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A SIMPLE AND EFFICIENT EXTENSION OF A CLASS OF SUBSTRUCTURE BASED PRECONDITIONERS TO HETEROGENEOUS STRUCTURAL MECHANICS PROBLEMS

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SUMMARY

Several domain decomposition methods with Lagrange multipliers have been recently designed for solving iteratively large-scale systems of finite element equations. While these methods differ typically by implementational details, they share in most cases the same substructure based preconditioners that were originally developed for the FETI method. The success of these preconditioners is due to the fact that, for homogeneous structural mechanics problems, they ensure a computational performance that scales with the problem size. In this paper, we address the suboptimal behaviour of these preconditioners in the presence of material and/or discretization heterogeneities. We propose a simple and virtually no-cost extension of these preconditioners that exhibits scalability even for highly heterogeneous systems of equations. We consider several intricate structural analysis problems, and demonstrate numerically the optimal performance delivered by the new preconditioners for problems with discontinuities. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: domain decomposition; heterogeneities; preconditioning; scalability

1. INTRODUCTION

Iterative solvers, have slowly but finally made their debut in commercial finite element structural and thermal software.[1–3] The key factors for this change in culture have been (a) the pressing need for higher-fidelity finite element structural models with as many as five million degrees of freedom[4] and, for such large problems, the extreme demands placed on computer resources by direct solvers, (b) the significant progress achieved during the last decade in the development of fast and robust iterative algorithms for the solution of solid mechanics and plate and shell structural problems, and (c) the advent of commercial parallel hardware and the fact that iterative methods are more amenable to parallel processing than direct algorithms.

Early work on preconditioning the conjugate gradient method[5] has focused on incomplete Cholesky factorization procedures[6, 7] to provide a bridge that spans the gap between direct and iterative methods. Then, attention shifted to element-by-element preconditioning techniques[8]

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because these utilize the structure inherent in finite element formulations and implementations, and to (algebraic) multigrid algorithms[9, 10] because multigrid theories address the concept of numerical scalability—that is, the achievement of a convergence rate that is asymptotically independent of the size of the problem to be solved. With the advent of parallel processing, substructure-based iterative methods,[11] also known as Domain Decomposition (DD) methods,[12] emerged as powerful contenders for both sequential and parallel computing platforms.[13, 14] When equipped with an appropriate substructure level preconditioner, a DD method is usually scalable with respect to the mesh size h (or number of elements) of the given problem. In order to be scalable with respect to the substructure size H (or number of substructures), it must also be equipped with a 'coarse space' preconditioner[15] whose mathematical foundation is similar to that of the 'coarse grids' encountered in multigrid methods. It should be noted that scalability with respect to the number of substructures is a necessary condition for achieving parallel scalability—that is, delivering larger speedups when a larger number of processors is used for solving a given problem.

The FETI[16, 17] and related Balancing[18] methods are among the first non-overlapping DD methods that have demonstrated numerical scalability with respect to both the mesh and substructure sizes, for both second-order elasticity[19] and fourth-order plate and shell problems.[20–22] In particular, the parallel scalability of the FETI method and its ability to outperform several popular direct and iterative algorithms on both sequential and parallel computers have been extensively demonstrated.[2, 14]

The FETI method is a Lagrange multiplier based DD method. It can be viewed as a two-step Preconditioned Conjugate Gradient (PCG) iterative procedure where substructure problems with Dirichlet boundary conditions are solved in the preconditioning step, and related substructure problems with Neumann boundary conditions are solved in a second step. This method is also known as the dual Schur complement method because on the outset, it constructs a dual Schur complement operator.[14] When the FETI method is equipped with the so-called Dirichlet preconditioner[19] and with the optional corner Lagrange multipliers for plate[21] and shell[22] problems, and is applied to homogeneous structural problems, the condition number of its interface problem grows asymptotically only as

$$\kappa = O\left(1 + \log^m\left(\frac{H}{h}\right)\right), \quad m \leqslant 3 \tag{1}$$

which establishes the numerical scalability of this DD method with respect to both the mesh size h and the substructure size H.

The FETI method has been successfully extended to free-vibration analysis,[2] component mode synthesis,[23] transient response analysis,[24] and systems with multiple and/or repeated right hand sides.[25] It has also inspired many variants, among which we note those described in References 26 and 32. All these DD methods share the so-called Dirichlet and lumped preconditioners that were originally developed for the FETI method. For homogeneous second-order elasticity as well as fourth-order plate and shell problems, the Dirichlet preconditioner ensures scalability with respect to the mesh size h of most FETI-like DD methods. The lumped preconditioner is a more economical version of the Dirichlet preconditioner that, for homogeneous second-order elasticity problems, delivers a scalable computational performance if not a scalable iteration count.[14]

For arbitrarily heterogeneous problems, the optimal conditioning result (1) does not hold for most otherwise scalable DD methods, and the performance of the Dirichlet, lumped, and similar preconditioners can be expected to deteriorate.[33] Various techniques for preconditioning substructure problems with coefficient jumps can be found in the domain decomposition literature (for

example, see Reference 13.) However, most if not all of these techniques are applicable only to domain decomposed problems where each substructure is characterized by a single coefficient. In other words, to problems that are globally heterogeneous but locally homogeneous. Using a Rayleigh–Ritz based interface smoothing procedure,[34] we have made in Reference 33 a first attempt to extend the Dirichlet and lumped preconditioners to the case of arbitrarily heterogeneous structural problems. However, our initial attempt resulted in a rather computationally expensive preconditioning algorithm. In this paper, we simplify the approach described in Reference 33 and derive a virtually no-cost extension of the Dirichlet and lumped preconditioners to arbitrarily heterogeneous structural problems. We report some amazing performance results that demonstrate numerically that the FETI method equipped with the extended Dirichlet preconditioner is numerically scalable for arbitrarily heterogeneous structural problems.

The remainder of this paper is organized as follows. In Section 2, we overview the FETI method, the Dirichlet and lumped preconditioners, and an algebra for signed Boolean matrices. In Section 3, we introduce the concept of mechanically consistent preconditioners for homogeneous structural problems. In Section 3.1, we highlight the adverse effect of structural heterogeneities on the performance of the Dirichlet and lumped preconditioners. In Section 3.1, we construct a displacement-based interface smoothing procedure, and apply it to the extension of both preconditioners to arbitrarily heterogeneous problems. In Section 6, we demonstrate numerically the scalable convergence properties of the extended preconditioners, and in Section 8 we conclude this paper.

2. THE FETI METHOD FOR HOMOGENEOUS PROBLEMS

2.1. Substructuring with Lagrange multipliers

The solution of a problem of the form

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{2}$$

where $\mathbf{K}$ is a symmetric positive matrix arising from the discretization of some second- or fourth-order elliptic structural mechanics problem on a domain Ω, can be obtained by partitioning Ω into N_s substructures $\Omega^{(s)}$, and gluing these with discrete Lagrange multipliers $\boldsymbol{\lambda}$. Such an approach is governed by the saddle-point problem[†, 14, 19]

$$\delta\{\mathbf{v}^{(s)}, \boldsymbol{\mu}\}\mathscr{L}(\mathbf{v}^{(s)}, \boldsymbol{\mu}) = 0 \tag{3}$$

where

$$\mathscr{L}(\mathbf{v}^{(s)}, \boldsymbol{\mu}) = \sum_{s=1}^{N_s} \left(\tfrac{1}{2}\mathbf{v}^{(s)\mathrm{T}}\mathbf{K}^{(s)}\mathbf{v}^{(s)} - \mathbf{v}^{(s)\mathrm{T}}\mathbf{f}^{(s)} \right) + \boldsymbol{\mu}^{\mathrm{T}} \sum_{s=1}^{N_s} \mathbf{B}^{(s)}\mathbf{v}^{(s)} \tag{4}$$

$\mathbf{B}^{(s)}$ is a signed Boolean matrix constructed so that $\mathbf{B}^{(s)}\mathbf{v}^{(s)}$ is the restriction of the substructure displacement field $\mathbf{v}^{(s)}$ to the substructure interface boundary, and $\mathbf{f}^{(s)}$ is the substructure vector of

[†] Throughout this paper, matrices are denoted by uppercase and bold letters, and uni-column matrices by lowercase and bold letters. The T superscript designates a transpose and the (s) superscript specifies that the quantity pertains to the substructure $\Omega^{(s)}$.

specified forces. The stationarity condition on $\mathscr{L}$ yields the following set of equations:

$$\mathbf{K}^{(s)}\mathbf{u}^{(s)} + \mathbf{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda} = \mathbf{f}^{(s)}, \quad s = 1,\ldots, N_s \tag{5}$$

$$\sum_{s=1}^{s=N_s} \mathbf{B}^{(s)}\mathbf{u}^{(s)} = 0 \tag{6}$$

where $\mathbf{u}^{(s)}$ and $\boldsymbol{\lambda}$ are, respectively, the substructure displacements and the Lagrange multipliers satisfying the stationarity of $\mathscr{L}$. Equations (5) state that the substructure problems are in equilibrium when the pointwise Lagrange multipliers $\boldsymbol{\lambda}$ are applied on the interface boundaries. Equation (6) states that the substructure displacement fields are compatible on the interface boundaries. The system (5)–(6) defines the Finite Element Tearing and Interconnecting (FETI) version of domain decomposition: a given domain is first torn into subdomains, then these are 'glued' together by applying appropriate forces, i.e. Lagrange multipliers, between connecting degrees of freedom (d.o.f.).

For each substructure $\Omega^{(s)}$, the displacement $\mathbf{u}^{(s)}$ can be extracted from equation (5) as follows

$$\mathbf{u}^{(s)} = \mathbf{K}^{(s)^{+}}\left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda}\right) + \mathbf{R}^{(s)}\boldsymbol{\alpha}^{(s)} \tag{7}$$

where $\mathbf{K}^{(s)^{+}}$ denotes the inverse of $\mathbf{K}^{(s)}$ if $\Omega^{(s)}$ is not a floating substructure, or a generalized inverse of $\mathbf{K}^{(s)}$ if $\Omega^{(s)}$ is a floating substructure—that is, a substructure without sufficient essential boundary conditions to prevent $\mathbf{K}^{(s)}$ from being singular. In the latter case, $\mathbf{R}^{(s)} = \mathrm{Ker}(\mathbf{K}^{(s)})$ stores a basis of the null space of $\mathbf{K}^{(s)}$ (these are the substructure rigid-body modes), and $\boldsymbol{\alpha}^{(s)}$ stores the amplitudes specifying the contribution of the null space of $\mathbf{K}^{(s)}$ to the solution $\mathbf{u}^{(s)}$. However, if $\mathbf{K}^{(s)}$ is singular, then $(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda})$ must be in the range of $\mathbf{K}^{(s)}$, and, therefore,

$$\mathbf{R}^{(s)^{\mathrm{T}}}\left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda}\right) = 0 \tag{8}$$

The above condition expresses that, for each floating substructure, the specified forces and interface Lagrange multipliers must be self-equilibrated. Substituting equation (7) into the interface compatibility condition (6) and recalling condition (8), the interface problem can be re-written in terms of the Lagrange multipliers and the local null space coefficients as[16]

$$\begin{bmatrix} \mathbf{F}_I & -\mathbf{G}_I \\ -\mathbf{G}_I{}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ -\mathbf{e} \end{bmatrix} \tag{9}$$

where

$$\begin{aligned} \mathbf{F}_I &= \sum_{s=1}^{s=N_s} \mathbf{B}^{(s)}\mathbf{K}^{(s)^{+}}\mathbf{B}^{(s)^{\mathrm{T}}}; \quad \mathbf{d} = \sum_{s=1}^{s=N_s} \mathbf{B}^{(s)}\mathbf{K}^{(s)^{+}}\mathbf{f}^{(s)} \\ \mathbf{G}_I &= \left[\, \mathbf{B}^{(1)}\mathbf{R}^{(1)} \;\ldots\; \mathbf{B}^{(N_s)}\mathbf{R}^{(N_s)} \,\right]; \quad \alpha = \left[\, \alpha^{(1)^{\mathrm{T}}} \;\ldots\; \alpha^{(N_s)^{\mathrm{T}}} \,\right]^{\mathrm{T}} \\ \mathbf{e} &= \left[\, \mathbf{f}^{(1)^{\mathrm{T}}}\mathbf{R}^{(1)} \;\ldots\; \mathbf{f}^{(N_s)^{\mathrm{T}}}\mathbf{R}^{(N_s)} \,\right]^{\mathrm{T}} \end{aligned} \tag{10}$$

The dual interface problem defined by equation (9) is a saddle-point problem that reflects the manner by which the substructures interconnect. A parallel substructure-by-substructure methodology for solving this interface problem calls for an iterative solver. In the FETI method, a Preconditioned Conjugate Projected Gradient (PCPG) algorithm is employed for this purpose. More specifically,

Table I. The FETI PCPG method

1. Initialize

$$\boldsymbol{\lambda}^0 = \mathbf{G}_I\ (\mathbf{G}_I{}^{\mathrm{T}}\mathbf{G}_I)^{-1}\,\mathrm{e}$$
$$\mathbf{r}^0 = \left(\mathrm{d} - \mathbf{F}_I\boldsymbol{\lambda}^0\right)$$

2. Iterate $k = 1,\ 2,\ \ldots$ until convergence

$$\begin{aligned}
\text{Project } w^{k-1} &= \mathbf{P}^{\mathrm{T}}\mathbf{r}^{k-1}\\
\text{Precondition } \mathbf{z}^{k-1} &= \overline{\mathbf{F}}_I^{-1}\mathbf{w}^{k-1}\\
\text{Re-project } \mathbf{y}^{k-1} &= \mathbf{P}\mathbf{z}^{k-1}\\
\text{Conjugate } \zeta^k &= \mathbf{y}^{k-1^{\mathrm{T}}T}\mathbf{w}^{k-1}\ /\ \mathbf{y}^{k-2^{\mathrm{T}}}\mathbf{w}^{k-2} \quad (\zeta^1 = 0)\\
\mathbf{p}^k &= \ \mathbf{y}^{k-1} + \zeta^k\mathbf{p}^{k-1}\\
\text{Minimize } \eta^k &= \ \mathbf{p}^{k-1^{\mathrm{T}}}\mathbf{w}^{k-1}/\mathbf{p}^{k^{\mathrm{T}}}\mathbf{F}_I\mathbf{p}^k\\
\text{Update } \boldsymbol{\lambda}^k &= \boldsymbol{\lambda}^{k-1} + \eta^k\mathbf{p}^k\\
\mathbf{r}^k &= \mathbf{r}^{k-1} - \eta^k\mathbf{F}_I\mathbf{p}^k
\end{aligned}$$

the interface problem (9) is transformed into a semi-definite system of equations by the means of a projector $\mathbf{P}$. The objective of this projector is to enforce at each iteration the constraint $\mathbf{G}_I^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{e}$. Its general form is given by[14]

$$\mathbf{P} = \mathbf{I} - \mathbf{Q}\mathbf{G}_I(\mathbf{G}_I^{\mathrm{T}}\mathbf{Q}\mathbf{G}_I)^{-1}\mathbf{G}_I^{\mathrm{T}} \tag{11}$$

where $\mathbf{Q}$ is a full column rank matrix. Decomposing the Lagrange multipliers as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \mathbf{P}\overline{\boldsymbol{\lambda}} \tag{12}$$

where

$$\boldsymbol{\lambda}_0 = \mathbf{Q}\mathbf{G}_I(\mathbf{G}_I^{\mathrm{T}}\mathbf{Q}\mathbf{G}_I)^{-1}\mathbf{e} \tag{13}$$

and using the projector $\mathbf{P}$ defined above, the interface problem (9) can be transformed into

$$(\mathbf{P}^{\mathrm{T}}\mathbf{F}_I\mathbf{P})\ \overline{\boldsymbol{\lambda}} = \mathbf{P}^{\mathrm{T}}(\mathbf{d} - \mathbf{F}_I\boldsymbol{\lambda}_0) \tag{14}$$

$$\alpha = (\mathbf{G}_I^{\mathrm{T}}\mathbf{Q}\mathbf{G}_I)^{-1}\mathbf{G}_I^{\mathrm{T}}\mathbf{Q}(\mathbf{d} - \mathbf{F}_I\boldsymbol{\lambda}) \tag{15}$$

Hence, a solution of the original indefinite system of interface equations (9) can be obtained by applying a PCG algorithm to the alternative symmetric semi-definite interface problem (14). Such a procedure can also be viewed as a PCPG algorithm. It is summarized in Table I where $\mathbf{F}_I^{-1}$ denotes the chosen preconditioner.

Remarks:

(a) In practice, $\mathbf{Q}$ is set to the identity matrix $\mathbf{I}$, in which case $\mathbf{P}^{\mathrm{T}} = \mathbf{P}$ (for a discussion on the choice of $\mathbf{Q}$, see. References 14 and 35)
(b) In Table I, the projector $\mathbf{P}$ is applied to the directions of descent to enforce $\mathbf{G}_I^{\mathrm{T}}\mathbf{p}^k = \mathbf{0}$. This ensures that at each iteration k, $\mathbf{G}_I^{\mathrm{T}}\boldsymbol{\lambda}^k = \mathbf{G}_I^{\mathrm{T}}\boldsymbol{\lambda}^0 = \mathbf{e}$.
(c) The projected residual $\mathbf{w}^k$ can be written as

$$\begin{aligned}
\mathbf{w}^k &= \mathbf{P}^{\mathrm{T}}\left(\mathbf{d} - \mathbf{F}_I\boldsymbol{\lambda}^k\right)\\
&= \mathbf{d} - \mathbf{F}_I\boldsymbol{\lambda}^k - \mathbf{G}_I\alpha^k = \sum_{s=1}^{N_s}\mathbf{B}^{(s)}\mathbf{u}^{(s)^k}
\end{aligned} \tag{16}$$

where

$$\alpha^k = (\mathbf{G}_I^{\mathrm{T}}\mathbf{G}_I)^{-1}\mathbf{G}_I^{\mathrm{T}}\left(\mathbf{d} - \mathbf{F}_I\boldsymbol{\lambda}^k\right) \tag{17}$$

Hence $\mathbf{w}^k$ is the residual of the dual interface problem (9). It has the physical meaning of the jump of the displacement field across the substructure interface boundaries. The reader can check that at each iteration k, $\mathbf{w}^k$ satisfies $\mathbf{G}_I^{\mathrm{T}}\mathbf{w}^k = 0$. In other words, a weak form of the compatibility of the displacement field is enforced at each iteration. This weak form requires solving at each iteration the auxiliary problem (17). This auxiliary problem is called a coarse problem because, in general, its size is less than $6 \times N_{\mathrm{f}}$, where $N_{\mathrm{f}} \leqslant N_{\mathrm{s}}$ denotes the number of floating substructures. The coarse problem (17) couples all the substructures and was shown to be the main reason for the scalability of the FETI and related Balancing methods.[16, 18, 19]

(d) In general, the dual interface problem is better conditioned than the global problem, and therefore its residual $\mathbf{w}^k$ converges to zero faster than the global residual $\mathbf{K}\mathbf{u}^k - \mathbf{f}$. Hence, it is important to monitor the global residual $\mathbf{K}\mathbf{u}^k - \mathbf{f}$ for establishing the convergence of the FETI method. In practice, $\mathbf{K}\mathbf{u}^k - \mathbf{f}$ is not explicitly computed; rather, a high-fidelity estimator that requires little additional computation is evaluated.[14]

(e) In practice, the FETI solution algorithm summarized in Table I is always used with a reorthogonalization procedure to accelerate convergence. In Reference 14, it was shown that such a strategy is cost-effective for substructure problems because reorthogonalization is applied only to the interface Lagrange multiplier unknowns.

(f) For plate and shell problems, the FETI PCPG algorithm summarized in Table I can be improved by inserting after the 'Re-project' step another computational step where an additional set of Lagrange multipliers $\boldsymbol{\lambda}_{\mathrm{C}}$ are computed and superposed to the basic Lagrange multipliers $\boldsymbol{\lambda}$. The purpose of the additional multipliers $\boldsymbol{\lambda}_{\mathrm{C}}$ is to enforce at each iteration k the continuity of the displacement field at the substructure crosspoints.[21, 22]

2.2. *Substructure level preconditioners*

Two substructure-by-substructure preconditioners have been previously developed for the FETI method: (a) a numerically scalable Dirichlet preconditioner that can be written as[19]

$$\overline{\mathbf{F}}_I^{D^{-1}} = \sum_{s=1}^{s=N_s} \mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}} \tag{18}$$

where $\mathbf{S}_{bb}^{(s)}$ is the Schur complement of substructure $\Omega^{(s)}$

$$\mathbf{S}_{bb}^{(s)} = \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{ib}^{(s)^{\mathrm{T}}}\mathbf{K}_{ii}^{(s)^{-1}}\mathbf{K}_{ib}^{(s)} \tag{19}$$

and the subscripts i and b designate the internal and interface boundary unknowns, respectively, and (b) a 'lumped' preconditioner that lumps the Dirichlet operator on the substructure interface unknowns[16, 17]

$$\overline{\mathbf{F}}_I^{L^{-1}} = \sum_{s=1}^{s=N_s} \mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}} \tag{20}$$

Note that in equations (18) and (20) we have implicitly assumed that the internal d.o.f. are numbered first in order to simplify the notations. Unlike $\overline{\mathbf{F}}_I^{D^{-1}}$, the lumped preconditioner $\overline{\mathbf{F}}_I^{L^{-1}}$ is

not numerically scalable. However, it is more economical than the Dirichlet preconditioner and has often proved to be more efficient for second-order elasticity problems.[14, 19] For fourth-order plate and shell problems, the Dirichlet preconditioner is more efficient.[21, 22]

The mechanical interpretations of the FETI method and the Dirichlet preconditioner are straightforward. At each iteration k of the FETI solver, a new traction field $\boldsymbol{\lambda}^k$ is computed. Then, for each substructure $\Omega^{(s)}$, a load vector equal to $\mathbf{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda}^k$ is imposed on its interface boundary, and the resulting substructure displacement field $\mathbf{u}^{(s)^k}$ is obtained from the solution of the corresponding equilibrium equation (5). Except at convergence, the substructure solutions $\mathbf{u}^{(s)^k}$ are not compatible on the substructure interface boundaries, and their jump is evaluated during the computation of the projected residual $\mathbf{w}^k$

$$\mathbf{w}^k = \sum_{s=1}^{s=N_s} \mathbf{B}^{(s)}\mathbf{u}^{(s)^k} = \mathbf{P}^{\mathrm{T}}\mathbf{r}^k$$

The preconditioning of $\mathbf{w}^k$ by the Dirichlet operator corresponds to an inverse sequence of local manipulations. After the jump of the computed displacement field is evaluated between two connected substructures, it is imposed as a displacement boundary condition on the corresponding substructure interface boundaries, and a Dirichlet problem is solved in each substructure to evaluate the traction field necessary for maintaining such a jump. In other words, the Dirichlet preconditioner approximates the inverse of the sum (sum over the substructures) by the sum of the inverses. The lumped preconditioner has the same mechanical interpretation as the Dirichlet preconditioner, except that it assumes that all the stiffness of a substructure is lumped at its interface boundary. In Reference 14, it was shown that when either of these two preconditioners is employed, the preconditioned projected residual $\mathbf{z}^k$ becomes an excellent estimator of the global residual $\mathbf{K}\mathbf{u}^k - \mathbf{f}$, so that the convergence of the FETI method can be simply monitored by

$$\|\mathbf{z}^k\| < \varepsilon \, \|\mathbf{f}\|$$

where ε is a specified threshold.

The Dirichlet and lumped preconditioners are used in several other domain decomposition methods besides FETI. The main objective of this paper is to formulate numerically scalable extensions for heterogeneous problems.

2.3. *Definition and algebra of a class of signed Boolean matrices*

Let $\Gamma_I^{(s)}$ denote the interface boundary of a substructure $\Omega^{(s)}$, Γ_I the union of all substructure interface boundaries, and $\mathbf{b}^{(s)}$ the restriction of the signed boolean matrix $\mathbf{B}^{(s)}$ to $\Gamma_I^{(s)}$. For simplicity, we assume that all interface boundary d.o.f. are numbered last, and therefore $\mathbf{B}^{(s)}$ can be written as

$$\mathbf{B}^{(s)} = \begin{bmatrix} 0 & \mathbf{b}^{(s)} \end{bmatrix} \tag{21}$$

The interface boundary Γ_I can be partitioned into interface edges Γ_I^j using the following guidelines:

(1) *An interface edge is defined as a collection of connecting interface nodes.*
(2) *Each interface node is assigned to one and only one interface edge.*

Note that the above characterization of the edges is independent of the dimensionality of the interface. The crosspoint that is often referred to in the domain decomposition literature as a

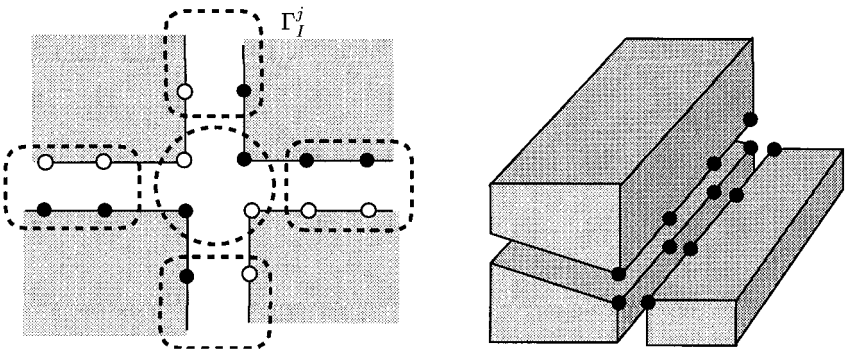

Figure 1. Edges and crosspoints for two- and three-dimensional substructure problems

'vertex' for two-dimensional problems is treated here as an interface edge with a single interface node (Figure 1). For three-dimensional problems, the crosspoint is an interface edge with an arbitrary number of interface nodes. In the sequel, the prefix interface is dropped for simplicity, and an interface edge is called an edge.

It follows that $\mathbf{b}^{(s)}$ can be partitioned as

$$\mathbf{b}^{(s)} = \left[\mathbf{b}^{(s),i} \ \mathbf{s}^{(s),j} \ \cdots \ \mathbf{b}^{(s),l} \right] \tag{22}$$

where $\mathbf{b}^{(s),j}$ is the restriction of $\mathbf{b}^{(s)}$ to Γ_I^j. From the definition of an edge Γ_I^j, it also follows that a Lagrange multiplier connects two d.o.f. that belong to a same edge and hence

$$\mathbf{b}^{(s),j^{\mathrm{T}}} \mathbf{b}^{(s),i} = 0, \quad j \neq i \tag{23}$$

Moreover, an edge Γ_I^j can be shared by multiple substructures (see Figure 1), and therefore $\mathbf{b}^{(s),j}$ can be partitioned as

$$\mathbf{b}^{(s),j} = \begin{bmatrix} 0 \\ \vdots \\ \mathbf{b}^{(sq),j} \\ 0 \\ \mathbf{b}^{(sr),j} \\ \vdots \\ 0 \end{bmatrix} \tag{24}$$

where $\mathbf{b}^{(sq),j}$ is the square submatrix of $\mathbf{b}^{(s),j}$ pertaining to the interconnectivity of substructures $\Omega^{(s)}$ and $\Omega^{(q)}$ along Γ_I^j.

In order to keep all subsequent discussions as simple as possible, it is assumed in the sequel that the d.o.f. attached to an edge have the same local ordering in every subdomain they belong to. This assumption affects only the notation and computer implementation of the ideas discussed in this paper. It does not restrict in any manner the scope of the proposed algorithms. With this assumption, the equation describing the compatibility of the displacement field along an edge Γ_I^j between two neighbouring substructures $\Omega^{(s)}$ and $\Omega^{(q)}$ can be written as

$$\mathbf{u}^{(s),j} = \mathbf{u}^{(q),j} \tag{25}$$

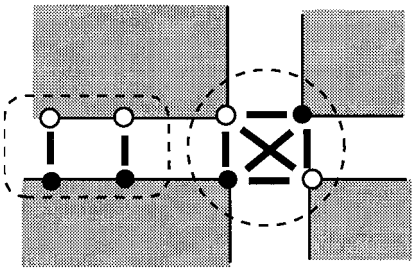

Figure 2. Redundant compatibility constraints on crosspoints

Next, we define the *multiplicity* m_j of an edge Γ_I^j as *the number of substructures it interconnects*. Hence, in general $m_j \geqslant 2$, and for a crosspoint $m_j > 2$. Given that in all FETI-like DD methods one and only one Lagrange multiplier is used to glue any pair of d.o.f. connecting on an edge Γ_I^j, there are exactly $(m_j - 1)$ Lagrange multipliers that are applied to each d.o.f. lying on an edge Γ_I^j (Figure 2). Recalling equation (24) and noting that $\mathbf{b}^{(sq),j^{\mathrm{T}}}\mathbf{b}^{(sq),j} = \mathbf{I}$, it follows that

$$\mathbf{b}^{(s),j^{\mathrm{T}}}\mathbf{b}^{(s),j} = (m_j - 1)\mathbf{I} \tag{26}$$

Finally, using equations (22) and (23), equation (26) can be re-written as

$$\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{(s)} + \mathbf{I} = \mathrm{diag}(m_j) \tag{27}$$

where $\mathrm{diag}(m_j)$ is the diagonal matrix of the multiplicity of the edges $\Gamma_I^j \subset \{\Gamma_I \bigcap \Gamma_I^{(s)}\}$.

Remark. Gluing each pair of connecting d.o.f. by a Lagrange multiplier introduces redundancies in the compatibility constraints at the crosspoints (see Figure 2). However, it will be argued later in this paper that keeping all redundant constraints in the definition of the interface problem is essential for constructing an efficient preconditioner.

3. MECHANICALLY CONSISTENT PRECONDITIONERS

As stated earlier, the projected residual $\mathbf{w}$ has the physical meaning of the jump of the displacement field across the substructure interface boundaries

$$\mathbf{w} = \textstyle\sum_{s=1}^{N_s} \mathbf{b}^{(s)}\mathbf{u}_b^{(s)} \tag{28}$$

Hence, from a mechanical viewpoint, the objective of a substructure-by-substructure preconditioner $\overline{\mathbf{F}}_I^{-1}$ can be stated as that of generating a Lagrange multiplier correction $\mathbf{z}$ and the corresponding substructure interface forces $\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z} = \mathbf{b}^{(s)^{\mathrm{T}}}[\overline{\mathbf{F}}_I^{-1}\mathbf{w}]$ that reduce as much as possible the jump $\mathbf{w}$.

Preconditioning the projected residual $\mathbf{w}$ by the Dirichlet (18) or lumped (20) operators yields

$$\mathbf{z} = \overline{\mathbf{F}}_I^{-1}\mathbf{w} = \sum_{s=1}^{N_s} \mathbf{b}^{(s)} \left(\mathbf{S}_{bb}^{(s)} \text{ or } \mathbf{K}_{bb}^{(s)} \right) \mathbf{b}^{(s)^{\mathrm{T}}} \mathbf{w} \tag{29}$$

The above expression of $\mathbf{z}$ suggests that preconditioning by the substructure-by-substructure Dirichlet and lumped operators can be understood as a three-step procedure for building a correction of the Lagrange multiplier field

1. First, displacement corrections $\Delta\mathbf{u}_b^{(s)}$ are imposed on the subdomain interfaces $\Gamma_I^{(s)}$ as follows

$$\Delta\mathbf{u}_b^{(s)} = \mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{w} \tag{30}$$

   This means that at every interface d.o.f., a displacement correction equal to the sum of the displacement jumps with neighboring d.o.f. is imposed.
2. Next, the corresponding interface nodal forces $\Delta\mathbf{f}_b^{(s)}$ are evaluated as

$$\Delta\mathbf{f}_b^{(s)} = \left(\mathbf{S}_{bb}^{(s)} \text{ or } \mathbf{K}_{bb}^{(s)} \right) \Delta\mathbf{u}_b^{(s)} \tag{31}$$

   Note that when the lumped operator $\mathbf{K}_{bb}^{(s)}$ is used instead of the Schur complement $\mathbf{S}_{bb}^{(s)}$, the computation of the interface forces $\Delta\mathbf{f}_b^{(s)}$ assumes implicitly that the internal d.o.f. are fixed.
3. Finally, the jump of interface nodal forces $\Delta\mathbf{f}_b^{(s)}$ are computed to obtain the Lagrange multiplier correction $\mathbf{z}$

$$\mathbf{z} = \sum_{s=1}^{N_s} \mathbf{b}^{(s)} \Delta\mathbf{f}_b^{(s)} \tag{32}$$

In this paper, we define a *mechanically consistent* preconditioner—that is, a preconditioner that respects mechanical principles—as a substructure-by-substructure preconditioner that fits the description of the above three-step procedure, but where the displacement increments $\Delta\mathbf{u}_b^{(s)}$ and Lagrange multiplier corrections $\mathbf{z}$ are chosen based on compatibility and near equilibrium concepts.

The steps for building a mechanically consistent preconditioner are graphically depicted in Figure 3. Figure 3(a) and (b) shows displacement increments $\Delta\mathbf{u}_b^{(s)}$ are constructed so that the corrected interface displacements $\tilde{\mathbf{u}}_b^{(s)}$ satisfy the interface compatibility. Figure 3(c) shows the nodal forces on the interface required to maintain the displacement increment $\Delta\mathbf{u}_b^{(s)}$ are computed for each subdomain as in equation (31). These forces $\Delta\mathbf{f}_b^{(s)}$ do not satisfy the interface equilibrium unless the corrected displacements $\tilde{\mathbf{u}}_b^{(s)}$ are the exact solution. Figure 3(d) is based on $\Delta\mathbf{f}_b^{(s)}$, Lagrange multipliers corrections $\mathbf{z}$ are defined. Interface Lagrange multipliers are naturally self-equilibrated in the sense that they result in interface forces $\mathbf{b}(s)^{\mathrm{T}}\mathbf{z}$ that are in equilibrium (Figure 3(e)). Hence, it is in general impossible to define $\mathbf{z}$ such that $\mathbf{b}(s)^{\mathrm{T}}\mathbf{z}$ restores the interface force corrections $\Delta\mathbf{f}_b^{(s)}$. Nevertheless, we require that $\mathbf{z}$ be constructed in such a way that as the $\Delta\mathbf{f}_b^{(s)}$ approach equilibrium, $\mathbf{b}(s)^{\mathrm{T}}\mathbf{z}$ exactly results in $\Delta\mathbf{f}_b^{(s)}$.

We therefore state that a mechanically consistent preconditioner implicitly results in

(1) *mechanically consistent displacement increments* $\Delta\mathbf{u}_b^{(s)}$ *such that*

$$\sum_{s=1}^{N_s} \mathbf{b}^{(s)} \tilde{\mathbf{u}}_b^{(s)} = \sum_{s=1}^{N_s} \mathbf{b}^{(s)} \left(\mathbf{u}_b^{(s)} + \Delta\mathbf{u}_b^{(s)}(\mathbf{w}) \right) = 0 \tag{33}$$

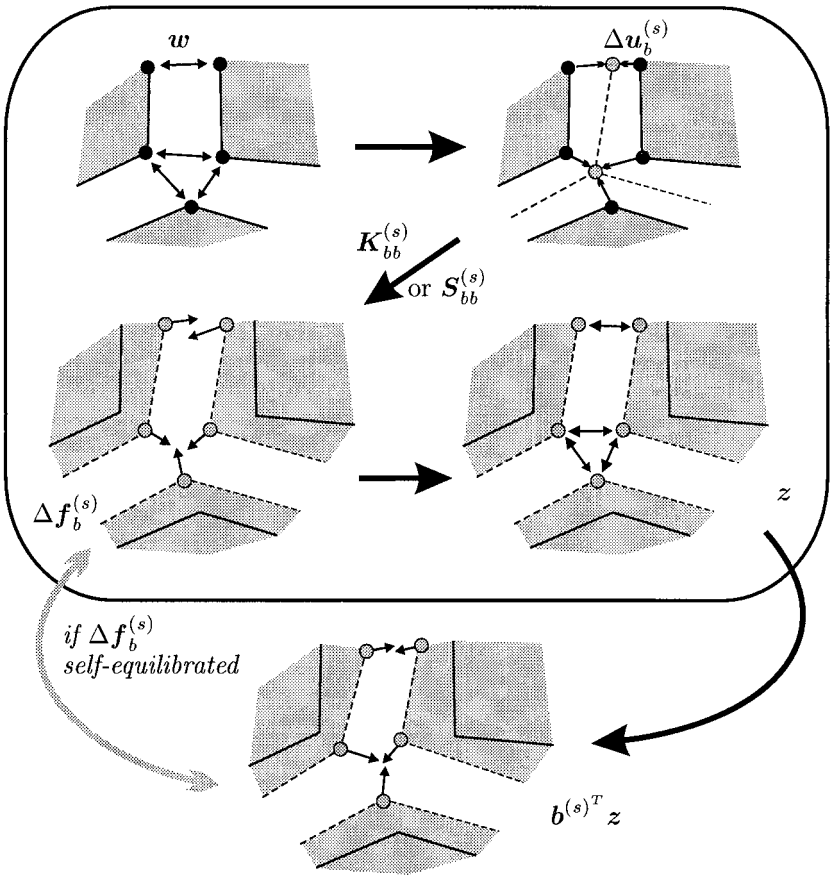

Figure 3. Mechanically consistent preconditioners

(2) *mechanically consistent Lagrange multiplier corrections* $\mathbf{z}$ such that

$$\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z}(\Delta\mathbf{f}_b^{(s)}) = \Delta\mathbf{f}_b^{(s)} \tag{34}$$

if

$$\Delta\mathbf{f}_b^{(s)} - \mathbf{b}^{(s)^{\mathrm{T}}} \sum_{\substack{r=1 \\ r\neq s}}^{N_s} \mathbf{b}^{(r)} \Delta\mathbf{f}_b^{(r)} = 0 \tag{35}$$

Equation (35) expresses that the sum of all forces acting on an interface d.o.f. is zero and is therefore the expression of interface equilibrium.

One can reasonably expect that for iterative substructuring methods, mechanically consistent preconditioners will deliver a better numerical performance than mechanically inconsistent ones. For this reason, we show next how, for homogeneous problems, the Dirichlet and lumped preconditioners can be modified to satisfy the two mechanical consistency requirements, which will pave the way for addressing heterogeneous problems.

Remark:

(a) The basic Dirichlet and lumped preconditioners are not mechanically consistent. Indeed, if $\Delta\mathbf{u}_b^{(s)}$ is constructed as in equation (30), then

$$\sum_{s=1}^{N_s}\mathbf{b}^{(s)}\left(\mathbf{u}_b^{(s)}+\Delta\mathbf{u}_b^{(s)}\right)=\mathbf{w}+\sum_{s=1}^{N_s}\mathbf{b}^{(s)}\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{w}\neq 0$$

which violates the mechanical consistency condition for the displacement field correction. Furthermore, if equation (35) holds and $\mathbf{z}$ is constructed as in equation (32), then

$$\begin{aligned}\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z}^k &= \mathbf{b}^{(s)^{\mathrm{T}}}\sum_{r=1}^{N_s}\mathbf{b}^{(r)}\Delta\mathbf{f}_b^{(r)}\\ &= \mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{s}\Delta\mathbf{f}_b^{(s)}+\mathbf{b}^{(s)^{\mathrm{T}}}\sum_{\substack{r=1\\ r\neq s}}^{N_s}\mathbf{b}^{(r)}\mathbf{f}_b^{(r)}\\ &= (\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{(s)}+\mathbf{I})\Delta\mathbf{f}_b^{(s)}\end{aligned}$$

and according to equation (27)

$$\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z}^k=\mathrm{diag}(m_j)\Delta\mathbf{f}_b^{(s)}\neq\Delta\mathbf{f}_b^{(s)} \tag{36}$$

(b) The natural self-equilibrium of the Lagrange multipliers can be expressed by substituting $(\mathbf{b}^{(r)^{\mathrm{T}}}\mathbf{z})$ for $\Delta\mathbf{f}_b^{(r)}$ in equation (35)

$$\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z}-\mathbf{b}^{(s)^{\mathrm{T}}}\sum_{\substack{r=1\\ r\neq s}}^{N_s}\mathbf{b}^{(r)}\mathbf{b}^{(r)^{\mathrm{T}}}\mathbf{z}=0 \tag{37}$$

Hence, we find that

$$\mathbf{b}^{(s)^{\mathrm{T}}}=\mathbf{b}^{(s)^{\mathrm{T}}}\sum_{\substack{r=1\\ r\neq s}}^{N_s}\mathbf{b}^{(r)}\mathbf{b}^{(r)^{\mathrm{T}}} \tag{38}$$

which is a direct consequence of the fact that compatibility constraints are defined between any pair of connecting d.o.f. (see Section 2.3).

3.1. Mechanically consistent preconditioners for homogeneous problems

For homogeneous problems, mechanically consistent preconditioners can be constructed in a straightforward manner using a simple averaging scheme. For the sake of clarity, we first describe the averaging scheme for the four-substructure problem depicted graphically in Figure 4. Then, we generalize it to arbitrary mesh decompositions.

All four substructures of the problem depicted in Figure 4 are assumed to have similar material, geometrical, and discretization properties, and therefore similar stiffnesses. Hence, it is mechanically sound to construct for this problem a compatible interface displacement as follows:

$$\tilde{\mathbf{u}}_1^{(1)}=\tilde{\mathbf{u}}_1^{(2)}=\tilde{\mathbf{u}}_1^{(3)}=\tilde{\mathbf{u}}_1^{(4)}=\frac{\mathbf{u}_1^{(1)}+\mathbf{u}_1^{(2)}+\mathbf{u}_1^{(3)}+\mathbf{u}_1^{(4)}}{4}$$

$$\tilde{\mathbf{u}}_2^{(1)}=\tilde{\mathbf{u}}_2^{(2)}=\frac{\mathbf{u}_2^{(1)}+\mathbf{u}_2^{(2)}}{2}$$

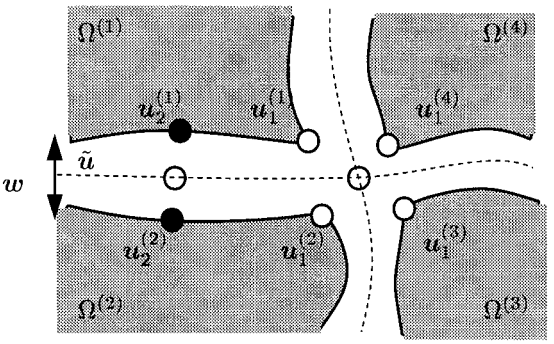

Figure 4. A four-substructure homogeneous problem

which implies that the following substructure displacement corrections are mechanically consistent:

$$\Delta\mathbf{u}_1^{(1)} = \tilde{\mathbf{u}}_1^{(1)} - \mathbf{u}_1^{(1)}$$

$$= \frac{\left(\mathbf{u}_1^{(2)} - \mathbf{u}_1^{(1)}\right) + \left(\mathbf{u}_1^{(3)} - \mathbf{u}_1^{(1)}\right) + \left(\mathbf{u}_1^{(4)} - \mathbf{u}_1^{(1)}\right)}{4}$$

$$\Delta\mathbf{u}_1^{(2)} = \frac{\left(\mathbf{u}_1^{(1)} - \mathbf{u}_1^{(2)}\right) + \left(\mathbf{u}_1^{(3)} - \mathbf{u}_1^{(2)}\right) + \left(\mathbf{u}_1^{(4)} - \mathbf{u}_1^{(2)}\right)}{4}$$

$$\cdots$$

$$\Delta\mathbf{u}_2^{(1)} = \frac{\left(\mathbf{u}_2^{(2)} - \mathbf{u}_2^{(1)}\right)}{2}$$

$$\Delta\mathbf{u}_2^{(2)} = \frac{\left(\mathbf{u}_2^{(1)} - \mathbf{u}_2^{(2)}\right)}{2} \tag{39}$$

If the interface nodal forces associated with the above substructure displacement corrections are computed as in equation (31), these forces will not be in general in equilibrium. For example, on the interface edge of multiplicity 2 (see Figure 5), $\Delta\mathbf{f}_2^{(1)} \neq \Delta\mathbf{f}_2^{(2)}$. For this reason, the Lagrange multiplier correction is constructed here by averaging the computed interface forces as follows:

$$\mathbf{z}_7 = \frac{-\Delta\mathbf{f}_2^{(1)} + \Delta\mathbf{f}_2^{(2)}}{2} \tag{40}$$

The minus sign in equation (40) is due to the fact that the force corrections $\Delta\mathbf{f}_2^{(1)}$ and $\Delta\mathbf{f}_2^{(2)}$ have opposite directions because by convention, the entries of $\mathbf{b}^{(sq),j}$ are set to $+1$ if $s > q$ and to -1 otherwise. For the crosspoint of multiplicity 4 (see Figure 5), the Lagrange multiplier correction can be constructed as

$$\mathbf{z}_1 = \frac{-\Delta\mathbf{f}_1^{(1)} + \Delta\mathbf{f}_1^{(2)}}{4}, \quad \mathbf{z}_2 = \frac{-\Delta\mathbf{f}_1^{(2)} + \Delta\mathbf{f}_1^{(3)}}{4}$$

$$\mathbf{z}_3 = \frac{-\Delta\mathbf{f}_1^{(3)} + \Delta\mathbf{f}_1^{(4)}}{4}, \quad \mathbf{z}_4 = \frac{-\Delta\mathbf{f}_1^{(1)} + \Delta\mathbf{f}_1^{(4)}}{4}$$

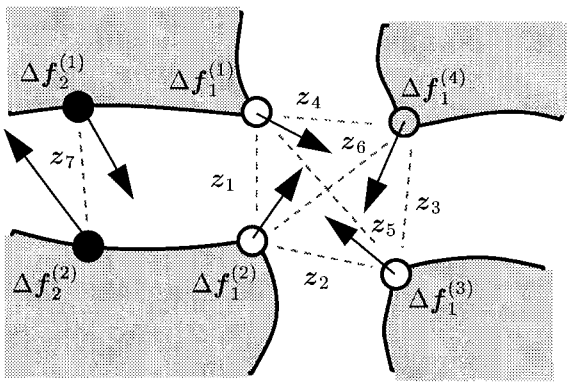

Figure 5. Lagrange multiplier corrections on homogeneous interfaces

$$\mathbf{z}_5 = \frac{-\Delta\mathbf{f}_1^{(1)} + \Delta\mathbf{f}_1^{(3)}}{4}, \quad \mathbf{z}_6 = \frac{-\Delta\mathbf{f}_1^{(2)} + \Delta\mathbf{f}_1^{(4)}}{4} \tag{41}$$

which completes the evaluation of $\mathbf{z}$.

The Lagrange multiplier correction defined by equations (40) and (41) is mechanically consistent. Indeed, from equations (40), (41) and (43) it follows that

$$\begin{aligned} \mathbf{b}^{(1)^{\mathrm{T}}}\mathbf{z} &= \begin{bmatrix} -\mathbf{z}_1 - \mathbf{z}_4 - \mathbf{z}_5 \\ -\mathbf{z}_7 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{4}\Delta\mathbf{f}_1^{(1)} - \frac{1}{4}\left(\Delta\mathbf{f}_1^{(2)} + \Delta\mathbf{f}_1^{(3)} + \Delta\mathbf{f}_1^{(4)}\right) \\ \frac{\Delta\mathbf{f}_2^{(1)} - \Delta\mathbf{f}_2^{(2)}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \Delta\mathbf{f}_1^{(1)} \\ \Delta\mathbf{f}_2^{(1)} \end{bmatrix} \end{aligned} \tag{42}$$

if

$$\sum_{s=1}^{4} \Delta\mathbf{f}_1^{(s)} = 0 \quad \text{and} \quad \sum_{s=1}^{2} \Delta\mathbf{f}_2^{(s)} = 0 \tag{43}$$

The reader can check that a similar result holds for all other substructures.

Hence, for the four-substructure problem depicted in Figure 4, the averaging procedure described herein shows how to construct mechanically consistent Dirichlet and lumped preconditioners. For homogeneous problems with an arbitrary number and configuration of substructures, the averaging procedure can be generalized as follows. On each edge Γ_I^i, a compatible displacement field $\tilde{\mathbf{u}}_b^{(s),i}$ can be constructed by averaging the displacements of all nodes belonging to Γ_I^i

$$\tilde{\mathbf{u}}_b^{(s),i} = \sum_{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\}} \frac{1}{m_i} \mathbf{u}_b^{(r),i} \tag{44}$$

where, as before, m_i is the multiplicity of edge Γ_I^i. This leads to the following mechanically consistent substructure displacement corrections:

$$\begin{aligned}\Delta\mathbf{u}_b^{(s),i} &= \tilde{\mathbf{u}}_b^{(s),i} - \mathbf{u}_b^{(s),i} \\ &= \sum_{\substack{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\} \\ r \neq s}} \frac{1}{m_i}(\mathbf{u}_b^{(r),i} - \mathbf{u}_b^{(s),i})\end{aligned} \tag{45}$$

The above equality can be written in global form as

$$\Delta\mathbf{u}_b^{(s)} = -\mathbf{b}^{(s)^{\mathrm{T}}} \operatorname{diag}\left(\frac{1}{m_i}\right) \sum_{r=1}^{N_s} \mathbf{b}^{(r)}\mathbf{u}_b^{(r)} \tag{46}$$

$$= -\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{A}^{(s)}\mathbf{w} \tag{47}$$

where $\mathbf{A}^{(s)}$ is a diagonal matrix storing the multiplicity of the edges that intersect substructure $\Omega^{(s)}$. It follows that $\mathbf{A}^{(s)}$ can be viewed as a *scaling* matrix.

The interface forces $\Delta\mathbf{f}_b^{(s)}$ associated with the displacement corrections $\Delta\mathbf{u}_b^{(s)}$ given above can be computed as in equation (31), and the Lagrange multiplier correction is constructed by generalizing equations (40) and (41)

$$\mathbf{z} = \operatorname{diag}\left(\frac{1}{m_i}\right) \sum_{r=1}^{N_s} \mathbf{b}^{(r)}\Delta\mathbf{f}_b^{(r)} \tag{48}$$

Assuming the interface equilibrium condition (35) and using equations (48), (35) and (27), the substructure interface forces associated with the Lagrange multiplier correction given above can be written as

$$\begin{aligned}\mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{z} &= \operatorname{diag}\left(\frac{1}{m_i}\right) \sum_{r=1}^{N_s} \mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{(r)}\Delta\mathbf{f}_b^{(r)} \\ &= \operatorname{diag}\left(\frac{1}{m_i}\right) \mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{(s)}\Delta\mathbf{f}_b^{(s)} + \operatorname{diag}\left(\frac{1}{m_i}\right) \sum_{\substack{r=1 \\ r \neq s}}^{N_s} \mathbf{b}^{(s)^{\mathrm{T}}}\mathbf{b}^{(r)}\Delta\mathbf{f}_b^{(r)} \\ &= \operatorname{diag}\left(\frac{1}{m_i}\right) (\operatorname{diag}(m_i) - \mathbf{I})\,\Delta\mathbf{f}_b^{(s)} + \operatorname{diag}\left(\frac{1}{m_i}\right) \Delta\mathbf{f}_b^{(s)} = \Delta\mathbf{f}_b^{(s)}\end{aligned} \tag{49}$$

if equation (35) which establishes the mechanical consistency of the proposed Lagrange multiplier correction $\mathbf{z}$.

In summary, mechanically consistent versions of the Dirichlet and lumped preconditioners can be constructed as follows

$$\begin{aligned}\overline{\mathbf{F}}_I^{D^{-1}} &= \sum_{s=1}^{s=N_s} \mathbf{A}^{(s)}\mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}}\mathbf{A}^{(s)} \\ \overline{\mathbf{F}}_I^{L^{-1}} &= \sum_{s=1}^{s=N_s} \mathbf{A}^{(s)}\mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}}\mathbf{A}^{(s)}\end{aligned} \tag{50}$$

These preconditioners have almost the same computational complexity as the basic Dirichlet and lumped preconditioners because the scaling matrices $\mathbf{A}^{(s)}$ are diagonal.

Remark:

(1) Symmetry arguments can be invoked to explain why the same scaling matrix should be used to scale **w** and **z** when defining the scaled preconditioners (50). In particular, the preconditioner should remain symmetric for usage in a Conjugate Gradient algorithm. Nevertheless, mechanical arguments must be introduced in order to choose the scaling coefficients.
(2) Scaling matrices based on the multiplicity of the edges have been previously introduced in the domain decomposition literature (for example, see References 13, 14, 18, 21, 22 and 36). However, the usage of such matrices in the design of substructure-by-substructure preconditioners has always been justified by a "partitioning of unity" argument. To the best of our knowledge, this work presents for the first time the mechanical justification of these matrices and the mechanical interpretation of their effect on substructure-by-substructure preconditioning. It also demonstrates that the redundant constraints at the crosspoints not only do not harm the solution method, but indeed are necessary to construct mechanically consistent and therefore efficient preconditioners, a point that has apparently been misunderstood in.[30]

4. ADVERSE EFFECTS OF STRUCTURAL HETEROGENEITIES ON CONVERGENCE RATE

Various heterogeneities can be encountered in a substructure based finite element structural model. We note the following cases

- Different substructures may have different material properties.
- Different substructures may have different mesh sizes h.
- Some substructure interfaces may coincide with structural intersections. Furthermore, the substructures on the different sides of an interface may have different orientations, in which case the stiffnesses associated with the interface d.o.f. can be highly discontinuous.

For substructure problems characterized by these and other heterogeneities, the multiplicity based averaging procedure discussed in the previous section is not mechanically sound, and leads to less than optimal Dirichlet and lumped preconditioners. This is demonstrated by the following four examples, each of which is representative of a class of heterogeneous structural mechanics problems.

Problem A: is a plane stress problem defined over a square domain clamped on one side (Figure 6). The domain includes four square substructures, each discretized by 20×20 four-noded finite elements. The two clamped substructures are made of steel, whereas the two remaining ones are made of a much softer material with a Young modulus 4098 times smaller than that of steel.

Problem B: is another plane stress problem defined over a square domain made of steel and clamped on one side. Here, the structural heterogeneity is induced by some soft inclusions that have a Young modulus that is 100 times smaller than that of the steel matrix (Figure 7). The domain is discretized by 64×64 four-noded elements, and is decomposed into 2×2, then 4×4, then 8×8 regular substructures.

Problem C: considers the idealization of a wing-box structure (Figure 8). The upper and lower surfaces, defined in this example as two separate substructures, represent the upper and lower skins of a given wing. They are clamped at one end, free at the other. The two vertical substructures

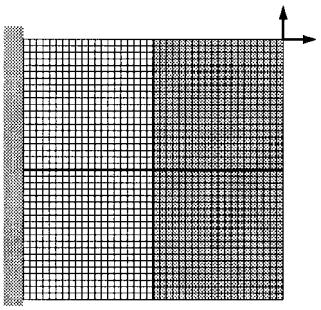

Figure 6. Problem A:—a heterogeneous plane stress problem with homogeneous substructure

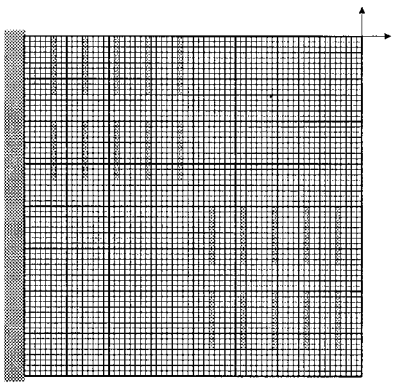

Figure 7. Problem B—a structure with soft inclusions

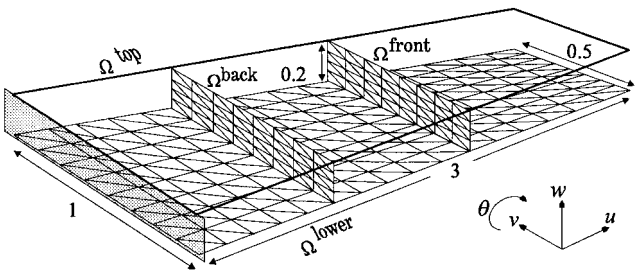

Figure 8. Problem C—an idealized wing-box structure

represent two of its stiffeners. The upper and lower skins are subjected to an aerodynamic pressure load. All four substructures have the same thickness and material properties and are discretized by three-noded triangular DKT[37] based shell elements with six d.o.f. per node.[38] In this example, the problem heterogeneities are not induced by different substructure material properties, but by different substructure geometries, orientations, and mesh resolutions.

Problem D: corresponds to the stress analysis of a composite wing with the spars, ribs, skin, hinges, and control surfaces. An initial detailed finite element structural model is constructed (Figure 9) using 308 bar elements, 304 beam elements, 806 triangular membrane elements, 338 triangular

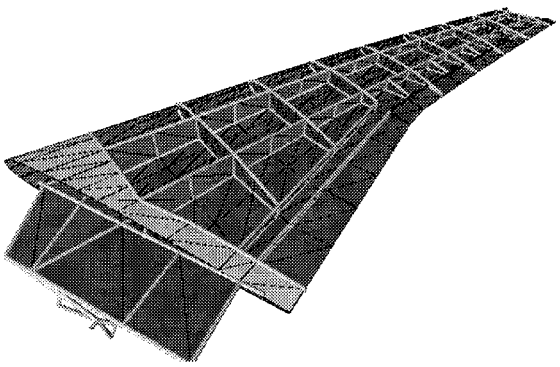

Figure 9. Problem D—a detailed finite element wind model

shell elements, 456 nodes, and 2556 active d.o.f. It is refined a first time to contain 11 838 d.o.f. and a second time to contain 49 950 d.o.f. The initial finite element model is decomposed into 8 substructures, and both refined versions into 32 substructures. Hence, this problem contains all of material, discretization, and geometrical heterogeneities.

Note that in problems A and C, the substructure interfaces coincide with the structural interfaces (interfaces between two materials in Problem A, interfaces between the skin and a stiffener in Problem C). In problems B and D, they do not in general.

We also consider a variant of problem A where all four substructures are made of steel, and a variant of problem B where the soft inclusions are replaced by steel elements. We refer to both variants as the homogeneous counterparts of problems A and B. For each problem, we conduct two linear stress analyses using as an equation solver the FETI method equipped with the mechanically consistent Dirichlet and lumped preconditioners (50), and with the corner Lagrange multipliers for problems C and D. For problem D, we consider only the mechanically consistent Dirichlet preconditioner. For all cases, we set the convergence criterion to

$$\|\mathbf{Ku} - \mathbf{f}\|_2 \leqslant 10^{-6}\|\mathbf{f}\|_2 \tag{51}$$

We report in Table II the number of iterations required for convergence.

The performance results reported for problem B and its variant illustrate the well-established numerical scalability of the FETI method equipped with the mechanically consistent Dirichlet and lumped preconditioners for homogeneous problems. However, they also illustrate the performance degradation that this method can exhibit when applied to structural problems with material heterogeneities. Problems C and D do not have homogeneous counterparts. Nevertheless, the numbers of iterations reported for the solution of problem C are relatively high, and the performance results reported for problem D highlight a loss of numerical scalability with respect to both the mesh and subdomain sizes, which exemplifies the adverse effects of heterogeneities on the convergence of the basic FETI method. One can reasonably argue that the main reason for this behaviour of the FETI method is that the multiplicity based averaging procedure is not mechanically sound for heterogeneous substructure problems. Next, we address this specific issue.

Table II. Effect of various heterogeneities on the performance of FETI

	Homogeneous		Heterogeneous	
	Dirichlet	lumped	Dirichlet	lumped
Problem A	10	21	68	82
Problem B (2×2)	10	25	20	37
(4×4)	15	29	79	92
(8×8)	16	25	129	141
Problem C	—	—	122	128
Problem D (2736 d.o.f., $N_s = 8$)	—	—	211	—
(11838 d.o.f., $N_s = 32$)	—	—	266	—
(49950 d.o.f., $N_s = 32$)	—	—	491	—

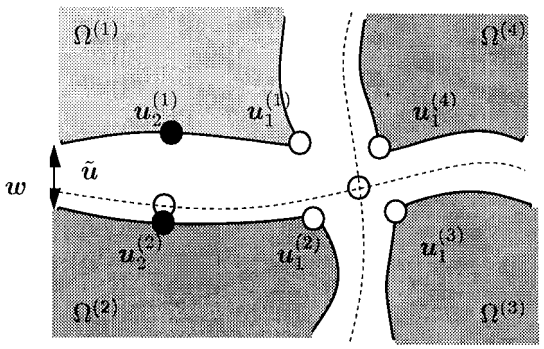

Figure 10. A four-substructure heterogeneous problem

5. EXTENSION OF MECHANICALLY CONSISTENT PRECONDITIONERS TO HETEROGENEOUS PROBLEMS

5.1. Generalization of the averaging procedure

Consider first a variation of the four-substructure problem depicted in Figure 4, where substructure $\Omega^{(1)}$ is assumed to be made of a softer material than the other substructures. For such a heterogeneous problem, common sense suggests that the substructure displacement corrections should be more important for $\Omega^{(1)}$, and that the compatible displacement field should be closer to the stiffer substructures (Figure 10). For this purpose, the compatible displacement field can be constructed as

$$\begin{aligned}\tilde{\mathbf{u}}_1^{(1)} &= \tilde{\mathbf{u}}_1^{(2)} = \tilde{\mathbf{u}}_1^{(3)} = \tilde{\mathbf{u}}_1^{(4)} \\ &= \beta_1^{(1)}\mathbf{u}_1^{(1)} + \beta_1^{(2)}\mathbf{u}_1^{(2)} + \beta_1^{(3)}\mathbf{u}_1^{(3)} + \beta_1^{(4)}\mathbf{u}_1^{(4)} \qquad (52)\\ \tilde{\mathbf{u}}_2^{(1)} &= \tilde{\mathbf{u}}_2^{(2)} \\ &= \beta_2^{(1)}\mathbf{u}_2^{(1)} + \beta_2^{(2)}\mathbf{u}_2^{(2)} \qquad (53)\end{aligned}$$

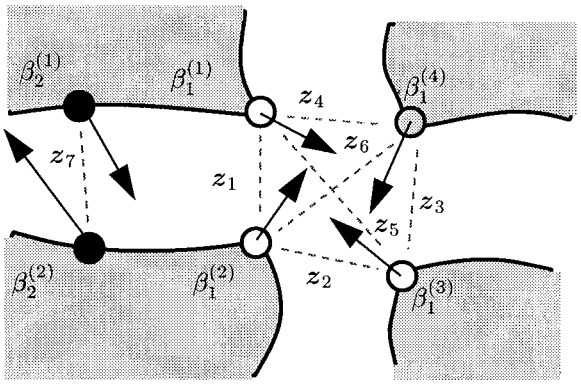

Figure 11. Lagrange multiplier corrections on heterogeneous interfaces

where the coefficients $\beta_k^{(s)}$ are smoothing (or weighting) coefficients that are defined for each interface boundary d.o.f. k of substructure $\Omega^{(s)}$. Furthermore, these coefficients should be constrained by

$$\begin{aligned} \beta_1^{(1)} + \beta_1^{(2)} + \beta_1^{(3)} + \beta_1^{(4)} &= 1 \\ \beta_2^{(1)} + \beta_2^{(2)} &= 1 \end{aligned} \tag{54}$$

so that the displacement corrections vanish when the solution before smoothing is already compatible. It follows that, for this problem, the mechanically consistent substructure displacement corrections can be written as

$$\begin{aligned} \Delta\mathbf{u}_1^{(1)} &= \tilde{\mathbf{u}}_1^{(1)} - \mathbf{u}_1^{(1)} \\ &= \beta_1^{(2)} \left(\mathbf{u}_1^{(2)} - \mathbf{u}_1^{(1)}\right) + \beta_1^{(3)} \left(\mathbf{u}_1^{(3)} - \mathbf{u}_1^{(1)}\right) + \beta_1^{(4)} \left(\mathbf{u}_1^{(4)} - \mathbf{u}_1^{(1)}\right) \\ &\vdots \\ \Delta\mathbf{u}_2^{(1)} &= \beta_2^{(2)} \left(\mathbf{u}_2^{(2)} - \mathbf{u}_2^{(1)}\right) \\ \Delta\mathbf{u}_2^{(2)} &= \beta_2^{(1)} \left(\mathbf{u}_2^{(1)} - \mathbf{u}_2^{(2)}\right) \end{aligned}$$

If the interface forces $\Delta\mathbf{f}_b^{(s)}$ associated with the above interface displacement corrections are computed as in equation (31), common sense also suggests that the Lagrange multiplier corrections should computed using the same weighting coefficients $\beta_k^{(s)}$ (Figure 11):

$$\begin{aligned} \mathbf{z}_1 &= -\beta_1^{(2)}\Delta\mathbf{f}_1^{(1)} + \beta_1^{(1)}\Delta\mathbf{f}_1^{(2)}, \quad \mathbf{z}_2 = -\beta_1^{(3)}\Delta\mathbf{f}_1^{(2)} + \beta_1^{(2)}\Delta\mathbf{f}_1^{(3)} \\ \mathbf{z}_3 &= -\beta_1^{(4)}\Delta\mathbf{f}_1^{(3)} + \beta_1^{(3)}\Delta\mathbf{f}_1^{(4)}, \quad \mathbf{z}_4 = -\beta_1^{(4)}\Delta\mathbf{f}_1^{(1)} + \beta_1^{(1)}\Delta\mathbf{f}_1^{(4)} \\ \mathbf{z}_5 &= -\beta_1^{(3)}\Delta\mathbf{f}_1^{(1)} + \beta_1^{(1)}\Delta\mathbf{f}_1^{(3)}, \quad \mathbf{z}_6 = -\beta_1^{(4)}\Delta\mathbf{f}_1^{(2)} + \beta_1^{(2)}\Delta\mathbf{f}_1^{(4)} \\ \mathbf{z}_7 &= -\beta_2^{(2)}\Delta\mathbf{f}_1^{(1)} + \beta_2^{(1)}\Delta\mathbf{f}_1^{(2)} \end{aligned} \tag{55}$$

The above Lagrange multiplier corrections satisfy the mechanical consistency condition (34). Indeed, assuming that the interface forces $\Delta\mathbf{F}_b^{(s)}$ are in equilibrium (equation (43)) and taking into

account equation (54), we can write for the first subdomain,

$$\begin{aligned}\mathbf{b}_1^{\mathrm{T}}\mathbf{z} &= \begin{bmatrix} -\mathbf{z}_1 - \mathbf{z}_4 - \mathbf{z}_5 \\ -\mathbf{z}_7 \end{bmatrix} \\ &= \begin{bmatrix} \left(\beta_1^{(2)} + \beta_1^{(4)} + \beta_1^{(3)}\right)\Delta\mathbf{f}_1^{(1)} - \beta_1^{(1)}\left(\Delta\mathbf{f}_1^{(2)} + \Delta\mathbf{f}_1^{(3)} + \Delta\mathbf{f}_1^{(4)}\right) \\ \beta_2^{(2)}\Delta\mathbf{f}_1^{(1)} - (1-\beta_2^{(2)})\Delta\mathbf{f}_1^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} \Delta\mathbf{f}_1^{(1)} \\ \Delta\mathbf{f}_2^{(1)} \end{bmatrix} \end{aligned} \tag{56}$$

if

$$\sum_{s=1}^{4}\Delta\mathbf{f}_1^{(s)} = \mathbf{0} \quad \text{and} \quad \sum_{s=1}^{2}\Delta\mathbf{f}_2^{(s)} = \mathbf{0} \tag{57}$$

The reader can check that similar expressions hold for the three other substructures.

For the four-substructure problem depicted in Figure 10, Equations (53)–(56) define a generalization of the multiplicity-based averaging procedure to a smoothing procedure. For an arbitrary number and configuration of substructures, this generalization goes as follows.

Using the notation introduced in Section 2.3, the general form of a compatible interface solution on an edge Γ_I^i obtained by a smoothing procedure can be expressed as

$$\tilde{\mathbf{u}}_b^{(s),i} = \sum_{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\}} \boldsymbol{\beta}^{(r),i}\mathbf{u}_b^{(r),i} \tag{58}$$

where $\boldsymbol{\beta}^{(r),i}$ is the diagonal matrix of the smoothing coefficients associated with the d.o.f. of $\Omega^{(r)}$ and belonging to Γ_I^i. These coefficients satisfy

$$\sum_{r:\Gamma_I^i \subset \Gamma_I^{(r)}} \boldsymbol{\beta}^{(r),i} = \mathbf{I} \tag{59}$$

The mechanically consistent substructure displacement corrections are then given by

$$\begin{aligned}\Delta\mathbf{u}_b^{(s),i} &= \tilde{\mathbf{u}}_b^{(s),i} - \mathbf{u}_b^{(s),i} \\ &= \sum_{\substack{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\} \\ r \neq s}} \boldsymbol{\beta}^{(r),i}(\mathbf{u}_b^{(r),i} - \mathbf{u}_b^{(s),i}) \end{aligned} \tag{60}$$

Note that the above displacement correction of substructure $\Omega^{(s)}$ uses the smoothing coefficients of the interface d.o.f. belonging to the neighbouring substructures. Equation (60) can also be written in global form as

$$\Delta\mathbf{u}_b^{(s)} = -\mathbf{b}^{(s)^{\mathrm{T}}}\boldsymbol{\beta}^{(s)}\mathbf{w}^k \tag{61}$$

where $\boldsymbol{\beta}^{(s)}$ is the diagonal matrix of the smoothing coefficients associated with the interface d.o.f. belonging to the neighbours of $\Omega^{(s)}$.

Equations (58)–(60) generalize the averaging scheme given in equations (44)–(47) to a smoothing scheme for an arbitrary number and configuration of substructures. Indeed, the multiplicity

based averaging equations (44) and (47) can be derived from the smoothing relations (58) and (60) by setting $\boldsymbol{\beta}^{(r),i} = \mathrm{diag}(1/m_i)$.

For the general case, a mechanically consistent correction of the Lagrange multiplier field can be computed for each edge Γ_I^i connecting two substructures $\Omega^{(s)}$ and $\Omega^{(r)}$ as follows:

$$\mathbf{z}^{sr,i} = \boldsymbol{\beta}^{(r),i} \Delta \mathbf{f}_b^{(s),i} - \boldsymbol{\beta}^{(s),i} \Delta \mathbf{f}_b^{(r),i} \tag{62}$$

Indeed, recalling equation (59) and noting that at the edge level, the equilibrium assumption (35) can be written as

$$\Delta \mathbf{f}_b^{(s),i} + \sum_{\substack{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\} \\ r \neq s}} \Delta \mathbf{f}_b^{(r),i} = 0 \tag{63}$$

we have

$$\begin{aligned} \sum_{\substack{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\} \\ r \neq s}} \mathbf{z}^{sr,i} &= \sum_{\substack{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\} \\ r \neq s}} \left(\boldsymbol{\beta}^{(r),i} \Delta \mathbf{f}_b^{(s),i} - \boldsymbol{\beta}^{(s),i} \Delta \mathbf{f}_b^{(r),i} \right) \\ &= \Delta \mathbf{f}_b^{(s),i} \end{aligned} \tag{64}$$

The global form expression of the Lagrange multiplier correction (62) is given by

$$\mathbf{z} = \sum_{s=1}^{N_s} \boldsymbol{\beta}^{(s)} \mathbf{b}^{(s)} \Delta \mathbf{f}_b^{(s)} \tag{65}$$

where, as in Equation (61), $\boldsymbol{\beta}^{(s)}$ is the diagonal matrix of the smoothing coefficients defined on the neighbouring substructures. Equation (65) above is a generalization of the averaging formula (48).

From the definition of the smoothed substructure displacement fields (61), the computation of the interface forces (31) and the definition of the Lagrange multiplier corrections (65), if follows that an extension of the mechanically consistent Dirichlet and lumped preconditioners to heterogeneous problems can be written as

$$\begin{aligned} \overline{\mathbf{F}}_{I,sm}^{D^{-1}} &= \sum_{s=1}^{s=N_s} \boldsymbol{\beta}^{(s)} \mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}} \boldsymbol{\beta}^{(s)} \\ \overline{\mathbf{F}}_{I,sm}^{L^{-1}} &= \sum_{s=1}^{s=N_s} \boldsymbol{\beta}^{(s)} \mathbf{B}^{(s)} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \mathbf{B}^{(s)^{\mathrm{T}}} \boldsymbol{\beta}^{(s)} \end{aligned} \tag{66}$$

The above two preconditioners, labelled here *smoothed preconditioners*, are similar to the preconditioners (50) that embed multiplicity-based scaling factors; however, instead of scaling the residual and the multiplier corrections by the interface multiplicity, they apply to them a smoothing procedure where the underlying coefficients $\boldsymbol{\beta}^{(s)}$ can take different values in each substructure. Next, we propose an explicit algorithm for determining the smoothing coefficients $\boldsymbol{\beta}^{(s)}$.

5.2. A superlumped smoothing procedure

The smoothing coefficients introduced in the previous section can be interpreted as generalized displacement variables that permit the construction of compatible substructure displacement fields $\tilde{\mathbf{u}}^{(s)}$. The smoothed substructure displacement fields parameterized by $\boldsymbol{\beta}^{(r)}$ can thus be considered as kinematically admissible functions for minimizing the energy of the global structural system.

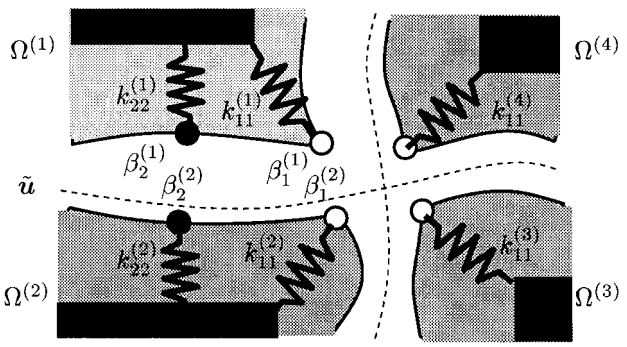

Figure 12. Schematic interpretation of the superlumped smoothing procedure

However, determining the coefficients $\boldsymbol{\beta}^{(r)}$ that minimize the energy of the system is as computationally expensive as solving the interface problem. Hence, some approximation must be injected in this reasoning in order to determine the smoothing coefficients.

In a previous attempt[33] to address this problem, we have assumed that the coefficients $\boldsymbol{\beta}^{(r)}$ are constant on each interface edge, because this hypothesis reduces significantly the number of unknown smoothing coefficients. In that case, the energy minimization principle leads to a coarse problem whose solution determines the smoothing coefficients $\boldsymbol{\beta}^{(r)}$.[34] Unfortunately, such an approach was found out to be still computationally expensive.

In this paper, we adopt a simpler approach that was first proposed even though not fully developed in Reference 14. We decouple all the interface d.o.f. by assuming that each one of them is connected to a stiffness-free home substructure by a linear spring (Figure 12). In other words, we assume that the stiffness of each substructure is lumped on its interface, and that the lumped stiffness matrix is diagonal and equal to $\mathrm{diag}(\mathbf{K}_{bb})$. Hence the name *superlumped* smoothing procedure. The advantage of this approach is that the smoothing coefficients can be computed independently from each other as the ratio between the spring stiffness of an interface d.o.f. in a given substructure, and the sum of all the spring stiffnesses connected to that same d.o.f. in all neighbouring substructures. For example, for the problem graphically depicted in Figure 12, the smoothing coefficients are given by

$$\begin{aligned}
\beta_1^{(1)} &= \frac{k_{11}^{(1)}}{k_{11}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)} + k_{11}^{(4)}} \\
\beta_1^{(2)} &= \frac{k_{11}^{(2)}}{k_{11}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)} + k_{11}^{(4)}} \\
&\vdots \\
\beta_2^{(1)} &= \frac{k_{22}^{(1)}}{k_{22}^{(1)} + k_{22}^{(2)}} \\
\beta_2^{(2)} &= \frac{k_{22}^{(2)}}{k_{22}^{(1)} + k_{22}^{(2)}}
\end{aligned} \tag{67}$$

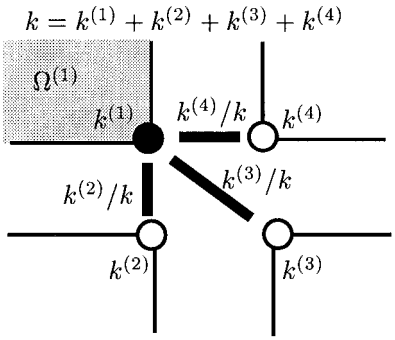

Figure 13. Implementation of super-lumped smoothing

For an arbitrary number and configuration of substructures, the coefficients $\boldsymbol{\beta}^{(s),i}$ associated with the superlumped smoothing procedure and an edge Γ_I^i are given by

$$\beta^{(s),i} = \mathrm{diag}(\mathbf{K}_{bb}^{(s),i}) \left\{ \sum_{r:\Gamma_I^i \subset \{\Gamma_I \cap \Gamma_I^{(r)}\}} \mathrm{diag}(\mathbf{K}_{bb}^{(r),j}) \right\}^{-1} \tag{68}$$

The superlumped smoothing procedure described above is easy to implement. For example, in order to evaluate $\beta^{(r)}$ at a crosspoint, one proceeds as illustrated in Figure 13.

(i) For every d.o.f. on the interface, the assembled stiffness is evaluated by gathering the diagonal stiffnesses from all neighbouring d.o.f.
(ii) The smoothing coefficient in substructure $\Omega^{(s)}$ associated with a Lagrange multiplier connecting two interface d.o.f. at the intersection of $\Omega^{(s)}$ and $\Omega^{(r)}$ is given by the ratio between the diagonal stiffness in $\Omega^{(r)}$ and the assembled stiffness. The computation of these coefficients needs be performed only once and requires only the communication of the diagonal stiffnesses between neighbouring d.o.f.

It is also computationally efficient as it increases the computational complexity of the basic Dirichlet and lumped preconditioners by a negligible amount (diagonal scaling).

Remarks:

(1) The superlumped smoothing procedure described here for extending the preconditioners of the FETI method to heterogeneous problems can be adapted in a straightforward manner to the preconditioners of Schur complement domain decomposition methods, also known as primal domain decomposition methods. A similar scaling was proposed for the primal domain decomposition method in Reference 36, but its application was restricted to Poisson problems with homogeneous subdomains (globally heterogeneous, locally homogeneous). In that case, the superlumping scaling is equivalent to the scaling proposed in Reference 36 and takes care of inter-subdomain coefficient jumps as well as discretization discrepancies in the vicinity of the interface.
(2) For the particular case where the problem is globally heterogeneous but locally homogeneous (i.e. Problem A, Figure 6), and when the discretization in the vicinity of the interface is identical in every substructure, the superlumped smoothing procedure described here

Table III. Performance of the superlumped smoothed preconditioners

	Dirichlet	lumped	Dirichlet	lumped
	Multiplicity scaling (50)		Superlumped scaling (66)	
Problem A	68	82	11	25
Problem B (2×2)	20	37	17	36
(4×4)	79	92	26	47
(8×8)	129	141	25	44
Problem C	122	128	25	50
Problem D (2736 d.o.f., $N_s = 8$)	211	—	96	—
(11838 d.o.f., $N_s = 32$)	266	—	130	—
(49950 d.o.f., $N_s = 32$)	491	—	128	—

generates the same scaling coefficients as those proposed in Reference 13 for primal domain decomposition methods.

6. APPLICATIONS

In order to assess the performance for heterogeneous problems of the smoothed Dirichlet and lumped preconditioners using the superlumped smoothing procedure, we consider again the solution by FETI of the four example problems introduced in Section 3.1 and illustrated in Figures 6–9. We set the convergence criterion as before, and report in Table III the performance results obtained for both the interface multiplicity-based scaled preconditioners, and the superlumping smoothed preconditioners.

All reported performance results not only demonstrate that the proposed extensions to heterogeneous problems of the Dirichlet and lumped preconditioners are effective, but also indicate that they restore the well established numerical scalability of the FETI method.

7. INTERFACE POSITIONING IN HETEROGENEOUS STRUCTURES

As shown in the examples of the previous section, the superlumped smoothing procedure appears to be very efficient in treating interface heterogeneities. When having to solve a heterogeneous problem, the question thus arises whether or not the subdomain interface should be made to coincide with the heterogeneity boundaries (as in problem A and C, Figures 6 and 8).

To answer this question, let us once more analyse a square structure made of steel and decomposed into 3×3 square subdomains. Every subdomain is discretized by 6×6 plane stress finite elements (Figure 14). First, we consider a configuration I where two subdomains along the right end ($\Omega^{(8)}$ and $\Omega^{(9)}$) have a Young's modulus 4094 times smaller than the rest of the structure. In this configuration, the subdomain interface coincides with the heterogeneity boundary. We then consider a slightly different configuration where all the elements along the interface surrounding subdomains $\Omega^{(8)}$ and $\Omega^{(9)}$ are made of the soft material (Figure 14, configuration II). This configuration corresponds to the case when the interface is positioned within the soft portion of the

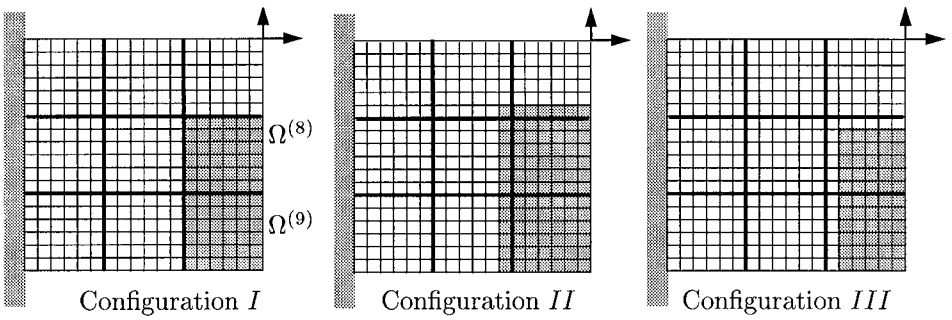

Figure 14. Effect of interface position with respect to heterogeneity boundary

Table IV. Effect of interface positioning (Figure 14)

	Dirichlet	lumped	Dirichlet	lumped
	Multiplicity scaling (50)		Superlumped smoothing (66)	
Configuration I	45	51	18	23
Configuration II	18	23	18	23
Configuration III	17	22	17	22

structure. In configuration III, all elements along the interface surrounding $\Omega^{(8)}$ and $\Omega^{(9)}$ are made of steel, simulating the case when the interface is positioned in the stiff portion of the structure.

The number of FETI iterations for the three configurations of Figure 14 are reported in Table IV where we compare the number of iterations when applying preconditioners equipped with multiplicity scaling or with superlumped smoothing. We observe that using the right preconditioners for heterogeneous problems, the number of iteration is virtually not affected by the position of the interface.

Note that when the subdomain interface lies beyond the material boundary (configurations II and III), the superlumped smoothing is equivalent to the multiplicity scaling. Although heterogeneity is present inside the subdomains, it barely affects the interface problem and the results in Table IV highlight the efficiency of the multiplicity scaling even in this case. Therefore, at least for the plane stress problem described here, heterogeneity affects the interface problem only when located in the direct vicinity of the interface.

From our example, we see that the position of the decomposition interface with respect to heterogeneities does not significantly affect the convergence of the FETI iterations as long as the superlumped preconditioners are used. This suggests that in general, the choice of substructure interfaces should be made regardless of heterogeneity boundaries and according to optimum subdomain aspect ratios, and load balancing.

8. CONCLUSION

In this paper, we have considered the solution of highly heterogeneous structural problems by FETI-like domain decomposition methods. We have revisited the Dirichlet and lumped preconditioners that are now popular in substructure based iterative methods, and offered a mechanical

justification for the interface multiplicity based scaling procedure that is usually used with these preconditioners. We have also highlighted the adverse effects of various structural heterogeneities on the performance of these preconditioners and that scaling procedure. Using the concept of smoothing, we have developed a virtually no-cost extension to heterogeneous problems of the Dirichlet and lumped preconditioners that we have applied to the solution by FETI of various intricate structural problems. We have shown that in all cases, the new preconditioners are effective at reducing the iteration count. The performance results we have obtained for problems that are representative of different classes of heterogeneous structural mechanics problems indicate that the preconditioners proposed in this paper restore the numerical scalability of the FETI method that is well established for homogeneous structural problems.

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