{\mathsf{T}})^{-1}\mu\right|_S^2 = \langle \widehat{M}^{-1}\mu, \mu\rangle ,$$

where $|w|_S := \sqrt{\langle Sw, w\rangle}$ is the seminorm on the space W induced by the Schur complement S. We have

LEMMA 4.4 *$\|\cdot\|_{V'}$ defines a norm on V'.*

PROOF: Since $\|\cdot\|_{V'}$ is clearly a seminorm, we only need to show that $\|\mu\|_{V'} = 0$ implies $\mu = 0$. Consider any $\mu \in V'$ with $\|\mu\|_{V'} = 0$. By Lemma 4.1, $\mu = B\widehat{w}$ for some $\widehat{w} \in \operatorname{range}(P_D)$. Since $P_D\widehat{w} = \widehat{w}$, we obtain

$$0 = \|\mu\|_{V'}^2 = \|B\widehat{w}\|_{V'}^2 = \left|D^{-1}B^{\mathsf{T}}(BD^{-1}B^{\mathsf{T}})^{-1}B\widehat{w}\right|_S^2 = |\widehat{w}|_S^2 .$$

Thus, $\widehat{w} \in \ker(S)$ and by the definition of V', we find that $\mu = 0$ since

$$\|\mu\|_Q^2 = \langle \mu, Q\mu\rangle = \langle \mu, QB\widehat{w}\rangle = 0 .$$

$\square$

We can now show that

$$P\widehat{M}^{-1} : V' \to V$$

is symmetric and positive definite. Symmetry is easy to establish and positive definiteness follows immediately from Lemma 4.4 and the fact that, with $\lambda \in V' = \operatorname{range}(P^{\mathsf{T}})$, $\langle P\widehat{M}^{-1}\lambda, \lambda\rangle = \langle \widehat{M}^{-1}\lambda, \lambda\rangle = \|\lambda\|_{V'}^2$.

We equip the space of admissible increments V with a norm

$$\|\lambda\|_V := \sup_{\mu\in V'} \frac{\langle \lambda, \mu\rangle}{\|\mu\|_{V'}} .$$

We note that V' is isomorphic to the dual space of V. Since

$$\|\mu\|_{V'}^2 = \langle \widehat{M}^{-1}\mu, \mu\rangle , \quad \mu \in V' , \tag{4.2}$$

we find by a simple computation that

$$\|\lambda\|_V^2 = \langle \widehat{M}\lambda, \lambda\rangle , \quad \lambda \in V . \tag{4.3}$$

We also find that

$$P^{\mathsf{T}}\widehat{M} : V \to V'$$

is symmetric and positive definite. We can effectively view $\widehat{M}^{-1}$ and $\widehat{M}$ as symmetric, positive definite operators from V' to V and V to V', respectively.

The next result is needed in the proofs of Lemmas 4.8 and 4.10 and indirectly for our theorems.

LEMMA 4.5 *For any $w \in W$, there exists a unique $z_w \in \ker(S)$ such that $\tilde{w} := w + z_w$ with $B\tilde{w} \in V'$. Moreover,*

$$\|Bz_w\|_Q \le \|Bw\|_Q .$$

PROOF: We recall that $\tilde{w} := w + z_w$, with $z_w \in \ker(S)$ and $B\tilde{w} \in V'$, means that $B^\top QB\tilde{w} \perp \ker(S)$. This element can be found by minimizing $\|B(w+z)\|_Q^2$, $z \in \ker(S)$. The uniqueness of z_w follows from the fact that $\ker(S) \cap \ker(B) = \{0\}$. The inequality follows from elementary variational arguments. $\square$

We will now establish an important stability estimate for P_D, which is at the core of the proof of our main results. It is closely related to a well-known result from the convergence theory of the Neumann-Neumann algorithms. For the choice $Q = \widehat{M}^{-1}$, we are then almost ready to prove one of our main results, Theorem 4.9. After its proof, we will also consider a choice of a diagonal Q that will require a more detailed analysis. We include a proof of Lemma 4.7 to make this paper more complete and to prepare for the proof of Lemma 4.10.

Throughout the rest of this section and in Section 5, we will add an extra assumption on the geometry of the subregions; this is not necessary in the Neumann-Neumann theory; cf. Dryja and Widlund [11].

ASSUMPTION 4.6 There is no subregion Ω_i with a boundary that intersects $\partial\Omega_{D,h}$ in only one or a few points.

LEMMA 4.7 *For any $w \in \mathrm{range}(S)$, we have*

$$|P_D w|_S^2 \leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 |w|_S^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

PROOF: We will work with the $H^{1/2}(\partial\Omega_i)$ seminorm, since $\rho_i|\cdot|^2_{H^{1/2}(\partial\Omega_i)}$ is equivalent to $|\cdot|^2_{S^{(i)}}$; see the appendix for a short discussion and references to the literature. Let us introduce the notation

$$P_D w =: v = (v_i)_{i=1,\ldots,N}\,, \quad v_i = (P_D w)_i\,.$$

It is then sufficient to show that

$$\rho_i |v_i|^2_{H^{1/2}(\partial\Omega_i)} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \sum_{j \in I_i} \rho_j |w_j|^2_{H^{1/2}(\partial\Omega_j)}\,,$$

where I_i includes i and the indices of all subregions that are neighbors of Ω_i.

We begin our proof by using Lemma 4.3 to obtain a formula for $P_D w$ for an arbitrary element $w \in W$. We find, by using formula (2.5), that for $i = 1, \ldots, N$,

$$(P_D w(x))_i = v_i(x) = \sum_{j \in \mathcal{N}_x} \rho_j^\gamma \mu_j^\dagger (w_i(x) - w_j(x))\,, \quad x \in \partial\Omega_{i,h}\,. \tag{4.4}$$

Here $\mathcal{N}_x$ is again the set of indices of the subregions that have x on their boundaries. We note that the coefficients in this expression are constant on each face and on each edge of $\partial\Omega_i$ and that they are independent of the particular choice of B. The coefficients will often differ between different faces and edges, and it is therefore natural to write v_i as a sum of terms that vanish at all the interface nodes outside individual faces, edges, and vertices, respectively. The norms of the individual terms of this sum are then estimated; cf., e.g., [8, 10, 11]. We will use the characteristic finite element function $\theta_{\mathcal{F}^{ij}}$ of a face $\mathcal{F}^{ij}$, described in the appendix, for this purpose, and also the characteristic finite element functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{V}^{il}}$ of an edge $\mathcal{E}^{ik}$ and a vertex $\mathcal{V}^{il}$, respectively. $\theta_{\mathcal{V}^{il}}$ is simply the standard nodal basis function, and $\theta_{\mathcal{E}^{ik}}$ the sum of such basis functions over the nodes of $\mathcal{E}_h^{ik}$. These functions define a partition of unity, and we find that

$$v_i = \sum_{\mathcal{F}^{ij}\subset\partial\Omega_i} I^h(\theta_{\mathcal{F}^{ij}} v_i) + \sum_{\mathcal{E}^{ik}\subset\partial\Omega_i} I^h(\theta_{\mathcal{E}^{ik}} v_i) + \sum_{\mathcal{V}^{il}\subset\partial\Omega_i} I^h(\theta_{\mathcal{V}^{il}} v_i)\,, \tag{4.5}$$

where I^h denotes the linear interpolation operator onto the finite element space $W^h(\Omega_i)$.

We find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}} v_i) = I^h\big(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger (w_i - w_j)\big),$$

and we estimate its $H_{00}^{1/2}(\mathcal{F}^{ij})$ norm since $\|I^h(\theta_{\mathcal{F}^{ij}} v_i)\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}$ is equivalent to $|I^h(\theta_{\mathcal{F}^{ij}} v_i)|_{H^{1/2}(\partial\Omega_i)}$; see the discussion in the appendix. We also note, for future reference, that Bw equals $w_i - w_j$ or $w_j - w_i$ at all the nodes on $\mathcal{F}^{ij}$.

With $\gamma \geq \frac{1}{2}$, we can easily prove that

$$\rho_i(\rho_j^\gamma \mu_j^\dagger)^2 \leq \min(\rho_i, \rho_j)\,. \tag{4.6}$$

Using this inequality and Lemma A.2, we obtain

$$\begin{aligned}
&\rho_i \|I^h(\theta_{\mathcal{F}^{ij}} v_i)\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\
&\quad = \rho_i \left\| I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger (w_i - w_j)) \right\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \\
&\quad \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \min(\rho_i, \rho_j) \\
&\qquad \times \left[|w_i - w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|w_i - w_j\|^2_{L_2(\mathcal{F}^{ij})} \right] \\
&\quad \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \left[2\rho_i\left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}||w_i||^2_{L_2(\mathcal{F}^{ij})} \right) \right. \\
&\qquad \left. + 2\rho_j\left(|w_j|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_j}||w_j||^2_{L_2(\mathcal{F}^{ij})} \right) \right].
\end{aligned} \tag{4.7}$$

We note that H_i and H_j are comparable since our subdomains Ω_i and Ω_j are, by assumption, shape-regular and share an entire face.

By using Lemma A.3, we can estimate the contributions of the edges of Ω_i to the energy in terms of L_2 norms over the edges. We obtain

$$|I^h(\theta_{\mathcal{E}^{ik}} v_i)|^2_{H^{1/2}(\partial\Omega_i)} \leq C\|I^h(\theta_{\mathcal{E}^{ik}} v_i)\|^2_{L_2(\mathcal{W}^i)} = C\|I^h(\theta_{\mathcal{E}^{ik}} v_i)\|^2_{L_2(\mathcal{E}^{ik})}\,.$$

If four subdomains, e.g., Ω_i, Ω_j, Ω_k, and Ω_ℓ, have an edge $\mathcal{E}^{ik}$ in common, then, according to (4.4), there are three edge contributions to the estimate of $\rho_i |v_i|^2_{H^{1/2}(\partial\Omega_i)}$, namely,

$$\rho_i \|\rho_j^\gamma \mu_j^\dagger (w_i - w_j)\|^2_{L_2(\mathcal{E}^{ik})} + \rho_i \|\rho_k^\gamma \mu_k^\dagger (w_i - w_k)\|^2_{L_2(\mathcal{E}^{ik})} \\ + \rho_i \|\rho_\ell^\gamma \mu_\ell^\dagger (w_i - w_\ell)\|^2_{L_2(\mathcal{E}^{ik})} .$$

We first consider the second term in detail, assuming that Ω_i shares a face with each of Ω_j and Ω_ℓ but only an edge with Ω_k as in Figure 4.1. We apply formula (4.6) and Lemma A.4 and obtain

$$\begin{aligned} &\rho_i \|\rho_k^\gamma \mu_k^\dagger (w_i - w_k)\|^2_{L_2(\mathcal{E}^{ik})} \\ &\le 2\big(\rho_i \|w_i\|^2_{L_2(\mathcal{E}^{ik})} + \rho_k \|w_k\|^2_{L_2(\mathcal{E}^{ik})}\big) \\ &\le C\left(1 + \log\left(\frac{H}{h}\right)\right)\left[\rho_i\left(|w_i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|w_i\|^2_{L_2(\mathcal{F}^{ij})}\right)\right. \\ &\qquad\qquad \left. + \rho_k\left(|w_k|^2_{H^{1/2}(\mathcal{F}^{kj})} + \frac{1}{H_k}\|w_k\|^2_{L_2(\mathcal{F}^{kj})}\right)\right], \end{aligned} \tag{4.8}$$

since $\mathcal{F}^{ij}$ is a face of Ω_i and $\mathcal{F}^{kj}$ a face of Ω_k that have the edge $\mathcal{E}^{ik}$ in common. We have now obtained a bound which, in fact, is better than that given above for the face contributions since there is only one logarithmic factor. We note that since Ω_i and Ω_j, as well as Ω_i and Ω_ℓ, have a face in common, the argument given above can be simplified for the first and third edge contributions.

As for the contributions from a vertex $\mathcal{V}^{i\ell}$ of a substructure, we can use an elementary argument to show that $|I^h(\theta_{\mathcal{V}^{il}} v_i)|^2_{H^{1/2}(\partial\Omega_i)}$ is bounded by $Ch_i|v_i(\mathcal{V}^{i\ell})|^2$. We just have to estimate the $H^1(\Omega_i)$ norm of the nodal basis function $\theta_{\mathcal{V}^{il}}$. It is on the order of h_i since the gradient is bounded by C/h_i and the volume of its support by Ch_i^3. We complete the argument by using a standard trace theorem.

Such a term can be trivially estimated by the square of the L_2 norm over any edge that has the vertex as an endpoint. With $\mathcal{V}^{il}$ an endpoint of the edge $\mathcal{E}^{ik}$, we obtain

$$h_i|v_i(\mathcal{V}^{il})|^2 \le h_i \sum_{x\in\overline{\mathcal{E}_h^{ik}}} |v_i(x)|^2 \le C\|v_i\|^2_{L_2(\mathcal{E}^{ik})} .$$

The rest of the argument is then the same as for the edge contributions.

Before we discuss a final case of special boundary subregions, we will show how the $L_2(\mathcal{F}^{ij})$ terms in our estimates can be eliminated. They can, of course, be bounded in terms of the $L_2(\partial\Omega_i)$ norm. For the interior subregions, we use a Poincaré inequality and the fact that $w_i \in \text{range}(S^{(i)})$. The simplest way of deriving such an inequality is to consider the generalized Rayleigh quotient defined by the matrix $S^{(i)}$ and the mass matrix $M^{(i)}$ corresponding to the trace space W_i. Since, by assumption, we only consider elements orthogonal to the null space of

$S^{(i)}$, the Rayleigh quotient is bounded from below by the second eigenvalue of the pencil defined by the two matrices. This follows from the Courant-Fischer min-max theorem. A simple scaling argument shows that this eigenvalue is bounded from below by C/H_i. Finally, we use the equivalence of the seminorm defined by $S^{(i)}$ and that of $H^{1/2}(\partial\Omega_i)$. For the subregions that have at least a substantial part of a face in common with $\partial\Omega_D$, we use the standard Friedrichs inequality.

We finally consider the remaining, special boundary subregions. By Assumption 4.6, there are no subregions that touch $\partial\Omega_D$ in just isolated nodes. The final case, consistent with our assumptions, is therefore that of a subregion Ω_i with only an edge intersecting $\partial\Omega_D$; the argument can easily be extended to the case when the measure of the intersection of an edge with $\partial\Omega_D$ is bounded from below in terms of the subdomain diameter. The arguments above have to be modified, and we will use a variant of Friedrichs' inequality given as Lemma A.5 of the appendix; this type of work was done already in [11, lemma 7]. The terms attributable to the edges of such a subdomain, in the estimate of $\rho_i |v_i|^2_{H^{1/2}(\partial\Omega_i)}$, create no problems since we only obtain one logarithmic factor from the basic estimates given above (see (4.8)) and then only one additional logarithmic factor from Lemma A.5 when removing the $L_2(\mathcal{F}^{ij})$ term. We have also shown that the estimates for the vertex terms can be traced back to those for the edges without introducing any additional logarithmic factor. Thus, we are left with the terms related to the faces of Ω_i. We define an average $\overline{w}$ of a finite element function w over the face $\mathcal{F}^{ij}$ by

$$\overline{w} := \frac{1}{m_{ij}} \int_{\mathcal{F}^{ij}} w(x)dx \quad \text{with } m_{ij} := \int_{\mathcal{F}^{ij}} 1\,dx.$$

Reexamining the estimate given in (4.7), we write

$$w_i - w_j = \big((w_i - w_j) - \overline{(w_i - w_j)}\big) + \overline{(w_i - w_j)}$$

and obtain

$$\begin{aligned}\rho_i \| I^h(\theta_{\mathcal{F}^{ij}} v_i)\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \le 2\rho_i \left\| I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger \big((w_i - w_j) - \overline{(w_i - w_j)}\big) \right\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \\ + 2\rho_i \left\| I^h\big(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger \overline{(w_i - w_j)}\big) \right\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}.\end{aligned}$$

We can now apply Lemma A.2 directly to the first term on the right-hand side and obtain, by using a Poincaré inequality,

$$\begin{aligned}2\rho_i \left\| I^h\big(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger \big((w_i - w_j) - \overline{(w_i - w_j)}\big)\big) \right\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \le \\ C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \min(\rho_i, \rho_j) |w_i - w_j|^2_{H^{1/2}(\mathcal{F}^{ij})},\end{aligned}$$

since $(w_i - w_j) - (\overline{w_i - w_j})$ has a zero average; a Friedrichs inequality is not required. The second term can be estimated by using Lemma A.1 and the elementary inequality

$$\overline{(w_i - w_j)}^2 \leq C \frac{1}{H_i^2} \| w_i - w_j \|^2_{L_2(\mathcal{F}^{ij})} ,$$

which is a direct consequence of the Cauchy-Schwarz inequality. Combining these two arguments with Lemma A.5, we obtain

$$\begin{aligned} \rho_i \left\| I^h \left(\theta_{\mathcal{F}^{ij}} \rho_j^{\gamma} \mu_j^{\dagger} \overline{(w_i - w_j)} \right) \right\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \\ \leq C \left(1 + \log \left(\frac{H_i}{h_i} \right) \right) \min(\rho_i, \rho_j) \frac{1}{H_i} \| w_i - w_j \|^2_{L_2(\mathcal{F}^{ij})} \\ \leq C \left(1 + \log \left(\frac{H_i}{h_i} \right) \right)^2 \min(\rho_i, \rho_j) | w_i - w_j |^2_{H^{1/2}(\mathcal{F}^{ij})} . \end{aligned}$$

As in (4.7), we can therefore conclude that

$$\begin{aligned} \rho_i \| I^h (\theta_{\mathcal{F}^{ij}} v_i) \|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \leq \\ C \left(1 + \log \left(\frac{H}{h} \right) \right)^2 \left(2\rho_i | w_i |^2_{H^{1/2}(\mathcal{F}^{ij})} + 2\rho_j | w_j |^2_{H^{1/2}(\mathcal{F}^{ij})} \right) . \end{aligned}$$

$\square$

In preparation for our first theorem, we combine the results of Lemmas 4.5 and 4.7.

LEMMA 4.8 *For any* $w \in \operatorname{range}(S)$*, for the unique* $z_w \in \ker(S)$ *given in Lemma* 4.5*, and for* $Q = \widehat{M}^{-1}$*, we have*

$$| P_D z_w |^2_S \leq C \left(1 + \log \left(\frac{H}{h} \right) \right)^2 | w |^2_S .$$

Here C *is independent of* h*,* H*,* γ*, and the values of the* ρ_i*.*

PROOF: For any $u \in W$, we have

$$| P_D u |^2_S = \langle S P_D u, P_D u \rangle = \langle \widehat{M}^{-1} B u, B u \rangle = \| B u \|^2_Q .$$

According to Lemma 4.5, for any $w \in \operatorname{range}(S)$, the unique $z_w \in \ker(S)$ such that $w + z_w \in V'$ satisfies

$$\| B z_w \|_Q \leq \| B w \|_Q .$$

The proof is completed by combining these results with Lemma 4.7. $\square$

We are now ready to prove a condition number estimate for the preconditioned FETI operator $P \widehat{M}^{-1} P^{\top} F$.

THEOREM 4.9 *The condition number of the FETI method, with the new preconditioner* $\widehat{M}$ *and with* $Q = \widehat{M}^{-1}$*, satisfies*

$$\kappa(P \widehat{M}^{-1} P^{\top} F) \leq C \left(1 + \log \left(\frac{H}{h} \right) \right)^2 .$$

Here C *is independent of* h*,* H*,* γ*, and the values of the* ρ_i*.*

PROOF: We have to estimate the smallest eigenvalue $\lambda_{\min}(P\widehat{M}^{-1}P^{\mathsf{T}}F)$ from below and the largest eigenvalue $\lambda_{\max}(P\widehat{M}^{-1}P^{\mathsf{T}}F)$ from above. We will show that

$$\langle \widehat{M}\lambda, \lambda\rangle \leq \langle F\lambda, \lambda\rangle \leq C\left(1+\log\left(\frac{H}{h}\right)\right)^2 \langle \widehat{M}\lambda, \lambda\rangle \quad \forall \lambda \in V. \tag{4.9}$$

Lower Bound. We note that this bound is optimal in the sense that it is independent of h and H and possible coefficient jumps. It is derived using purely algebraic arguments.

Following Mandel and Tezaur [24, proof of lemma 3.11], we will use the formula

$$\langle F\lambda, \lambda\rangle = \sup_{w\in \mathrm{range}(S)} \frac{\langle \lambda, Bw\rangle^2}{|w|_S^2}, \quad \lambda \in V\,. \tag{4.10}$$

For completeness, we provide a short proof of (4.10). We first note that $S^{-1/2}B^{\mathsf{T}}\lambda \in \mathrm{range}(S)$ has a good meaning since $\lambda \in V$ means that $B^{\mathsf{T}}\lambda \in \mathrm{range}(S)$. We find that

$$\begin{aligned}\langle F\lambda, \lambda\rangle &= \langle S^{\dagger}B^{\mathsf{T}}\lambda, B^{\mathsf{T}}\lambda\rangle = \|S^{-1/2}B^{\mathsf{T}}\lambda\|^2 \\ &= \sup_{v\in\mathrm{range}(S)} \frac{\langle S^{-1/2}B^{\mathsf{T}}\lambda, v\rangle^2}{\|v\|^2} = \sup_{w\in\mathrm{range}(S)} \frac{\langle B^{\mathsf{T}}\lambda, w\rangle^2}{|w|_S^2}\,.\end{aligned}$$

Let $\mu \in V'$ be arbitrary. It follows from Lemma 4.1 that there exists a $\widehat{w} \in W$ such that $\mu = B\widehat{w}$ with $\widehat{w} \in \mathrm{range}(P_D)$. We denote by $\widehat{w}_\perp$ the component of $\widehat{w}$ that is orthogonal to $\ker(S)$. Clearly, we have

$$\sup_{w\in\mathrm{range}(S)} \frac{\langle \lambda, Bw\rangle^2}{|w|_S^2} \geq \frac{\langle \lambda, B\widehat{w}_\perp\rangle^2}{|\widehat{w}_\perp|_S^2}\,.$$

We also observe that for all $\widehat{w}$,

$$\langle S\widehat{w}_\perp, \widehat{w}_\perp\rangle = \langle S\widehat{w}, \widehat{w}\rangle\,, \tag{4.11}$$

and it also follows, from the definition of V, that

$$\langle \lambda, B\widehat{w}_\perp\rangle = \langle \lambda, B\widehat{w}\rangle, \quad \lambda \in V\,. \tag{4.12}$$

Using (4.11) and (4.12), we obtain, since $\widehat{w} = P_D\widehat{w}$,

$$\frac{\langle \lambda, B\widehat{w}_\perp\rangle^2}{|\widehat{w}_\perp|_S^2} = \frac{\langle \lambda, B\widehat{w}\rangle^2}{|\widehat{w}|_S^2} = \frac{\langle \lambda, \mu\rangle^2}{|D^{-1}B^{\mathsf{T}}(BD^{-1}B^{\mathsf{T}})^{-1}\mu|_S^2} = \frac{\langle \lambda, \mu\rangle^2}{\|\mu\|_{V'}^2}\,.$$

The proof of the left inequality of (4.9) concludes by using the definition of the norm $\|\cdot\|_V$ and formula (4.3).

Upper Bound. We will derive an upper bound for $\langle F\lambda, \lambda\rangle$ that depends only polylogarithmically on H/h and is independent of possible coefficient jumps.

Let $w \in \text{range}(S)$ be arbitrary. By Lemma 4.5, there exists a unique $z_w \in \ker(S)$ such that $B(w + z_w) \in V'$. By using Lemmas 4.7 and 4.8, we obtain

$$|P_D(w + z_w)|_S^2 \leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 |w|_S^2 . \tag{4.13}$$

Combining this formula with (4.10), we obtain, for all $\lambda \in V$,

$$\begin{aligned}
\langle F\lambda, \lambda\rangle = \sup_{w\in\text{range}(S)} \frac{\langle\lambda, Bw\rangle^2}{|w|_S^2} &\leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sup_{w\in\text{range}(S)} \frac{\langle\lambda, Bw\rangle^2}{|P_D(w + z_w)|_S^2} \\
&= C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sup_{w\in\text{range}(S)} \frac{\langle\lambda, B(w + z_w)\rangle^2}{\|B(w + z_w)\|_{V'}^2} \\
&= C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sup_{\substack{\tilde{w}\in W \\ B\tilde{w}\in V'}} \frac{\langle\lambda, B\tilde{w}\rangle^2}{\|B\tilde{w}\|_{V'}^2} \\
&= C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sup_{\mu\in V'} \frac{\langle\lambda, \mu\rangle^2}{\|\mu\|_{V'}^2} \\
&= C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \|\lambda\|_V^2 .
\end{aligned}$$

The proof of the right inequality of (4.9) concludes by using (4.3). $\square$

It is important to note that the special choice of Q enters the proof of this theorem only via (4.13), which in turn depends on Lemma 4.8. Therefore, if we can prove an equally strong bound for $|P_D z_w|_S^2$ for another choice of Q, then we immediately obtain a result as strong as Theorem 4.9. The following recipe for Q is successful for arbitrary values of the coefficients ρ_i, provided that the operator B is chosen in the particular way illustrated in Figure 4.1. This figure shows, without loss of generality, an edge and four subregions, Ω_i, Ω_j, Ω_k, and Ω_ℓ, which have that edge in common. The subregion with the largest coefficient ρ_k plays a special role as indicated in the figure. For a vertex, we select the constraints, i.e., the rows of B, in the same way.

The elements of the diagonal matrix Q are chosen as follows for the case of arbitrary coefficients:

$$
\begin{aligned}
q_{\mathcal{F}^{ij}} &= \min(\rho_i, \rho_j)\left(1+\log\left(\frac{H_i}{h_i}\right)\right)\frac{h_i^2}{H_i}\,, \\
q_{\mathcal{E}^{ik}} &= \min(\rho_i, \rho_k) h_i\,, \qquad (4.14)\\
q_{\mathcal{V}^{i\ell}} &= \min(\rho_i, \rho_\ell) h_i\,.
\end{aligned}
$$

Thus, the same scale factor is used for all the nodes on a face. For the edge shown in Figure 4.1, there are three different edge weights, $q_{\mathcal{E}^{ik}}$, $q_{\mathcal{E}^{jk}}$, and $q_{\mathcal{E}^{\ell k}}$ corresponding to the three sets of constraints across that edge.

LEMMA 4.10 *For any* $w \in \mathrm{range}(S)$ *and the unique* $z_w \in \ker(S)$, *given in Lemma* 4.5, *and the diagonal scaling matrix* Q *given by* (4.14), *we have*

$$
|P_D z_w|_S^2 \le C\left(1+\log\left(\frac{H}{h}\right)\right)^2 |w|_S^2\,.
$$

Here C *is independent of* h, H, γ, *and the values of the* ρ_i.

PROOF: We recall that any element of $\ker(S)$, in particular z_w as constructed in Lemma 4.5, is constant in each subdomain; we denote the component of z_w associated with $\partial\Omega_i$ by z_i.

As in the proof of Lemma 4.7, we will focus on the contribution to $|P_D z_w|_S^2$ from one subdomain Ω_i. We note that for any nodal point on a face $\mathcal{F}^{ij}$, the value of Bz is $z_i - z_j$ or $z_j - z_i$. For the choice of B as indicated in Figure 4.1, there will be three components of Bz associated with any node on the common edge, namely, $z_i - z_k$, $z_j - z_k$, and $z_\ell - z_k$. The strategy is now to estimate the contributions to $|P_D z_w|_S^2$

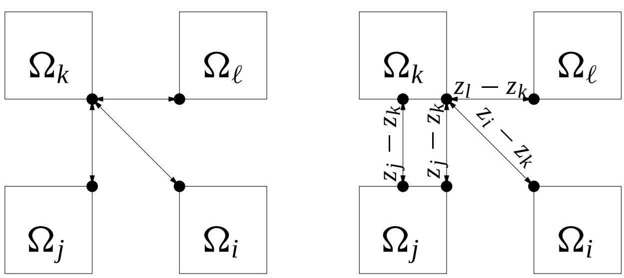

FIGURE 4.1. Left figure: Four subdomains meeting at an edge and a "fork" choice of the Lagrange multipliers for $\rho_k \ge \rho_i, \rho_j, \rho_\ell$. Right figure: Displacement differences for this choice.

from individual faces, edges, and vertices of the substructure Ω_i in terms of jumps of z_w across the interface. We then interpret the jumps as elements of Bz_w and show that we can bound the contributions to the left-hand side of the inequality of the current lemma by appropriate terms in the quadratic form $\|Bz_w\|_Q^2$ if the weights are chosen as in (4.14). We then use Lemma 4.5 and obtain a uniform bound in terms of $\|Bw\|_Q^2$. Finally, we show that this expression can be bounded by certain L_2 terms that have already been estimated successfully in the proof of Lemma 4.7.

We thus consider $v_i := (P_D z_w)_i$ (cf. (4.4)) and use the partition-of-unity functions $\theta_{\mathcal{F}^{ij}}$, $\theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ to decompose v_i as in (4.5).

We first consider the contribution to $|P_D z_w|_S^2$ from the face $\mathcal{F}^{ij}$ of Ω_i. As in the proof of Lemma 4.7, we find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}} v_i) = I^h(\theta_{\mathcal{F}^{ij}} \rho_j^\gamma \mu_j^\dagger (z_i - z_j)) .$$

By using Lemma A.1 and the fact that $z_i - z_j$ is constant on the face, we can easily see that

$$\rho_i \| I^h(\theta_{\mathcal{F}^{ij}} v_i)\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \le C \min(\rho_i, \rho_j) \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) H_i (z_i - z_j)^2 .$$

We now write $z_i - z_j$ as the components of Bz_w corresponding to the nodal points on $\mathcal{F}^{ij}$. To see that this expression can be bounded by the corresponding terms of $\|Bz_w\|_Q^2$, we just have to use the expression given for $q_{\mathcal{F}^{ij}}$ and note that there are on the order of $(H_i/h_i)^2$ equal contributions to $\|Bz_w\|_Q^2$ attributable to the face $\mathcal{F}^{ij}$. We note that with the logarithmic factor included in the definition of $q_{\mathcal{F}^{ij}}$, we obtain a uniform bound of this contribution to $|P_D z_w|_S^2$ in terms of the corresponding terms of $\|Bz_w\|_Q^2$.

We next consider the edge contributions. As in the proof of Lemma 4.7, we can use Lemma A.3 to bound the $H^{1/2}(\partial\Omega_i)$ seminorm of the three relevant edge terms of $P_D z_w$ by their $L_2(\mathcal{E}^{ik})$ norms. By using that the z_i are constant, we obtain an upper bound of the edge contributions of the form

$$CH_i\big(\min(\rho_i, \rho_j)(z_i - z_j)^2 + \min(\rho_i, \rho_k)(z_i - z_k)^2 + \min(\rho_i, \rho_\ell)(z_i - z_\ell)^2\big). \tag{4.15}$$

We can absorb the first and third terms into the expressions for the faces $\mathcal{F}^{ij}$ and $\mathcal{F}^{i\ell}$, respectively. We therefore need only consider the second term and note that $z_i - z_k$ is an element of Bz_w. We count the number of contributions of these components of Bz_w to $\|Bz_w\|_Q^2$, and we find that we obtain a uniform bound if the weights $q_{\mathcal{E}^{ik}}$ are chosen as in (4.14).

Since the choice of B introduces a certain nonsymmetry, we will also examine the contributions from Ω_k and Ω_ℓ. We note that the edge terms related to Ω_k all contain factors that can be found among the elements of Bz_w; this is an easy case and no new ideas are needed. The subdomain Ω_ℓ gives rise to the expression

$$CH_\ell\big(\min(\rho_\ell, \rho_i)(z_\ell - z_i)^2 + \min(\rho_\ell, \rho_j)(z_\ell - z_j)^2 + \min(\rho_\ell, \rho_k)(z_\ell - z_k)^2\big) .$$

Of these, the first and third terms can be absorbed into face terms related to $\mathcal{F}^{\ell i}$ and $\mathcal{F}^{\ell k}$, respectively, but the second term requires special attention since $z_\ell - z_j$ is neither an element of Bz_w, nor do Ω_j and Ω_ℓ share a face. Here we can instead use our assumption that ρ_k is at least as large as ρ_j and ρ_ℓ. The second term can then be bounded from above by

$$C H_\ell \bigl(2 \min(\rho_\ell, \rho_k)(z_\ell - z_k)^2 + 2 \min(\rho_j, \rho_k)(z_j - z_k)^2\bigr) ;$$

this alternative expression contains elements of $B z_w$ only.

The vertex contributions to the norm of $P_D z_w$ can be handled as those from the edges without introducing any new ideas; cf. the proof of Lemma 4.7.

We have now arrived at the estimate

$$|P_D z_w|_S^2 \le C \| B z_w \|_Q^2 ,$$

and the next step is to use Lemma 4.5 to obtain

$$|P_D z_w|_S^2 \le C \| B w \|_Q^2 .$$

By considering the contributions of faces, edges, and vertices to the weighted ℓ_2 norm on the right, and the definition of the weights, we first find that the contribution from the face $\mathcal{F}^{ij}$ can be bounded from above by

$$C \min(\rho_i, \rho_j) \left(1 + \log\left(\frac{H}{h}\right)\right) \frac{1}{H_i} \| w_i - w_j \|_{L_2(\mathcal{F}^{ij})}^2 ,$$

and that of one of the edge contributions by

$$C \min(\rho_i, \rho_k) \| w_i - w_k \|_{L_2(\mathcal{E}^{ik})}^2 .$$

What remains is just to repeat certain arguments in the proof of Lemma 4.7, where appropriate estimates already have been given for these L_2 terms. □

We have now completed all the work necessary for the proof of the following result. As already indicated just after the proof of Theorem 4.9, we can use Lemma 4.10 instead of Lemma 4.8 to derive an equally strong result.

THEOREM 4.11 *The condition number of the FETI method, with the preconditioner $\widehat{M}$, with Q as given in* (4.14) *and with B chosen as in Figure* 4.1, *satisfies*

$$\kappa(P \widehat{M}^{-1} P^{\mathsf{T}} F) \le C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

We conclude this section by considering simplifications possible if the collection of coefficients is less general. We first note that if the ρ_i are the same or are all of the same order of magnitude, then we can choose B arbitrarily and write any difference $z_i - z_m$ as the sum of such terms corresponding to faces; these terms can then be absorbed into face contributions to $B z_w$ without any complications related to the values of the ρ_i. In such a case, the matrix Q can then be chosen as a multiple of the identity matrix. In a more general case, we can show that the special choice of B, used above in the discussion of the contributions of Ω_ℓ, is not necessary, if between any pair of subdomains, with an edge or vertex in common, there is a path through faces of neighboring subdomains such that the coefficients are monotonically nonincreasing or nondecreasing along the path. We note that this condition resembles but is not the same as the concept of quasi-monotone coefficients introduced in [7]. While the elements of Q corresponding to the faces still generally must depend on the coefficients, we note that in such a case we can decrease the values of the scale factors corresponding to edges and vertices quite arbitrarily.

5 FETI with Redundant Lagrange Multipliers

In this section, we extend our analysis to the case of redundant Lagrange multipliers. For a detailed algorithmic description of FETI preconditioners in this case, with $\gamma = 1$, and an analysis based on mechanics, see Rixen and Farhat [27, 28]. In those papers, $Q_r = I$; to distinguish the redundant from the nonredundant case, we will write Q_r instead of Q, etc., in this section. We will first choose the Dirichlet preconditioner as Q_r and note that the resulting algorithm has proven successful for difficult industrial problems; cf. Bhardwaj et al. [1]. For this choice of Q_r, we show, in Theorem 5.6, a condition number estimate that is independent of the jumps in the coefficients. At the end of this section, we also consider a diagonal Q_r constructed using the recipe given in (4.14) and again prove a condition number estimate that is independent of the values of the ρ_i; see Theorem 5.7.

Following Rixen and Farhat, we consider the case where a maximum number of redundant Lagrange multipliers are introduced, i.e., when all possible pairs of degrees of freedom of the primal variables u that belong to the same nodal point $x \in \Gamma_h$ are connected by a Lagrange multiplier. Any edge or vertex node where at least three subregions meet will then contribute at least one additional Lagrange multiplier in comparison with the nonredundant case. An illustration of an edge common to four subregions is given in Figure 5.1.

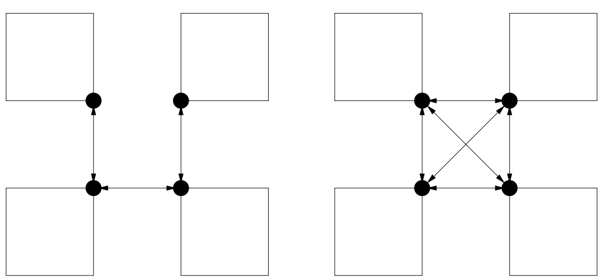

FIGURE 5.1. Left: U-shaped distribution of Lagrange multipliers for an edge in a nonredundant case. Right: Distribution of Lagrange multipliers for an edge in the fully redundant case.

We denote the new jump operator, similar to the one given by (3.4) but with additional rows, by B_r, and the new vector of Lagrange multipliers by λ_r. The space of Lagrange multipliers is chosen as $U_r := \mathrm{range}(B_r)$. This guarantees uniqueness of the Lagrange multiplier solution. Otherwise the solution of equation (5.1), given below, would only be unique up to an additive term from $\ker(B_r^\top)$.

We also introduce scaling matrices $D_r^{(i)}$ that operate on the Lagrange multiplier space. This is in contrast to the matrix D of the nonredundant case, discussed in Section 4, which maps the space of primal variables W onto itself. The Lagrange multiplier that enforces continuity between the nodal values of $w_i \in W_i$ and $w_j \in W_j$ at $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$ is scaled by $\rho_j^{\gamma}(x)\mu_j^{\dagger}(x)$, and this scale factor defines the corresponding element of $D_r^{(i)}$. Finally, we define a scaled jump operator by

$$B_{D_r} := \left[D_r^{(1)} B_r^{(1)}, \ldots, D_r^{(N)} B_r^{(N)}\right]$$

and the FETI preconditioner by

$$\widehat{M}_r^{-1} := \sum_i D_r^{(i)} B_r^{(i)} S^{(i)} B_r^{(i)t} D_r^{(i)} = B_{D_r} S B_{D_r}^{\top} .$$

This preconditioner, with $\gamma = 1$ and a different scaling, was introduced in Rixen and Farhat [28, section 5]. We also note that in the special case of continuous coefficients, we obtain the multiplicity scaling described in [28, section 3].

The matrix of the reduced linear system can be written as

$$F_r := B_r S^{\dagger} B_r^{\top} ,$$

and we now have to solve the preconditioned system

$$P_r \widehat{M}_r^{-1} P_r^{\top} F_r \lambda_r = P_r \widehat{M}_r^{-1} P_r^{\top} d_r \tag{5.1}$$

with $P_r := I - Q_r G_r (G_r^{\top} Q_r G_r)^{-1} G_r^{\top}$, $G_r := B_r R$, and $d_r := B_r S^{\dagger} f$. Here Q_r will be chosen either as $\widehat{M}_r^{-1}$ or as the diagonal matrix defined in (4.14). We denote the inner product induced by Q_r by $\langle \lambda_r, \mu_r \rangle_{Q_r}$.

The next lemma shows that the redundant and the nonredundant implementations of the Lagrange multiplier methods yield the same corrections of the primal variables.

LEMMA 5.1 *The operator $B_{D_r}^{\top} B_r$, with its two factors just defined in this section, and the operator P_D, defined in* (4.1)*, are the same*:

$$B_{D_r}^{\top} B_r = P_D .$$

PROOF: We first note that range(B_r) contains all possible Lagrange multipliers for every $x \in \Gamma_h$. By construction, each nonzero entry of $D_r^{(i)}$ corresponds to a Lagrange multiplier and to a point on $\partial\Omega_{i,h} \cap \partial\Omega_{j,h}$, for some other subregion Ω_j, and it is equal to $\rho_j^{\gamma}(x)\mu_j^{\dagger}(x)$. Applying $(B_r^{(i)})^{\top}$ to the vector of Lagrange multipliers given by $D_r^{(i)} B_r w$ yields a vector $v_i \in W_i$, with the components

$$v_i(x) = (B_r^{(i)})^{\top} D_r^{(i)} B_r w(x) = \sum_{j \in \mathcal{N}_x} \rho_j^{\gamma}(x)\mu_j^{\dagger}(x)(w_i(x) - w_j(x)), x \in \partial\Omega_{i,h} .$$

This is the same formula as (4.4). □

A consequence of this result is that we can still use Lemma 4.7 in the redundant case. We also note that $B_{D_r}^{\top} B_r$ is not symmetric unless the ρ_i are all the same.

Informally, one can say that the Lagrange multipliers, of the two variants could be viewed as temporary variables that can be hidden in two otherwise identical iterative methods, both written in terms of the primal variables.

A formal analysis of this FETI variant, with redundant Lagrange multipliers, can now be carried out using Lemma 5.1, adapting the arguments of Section 4 to the current context step by step. From Lemmas 4.3 and 5.1, we obtain

$$B_r B_{D_r}^{\top} B_r = B_r P_{D} = B_r \, .$$

As in Section 4, we get several results by using such an identity.

LEMMA 5.2 *For any $\mu_r \in U_r$, there exists a $\widehat{w} \in \mathrm{range}(B_{D_r}^{\top} B_r)$ such that $\mu_r = B_r \widehat{w}$.*

The proof proceeds exactly as that of Lemma 4.1.

As in Section 4, we define a space of admissible increments

$$V_r := \{\lambda_r \in U_r : \langle \lambda_r, B_r z \rangle = 0 \ \forall z \in \ker(S)\} = \mathrm{range}(P_r)$$

and the space

$$V_r' := \{\mu_r \in U_r : \langle \mu_r, B_r z \rangle_{Q_r} = 0 \ \forall z \in \ker(S)\} = \mathrm{range}(P_r^{\top}) \, .$$

We equip V_r' with the norm

$$\|\mu_r\|_{V_r'} := |B_{D_r}^{\top} \mu_r|_S \, , \quad \mu_r \in V_r' \, ,$$

and V_r with the norm

$$\|\lambda_r\|_{V_r} := \sup_{\mu_r \in V_r'} \frac{\langle \lambda_r, \mu_r \rangle}{\|\mu_r\|_{V_r'}} \, .$$

The fact that $\|\cdot\|_{V_r'}$ is a norm is established exactly as in the nonredundant case by using Lemmas 5.1 and 5.2.

LEMMA 5.3 *$\|\cdot\|_{V_r'}$ defines a norm on V_r'.*

As in the nonredundant case, we find that

$$\langle \widehat{M}_r \lambda_r, \lambda_r \rangle = \|\lambda_r\|_{V_r}^2 \, , \quad \lambda_r \in V_r \, , \tag{5.2}$$

by a simple computation. Again, as in the nonredundant case, it is immediate that $P_r \widehat{M}_r^{-1} : V_r' \to V_r$ and $P_r^{\top} \widehat{M}_r : V_r \to V_r'$ are symmetric, positive definite operators. Thus, we can view $\widehat{M}_r^{-1}$ and $\widehat{M}_r$ as symmetric, positive definite operators from V_r' to V_r and V_r to V_r', respectively. We now formulate a result analogous to Lemma 4.5.

LEMMA 5.4 *For any $w \in W$, there exists a unique $z_w \in \ker(S)$ such that $\tilde{w} := w + z_w$ with $B_r \tilde{w} \in V_r'$. Moreover,*

$$\|B_r z_w\|_{Q_r} \leq \|B_r w\|_{Q_r} \, .$$

The proof of this lemma proceeds like that of Lemma 4.5.

Using Lemmas 4.7, 5.1, and 5.4, we obtain an exact counterpart of Lemma 4.8.

LEMMA 5.5 *For any $w \in \mathrm{range}(S)$, and the unique $z_w \in \ker(S)$ given in Lemma* 5.4*, and for $Q_r = \widehat{M}_r^{-1}$, we have*

$$|B_{D_r}^\top B_r z_w|_S^2 \leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 |w|_S^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

Continuing as in the nonredundant case, after substituting $B_{D_r}^\top B_r$ for P_D and the auxiliary results of this section for those of the previous section, we obtain the following:

THEOREM 5.6 *The condition number of the FETI method defined by $\widehat{M}_r$ and with $Q_r = \widehat{M}_r^{-1}$ satisfies*

$$\kappa(P_r \widehat{M}_r^{-1} P_r^\top F_r) \leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

PROOF: The proof proceeds, line by line, exactly as in the proof of Theorem 4.9. □

In the rest of this section, we consider a diagonal Q_r given by the recipe in (4.14). As in Section 4, we only have to prove a result equivalent to Lemma 4.10 for the fully redundant case. Examining the proof of that lemma, we see that we need only reexamine the edge and vertex contributions since there are no redundant Lagrange multipliers associated with the faces. The estimates of the vertex contributions can be reduced to those for the edge contributions, and it is therefore sufficient to consider only the latter.

In the fully redundant case, we have all possible Lagrange multipliers available, and any formula such as (4.15) therefore already contains only elements of $B_r z_w$. The arguments in the proof of Lemma 4.10 can then be somewhat simplified, and we readily obtain a result analogous to Theorem 4.11.

THEOREM 5.7 *The condition number of the FETI method using the Dirichlet preconditioner $\widehat{M}_r$ and the diagonal matrix Q_r defined as in* (4.14) *satisfies*

$$\kappa(P_r \widehat{M}_r^{-1} P_r^\top F_r) \leq C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

Finally, as in Section 4, we see that Theorem 5.7 still holds with Q_r chosen as a multiple of the identity matrix if the ρ_i are the same or are all of the same order of magnitude. As in Section 4, we can also obtain other results for other special coefficient patterns.

6 The Neumann-Neumann Balancing Method

The purpose of this section is to redevelop the theory for the balancing variant of the Neumann-Neumann methods. There are no new results, but the theory is presented differently from before, and we will also show some close connections, algorithmically and theoretically, to the FETI methods. For earlier work on the theory for Neumann-Neumann methods, see in particular Mandel and Brezina [23] and Dryja and Widlund [11]. There are also a number of other studies of primal iterative substructuring methods of the same vintage. Many of these results, which provide bounds that are independent of the ρ_i and grow only polylogarithmically with H/h, are summarized in [8, 32]. In many of these studies the algorithms are viewed as *Schwarz methods* that are specified in terms of a family of subspaces of the given finite element space, related projections or projectionlike operators, and a polynomial in these operators. We will work in this framework, described in detail in [8, 32] and by now quite well known, but we will carry out our analysis directly rather than in the framework of the abstract Schwarz theory.

In this section, we will work primarily with finite element functions that are continuous across the interface Γ, i.e., belonging to the space $\widehat{W}$; the Neumann-Neumann iterates are all continuous functions. We recall that all elements of W and $\widehat{W}$ are piecewise discrete harmonic functions; see Section 3. We also recall that we identify elements in the product space $\widehat{W}$ with the corresponding continuous, piecewise, discrete harmonic finite element functions.

We introduce the standard finite element assembly matrices $L^{(i)}, i = 1, \ldots, N$, which map local into global degrees of freedom. $L^{(i)}$ is a Boolean matrix that extends elements in the local space W_i into elements in an isomorphic space $\widehat{W}_i := L^{(i)} W_i \subset \widehat{W}$, which will be used in the construction of our Schwarz algorithm. Under the mapping $L^{(i)}$, the values at the nodes of $\partial\Omega_{i,h}$ are retained, and those at all other nodes of Γ_h are set to zero. Its transpose $L^{(i)^\top}$ is a restriction operator mapping $\widehat{W}$ onto W_i. We also introduce the matrix L formed by all the columns of the $L^{(i)}$; L maps W onto $\widehat{W}$.

The assembled Schur complement matrix $\widehat{S}$ is obtained by subassembly

$$\widehat{S} = \sum_{i=1}^{N} L^{(i)} S^{(i)} L^{(i)^\top} = L S L^\top ,$$

and we denote the corresponding bilinear form by $\widehat{s}(u, v)$:

$$\widehat{s}(u, v) := \langle \widehat{S} u, v \rangle = \sum_{i=1}^{N} \langle S^{(i)} L^{(i)^\top} u, L^{(i)^\top} v \rangle \quad \forall u, v \in \widehat{W} .$$

The Neumann-Neumann balancing method is a two-level method with a coarse global space, $\widehat{W}_0 \subset \widehat{W}$. Each interior substructure, i.e., a substructure that does not intersect $\partial\Omega_D$, contributes one basis function $\rho_i^\gamma \mu_i^\dagger$ to $\widehat{W}_0$. In addition, any substructure that touches $\partial\Omega_D$ in only one or a few points should also contribute a basis function. A detailed discussion of this matter is given in [11]; we believe these details are of no real importance for our current discussion, and we will therefore confine our study to the case where Assumption 4.6 is satisfied.

We solve one coarse problem in each iteration and use an exact solver for this relatively small subspace, which has a dimension less than N. We denote the projection onto $\widehat{W}_0$, which is orthogonal with respect to $\widehat{s}(\cdot,\cdot)$, by $P_0 : \widehat{W} \to \widehat{W}_0$. It is defined by

$$\widehat{s}(P_0 u, v) = \widehat{s}(u, v) \quad \forall v \in \widehat{W}_0 .$$

We can also use matrices to describe this space and the projection P_0. Thus, using the matrix R of null space elements of S and the scaling matrix D, defined in Sections 3 and 4, respectively, we find that

$$\widehat{W}_0 := \text{range}(LDR) = \sum_{i=1}^{N} \text{range}(L^{(i)} D^{(i)} R^{(i)}) .$$

It is in fact easy to see that each of the terms in this sum contributes a vector representing the basis function $\rho_i^{\gamma} \mu_i^{\dagger}$ introduced above. The matrix form of P_0 is

$$P_0 = L_0 S_0^{-1} {L_0}^{\mathsf{T}}$$

where $L_0 := LDR$ and $S_0 := {L_0}^{\mathsf{T}} \widehat{S} L_0$.

In addition, just as for the FETI algorithms, there are local problems to solve. They are built from Neumann and Dirichlet solvers for the individual substructures and are associated with the spaces $\widehat{W}_i$ introduced above. A bilinear form $\tilde{s}_i(u, v)$ is defined for $u, v \in \widehat{W}_i$ by

$$\tilde{s}_i(u, v) := \left\langle S^{(i)} (D^{(i)})^{-1} {L^{(i)}}^{\mathsf{T}} u, (D^{(i)})^{-1} {L^{(i)}}^{\mathsf{T}} v \right\rangle .$$

We note that the matrix $(D^{(i)})^{-1} S^{(i)} (D^{(i)})^{-1}$ also appears as a block in one of the matrices of our FETI preconditioner $\widehat{M}^{-1}$. The bilinear form is used to define a projectionlike operator $T_i : \widehat{W} \to \widehat{W}_i$ given by

$$\tilde{s}_i(T_i u, v) = \widehat{s}(u, v) \quad \forall v \in \widehat{W}_i . \tag{6.1}$$

This operator is well defined only for elements $u \in \widehat{W}$ such that $\widehat{s}(u, \mu_i^{\dagger}) = 0$, and since the $\mu_i^{\dagger}$ span $\widehat{W}_0$, we find that we can compute all of the $T_i u$ provided that $u \in \text{range}(I - P_0)$. This condition could also be expressed in terms of vectors, but that does not appear to provide any additional insight.

In our algorithm, we will form the sum $Tu = \sum_{i=1}^{N} T_i u$. Typically this is done by adding the contributions of the $T_i u$ at all nodes of Γ_h. After that, a Dirichlet problem is solved on each subdomain, and we are then ready to apply the operator $\widehat{S}$ to the vector. Given that the piecewise discrete harmonic extension has been computed, this can be carried out using the original stiffness matrix.

We choose to make the solution $T_i u$ of (6.1) unique by making $(D^{(i)})^{-1} {L^{(i)}}^{\mathsf{T}} T_i u$ orthogonal to the null space of $S^{(i)}$. The matrix form of the operator T_i is given by

$$T_i = L^{(i)} D^{(i)} (S^{(i)})^{\dagger} D^{(i)} {L^{(i)}}^{\mathsf{T}} \widehat{S} .$$

The scaling $D^{(i)}$ introduced in the definition of the bilinear form $\tilde{s}_i(\cdot,\cdot)$, and indeed already in Section 4, also results in a convenient decomposition of any $u \in \widehat{W}$,

$$u = \sum_{i=1}^{N} u^{(i)} \quad \text{with } u^{(i)} = L^{(i)} D^{(i)} L^{(i)^{\mathsf{T}}} u \in \widehat{W}_i \,. \tag{6.2}$$

This formula can be derived by using (2.5). One can also show, straightforwardly, the following:

LEMMA 6.1 *For any* $u \in \widehat{W}$ *and with* $u^{(i)} := L^{(i)} D^{(i)} L^{(i)^{\mathsf{T}}} u$, $i = 1, \ldots, N$, *we have*

$$\sum_{i=1}^{N} \tilde{s}_i(u^{(i)}, u^{(i)}) = \widehat{s}(u,u)\,.$$

With $T := \sum_{i=1}^{N} T_i$, we obtain the following:

LEMMA 6.2 *For any* $u \in \text{range}(I - P_0)$ *and with* $w \in W$ *defined by its components* $w_i := (D^{(i)})^{-1} L^{(i)^{\mathsf{T}}} T_i u$, *we have*

$$|w|_S^2 = \widehat{s}(Tu, u)\,.$$

PROOF: By using the definitions of the $\tilde{s}_i(\cdot,\cdot)$ and the T_i, we find that

$$\begin{aligned} |w|_S^2 &= \sum_{i=1}^{N} \langle S^{(i)} w_i, w_i \rangle = \sum_{i=1}^{N} \langle S^{(i)} (D^{(i)})^{-1} L^{(i)^{\mathsf{T}}} T_i u, (D^{(i)})^{-1} L^{(i)^{\mathsf{T}}} T_i u \rangle \\ &= \sum_{i=1}^{N} \tilde{s}_i(T_i u, T_i u) = \sum_{i=1}^{N} \widehat{s}(u, T_i u) = \widehat{s}(u, Tu) = \widehat{s}(Tu, u)\,. \end{aligned}$$

□

We will now use P_0 and the T_i to construct a special hybrid Schwarz operator; see [22] and also [32, chapter 5.1], with the error propagation operator

$$\left(I - \sum_{i=1}^{N} T_i \right) (I - P_0)\,,$$

or after an additional coarse solve,

$$(I - P_0) \left(I - \sum_{i=1}^{N} T_i \right) (I - P_0)\,. \tag{6.3}$$

This operator is symmetric with respect to $\widehat{s}(\cdot,\cdot)$, and we can work with it, in a Krylov space iteration, without any extra real computational cost, since $(I - P_0)^2 = (I - P_0)$.

As already noted, the condition on the right-hand side of (6.1) is satisfied for all elements in range$(I - P_0)$. The Schwarz operator is therefore well defined. While we have chosen to make the range$((D^{(i)})^{-1} L^{(i)^{\mathsf{T}}} T_i) \subset$ range$(S^{(i)})$, we note that we could also use any solution of (6.1) in our computations since any two such solutions will differ only by an element in $\ker(I - P_0)$.

Subtracting the operator in (6.3) from I, we obtain the operator

$$T_{\text{hyb}} = P_0 + (I - P_0)\left(\sum_{i=1}^{N} T_i\right)(I - P_0)\,. \tag{6.4}$$

This represents the preconditioned operator and is the operator relevant for the conjugate gradient iteration; see, e.g., [32, chapter 5.1]. Our main result of this section is the following:

THEOREM 6.3 *The condition number of* T_{hyb} *of the balancing Neumann-Neumann method satisfies*

$$\kappa(T_{\text{hyb}}) \le C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 .$$

Here C is independent of h, H, γ, and the values of the ρ_i.

PROOF: The preconditioned operator T_{hyb} is symmetric in the $\widehat{s}(\cdot,\cdot)$ inner product, and we have

$$\widehat{s}(T_{\text{hyb}}u, u) = \widehat{s}(P_0u, u) + \widehat{s}(T(I - P_0)u, (I - P_0)u)$$

with $T = \sum_{i=1}^{N} T_i$. Since the first term in this expression $\widehat{s}(P_0u, u) = \widehat{s}(P_0u, P_0u)$ and $\widehat{s}(u, u) = \widehat{s}(P_0u, P_0u) + \widehat{s}((I - P_0)u, (I - P_0)u)$, it is easy to see that all that is required to estimate the condition number of T_{hyb} are upper and lower bounds of T restricted to $\operatorname{range}(I - P_0)$. We will prove the following estimates:

$$\widehat{s}(u, u) \le \widehat{s}(Tu, u) \le C\left(1 + \log\left(\frac{H}{h}\right)\right)^2 \widehat{s}(u, u) \quad \forall u \in \operatorname{range}(I - P_0)\,. \tag{6.5}$$

We choose to prove these bounds directly rather than in the framework of the abstract Schwarz theory as developed in [8, 11, 32] and in a number of other papers. As for the FETI methods, we obtain the lower bound by purely algebraic arguments. The upper bound is an almost direct consequence of Lemma 4.7.

Lower Bound. The left inequality of (6.5) is obtained quite easily by using (6.2), (6.1), Lemma 6.1, and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\widehat{s}(u, u) = \sum_i \widehat{s}(u, u^{(i)}) &= \sum_i \tilde{s}_i(T_iu, u^{(i)}) \\
&\le \left(\sum_i \tilde{s}_i(T_iu, T_iu)\right)^{1/2} \left(\sum_i \tilde{s}_i(u^{(i)}, u^{(i)})\right)^{1/2} \\
&= \left(\sum_i \widehat{s}(u, T_iu)\right)^{1/2} \widehat{s}(u, u)^{1/2} = \widehat{s}(Tu, u)^{1/2}\widehat{s}(u, u)^{1/2} .
\end{aligned}$$

Therefore, squaring and canceling a common factor, we find that $\widehat{s}(u, u) \le \widehat{s}(Tu, u)$.

Upper Bound. To prove the right inequality of (6.5), we use the element $w \in W$ with $w_i := (D^{(i)})^{-1} L^{(i)^\top} T_i u$ as in Lemma 6.2. By the definition of the operator E_D (see Lemma 4.3), we find that

$$E_D w = \sum_i L^{(i)^\top} T_i u = L^\top T u \, .$$

Since, by Lemma 4.3, $E_D w = w - P_D w$, we are able to bound the $\widehat{S}$ norm of Tu by using Lemmas 4.7 and 6.2. We note that since by definition $w_i \in \mathrm{range}(S^{(i)})$, we have $w \in \mathrm{range}(S)$, as required in Lemma 4.7. We find

$$\begin{aligned} \widehat{s}(Tu, Tu) &= \langle S L^\top T u, L^\top T u \rangle = \langle S E_D w, E_D w \rangle = |E_D w|_S^2 \\ &\leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 |w|_S^2 = C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \widehat{s}(Tu, u) \, . \end{aligned}$$

By using the Cauchy-Schwarz inequality and canceling a common factor, we find

$$\widehat{s}(Tu, Tu)^{1/2} \leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \widehat{s}(u, u)^{1/2} \, . \tag{6.6}$$

The upper bound now follows immediately by using the Cauchy-Schwarz inequality and (6.6):

$$\widehat{s}(Tu, u) \leq (\widehat{s}(Tu, Tu))^{1/2} (\widehat{s}(u, u))^{1/2} \leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \widehat{s}(u, u) \, .$$

□

Appendix: Some Auxiliary Results

The purpose of this appendix is to provide, without proofs, the few auxiliary results that are required for complete proofs of Lemmas 4.7 and 4.10. These results are all borrowed from [8, 10, 11]. Here we formulate them using trace spaces on the subdomain boundaries, i.e., $H^{1/2}(\partial\Omega_i)$, instead of the spaces $H^1(\Omega_i)$ and discrete harmonic extensions; given the well-known equivalence of the norms, nothing essentially new needs to be proven. The equivalence of the $S^{(i)}$ and the $H^{1/2}(\partial\Omega_i)$ seminorms of elements of W_i was established already in [2] for the case of piecewise linear elements and two dimensions. The tools necessary to extend this result to more general finite elements were provided in [35]; in our case, we of course have to multiply $|w_i|^2_{H^{1/2}(\partial\Omega_i)}$ by the factor ρ_i.

We also recall that we can define the $H_{00}^{1/2}(\tilde{\Gamma})$ norm of an element of W_i, $\tilde{\Gamma} \subset \partial\Omega_i$, as the $H^{1/2}(\partial\Omega_i)$ norm of the function extended by zero onto the rest of $\partial\Omega_i$.

The next lemma can essentially be found in Dryja, Smith, and Widlund [8, lemma 4.4].

LEMMA A.1 *Let $\theta_{\mathcal{F}^{ij}}$ be the finite element function that is equal to* 1 *at the nodal points on the face $\mathcal{F}^{ij}$ that is common to two subregions Ω_i and Ω_j and that vanishes on $(\partial\Omega_{i,h} \cup \partial\Omega_{j,h}) \setminus \mathcal{F}_h^{ij}$. Then*

$$|\theta_{\mathcal{F}^{ij}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)H_i\,.$$

The same bounds also hold for the other subregion Ω_j.

We remark that the proof of Lemma A.1 involves the explicit construction of a partition of unity from functions $\vartheta_{\mathcal{F}^{ij}}$, with the same boundary conditions as the $\theta_{\mathcal{F}^{ij}}$ and which satisfy the bound of the lemma. This set of functions is well defined in the interior of the substructure where the functions form a partition of unity. The discrete harmonic function $\theta_{\mathcal{F}^{ij}}$ will have a smaller energy than $\vartheta_{\mathcal{F}^{ij}}$. Further details are not provided here; see, e.g., [8], [32, chapter 5.3.2]. The following result can essentially be found in Dryja, Smith, and Widlund [8, lemma 4.5] or in Dryja [6, lemma 3]:

LEMMA A.2 *Let $\theta_{\mathcal{F}^{ij}}(x)$ be the function introduced in Lemma* A.1, *and let I^h denote the interpolation operator onto the finite element space $W^h(\Omega_i)$. Then $\forall u \in W_i$,*

$$\|I^h(\theta_{\mathcal{F}^{ij}}u)\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2\left(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\right).$$

We will also need two additional results that are used to estimate the contributions to our bounds from the values on the wire basket. For the next lemma, see Dryja, Smith, and Widlund [8, lemma 4.7].

LEMMA A.3 *Consider all finite element functions $u \in W_i$ that vanish at all the nodal points on the faces of Ω_i. Then*

$$|u|^2_{H^{1/2}(\partial\Omega_i)} \leq C\|u\|^2_{L_2(\mathcal{W}^i)}\,.$$

This result follows by estimating the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy.

We will also need a Sobolev-type inequality for finite element functions; see Dryja and Widlund [10, lemma 3.3] or Dryja [6, lemma 1].

LEMMA A.4 *Let $\mathcal{E}^{ik}$ be any edge of Ω_i that forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then $\forall u \in W_i$,*

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)\left(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\right).$$

Finally, we state a nonstandard version of Friedrichs' inequality that is given in a somewhat different form in [11, lemma 6].

LEMMA A.5 *Consider all finite element functions $u \in W_i$ that vanish on an edge $\mathcal{E}^{ik}$ of $\mathcal{F}^{ij}$. Then*

$$\|u\|^2_{L_2(\mathcal{F}^{ij})} \leq CH_i\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)|u|^2_{H^{1/2}(\mathcal{F}^{ij})}\,.$$

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